Exponential Synchronization of Impulsive Complex Networks with Output Coupling

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Abstract: This paper proposes a new impulsive complex delayed dynamical network model with output coupling, which is totally different from some existing network models. Then, by employing impulsive delay differential inequalities, some sufficient conditions are obtained to guarantee the global exponential state synchronization and output synchronization of the impulsive complex delayed dynamical network. Finally, two numerical examples are given to demonstrate the effectiveness of the obtained results.

Keywords: Impulsive complex networks, output coupling, state synchronization, output synchronization, impulsive delay differential inequalities.

1 Introduction

In the past ten years, there have been many researchers studying the topology and dynamical behavior of complex networks. The main reason is that many practical systems can be characterized by various models of complex networks. Some well-known examples include food webs, communication networks, social networks, power grids, cellular networks, world wide web, metabolic systems, disease transmission networks, etc.^[1, 2] In particular, the synchronization problem has received much of the focus in recent years. A wide variety of synchronization criteria have been presented for various complex networks^[3–25]. Wang et al.^[4] considered a generalized complex network possessing general topology. Some criteria on local and global exponential synchronization were derived by constructing suitable Lyapunov functionals. Li and Lai^[19] studied the adaptive-impulsive synchronization of uncertain complex networks. Based on the stability analysis of impulsive system, several synchronization criteria for local and global adaptive-impulsive synchronization were established. But, it should be noticed that the complex networks with state coupling were considered in these articles.

Jiang et al.^[26] first introduced a complex dynamical network model with output coupling without time delays. Practically, the time delays always exist in complex networks due to the finite speeds of transmission and/or the traffic congestion, and most of the delays are notable. Moreover, in practical evolutionary processes of the networks, absolute constant delay may be scarce and delays are frequently varying with time. Unfortunately, only a few authors have considered the synchronization for complex networks with time-varying delay and output coupling^[27]. Therefore, it is essential to further study the synchroniza-

tion of complex dynamical networks with time-varying delay and output coupling.

Recently, many researchers have studied the stability of impulsive systems^[28–36]. Yang and Xu^[29] investigated a class of impulsive control systems with time-varying delays. They analyzed the global exponential stability of the impulsive delay systems and estimated the exponential convergence rate. Yang and Cao^[35] considered the exponential synchronization of complex dynamical network with a coupling delay and impulses. Several criteria for exponential synchronization were derived by the geometrical decomposition of network states and linear matrix inequality method. To the best of our knowledge, the state synchronization of impulsive complex delayed networks with output coupling has not yet been established.

State synchronization is an important direction in the complex networks. However, the node state in complex networks is difficult to be observed or measured, even the node state cannot be observed or measured at all. Moreover, in many circumstances, only part states are needed to make the synchronization to come true^[27]. Therefore, it is very important and interesting to study the output synchronization of complex networks.

Motivated by the above discussions, a new impulsive complex delayed dynamical network model with output coupling is proposed. The objective of this paper is to study the state synchronization and output synchronization of the proposed network model. Some sufficient conditions ensuring the state synchronization and output synchronization are obtained by using impulsive delay differential inequalities.

The rest of this paper is organized as follows. In Section 2, our mathematical model of the complex network is presented and some preliminaries are given. Several state synchronization and output synchronization criteria are established in Section 3. In Section 4, numerical examples are given to illustrate the effectiveness of the proposed results. Concluding remarks are collected in Section 5.

Manuscript received June 8, 2012; revised October 22, 2012

This work was supported by Key Project of Chinese Education Ministry (No. 212138), Natural Science Foundation of Chongqing (No. CQ CSTC 2011BB0117), Foundation of Science and Technology Project of Chongqing Education Commission (No. KJ120630), Innovation Foundation of BUAA for PhD Graduates (No. YWF-12-RBYJ-005).

2 Network model and preliminaries

Let $\mathbf{R}^+ = [0, +\infty)$, $\mathbf{N} = \{1, 2, 3, \dots\}$, $\mathbf{R}^n$ be the n -dimensional Euclidean space, and $\mathbf{R}^{n \times m}$ be the space of $n \times m$ real matrices. $P \in \mathbf{R}^{n \times n} > 0$ ($P \in \mathbf{R}^{n \times n} < 0$) means that matrix P is symmetric and positive (negative) definite. I_n denotes the $n \times n$ real identity matrix. $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote the minimum and the maximum eigenvalues of the corresponding matrix, respectively. B^T denotes the transpose of a square matrix B . $\|\cdot\|$ denotes the Euclidean norm for vector or the spectral norm of matrix. $C([-\tau, 0], \mathbf{R}^n)$ is a Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbf{R}^n$ with the norm $\|\phi\|_\tau = \sup_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$. $PC[I, \Omega] = \{\phi : [-\tau, +\infty) \rightarrow \mathbf{R}, \phi(t) \text{ is continuous everywhere except for the points } t_k (k \in \mathbf{N}) \text{ at which } \phi(t_k^+) = \phi(t_k) \text{ and } \phi(t_k^-) \text{ exists}\}$.

Throughout this paper, we consider an impulsive complex delayed dynamical network consisting of N identical nodes with diffusive and output coupling. The mathematical model of the coupled network can be described as

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + \frac{a}{k_i^{\beta_\omega}} \sum_{j=1}^N L_{ij} \Gamma y_j(t - \tau(t)), \\ \quad t \neq t_k, \quad t \geq 0 \\ y_i(t) = C x_i(t) \\ \Delta x_i = x_i(t_k^+) - x_i(t_k^-) = B_{ik} x_i(t_k^-), \\ \quad t = t_k, \quad k \in \mathbf{N} \end{cases} \quad (1)$$

where $i = 1, 2, \dots, N$, and $\tau(t)$ is the time-varying delay with $0 \leq \tau(t) \leq \tau$.

The function $f(\cdot)$, describing the local dynamics of the nodes, is continuously differentiable and capable of producing various rich dynamical behaviors. $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^T \in \mathbf{R}^n$ is the state variable of node i , $y_i(t) \in \mathbf{R}^q$ ($1 \leq q \leq n$) is the output of node i , C and B_{ik} are known matrices with appropriate dimensions, $\Gamma \in \mathbf{R}^{n \times q}$ is inner-coupling matrix which describes the individual coupling between two connected nodes of the network, a is a positive real number, which represents the overall coupling strength, the fixed moments t_k satisfy $0 = t_0 < t_1 < t_2 < \dots, \lim_{k \rightarrow +\infty} t_k = +\infty, k \in \mathbf{N}$, k_i is the degree of node i , β_ω is a tunable weight parameter, the real matrix $L = (L_{ij})_{N \times N}$ is a symmetric matrix with diagonal entries $L_{ii} = -k_i$ and off-diagonal entries $L_{ij} = 1$ if node i and node j are connected by a link, and $L_{ij} = 0$ otherwise.

In this paper, we always assume that complex network (1) is connected, and $x_i(t)$ is right continuous at $t = t_k$, i.e., $x_i(t_k) = x_i(t_k^+)$. Let $x(t) = (x_1^T(t), x_2^T(t), \dots, x_N^T(t))^T$, $y(t) = (y_1^T(t), y_2^T(t), \dots, y_N^T(t))^T$. The initial condition associated with the network (1) is given in the form:

$$\begin{aligned} x(s) &= \Phi(s), \quad s \in [-\tau, 0], \phi_i \in C([-\tau, 0], \mathbf{R}^n) \\ \Phi(s) &= (\phi_1^T(s), \phi_2^T(s), \dots, \phi_N^T(s))^T \\ y(s) &= \text{diag}(C, C, \dots, C) \Phi(s), \quad s \in [-\tau, 0]. \end{aligned}$$

Next, we give several useful definitions and lemmas.

Definition 1. The complex network (1) is said to achieve state synchronization if

$$\lim_{t \rightarrow +\infty} \|x_i(t) - x_j(t)\| = 0, \text{ for all } i, j = 1, 2, \dots, N.$$

Definition 2.^[27] The complex network (1) is said to

achieve output synchronization if

$$\lim_{t \rightarrow +\infty} \|y_i(t) - y_j(t)\| = 0, \text{ for all } i, j = 1, 2, \dots, N.$$

Definition 3. (see [1]) Let $A = (a_{ij})_{m \times n} \in \mathbf{R}^{m \times n}$ and $B = (b_{ij})_{p \times q} \in \mathbf{R}^{p \times q}$. Then the Kronecker product (or tensor product) of A and B is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \in \mathbf{R}^{mp \times nq}.$$

The Kronecker product has the following properties:

- 1) $(A \otimes B)^T = A^T \otimes B^T$
- 2) $(\alpha A) \otimes B = A \otimes (\alpha B)$
- 3) $(A + B) \otimes C = A \otimes C + B \otimes C$
- 4) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

where $\alpha \in \mathbf{R}$, C, D are matrices with suitable dimensions.

Lemma 1. (see [5]) Let $0 \leq \tau_i(t) \leq \tau$, $i = 1, 2, \dots, m$. If there exist real numbers $\sigma > 0$, $\vartheta_1 \geq 0, \vartheta_2 \geq 0, \dots, \vartheta_m \geq 0$ and ϑ such that

$$\begin{cases} D^+ u(t) \leq \vartheta u(t) + \hat{\vartheta}_1 u(t - \tau_1(t)) + \hat{\vartheta}_2 u(t - \tau_2(t)) + \cdots + \\ \quad \hat{\vartheta}_m u(t - \tau_m(t)), \quad t \geq 0 \\ u(t_k) \leq \sigma u(t_k^-), \quad k \in \mathbf{N} \end{cases}$$

$$\begin{cases} D^+ v(t) > \vartheta v(t) + \hat{\vartheta}_1 v(t - \tau_1(t)) + \hat{\vartheta}_2 v(t - \tau_2(t)) + \cdots + \\ \quad \vartheta_m v(t - \tau_m(t)), \quad t \geq 0 \\ v(t_k) = \sigma v(t_k^-), \quad k \in \mathbf{N} \end{cases}$$

then $u(t) \leq v(t)$, for $-\tau \leq t \leq 0$ implies

$$u(t) \leq v(t), \text{ for } t \geq 0$$

where $u(t), v(t) \in PC[-\tau, +\infty), \mathbf{R}]$.

Lemma 2. (see [35]) Suppose $p > q \geq 0$ and $u(t)$ satisfies scalar impulsive differential inequality

$$\begin{cases} D^+ u(t) \leq -pu(t) + q(\sup_{t-\tau \leq s \leq t} u(s)), \quad t \neq t_k, \quad t \geq 0, \\ u(t_k) \leq a_k u(t_k^-), \quad k \in \mathbf{N} \\ u(t) = \phi(t), \quad t \in [-\tau, 0] \end{cases}$$

where $u(t)$ is continuous at $t \neq t_k$, $u(t_k) = u(t_k^+) = \lim_{t \rightarrow t_k^+} u(t)$, and $u(t_k^-) = \lim_{t \rightarrow t_k^-} u(t)$, $\phi \in PC([-\tau, 0], \mathbf{R})$. Then,

$$u(t) \leq \left(\prod_{0 < t_k \leq t} \theta_k \right) e^{-\mu t} (\sup_{-\tau \leq s \leq 0} \phi(s))$$

where $\theta_k = \max\{1, |a_k|\}$ and $\mu > 0$ is a solution of the inequality $\mu - p + qe^{\mu\tau} \leq 0$.

Lemma 3. (see [29]) Let $P \in \mathbf{R}^{n \times n} > 0$ and $P = Q^T Q$. For any $x, \bar{x} \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n \times n}$

- 1) $x^T A^T P A x \leq \|Q A Q^{-1}\|^2 x^T P x$
- 2) $|x^T P \bar{x}| \leq \sqrt{x^T P x} \sqrt{\bar{x}^T P \bar{x}}$.

3 Main results

Lu et al.^[37,38] made the assumption that the function $f(\cdot)$ is in the QUAD function class.

Assumption 1. There exist positive definite diagonal matrix $P = \text{diag}(p_1, p_2, \dots, p_n)$ and a diagonal matrix $\Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ such that $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies the following inequality:

$$(x - y)^T P[f(x) - f(y) - \Delta(x - y)] \leq -\eta(x - y)^T(x - y)$$

for some $\eta > 0$ and all $x, y \in \mathbf{R}^n$.

It can be verified that Assumption 1 holds for many of the benchmark chaotic systems including the Lorenz system, the Chen system, the Lü system, and the unified chaotic system. In this paper, we always assume $f(0) = 0$. If we let $y = 0$ in Assumption 1, then we have

Assumption 2. There exists a positive definite diagonal matrix $P = \text{diag}(p_1, p_2, \dots, p_n)$ and a diagonal matrix $\Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ such that f satisfies the following inequality:

$$x^T P(f(x) - \Delta x) \leq -\eta x^T x$$

for some $\eta > 0$ and all $x \in \mathbf{R}^n$.

For convenient analysis, we let

$$\begin{aligned} G_{ij} &= \frac{L_{ij}}{k_i^{\beta\omega}} \\ Q &= P^{\frac{1}{2}} = \text{diag}(\sqrt{p_1}, \sqrt{p_2}, \dots, \sqrt{p_n}) \\ \hat{P} &= \text{diag}(P, P, \dots, P) \\ \hat{Q} &= \text{diag}(Q, Q, \dots, Q) \\ \hat{\Delta} &= \text{diag}(\Delta, \Delta, \dots, \Delta) \\ B_k &= \text{diag}(B_{1k}, B_{2k}, \dots, B_{Nk}) \\ x(t) &= (x_1^T(t), x_2^T(t), \dots, x_N^T(t))^T. \end{aligned}$$

$G = (G_{ij})_{N \times N}$ is a coupling matrix, accounting for the topology of the complex dynamical network.

Theorem 1. Let Assumption 2 hold.

1) If there exists a positive constant $\beta < 1$ such that

$$\|\hat{Q}(I_{nN} + B_k)\hat{Q}^{-1}\| \leq \beta, \quad k \in \mathbf{N} \quad (2)$$

$$\begin{aligned} &2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} + \\ &a + \frac{2\ln \beta}{\rho} < 0 \end{aligned} \quad (3)$$

where $\rho = \sup\{t_k - t_{k-1}, k \in \mathbf{N}\}$, then the network (1) achieves global exponential state synchronization in the following sense:

$$\|x(t)\| \leq \beta^{-1} \sqrt{\frac{\lambda_M(\hat{P})}{\lambda_m(\hat{P})}} \|\Phi\| e^{\frac{-\lambda t}{2}}, \quad t \geq 0$$

where $\lambda > 0$ is a unique solution of

$$\begin{aligned} &\lambda + 2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a + \frac{2\ln \beta}{\rho} + \\ &e^{\lambda\tau} \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} = 0. \end{aligned} \quad (4)$$

2) If there exists a positive constant $\beta \geq 1$ such that

$$\|\hat{Q}(I_{nN} + B_k)\hat{Q}^{-1}\| \leq \beta, \quad k \in \mathbf{N} \quad (5)$$

$$\begin{aligned} &2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a + \frac{2\ln \beta}{\rho} + \\ &\frac{a\beta^2 \lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} < 0 \end{aligned} \quad (6)$$

where $\rho = \inf\{t_k - t_{k-1}, k \in \mathbf{N}\}$, then the network (1) achieves global exponential state synchronization in the following sense:

$$\|x(t)\| \leq \beta \sqrt{\frac{\lambda_M(\hat{P})}{\lambda_m(\hat{P})}} \|\Phi\| e^{\frac{-\lambda t}{2}}, \quad t \geq 0$$

where $\lambda > 0$ is a unique solution of

$$\begin{aligned} &\lambda + 2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a + \frac{2\ln \beta}{\rho} + \\ &e^{\lambda\tau} \frac{a\beta^2 \lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} = 0. \end{aligned} \quad (7)$$

Proof. First, let $s(\lambda) = \lambda + 2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + e^{\lambda\tau} \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} + a + \frac{2\ln \beta}{\rho}$. From (3), we have $s(0) < 0$. Moreover, it is obvious that $s(+\infty) > 0$ and $s'(\lambda) > 0$. Using the continuity and the monotonicity of $s(\lambda)$, (4) has a unique solution $\lambda > 0$. Similarly, we can easily obtain that (7) also has a unique solution $\lambda > 0$. From (1), we can get

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + a \sum_{j=1}^N G_{ij} \Gamma y_j(t - \tau(t)), \\ \quad t \neq t_k, \quad t \geq 0 \\ y_i(t) = Cx_i(t) \\ \Delta x_i = x_i(t_k^+) - x_i(t_k^-) = B_{ik} x_i(t_k^-), \\ \quad t = t_k, \quad k \in \mathbf{N} \end{cases} \quad (8)$$

where $i = 1, 2, \dots, N$. We can rewrite system (8) in a compact form as

$$\begin{cases} \dot{x}(t) = F(x(t)) + a(G \otimes \Gamma C)x(t - \tau(t)), \\ \quad t \neq t_k, \quad t \geq 0 \\ \Delta x = x(t_k^+) - x(t_k^-) = B_k x(t_k^-), \\ \quad t = t_k, \quad k \in \mathbf{N} \end{cases} \quad (9)$$

where

$$\begin{aligned} \Delta x &= ((\Delta x_1)^T, (\Delta x_2)^T, \dots, (\Delta x_N)^T)^T \\ F(x(t)) &= (f^T(x_1(t)), f^T(x_2(t)), \dots, f^T(x_N(t)))^T. \end{aligned}$$

Let $V_1(t) = x^T(t)\hat{P}x(t)$. By Lemma 3, we have

$$\begin{aligned} \dot{V}_1(t) &= 2x^T(t)\hat{P}\dot{x}(t) = \\ &2x^T(t)\hat{P}F(x(t)) + 2ax^T(t)\hat{P}(G \otimes \Gamma C)x(t - \tau(t)) \leq \\ &2x^T(t)\hat{P}F(x(t)) + 2a\sqrt{x^T(t)\hat{P}x(t)} \times \\ &\sqrt{x^T(t - \tau(t))(G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C)x(t - \tau(t))} \leq \\ &2x^T(t)\hat{P}F(x(t)) + 2a\sqrt{x^T(t)\hat{P}x(t)} \times \\ &\sqrt{\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))\|x(t - \tau(t))\|} \leq \\ &2x^T(t)\hat{P}F(x(t)) + aV_1(t) + \\ &a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))\|x(t - \tau(t))\|^2 \leq \\ &2x^T(t)\hat{P}F(x(t)) + aV_1(t) + \\ &\frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})}V_1(t - \tau(t)), \quad t \neq t_k. \end{aligned}$$

According to Assumption 2, we can obtain

$$\begin{aligned} x^T(t)\hat{P}F(x(t)) &= \\ &\sum_{i=1}^N x_i^T(t)Pf(x_i(t)) \leq \\ &\sum_{i=1}^N [-\eta x_i^T(t)x_i(t) + x_i^T(t)P\Delta x_i(t)] = \\ &x^T(t)(-\eta I_{nN} + \hat{P}\hat{\Delta})x(t). \end{aligned} \quad (10)$$

It follows from (10) that

$$\begin{aligned} \dot{V}_1(t) &\leq 2x^T(t)(-\eta I_{nN} + \hat{P}\hat{\Delta})x(t) + aV_1(t) + \\ &\frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} \times V_1(t - \tau(t)), \quad t \neq t_k. \end{aligned}$$

Let $\nu \in \mathbf{R}^{nN} \neq 0$, $\psi \in \mathbf{R}^{nN}$ and $z = \hat{Q}\nu$. From the definition of matrix measure induced by vector norm^[39],

$$\begin{aligned} \frac{\nu^T(-\eta I_{nN} + \hat{P}\hat{\Delta})\nu}{\nu^T\hat{P}\nu} &= \\ \frac{z^T(-\eta\hat{P}^{-1} + \hat{\Delta})z}{z^Tz} &\leq \\ \sup_{\psi \neq 0} \frac{\psi^T(-\eta\hat{P}^{-1} + \hat{\Delta})\psi}{\psi^T\psi} &= \\ \lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) \end{aligned}$$

namely

$$\frac{\nu^T(-\eta I_{nN} + \hat{P}\hat{\Delta})\nu}{\nu \in \mathbf{R}^{nN} \neq 0} \leq \lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta})\nu^T\hat{P}\nu, \quad (11)$$

If $\nu = 0$, (11) is obvious. Then, we can obtain

$$\begin{aligned} \dot{V}_1(t) &\leq [2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a]V_1(t) + \\ &\frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} \times \\ &V_1(t - \tau(t)), \quad t \neq t_k. \end{aligned} \quad (12)$$

By Lemma 3, we can get

$$\begin{aligned} V_1(t_k) &= x^T(t_k^-)(I_{nN} + B_k)^T\hat{P}(I_{nN} + B_k)x(t_k^-) \leq \\ &\|\hat{Q}(I_{nN} + B_k)\hat{Q}^{-1}\|^2 x^T(t_k^-)\hat{P}x(t_k^-) \leq \\ &\beta^2 V_1(t_k^-), \quad k \in \mathbf{N}. \end{aligned}$$

For any $\varepsilon > 0$, let $\eta(t)$ be a unique solution of impulsive delay system.

$$\begin{cases} \dot{\eta}(t) = [2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a]\eta(t) + \epsilon + \\ \quad \frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} \times \eta(t - \tau(t)), \\ \quad t \neq t_k, \quad t \geq 0 \\ \eta(t_k) = \beta^2 \eta(t_k^-), \quad k \in \mathbf{N} \\ \eta(t) = \lambda_M(\hat{P})\|\Phi(t)\|^2, \quad -\tau \leq t \leq 0. \end{cases}$$

Since $V_1(t) \leq \lambda_M(\hat{P})\|\Phi(t)\|^2 = \eta(t)$ for $-\tau \leq t \leq 0$. According to Lemma 1, we have

$$\eta(t) \geq V_1(t) \geq 0, \quad t \geq 0.$$

By the formula for the variation of parameters^[40], we have

$$\begin{aligned} \eta(t) &= W(t, 0)\eta(0) + \\ &\int_0^t W(t, s) \left[\frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} \times \right. \\ &\quad \left. \eta(s - \tau(s)) + \epsilon \right] ds, \quad t \geq 0 \end{aligned}$$

where $W(t, s)$, $t, s \geq 0$ is the Cauchy matrix of linear system

$$\begin{cases} \dot{\varpi}(t) = [2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a]\varpi(t), \quad t \neq t_k \\ \varpi(t_k) = \beta^2 \varpi(t_k^-), \quad k \in \mathbf{N}. \end{cases} \quad (13)$$

Now, we prove the case 1.

Let $b = -2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) - a - \frac{2\ln\beta}{\rho}$. According to the representation of the Cauchy matrix^[40], we get the following estimation since $0 < \beta < 1$ and $\rho \geq t_k - t_{k-1}$:

$$\begin{aligned} W(t, s) &= e^{[2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a](t-s)} \left(\prod_{s < t_k \leq t} \beta^2 \right) \leq \\ &e^{[2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a](t-s)} \beta^{2(\frac{t-s}{\rho} - 1)} = \\ &e^{(-b - \frac{2\ln\beta}{\rho})(t-s)} \beta^{2(\frac{t-s}{\rho} - 1)} = \\ &\beta^{-2} e^{-b(t-s)}, \quad t \geq s \geq 0. \end{aligned}$$

Let $\gamma = \beta^{-2}\lambda_M(\hat{P})\|\Phi\|_\tau^2$. Accordingly

$$\begin{aligned} \eta(t) &\leq \beta^{-2} e^{-bt} \lambda_M(\hat{P})\|\Phi(0)\|^2 + \\ &\int_0^t e^{-b(t-s)} \left[\frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} \times \right. \\ &\quad \left. \eta(s - \tau(s)) + \frac{\epsilon}{\beta^2} \right] ds \leq \\ &\gamma e^{-bt} + \int_0^t e^{-b(t-s)} \left[\frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} \times \right. \\ &\quad \left. \eta(s - \tau(s)) + \frac{\epsilon}{\beta^2} \right] ds, \quad t \geq 0. \end{aligned} \quad (14)$$

Since $\epsilon, \lambda > 0$, $b - \frac{a\lambda_M((G \otimes \Gamma C)^T\hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} > 0$ and $0 < \beta < 1$,

we have

$$\eta(t) \leq \frac{\lambda_M(\hat{P})\|\Phi(t)\|^2}{\beta^2} < \gamma e^{-\lambda t} + \frac{\epsilon}{[b - \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})}]\beta^2}, \quad -\tau \leq t \leq 0. \quad (15)$$

In the following, we shall prove that

$$\eta(t) < \gamma e^{-\lambda t} + \frac{\epsilon}{[b - \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})}]\beta^2}, \quad t \geq 0. \quad (16)$$

If this is not true, according to the estimation (15) and $\eta(t) \in PC[-\tau, +\infty), \mathbf{R}^+$, then there obviously exists a $t^* > 0$ such that

$$\eta(t^*) \geq \gamma e^{-\lambda t^*} + \frac{\epsilon}{[b - \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})}]\beta^2} \quad (17)$$

$$\eta(t) < \gamma e^{-\lambda t} + \frac{\epsilon}{[b - \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})}]\beta^2}, \quad t < t^*. \quad (18)$$

Let $\chi = \frac{\epsilon}{[b - \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})}]\beta^2}$. According to (14) and (18), we get

$$\begin{aligned} \eta(t^*) &\leq \gamma \leq e^{-bt^*} + \int_0^{t^*} e^{-b(t^*-s)} \times \\ &\quad [\frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} \eta(s - \tau(s)) + \frac{\epsilon}{\beta^2}] ds < \\ &\quad e^{-bt^*} \{ \gamma + \chi + \int_0^{t^*} e^{bs} \times \\ &\quad [\frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} (\gamma e^{-\lambda(s-\tau(s))} + \\ &\quad \chi) + \frac{\epsilon}{\beta^2}] ds \} \leq \\ &\quad e^{-bt^*} [\gamma + \chi + \gamma e^{\lambda\tau} \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})} \times \\ &\quad \int_0^{t^*} e^{(b-\lambda)s} ds + b\chi \int_0^{t^*} e^{bs} ds] \leq \\ &\quad e^{-bt^*} [\gamma + \chi + \gamma(b-\lambda) \int_0^{t^*} e^{(b-\lambda)s} ds + b\chi \int_0^{t^*} e^{bs} ds] = \\ &\quad \gamma e^{-\lambda t^*} + \frac{\epsilon}{[b - \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\beta^2 \lambda_m(\hat{P})}]\beta^2}. \end{aligned}$$

This contradicts (17), so the estimation (16) holds. Letting $\epsilon \rightarrow 0$, we have

$$0 \leq V_1(t) \leq \eta(t) \leq \gamma e^{-\lambda t}, \quad t \geq 0.$$

It implies the case 1.

Case 2 can be easily proved in a way similar to case 1. $\square$

Theorem 2. Let Assumption 2 holds. If there exist

positive constants β_k, σ, ξ and μ such that

- 1) $\|\hat{Q}(I_{nN} + B_k)\hat{Q}^{-1}\| \leq \beta_k$
- 2) $\sigma \leq \inf\{t_k - t_{k-1}, k \in \mathbf{N}\}$
- 3) $\sup\{\theta_k, k \in \mathbf{N}\} \leq \xi < e^{\mu\sigma}$
- 4) $\mu + a + \frac{ae^{\mu\tau}\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} + 2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) \leq 0$

where $\theta_k = \max\{1, \beta_k^2\}$, then the network (1) achieves global exponential state synchronization in sense as

$$\|x(t)\| \leq \sqrt{\frac{\lambda_M(\hat{P})}{\lambda_m(\hat{P})}} \|\Phi\|_\tau e^{-\frac{\mu - \frac{\ln \xi}{\sigma}}{2} t}, \quad t \geq 0.$$

Proof. According to (12), we can obtain

$$\begin{aligned} \dot{V}_1(t) &\leq [2\lambda_M(-\eta\hat{P}^{-1} + \hat{\Delta}) + a]V_1(t) + \\ &\quad \frac{a\lambda_M((G \otimes \Gamma C)^T \hat{P}(G \otimes \Gamma C))}{\lambda_m(\hat{P})} \times \\ &\quad (\sup_{t-\tau \leq s \leq t} V_1(s)), \quad t \neq t_k. \end{aligned} \quad (19)$$

Furthermore,

$$\begin{aligned} V_1(t_k) &= x^T(t_k^-)(I_{nN} + B_k)^T \hat{P}(I_{nN} + B_k)x(t_k^-) \leq \\ &\quad \|\hat{Q}(I_{nN} + B_k)\hat{Q}^{-1}\|^2 x^T(t_k^-) \hat{P} x(t_k^-) \leq \\ &\quad \beta_k^2 V_1(t_k^-), \quad k \in \mathbf{N}. \end{aligned}$$

By Lemma 2, we can get

$$\begin{aligned} V_1(t) &\leq (\prod_{0 < t_k \leq t} \theta_k) e^{-\mu t} (\sup_{-\tau \leq s \leq 0} \Phi^T(s) \hat{P} \Phi(s)) \leq \\ &\quad (\prod_{0 < t_k \leq t} \theta_k) \lambda_M(\hat{P}) \|\Phi\|_\tau^2 e^{-\mu t}. \end{aligned} \quad (20)$$

Since $\sigma \leq \inf\{t_k - t_{k-1}, k \in \mathbf{N}\}$, one has $k-1 \leq \frac{t_{k-1} - t_0}{\sigma} = \frac{t_{k-1}}{\sigma}$, which implies

$$\xi^{k-1} \leq e^{\frac{\ln \xi}{\sigma} t}, \quad t \in [t_{k-1}, t_k), \quad k \in \mathbf{N}. \quad (21)$$

Then, we have

$$V_1(t) \leq \lambda_M(\hat{P}) \|\Phi\|_\tau^2 e^{-(\mu - \frac{\ln \xi}{\sigma})t}.$$

Therefore,

$$\|x(t)\| \leq \sqrt{\frac{\lambda_M(\hat{P})}{\lambda_m(\hat{P})}} \|\Phi\|_\tau e^{-\frac{\mu - \frac{\ln \xi}{\sigma}}{2} t}, \quad t \geq 0. \quad \square$$

In the above, two sufficient conditions are given to ensure the global exponential state synchronization of network (1). Next, we shall investigate the output synchronization of complex network (1).

There obviously exist two nonsingular matrices $U \in \mathbf{R}^{q \times q}$ and $H \in \mathbf{R}^{n \times n}$ such that

$$C = U \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} H$$

where $r = \text{rank}(C)$. Let $\hat{x}_i(t) = Hx_i(t)$ and $\hat{y}_i(t) = U^{-1}y_i(t)$, system (8) can be rewritten as

$$\begin{cases} \dot{\hat{x}}_i(t) = Hf(H^{-1}\hat{x}_i(t)) + a \sum_{j=1}^N G_{ij} H\Gamma U \hat{y}_j(t - \tau(t)), \\ t \neq t_k, \quad t \geq 0 \\ \hat{y}_i(t) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \hat{x}_i(t) \\ \Delta \hat{x}_i = \hat{x}_i(t_k^+) - \hat{x}_i(t_k^-) = HB_{ik}H^{-1}\hat{x}_i(t_k^-), t = t_k, \quad k \in \mathbf{N} \end{cases}$$

where $i = 1, 2, \dots, N$. Then, we can get

$$\begin{cases} \dot{\hat{y}}_i(t) = \hat{f}(\hat{x}_i(t)) + a \sum_{j=1}^N G_{ij} \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \times \\ H\Gamma U \hat{y}_j(t - \tau(t)), \quad t \neq t_k, \quad t \geq 0 \\ \Delta \hat{y}_i = \hat{y}_i(t_k^+) - \hat{y}_i(t_k^-) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \times \\ HB_{ik}H^{-1}\hat{x}_i(t_k^-), \quad t = t_k, \quad k \in \mathbf{N} \end{cases}$$

where $\hat{f}(\hat{x}_i(t)) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Hf(H^{-1}\hat{x}_i(t))$, $i = 1, 2, \dots, N$.

In order to obtain our main results, two assumptions are introduced.

Assumption 3. Assume that $\begin{pmatrix} I_r & 0 \end{pmatrix} HB_{ik}H^{-1} = (\hat{B}_{ik} \ 0)$, $\hat{B}_{ik} \in \mathbf{R}^{r \times r}$, and $\hat{f}(\hat{x}_i(t)) = \begin{pmatrix} \hat{f}_1(\hat{x}_i^1(t)) \\ 0 \end{pmatrix}$

where

$$\begin{aligned} \hat{f}_1(\hat{x}_i^1(t)) &\in \mathbf{R}^r \\ \hat{x}_i^1(t) &= (\hat{x}_{i1}(t), \hat{x}_{i2}(t), \dots, \hat{x}_{ir}(t))^T \\ \hat{x}_i^2(t) &= (\hat{x}_{i(r+1)}(t), \hat{x}_{i(r+2)}(t), \dots, \hat{x}_{in}(t))^T \\ \hat{x}_i(t) &= [(\hat{x}_i^1(t))^T, (\hat{x}_i^2(t))^T]^T, i = 1, 2, \dots, N. \end{aligned}$$

Assumption 4. There exists positive definite diagonal matrix $Z = \text{diag}(z_1, z_2, \dots, z_r)$ and a diagonal matrix $\Psi = \text{diag}(\psi_1, \psi_2, \dots, \psi_r)$ such that $\hat{f}_1 : \mathbf{R}^r \rightarrow \mathbf{R}^r$ satisfies the following inequality:

$$x^T Z [\hat{f}_1(x) - \Psi x] \leq -\zeta x^T x$$

for some $\zeta > 0$ and all $x \in \mathbf{R}^r$.

Remark 1. To the best of our knowledge, this is the first paper to consider the output synchronization of impulsive complex delayed dynamical networks with output coupling, which is a very important and challenging problem. In order to obtain our main results, it is necessary to make the assumption Assumption 3. Practically, Wang and Wu^[27] have studied the output synchronization of complex delayed dynamical networks with output coupling by using this assumption.

According to Assumption 3, we can obtain

$$\begin{cases} \dot{\hat{x}}_i^1(t) = \hat{f}_1(\hat{x}_i^1(t)) + a \sum_{j=1}^N G_{ij} M \hat{x}_j^1(t - \tau(t)), \\ t \neq t_k, \quad t \geq 0 \\ \Delta \hat{x}_i^1 = \hat{x}_i^1(t_k^+) - \hat{x}_i^1(t_k^-) = \hat{B}_{ik} \hat{x}_i^1(t_k^-), \\ t = t_k, \quad k \in \mathbf{N} \end{cases} \quad (22)$$

where $M = \begin{pmatrix} I_r & 0 \end{pmatrix}_{r \times n} H\Gamma U \begin{pmatrix} I_r \\ 0 \end{pmatrix}_{q \times r}$, $i = 1, 2, \dots, N$.

It is clear that the complex network (1) achieves global exponential output synchronization if and only if system (22) is globally exponentially stable.

For convenient analysis, we denote

$$\begin{aligned} E &= Z^{\frac{1}{2}} = \text{diag}(\sqrt{z_1}, \sqrt{z_2}, \dots, \sqrt{z_r}) \\ \hat{Z} &= \text{diag}(Z, Z, \dots, Z) \\ \hat{E} &= \text{diag}(E, E, \dots, E) \\ \hat{\Psi} &= \text{diag}(\Psi, \Psi, \dots, \Psi) \\ \hat{B}_k &= \text{diag}(\hat{B}_{1k}, \hat{B}_{2k}, \dots, \hat{B}_{Nk}) \\ y(t) &= (y_1^T(t), y_2^T(t), \dots, y_N^T(t))^T. \end{aligned}$$

By the similar proofs of Theorems 1 and 2, we can obtain the following conclusions. Here, we omit their proof to avoid the repetition.

Theorem 3. Let Assumptions 3 and 4 hold.

1) If there exists a positive constant $\beta < 1$ such that

$$\|\hat{E}(I_{rN} + \hat{B}_k)\hat{E}^{-1}\| \leq \beta, \quad k \in \mathbf{N} \quad (23)$$

$$2\lambda_M(-\zeta \hat{Z}^{-1} + \hat{\Psi}) + \frac{a\lambda_M((G \otimes M)^T \hat{Z}(G \otimes M))}{\beta^2 \lambda_m(\hat{Z})} + a + \frac{2\ln \beta}{\rho} < 0 \quad (24)$$

where $\rho = \sup\{t_k - t_{k-1}, k \in \mathbf{N}\}$, then network (1) achieves global exponential output synchronization in the following sense:

$$\begin{aligned} \|y(t)\| &\leq \beta^{-1} \sqrt{\frac{\lambda_M(\hat{Z})}{\lambda_m(\hat{Z})}} \|I_N \otimes U\| \times \\ &\|I_N \otimes [\begin{pmatrix} I_r & 0 \end{pmatrix} U^{-1}]\| \times \\ &\|\text{diag}(C, C, \dots, C)\Phi\|_{\tau} e^{\frac{-\lambda t}{2}}, \quad t \geq 0 \end{aligned}$$

where $\lambda > 0$ is a unique solution of

$$\lambda + 2\lambda_M(-\zeta \hat{Z}^{-1} + \hat{\Psi}) + a + \frac{2\ln \beta}{\rho} + e^{\lambda \tau} \frac{a\lambda_M((G \otimes M)^T \hat{Z}(G \otimes M))}{\beta^2 \lambda_m(\hat{Z})} = 0. \quad (25)$$

2) If there exists a positive constant $\beta \geq 1$ such that

$$\|\hat{E}(I_{rN} + \hat{B}_k)\hat{E}^{-1}\| \leq \beta, \quad k \in \mathbf{N} \quad (26)$$

$$2\lambda_M(-\zeta \hat{Z}^{-1} + \hat{\Psi}) + \frac{a\beta^2 \lambda_M((G \otimes M)^T \hat{Z}(G \otimes M))}{\lambda_m(\hat{Z})} + a + \frac{2\ln \beta}{\rho} < 0 \quad (27)$$

where $\rho = \inf\{t_k - t_{k-1}, k \in \mathbf{N}\}$, then network (1) achieves global exponential output synchronization in the following sense:

$$\begin{aligned} \|y(t)\| &\leq \beta \sqrt{\frac{\lambda_M(\hat{Z})}{\lambda_m(\hat{Z})}} \|I_N \otimes U\| \|I_N \otimes [\begin{pmatrix} I_r & 0 \end{pmatrix} U^{-1}]\| \times \\ &\|\text{diag}(C, C, \dots, C)\Phi\|_{\tau} e^{\frac{-\lambda t}{2}}, \quad t \geq 0 \end{aligned}$$

where $\lambda > 0$ is a unique solution of

$$\lambda + 2\lambda_M(-\zeta\hat{Z}^{-1} + \hat{\Psi}) + \frac{2\ln\beta}{\rho} + a + e^{\lambda\tau} \frac{a\beta^2\lambda_M((G \otimes M)^T \hat{Z}(G \otimes M))}{\lambda_m(\hat{Z})} = 0. \quad (28)$$

Theorem 4. Let Assumptions 3 and 4 hold. If there exist positive constants β_k, σ, ξ and μ such that

- 1) $\|\hat{E}(I_{rN} + \hat{B}_k)\hat{E}^{-1}\| \leq \beta_k$
- 2) $\sigma \leq \inf\{t_k - t_{k-1}, k \in \mathbf{N}\}$
- 3) $\sup\{\theta_k, k \in \mathbf{N}\} \leq \xi < e^{\mu\sigma}$
- 4) $\mu + \frac{ae^{\mu\tau}\lambda_M((G \otimes M)^T \hat{Z}(G \otimes M))}{\lambda_m(\hat{Z})} + a + 2\lambda_M(-\zeta\hat{Z}^{-1} + \hat{\Psi}) \leq 0$

where $\theta_k = \max\{1, \beta_k^2\}$, then network (1) achieves global exponential output synchronization in the following sense:

$$\|y(t)\| \leq \sqrt{\frac{\lambda_M(\hat{Z})}{\lambda_m(\hat{Z})}} \|I_N \otimes U\| \|I_N \otimes [(I_r \ 0) U^{-1}]\| \times \|\text{diag}(C, C, \dots, C)\Phi\| e^{-\frac{\mu - \frac{\ln \xi}{\sigma}}{2} t}, \quad t \geq 0.$$

Remark 2. To the best of our knowledge, only few researchers have considered the output synchronization for complex networks with time-varying delay and output coupling^[27]. Wang and Wu^[27] always assumed that the time delay is a differentiable function, and its derivative is less than 1. The above results remove these conservative restrictions.

4 Numerical examples

In this section, two illustrative examples are provided to verify the effectiveness of the proposed theoretical results.

Example 1. Consider a complex network model with 10 nodes, in which each node is a three-dimensional nonlinear system described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -10x_1 + 2x_2 \\ 2x_1 - 10x_2 - x_1x_3 \\ x_1x_2 - 6x_3 \end{pmatrix}.$$

It is easy to verify that Assumption 2 is satisfied with $\eta = 6$, $P = \text{diag}(1, 1, 1)$ and $\Delta = \text{diag}(0, 0, 0)$. Take

$$\Gamma = \begin{pmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}, C = \begin{pmatrix} 0.3 & 0.1 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{pmatrix},$$

$a = 0.1, \beta_\omega = 1, B_{ik} = -\frac{1}{2}I_3, t_k = 0.1k, k \in \mathbf{N}, i = 1, 2, \dots, 10$. Matrix L is chosen as

$$L = \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -5 \end{pmatrix}.$$

We can easily obtain that such complex network is connected, and matrix G is

$$G = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & -1 & \frac{1}{4} \\ \frac{1}{5} & 0 & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & \frac{1}{5} & -1 \end{pmatrix}.$$

Set $\tau(t) = 0.3 - 0.3e^{-t}$. Then, we have $0 \leq \tau(t) \leq \tau = 0.3$, for $t \geq 0$. We can find constant $\beta = 0.5$ satisfying (2) and (3). According to Theorem 1, we know that network (1) with the above given parameters achieves global exponential state synchronization. The simulation results are shown in Fig. 1.

Example 2. Consider a complex network model with 10 nodes, in which each node is a three-dimensional nonlinear system described by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -5x_1 + 0.8x_2 \\ -0.8x_1 - 4x_2 \\ 2x_3 + 2x_1^2 + x_1x_2 \end{pmatrix}.$$

Take

$$\Gamma = \begin{pmatrix} 0.3 & 0.6 \\ 0.8 & 0.3 \\ 0.9 & 0.2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$a = 0.2, \beta_\omega = 1, B_{ik} = \frac{1}{2}I_3, t_k = k, k \in \mathbf{N}, i = 1, 2, \dots, 10$. Matrix L is chosen as

Fig. 1 The change processes of the state variables and output variables

$$L = \begin{pmatrix} -9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

We can easily obtain that such complex network is connected, and matrix G is

$$G = \begin{pmatrix} -1 & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is obvious that we can take $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \text{ Moreover, } M = \begin{pmatrix} 0.3 & 0.6 \\ 0.8 & 0.3 \end{pmatrix}, \quad \hat{B}_{ik} =$$

$$\frac{1}{2}I_2, \quad \hat{f}_1(\hat{x}_i^1(t)) = \begin{pmatrix} -5 & 0.8 \\ -0.8 & -4 \end{pmatrix} \hat{x}_i^1(t), \quad \hat{x}_i^1(t) \in \mathbf{R}^2, i = 1, 2, \dots, 10. \text{ It is easy to verify that Assumption 4 is satisfied with } \zeta = 4, Z = \text{diag}(1, 1) \text{ and } \Psi = \text{diag}(0, 0).$$

Set $\tau(t) = 0.4 - 0.4e^{-t}$. Then, we have $0 \leq \tau(t) \leq \tau = 0.4$. We can find constants $\beta_k = 1.5, \mu = 2, \xi = 2.5, \sigma = 1$ satisfying Theorem 4. Therefore, we know that network (1) with the above given parameters achieves global exponential output synchronization. The simulation results are shown in Fig. 2.

Fig. 2 The evolution of the state variables and output variables

Remark 3. According to the evolution of the state variables and output variables in Fig. 1, we can see clearly that the complex network realizes state synchronization and output synchronization. However, the outputs are synchronous but the states are asynchronous in Fig. 2. The main reason is that state variables are only related to states. In other words, only part of states are needed to make output synchronization to come true. Therefore, it is important and interesting to study the output synchronization of the complex networks.

5 Conclusions

A new impulsive complex delayed dynamical network with output coupling has first been introduced. We have considered the state synchronization and output synchronization of the proposed network model. Some global exponential state synchronization and output synchronization criteria have been established by employing impulsive

delay differential inequalities. Illustrative simulations have been provided to verify the correctness and effectiveness of the obtained results.

Acknowledgement

The authors would like to thank the associate editors and anonymous reviewers for their valuable comments and suggestions.

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