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Improved Stability Criteria on Discrete-time Systems with Time-varying and Distributed Delays

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Abstract: In this paper, through constructing some novel Lyapunov-Krasovskii functional (LKF) terms and using some effective techniques, two sufficient conditions are derived to guarantee a class of discrete-time time-delay systems with distributed delay to be asymptotically and robustly stable, in which the linear fractional uncertainties are involved and the information on the time-delays is fully utilized. By employing the improved reciprocal convex technique, some important terms can be reconsidered when estimating the time difference of LKF, and the criteria can be presented in terms of linear matrix inequalities (LMIs). Especially, these derived conditions heavily depend on the information of time-delay of addressed systems. Finally, three numerical examples demonstrate that our methods can reduce the conservatism more efficiently than some existing ones.

Keywords: Discrete-time system, global stability, interval time-varying delay, distributed delay, reciprocal convex technique.

1 Introduction

Since Lyapunov functional approach presented some simple and delay-independent results, the Lyapunov-Krasovskii functional (LKF) one has been widely utilized, and its analyzing procedure could fully utilize the information of time-delay system. Thus for the past decades, the delay-dependent stability has become a topic of primary significance, in which the main purpose is to derive the maximum allowable upper bound (MAUB) on time-delay, such that the addressed systems keep to be convergent in different ways[1−22].

Meanwhile, in order to implement the continuous-time system for simulation or computation, it is important to formulate discrete-time systems which are the analogues of the continuous-time systems. The discrete-time models are usually obtained from the continuous-time ones by using a discretization technique. Ideally, the discrete-time analogue should inherit the dynamical behaviors of the continuous-time models and maintain the functional similarity to continuous-time models. Unfortunately, the discretization cannot always preserve the dynamics of the continuous-time counterpart even for a small sampling period[23]. Thus, many elegant results have been reported to study the stability for various discrete-time time-delay systems[9−22]. In [9, 10], by using finite sum inequality approach, the stability was studied for discrete-time delay systems, and the delayed controller design was also presented based on linear matrix inequality (LMI) technique. Some researchers have also investigated the robust stability for the systems with uncertain parameters[11−13]. At the same time, to aim additive time-varying delays, some LMI results have been derived for discrete-time systems[14, 15]. Especially, Zhao et al. have discussed the global stability for the discrete-time networked control system[16], descriptor system[17], Lur′e system[18], and neural networks[22]. Recently, by utilizing some developed techniques such as reciprocal convex technique, some less conservative results have been given, as compared with these previous ones[19, 20]. Furthermore, since the delay-partitioning idea is efficient in reducing the conservatism, this idea and improved one were also used to study the stability for discrete-time systems, and the conservatism can be greatly reduced by thinning the delay intervals[21, 22].

Though those aforementioned results were elegant, there still exist some points waiting for improvements. Firstly, those constructions of LKF and utilized techniques still need some improvement since they cannot employ the whole information of the addressed systems. Secondly, as for $\tau(t) \in [0, \tau_m]$ in the continuous time-delay system, the LKF term $\frac{\tau_m^2}{2}\int_{-\tau_m}^{0}\int_{\theta}^{0}\int_{t+\nu}^{t}\dot{x}^{\mathrm{T}}(s)\mathrm{Q}\dot{x}(s)\mathrm{d}s\mathrm{d}\nu\mathrm{d}\theta$ was first chosen and played an important role in reducing the conservatism[7]. Yet, some important terms have been ignored when estimating its upper bound on its time derivative. Thus, some better methods should be employed to deal with this problem. Especially, few works have extended this LKF term to the discrete-time time-delay system.

In this paper, we will study the robust stability for a class of discrete-time time-delay systems, in which the linear fractional uncertainties and the distributed delay are involved. Through constructing a novel Lyapunov-Krasovskii functional and utilizing some elegant techniques, two novel conditions are presented in terms of LMIs, and the feasibility can be easily checked. Finally, numerical examples show that the proposed ideas are less conservative.

Notations. I_n denotes an $n \times n$ identity matrix and $0_{m\times n}$ means an $m \times n$ zero matrix.

2 Problem formulations

In this paper, we consider the uncertain discrete-time time-delay system described by

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$$x(k+1) = A(k)x(k) + B(k)x(k-\tau(k)) + D(k) \times \sum_{i=1}^{+\infty}\delta(i)x(k-i) \quad (1)$$

$$x(k) = \phi(k), \quad k = -\tau_m, -\tau_m+1, \cdots, 0 \quad (2)$$

for $k = 1, 2, \cdots$, where $x(k) = [x_1(k), \cdots, x_n(k)]^{\mathrm{T}} \in \mathbf{R}^n$ is the system state vector, $A(k) = A + \Delta A(k), B(k) = B + \Delta B(k)$, $D(k) = D + \Delta D(k)$ are the time-variant matrices of appropriate dimensions, and $\phi(k)$ is the initial condition of system (1).

The following assumptions are made for system (1) throughout the paper.

Assumption 1. The time-varying delay $\tau(k)$ satisfies

$$\tau_0 \leqslant (k) \leqslant \tau_m \quad (3)$$

where τ_0 and τ_m are known positive integers. Here, we denote $\bar{\tau}_m = \tau_m - \tau_0$.

Assumption 2. There exists a constant scalar $\xi > 0$ such that function $\delta(i)$ satisfies $\sum_{i=1}^{+\infty}\delta(i) = \xi < +\infty$.

Assumption 3. $\Delta A(k), \Delta B(k)$, and $\Delta D(k)$ represent the time-varying parameter uncertainties, and are assumed to satisfy the following linear fractional forms:

$$[\Delta A(k)\ \Delta B(k)\ \Delta D(k)] = F\Delta(k)\,[E_a\ E_b\ E_d] \quad (4)$$

$$\Delta(k) = \Lambda(k)(I - J\Lambda(k))^{-1},\ I - J^{\mathrm{T}}J > 0 \quad (5)$$

where F, J, E_a, E_b, E_d are known constant matrices of the appropriate dimensions, and $\Lambda(k)$ is an unknown time-varying matrix function satisfying $\Lambda^{\mathrm{T}}(k)\Lambda(k) \leqslant I$.

Remark 1. From Assumption 3, it is easy to check that the structured linear fraction in (5) includes the widely used norm-bounded uncertainty as its special case when $J = 0$.

3 Delay-dependent stability results

Firstly, denoting $y(k) = x(k+1) - x(k)$, we can represent the nominal system of (1) as the following form:

$$x(k+1) = Ax(k) + Bx(k-\tau(k)) + D\sum_{i=1}^{+\infty}\delta(i)x(k-i). \quad (6)$$

In what follows, some lemmas are introduced to help derive our main results.

Lemma 1.[9] Let $\mathrm{M} \in \mathbf{R}^{n\times n}$ be a positive-definite matrix, $X_i \in \mathbf{R}^n$, $a_i \geqslant 0$ $(i = 1, 2, \cdots)$. If the sums concerned are well defined, then

$$\left[\sum_{i=1}^{+\infty} a_i X_i\right]^{\mathrm{T}} \mathrm{M} \left[\sum_{i=1}^{+\infty} a_i X_i\right] \leqslant \left[\sum_{i=1}^{+\infty} a_i\right] \sum_{i=1}^{+\infty} a_i X_i^{\mathrm{T}} \mathrm{M} X_i$$

$$\left[\sum_{i=-N}^{-1}\sum_{i=k+j}^{k-1} X_j\right]^{\mathrm{T}} \mathrm{M} \left[\sum_{i=-N}^{-1}\sum_{i=k+j}^{k-1} X_j\right] \leqslant \frac{N^2}{2}\sum_{i=-N}^{-1}\sum_{i=k+j}^{k-1} X_j^{\mathrm{T}} \mathrm{M} X_j.$$

Lemma 2.[20] For vectors ζ_1 and ζ_2, given constant matrices R, S, and real scalars $\alpha \geqslant 0, \beta \geqslant 0$ satisfying that $\begin{bmatrix} R & S \\ * & R \end{bmatrix} \geqslant 0$ and $\alpha + \beta = 1$, the following inequality holds:

$$-\frac{1}{\alpha}\zeta_1^{\mathrm{T}} R\zeta_1 - \frac{1}{\beta}\zeta_2^{\mathrm{T}} R\zeta_2 \leqslant -\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} R & S \\ * & R \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

Lemma 3.[22] If Ω, Ξ_1, and Ξ_2 are the constant matrices of appropriate dimensions, $\alpha \in [0, 1]$, then $\Omega + [\alpha\Xi_1 + (1-\alpha)\Xi_2] < 0$ holds, if the inequalities $\Omega + \Xi_1 < 0$ and $\Omega + \Xi_2 < 0$ hold simultaneously.

Lemma 4.[20] For the symmetric appropriate dimensional matrices $R > 0, \Xi$, matrix Γ, the following two statements are equivalent: 1) $\Xi - \Gamma^{\mathrm{T}} R\Gamma < 0$; 2) there exists a matrix of appropriate dimension Λ such that

$$\begin{bmatrix} \Xi + \Gamma^{\mathrm{T}}\Lambda + \Lambda^{\mathrm{T}}\Gamma & \Lambda^{\mathrm{T}} \\ * & -R \end{bmatrix} < 0.$$

Lemma 5.[22] Let $I - G^{\mathrm{T}}G > 0$, and define the set $\Upsilon = \left\{\Delta(t) = \Sigma(t)[I - G\Sigma(t)]^{-1}, \Sigma^{\mathrm{T}}(t)\Sigma(t) \leqslant I\right\}$. For given matrices H, Q, R and symmetrical matrix H, $H + Q\Delta(t)R + R^{\mathrm{T}}\Delta^{\mathrm{T}}(t)Q^{\mathrm{T}} < 0$, if and only if there exists a scalar $\rho > 0$ such that $H + \begin{bmatrix} \rho^{-1}R \\ \rho Q^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} I & -G \\ -G^{\mathrm{T}} & I \end{bmatrix}^{-1} \begin{bmatrix} \rho^{-1}R \\ \rho Q^{\mathrm{T}} \end{bmatrix} < 0$.

Theorem 1. For positive integers $0 \leqslant \tau_0 \leqslant \tau_m$, the origin of discrete-time system (6) is asymptotically stable, if there exist $9n \times n$ matrices Π_i $(i = 1, 2)$ making $\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}$, $n \times n$ matrices $P > 0, P_i > 0$ $(i = 1, 2, 3, 4, 5)$, $Q_j > 0, R_j > 0$ $(j = 4, 5)$, $Q > 0, R > 0$, $T > 0, S, X_5, Y_5, V_5, Z_5, N_1, N_2$ making $\begin{bmatrix} P_4 & H_4 \\ * & Q_4 \end{bmatrix} \geqslant 0$, $\begin{bmatrix} P_5 & H_5 & X_5 & Y_5 \\ * & Q_5 & V_5 & Z_5 \\ * & * & P_5 & H_5 \\ * & * & * & Q_5 \end{bmatrix} \geqslant 0$, $\begin{bmatrix} R & S \\ * & R \end{bmatrix} > 0$, such that the LMIs in (7) and (8) hold.

$$\begin{bmatrix} \Omega + \Upsilon_1^{\mathrm{T}}\Pi^{\mathrm{T}} + \Pi\Upsilon_1 & \Pi_1 & \Pi_2 \\ * & -R & -S \\ * & * & -R \end{bmatrix} < 0 \quad (7)$$

$$\begin{bmatrix} \Omega + \Upsilon_2^{\mathrm{T}}\Pi^{\mathrm{T}} + \Pi\Upsilon_2 & \Pi_1 & \Pi_2 \\ * & -R & -S \\ * & * & -R \end{bmatrix} < 0 \quad (8)$$

where

$$\Upsilon_1 = \begin{bmatrix} \bar{\tau}_m I_n & 0_{n\times 4n} & -I_n & 0_{n\times 3n} \\ 0_n & 0_{n\times 5n} & -I_n & 0_{n\times 2n} \end{bmatrix}$$

$$\Upsilon_2 = \begin{bmatrix} 0_n & 0_{n\times 4n} & -I_n & 0_{n\times 3n} \\ \bar{\tau}_m I_n & 0_{n\times 5n} & -I_n & 0_{n\times 2n} \end{bmatrix}$$

$$\Omega = \begin{bmatrix} \Omega_{11} & R_4 & \Omega_{13} & 0 & \Omega_{15} & 0 & 0 & \Omega_{18} & \Omega_{19} \\ * & \Omega_{22} & \Omega_{23} & Z_5 & Q_4^{\mathrm{T}} & -Q_5^{\mathrm{T}} & -Y_5^{\mathrm{T}} & 0 & 0 \\ * & * & \Omega_{33} & \Omega_{34} & 0 & \Omega_{36} & \Omega_{37} & 0 & \Omega_{39} \\ * & * & * & \Omega_{44} & 0 & V_5 & Q_5^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & \Omega_{55} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & X_5^{\mathrm{T}} & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77} & 0 & 0 \\ * & * & * & * & * & * & * & -\frac{1}{\xi}^{\mathrm{T}} & \Omega_{89} \\ * & * & * & * & * & * & * & * & \Omega_{99} \end{bmatrix}$$

with

$$\begin{aligned}
\Omega_{11} &= N_1^{\mathrm{T}}(A-I_n) + (A-I_n)^{\mathrm{T}}N_1 + P_1 - R_4 - \bar{\tau}_0^2 Q + \\
&\qquad (\bar{\tau}_m+1)P_3 + \tau_0^2 P_4 + \bar{\tau}_m^2 P_5 + \xi T \\
\Omega_{13} &= N_1^{\mathrm{T}}B \\
\Omega_{15} &= \tau_0 Q - Q_4^{\mathrm{T}} \\
\Omega_{18} &= N_1^{\mathrm{T}}D \\
\Omega_{19} &= P - N_1^{\mathrm{T}} + (A-I_n)^{\mathrm{T}}N_2 + \tau_0^2 Q_4 + \bar{\tau}_m^2 Q_5 \\
\Omega_{22} &= P_2 - P_1 - R_4 - R_5 \\
\Omega_{23} &= R_5 - Z_5^{\mathrm{T}} \\
\Omega_{33} &= -P_3 - 2R_5 + Z_5^{\mathrm{T}} + Z_5 \\
\Omega_{34} &= R_5 - Z_5^{\mathrm{T}} \\
\Omega_{36} &= Q_5^{\mathrm{T}} - V_5 \\
\Omega_{37} &= Y_5^{\mathrm{T}} - Q_5^{\mathrm{T}} \\
\Omega_{39} &= B^{\mathrm{T}}N_2 \\
\Omega_{44} &= -P_2 - R_5 \\
\Omega_{55} &= -Q - P_4 \\
\Omega_{66} &= -P_5 \\
\Omega_{77} &= -P_5 \\
\Omega_{89} &= D^{\mathrm{T}}N_2 \\
\Omega_{99} &= -N_2^{\mathrm{T}} - N_2 + P + \tau_0^2 R_4 + \bar{\tau}_m^2 R_5 + \\
&\qquad \frac{\tau_0^4}{4}Q + \frac{(\tau_m^2-\tau_0^2)^2}{4}R.
\end{aligned}$$

Proof. Firstly, together with Assumptions 1 and 2, we can construct the Lyapunov-Krasovskii functional as

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k)) \tag{9}$$

where

$$\begin{aligned}
V_1(x(k)) &= x^{\mathrm{T}}(k)Px(k) + \sum_{i=k-\tau_0}^{k-1} x^{\mathrm{T}}(i)P_1x(i) + \\
&\sum_{i=k-\tau_m}^{k-\tau_0-1} x^{\mathrm{T}}(i)P_2x(i) + \sum_{i=k-\tau(k)}^{k-1} x^{\mathrm{T}}(i)P_3x(i) + \\
&\sum_{i=k-\tau_m+1}^{k-\tau_0}\sum_{j=i}^{k-1} x^{\mathrm{T}}(i)P_3x(i) \\
V_2(x(k)) &= \frac{\tau_0^2}{2}\sum_{i=-\tau_0}^{-1}\sum_{j=i}^{0}\sum_{l=k+j}^{k-1} y^{\mathrm{T}}(l)Qy(l) + \\
&\frac{\tau_m^2-\tau_0^2}{2}\sum_{i=-\tau_m}^{-\tau_0-1}\sum_{j=i}^{0}\sum_{l=k+j}^{k-1} y^{\mathrm{T}}(l)Ry(l) \\
V_3(x(k)) &= \tau_0\sum_{i=-\tau_0}^{-1}\sum_{j=k+i}^{k-1}\eta(j)^{\mathrm{T}}\Phi_4\eta(j) + \\
&\bar{\tau}_m\sum_{i=-\tau_m}^{-\tau_0-1}\sum_{j=k+i}^{k-1}\eta(j)^{\mathrm{T}}\Phi_5\eta(j) + \\
&\sum_{i=1}^{+\infty}\delta(i)\sum_{j=k-i}^{k-1} x^{\mathrm{T}}(j)Tx(j) \\
\eta(j) &= \begin{bmatrix} x(j) \\ y(j) \end{bmatrix} \\
\Phi_l &= \begin{bmatrix} P_l & Q_l \\ * & R_l \end{bmatrix}, \quad (l=4,5).
\end{aligned}$$

Through directly computing, we can obtain the time difference of $V_i(x(k))(i=1,2)$ along the trajectories of system (6) as

$$\begin{aligned}
\Delta V_1(x(k)) \leqslant\, & 2x^{\mathrm{T}}(k)Py(k) + y^{\mathrm{T}}(k)Py(k) + \\
& x^{\mathrm{T}}(k)\left[P_1 + (\bar{\tau}_m+1)P_3\right]x(k) - \\
& x^{\mathrm{T}}(k-\tau_0)(P_1-P_2)x(k-\tau_0) - \\
& x^{\mathrm{T}}(k-\tau_m)P_2x(k-\tau_m) - \\
& x^{\mathrm{T}}(k-\tau(k))P_3x(k-\tau(k))
\end{aligned} \tag{10}$$

$$\begin{aligned}
\Delta V_2(x(k)) =\, & y^{\mathrm{T}}(k)\left[\frac{\tau_0^4}{4}Q + \frac{(\tau_m^2-\tau_0^2)^2}{4}R\right]y(k) - \\
& \frac{\tau_0^2}{2}\sum_{i=-\tau_0}^{-1}\sum_{i=k+j}^{k-1} y^{\mathrm{T}}(j)Qy(j) - \\
& \frac{\tau_m^2-\tau_0^2}{2}\sum_{i=-\tau_m}^{-\tau_0-1}\sum_{i=k+j}^{k-1} y^{\mathrm{T}}(j)Ry(j).
\end{aligned} \tag{11}$$

Based on Lemmas 1 and 2, we can derive two terms in (11) satisfying

$$\begin{aligned}
& -\frac{\tau_0^2}{2}\sum_{i=-\tau_0}^{-1}\sum_{i=k+j}^{k-1} y^{\mathrm{T}}(j)Qy(j) \leqslant \\
& \quad -\left[\sum_{i=-\tau_0}^{-1}\sum_{i=k+j}^{k-1} y(j)\right]^{\mathrm{T}} Q \left[\sum_{i=-\tau_0}^{-1}\sum_{i=k+j}^{k-1} y(j)\right] = \\
& \quad -\left[\tau_0 x(k) - \sum_{i=k-\tau_0}^{k-1} x(i)\right]^{\mathrm{T}} Q \left[\tau_0 x(k) - \sum_{i=k-\tau_0}^{k-1} x(i)\right]
\end{aligned} \tag{12}$$

$$\begin{aligned}
& -\frac{\tau_m^2-\tau_0^2}{2}\sum_{i=-\tau_m}^{-\tau_0-1}\sum_{j=k+i}^{k-1} y^{\mathrm{T}}(j)Ry(j) = \\
& \quad -\frac{\tau_m^2-\tau_0^2}{\tau^2(k)-\tau_0^2}\frac{\tau^2(k)-\tau_0^2}{2}\sum_{i=-\tau(k)}^{-\tau_0-1}\sum_{i=k+j}^{k-1} y^{\mathrm{T}}(j)Ry(j) -
\end{aligned}$$

$$\frac{\tau_m^2-\tau_0^2}{\tau_m^2-\tau^2(k)}\frac{\tau_m^2-\tau^2(k)}{2}\sum_{i=-\tau_m}^{-\tau(k)-1}\sum_{i=k+j}^{k-1}y^{\mathrm{T}}(j)Ry(j)\leqslant$$
$$-\left[\sum_{i=-\tau(k)}^{-\tau_0-1}\sum_{i=k+j}^{k-1}y(j)\right]^{\mathrm{T}}R\left[\sum_{i=-\tau(k)}^{-\tau_0-1}\sum_{i=k+j}^{k-1}y(j)\right]-$$
$$\left[\sum_{i=-\tau(k)}^{-\tau_0-1}\sum_{i=k+j}^{k-1}y(j)\right]^{\mathrm{T}}2S\left[\sum_{i=-\tau_m}^{-\tau(k)-1}\sum_{i=k+j}^{k-1}y(j)\right]-$$
$$\left[\sum_{i=-\tau_m}^{-\tau(k)-1}\sum_{i=k+j}^{k-1}y(j)\right]^{\mathrm{T}}R\left[\sum_{i=-\tau_m}^{-\tau(k)-1}\sum_{i=k+j}^{k-1}y(j)\right]=$$
$$-\left[[\tau_m-\tau(k)]x(k)-\sum_{i=k-\tau_m}^{k-\tau(k)-1}x(i)\right]^{\mathrm{T}}R\times$$
$$\left[[\tau_m-\tau(k)]x(k)-\sum_{i=k-\tau_m}^{k-\tau(k)-1}x(i)\right]-$$
$$\left[[\tau(k)-\tau_0]x(k)-\sum_{i=k-\tau(k)}^{k-\tau_0-1}x(i)\right]^{\mathrm{T}}2S\times$$
$$\left[[\tau_m-\tau(k)]x(k)-\sum_{i=k-\tau_m}^{k-\tau(k)-1}x(i)\right]-$$
$$\left[[\tau(k)-\tau_0]x(k)-\sum_{i=k-\tau(k)}^{k-\tau_0-1}x(i)\right]^{\mathrm{T}}R\times$$
$$\left[[\tau(k)-\tau_0]x(k)-\sum_{i=k-\tau(k)}^{k-\tau_0-1}x(i)\right]=$$
$$-\xi^{\mathrm{T}}(k)\Upsilon^{\mathrm{T}}(k)\begin{bmatrix}R & S\\ * & R\end{bmatrix}\Upsilon(k)\xi(k) \tag{13}$$

where

$$\xi^{\mathrm{T}}(k)=\Bigg[x^{\mathrm{T}}(k)\quad x^{\mathrm{T}}(k-\tau_0)\quad x^{\mathrm{T}}(k-\tau(k))\quad x^{\mathrm{T}}(k-\tau_m)\times$$
$$\left[\sum_{j=k-\tau_0}^{k-1}x(j)\right]^{\mathrm{T}}\quad\left[\sum_{j=k-\tau(k)}^{k-\tau_0-1}x(j)\right]^{\mathrm{T}}\times$$
$$\left[\sum_{j=k-\tau_m}^{k-\tau(k)-1}x(j)\right]^{\mathrm{T}}\quad\left[\sum_{i=1}^{+\infty}\delta(i)x(k-i)\right]^{\mathrm{T}}\quad y^{\mathrm{T}}(k)\Bigg]$$
$$\Upsilon(k)=\begin{bmatrix}(\tau(k)-\tau_0)I_n & 0_{n\times 4n} & -I_n & 0_{n\times 3n}\\ (\tau_m-\tau(k))I_n & 0_{n\times 5n} & -I_n & 0_{n\times 2n}\end{bmatrix}.$$

Furthermore, it follows from the conditions in Theorem 1, Lemma 2, and $\sum_{i=1}^{+\infty}\delta(i)=\xi$ that

$$\Delta V_3(x(k))\leqslant\begin{bmatrix}x(k)\\ y(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tau_0^2P_4 & \tau_0^2Q_4\\ * & \tau_0^2R_4\end{bmatrix}\begin{bmatrix}x(k)\\ y(k)\end{bmatrix}-$$
$$\begin{bmatrix}\sum\limits_{j=k-\tau_0}^{k-1}x(j)\\ x(k)-x(k-\tau_0)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}P_4 & Q_4\\ * & R_4\end{bmatrix}\times\begin{bmatrix}\sum\limits_{j=k-\tau_0}^{k-1}x(j)\\ x(k)-x(k-\tau_0)\end{bmatrix}+$$
$$\begin{bmatrix}x(k)\\ y(k)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\bar{\tau}_m^2P_5 & \bar{\tau}_m^2Q_5\\ * & \bar{\tau}_m^2R_5\end{bmatrix}\begin{bmatrix}x(k)\\ y(k)\end{bmatrix}-$$
$$\begin{bmatrix}\sum\limits_{j=k-\tau_m}^{k-\tau(k)-1}x(j)\\ x(k-\tau(k))-x(k-\tau_m)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}P_5 & Q_5\\ * & R_5\end{bmatrix}\times\begin{bmatrix}\sum\limits_{j=k-\tau_m}^{k-\tau(k)-1}x(j)\\ x(k-\tau(k))-x(k-\tau_m)\end{bmatrix}-$$
$$\begin{bmatrix}\sum\limits_{j=k-\tau(k)}^{k-\tau_0-1}x(j)\\ x(k-\tau_0)-x(k-\tau(k))\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}P_5 & Q_5\\ * & R_5\end{bmatrix}\times\begin{bmatrix}\sum\limits_{j=k-\tau(k)}^{k-\tau_0-1}x(j)\\ x(k-\tau_0)-x(k-\tau(k))\end{bmatrix}-$$
$$\begin{bmatrix}\sum\limits_{j=k-\tau_m}^{k-\tau(k)-1}x(j)\\ x(k-\tau(k))-x(k-\tau_m)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}2X_5 & 2Y_5\\ 2V_5 & 2Z_5\end{bmatrix}\times\begin{bmatrix}\sum\limits_{j=k-\tau(k)}^{k-\tau_0-1}x(j)\\ x(k-\tau_0)-x(k-\tau(k))\end{bmatrix}+x^{\mathrm{T}}(k)(\xi\mathrm{T})x(k)-$$
$$\left[\sum_{i=1}^{+\infty}\delta(i)x(k-i)\right]^{\mathrm{T}}\frac{1}{\xi}\left[\sum_{i=1}^{+\infty}\delta(i)x(k-i)\right]. \tag{14}$$

Moreover, it follows from (6) and $n\times n$ matrices N_1 and N_2 that

$$0=2\left[N_1x(k)+N_2y(k)\right]^{\mathrm{T}}\Bigg[-y(k)+(A-I)x(k)+Bx(k-\tau(k))+D\sum_{i=1}^{+\infty}\delta(i)x(k-i)\Bigg]. \tag{15}$$

Now replacing the terms (12) and (13) into (11), and employing the right-hand sides in (10), (11), (14), (15) for $\Delta V(x(k))$, we can deduce

$$\Delta V(x(k))\leqslant\xi^{\mathrm{T}}(k)\left[\Omega-\Upsilon^{\mathrm{T}}(k)\begin{bmatrix}R & S\\ * & R\end{bmatrix}\Upsilon(k)\right]\xi(k)$$

where Ω is presented in (7) and (8), $\Upsilon(k)$, and $\xi(k)$ are expressed in (13). The LMI results in (7) and (8) mean the inequalities $\Omega+\Upsilon_i^{\mathrm{T}}\Pi^{\mathrm{T}}+\Pi\Upsilon_i<0$ for $i=1,2$. Then utilizing Lemma 3, the terms $\Omega+\Upsilon_i^{\mathrm{T}}\Pi^{\mathrm{T}}+\Pi\Upsilon_i<0$ can guarantee

$$\Omega + \Upsilon^{\mathrm{T}}(k)\Pi^{\mathrm{T}} + \Pi\Upsilon(k) < 0.$$

Now, the LMIs in (7) and (8) can make

$$\begin{bmatrix} \Omega + \Upsilon^{\mathrm{T}}(k)\Pi^{\mathrm{T}} + \Pi\Upsilon(k) & \Pi_1 & \Pi_2 \\ * & -R & -S \\ * & * & -R \end{bmatrix} < 0. \tag{16}$$

Then together with Lemma 4, we can derive

$$\Omega - \Upsilon^{\mathrm{T}}(k)\begin{bmatrix} R & S \\ * & R \end{bmatrix}\Upsilon(k) < 0. \tag{17}$$

Thus, there must exist a positive scalar $\chi > 0$ such that

$$\Delta V(x(k)) \leqslant -\chi\|x(k)\|^2 < 0, \quad \forall \ \ x(t) \neq 0. \tag{18}$$

Thus, it follows from Lyapunov-Krasovskii stability theorem that system (6) is asymptotically stable. □

Remark 2. During the proof procedure in Theorem 1, we can easily check that our LKF has fully included the information of system (6), and some mathematical techniques were used in (13) and (16), which can reduce the conservatism more efficiently than ever. Especially, the LKF term $V_2(x(k))$ in (9) was first put forward and the reciprocal convex technique was used in (13) and (14), and one less conservative sufficient condition has been given in Theorem 1, whose feasibility can be easily checked without tuning any parameter by LMI in the Matlab toolbox.

In the following, through employing Lemma 5, Theorem 2 can be directly established for the robust stability of system (1) based on Theorem 1.

Theorem 2. For positive integers $0 \leqslant \tau_0 \leqslant \tau_m$, the origin of discrete-time time-delay system (1) and (2) is robustly stable, if there exist two positive scalars $\mu > 0, \nu > 0$, $9n \times n$ matrices Π_i $(i = 1, 2)$ making $\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}$, $n \times n$ matrices $P > 0, P_i > 0$ $(i = 1, 2, 3, 4, 5), Q_j > 0, R_j > 0$ $(j = 4, 5)$, $Q > 0, R > 0, T > 0, S, X_5, Y_5, V_5, Z_5, N_1, N_2$ making $\begin{bmatrix} P_4 & H_4 \\ * & Q_4 \end{bmatrix} \geqslant 0$, $\begin{bmatrix} P_5 & H_5 & X_5 & Y_5 \\ * & Q_5 & V_5 & Z_5 \\ * & * & P_5 & H_5 \\ * & * & * & Q_5 \end{bmatrix} \geqslant 0, \begin{bmatrix} R & S \\ * & R \end{bmatrix} > 0$, such that the LMIs in (18) and (19) hold.

$$\begin{bmatrix} \Omega + \Upsilon_1^{\mathrm{T}}\Pi^{\mathrm{T}} + \Pi\Upsilon_1 & \Pi_1 & \Pi_2 & \mu\Phi^{\mathrm{T}} & \Theta \\ * & -R & -S & 0 & 0 \\ * & * & -R & 0 & 0 \\ * & * & * & -\mu I_n & \mu J \\ * & * & * & * & -\mu I_n \end{bmatrix} < 0 \tag{19}$$

$$\begin{bmatrix} \Omega + \Upsilon_2^{\mathrm{T}}\Pi^{\mathrm{T}} + \Pi\Upsilon_2 & \Pi_1 & \Pi_2 & \nu\Phi^{\mathrm{T}} & \Theta \\ * & -R & -S & 0 & 0 \\ * & * & -R & 0 & 0 \\ * & * & * & -\nu I_n & \nu J \\ * & * & * & * & -\nu I_n \end{bmatrix} < 0 \tag{20}$$

where Ω, Υ_1, and Υ_2 are identical to the counterparts in Theorem 1, and

$$\Phi = [E_a \ \ 0_{n\times n} \ \ E_b \ \ 0_{n\times 5n} \ \ E_d]$$
$$\Theta = \left[F^{\mathrm{T}}N_1 \ \ 0_{n\times 7n} \ \ F^{\mathrm{T}}N_2\right]^{\mathrm{T}}.$$

Proof. Together with LMI results of Theorem 1, by replacing A, B, and D in (7) and (8) with $A + \Delta A(k), B + \Delta B(k)$, and $D + \Delta D(k)$ in (4), respectively, it follows from Lemma 5 and Schur-complement that there must exist two positive scalars $\rho > 0$ and $\upsilon > 0$ such that

$$\begin{bmatrix} \Omega + \Upsilon_1^{\mathrm{T}}\Pi^{\mathrm{T}} + \Pi\Upsilon_1 & \Pi_1 & \Pi_2 \\ * & -R & -S \\ * & * & -R \end{bmatrix} + \begin{bmatrix} \rho^{-1}\Phi^{\mathrm{T}} & \rho\Theta \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} I_n & -J \\ -J^{\mathrm{T}} & I_n \end{bmatrix}^{-1} \begin{bmatrix} \rho^{-1}\Phi^{\mathrm{T}} & \rho\Theta \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} < 0 \tag{21}$$

$$\begin{bmatrix} \Omega + \Upsilon_2^{\mathrm{T}}\Pi^{\mathrm{T}} + \Pi\Upsilon_2 & \Pi_1 & \Pi_2 \\ * & -R & -S \\ * & * & -R \end{bmatrix} + \begin{bmatrix} \upsilon^{-1}\Phi^{\mathrm{T}} & \upsilon\Theta \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \times$$
$$\begin{bmatrix} I_n & -J \\ -J^{\mathrm{T}} & I_n \end{bmatrix}^{-1} \begin{bmatrix} \upsilon^{-1}\Phi^{\mathrm{T}} & \upsilon\Theta \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} < 0. \tag{22}$$

Then by utilizing the definition of Schur-complement, the inequalities in (20) and (21) can be equivalent to the LMIs in (18) and (19) through setting $\mu = \rho^{-2}$ and $\nu = \upsilon^{-2}$. □

Remark 3. Based on [7], we can easily check that the Lemma 1 in this work is stemmed from the continuous-time counterpart. Park et al.[8] first utilized reciprocal convex approach to tackle the continuous-time time-delay system and it could reduce the conservatism more effectively than some previous convex ones[1, 12, 15, 19, 22]. Yet, it has come to our attention that the reciprocal convex one has not been utilized to study the discrete-time time-delay system, which has been fully addressed in our work.

Remark 4. Though in this work, some novel Lyapunov functional terms are constructed, such as $V_2(x(k))$ in (9), and the effective techniques are employed during the proof procedure, Theorems 1 and 2 are still rigorous and limited. In recent years, the delay-partitioning idea and improved ones have been widely used to further reduce the conservatism in [21, 22], and they could be used in our work to get better results. However, these techniques would add significantly to the complexities of the proof and results.

Remark 5. It is worth pointing out that, though in (13), we can estimate the time difference of $V_2(x(k))$ efficiently, the free-weighting matrix $\Pi = [\Pi_1 \ \Pi_2]$ is also introduced in (7), (8), (18) and (19), and will induce some computational complexity when checking the theorems. Thus, we will give some discussions on this point in future works. Moreover, we can extend the derived idea to more general cases.

4 Numerical examples

In this section, three examples are used to demonstrate the validity of the proposed methods.

Example 1. We consider the discrete-time system of (6) with the parameters as

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}$$

which has been extensively studied[11, 12, 19, 20]. By using Theorem 1 and LMI in Matlab toolbox, the aim of Example 1 is to find the maximum allowable upper bounds $\tau_{\max}$ with various τ_0's such that the system is asymptotically stable. The computational results are shown in Table 1, which can summarize the derived MAUBs. Table 1 also shows that the stability criterion expressed in Theorem 1 can be less conservative than the results proposed in [12, 19, 20].

Table 1 The maximum allowable upper bounds for various τ_0's in Example 1

Method	τ_0				
	6	7	10	15	20
Theorem 1[12]	16	16	18	21	25
Proposition 2[19]	16	16	18	21	25
Proposition 1[19]	18	18	20	23	27
Theorem 3[20]	17	17	19	22	26
Theorem 1[20]	18	18	20	23	27
Theorem 1	18	19	21	24	28

Example 2. We consider the discrete-time system of (1)

$$x(k+1) = A(k)x(k) + B(k)x(k-\tau(k))$$

with the following parameters

$$A(k) = \begin{bmatrix} 0.8+\alpha(k) & 0 \\ 0 & 0.9 \end{bmatrix}, \ B(k) = \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix}$$

which has been studied in [11−13]. If $|\ \alpha(k)\ | \leqslant \bar{\alpha}$, then $F = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}, E_a = \begin{bmatrix} 1 & 0 \end{bmatrix}, E_b = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $J = 0$. Now, given τ_0 and τ_m, we want to find the upper values of $\bar{\alpha}$ obtained by employing different techniques. It is clear from Table 2 that Theorem 2 can be less conservative and more applicable than the results in [12, 13].

Table 2 The upper bounds of $\bar{\alpha}$ for interval $[\tau_0, \tau_m]$ in Example 2

Method	τ_0			
	[2, 7]	[3, 9]	[5, 10]	[6, 12]
Theorem 4[12]	0.192	0.155	0.142	0.115
Theorem 2[13]	0.195	0.165	0.154	0.131
Theorem 2	0.205	0.172	0.161	0.138

Example 3. Consider a 2-dimensional discrete time-delay system of (6) as

$$x(k+1) = Ax(k) + Bx(k-\tau(k)) + D\sum_{i=1}^{+\infty} \mathrm{e}^{-2i}x(k-i)$$

with the following parameters

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, D = \begin{bmatrix} -0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}.$$

As for $\tau(t) = 24 + 4\cos(\frac{\pi k}{2})$, we have $\tau_0 = 20$ and $\tau_m = 28$. Based on Theorem 1 and Matlab LMI toolbox, we can easily verify that there exists the feasible solution to the LMIs in (7) and (8) that the system is asymptotically stable, which can also be further supported by Fig. 1 with the initial condition $x(0) = [0.5, -1]^{\mathrm{T}}$.

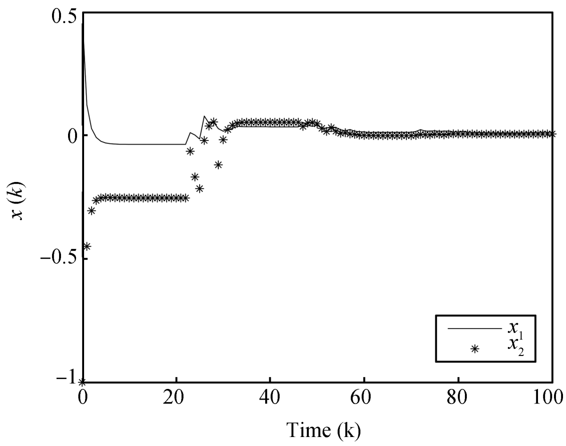

Fig. 1 The state trajectories of the system

5 Conclusions

The paper has studied the robust stability for discrete-time system with both interval time-varying and distributed delays. Through choosing some novel Lyapunov-Krasovskii functional terms, two obtained stability criteria with significantly reduced conservatism have been established in terms of LMIs. The proposed stability criteria benefit from the improved convex technique. Finally, three numerical examples have been given to demonstrate the effectiveness of the presented criteria and their improvement over some existing ones. Finally, it should be worth noting that the ideas presented in the paper is applicable to many application fields.

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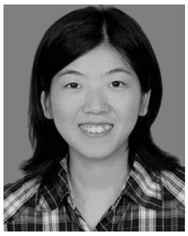

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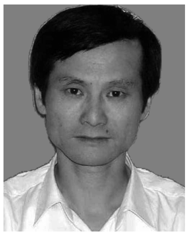

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