Optimization Letters
https://doi.org/10.1007/s11590-024-02111-2

ORIGINAL PAPER

Affine optimal *k*-proper connected edge colorings

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Received: 15 May 2022 / Accepted: 8 March 2024

Abstract

We introduce *affine optimal k-proper connected edge colorings* as a variation on Fujita's notion of *optimal k-proper connected colorings* (Fujita in Optim Lett 14(6):1371–1380, 2020. https://doi.org/10.1007/s11590-019-01442-9) with applications to the frequency assignment problem. Here, for a simple undirected graph G with edge set E_G, such a coloring corresponds to a decomposition of E_G into color classes $C_1, C_2, \ldots, C_n$, with associated weights $w_1, w_2, \ldots, w_n$, minimizing a specified affine function $\mathcal{A} := \sum_{i=1}^{n} \left(w_i \cdot |C_i|\right)$, while also ensuring the existence of k vertex disjoint *proper paths* (i.e., simple paths with no two adjacent edges in the same color class) between all pairs of vertices. In this context, we define $\zeta_{\mathcal{A}}^{k}(G)$ as the minimum possible value of $\mathcal{A}$ under a k-proper connectivity requirement. For any fixed number of color classes, we show that computing $\zeta_{\mathcal{A}}^{k}(G)$ is treewidth fixed parameter tractable. However, we also show that determining $\zeta_{\mathcal{A}'}^{k}(G)$ with the affine function $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$ is NP-hard for 2-connected planar graphs in the case where $k = 1$, cubic 3-connected planar graphs for $k = 2$, and k-connected graphs $\forall k \geq 3$. We also show that no fully polynomial-time randomized approximation scheme can exist for approximating $\zeta_{\mathcal{A}'}^{k}(G)$ under any of the aforementioned constraints unless $NP = RP$.

Keywords Optimal proper connection number · Proper connection · Edge coloring · Optimization hardness · Approximation hardness · FPRAS

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Published online: 13 April 2024

1 Introduction

An edge colored graph G is called *k-proper connected*, or stated to have a *k-proper connected edge coloring*, if there exist k vertex disjoint *proper paths* between all pairs of vertices having no two adjacent edges of the same coloration [3, 6]. Here, letting G be a graph with an initial monochromatic edge coloration, Fujita [16] introduced the discrete optimization problem of computing the minimum of the sum $p + q$ for the number of edges p that must be recolored using q new colors to ensure G is k-proper connected. In this context, $\min\{p + q\}$ is referred to as the *optimal k-proper connection number* of G, or $pc^k_{opt}(G)$, and any edge coloration minimizing $p + q$ is referred to as an *optimal k-proper connected coloring*. Moderating $pc^k_{opt}(G)$, Fujita also introduced a parameter $pc^k_{opt'}(G)$, corresponding to the minimum total number of edges that must be recolored using all distinct colors to ensure G is k-proper connected.

Inspired by Fujita's concept of optimal k-proper connectivity [16], we introduce the notion of *affine optimal k-proper connected edge colorings*. Letting G be a simple undirected graph with edge set E_G, such a coloring corresponds to a decomposition of E_G into n distinct color classes $C_1, C_2, \dots, C_n$, with associated weights $w_1, w_2, \dots, w_n$, under the dual objectives of minimizing an affine function $\mathcal{A} := \sum_{i=1}^{n} \left(w_i \cdot |C_i|\right)$ and ensuring G is at least k-proper connected. We likewise introduce the *affine optimal k-proper connection number*, $\zeta^k_{\mathcal{A}}(G)$, as the minimum possible value of $\mathcal{A}$ under a k-proper connectivity requirement for G. While we cannot directly express $pc^k_{opt}(G)$ in terms of $\zeta^k_{\mathcal{A}}(G)$, for $\mathcal{A}^* := 0 \cdot |C_1| + \sum_{i=2}^{|E_G|} |C_i|$ we can observe that $pc^k_{opt'}(G)$ is equivalent to $\zeta^k_{\mathcal{A}^*}(G)$.

In this work, we show that $\zeta^k_{\mathcal{A}}(G)$ is expressible in Monadic Second-order (MS_2) logic, and for any fixed number of color classes $n \in \mathbb{N}_{>0}$ for affine function $\mathcal{A}$, correspondingly treewidth Fixed-Parameter Tractable (FPT) to compute (Theorem 1). Furthermore, in the special case where we consider highly restricted affine functions of the form $\mathcal{A}^* := 0 \cdot |C_1| + \sum_{i=2}^{|E_G|} |C_i|$, we show that computing $\zeta^k_{\mathcal{A}^*}(G)$ is treewidth FPT without a bound on the number of color classes (Proposition 1), and that this correspondingly holds for the parameter $pc^k_{opt'}(G)$ (Corollary 1).

With regard to hardness results, we consider the problem of determining $\zeta^k_{\mathcal{A}'}(G)$ for highly restricted affine functions of the form $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$. In particular, we show that $\zeta^k_{\mathcal{A}'}(G)$ is NP-hard to compute if G is a 2-connected planar graph and $k = 1$ (Theorem 2), a cubic 3-connected planar graph and $k = 2$ (Theorem 4 in the Appendix), or a k-connected graph and $k \geq 3$ (Theorem 5 in the Appendix). Concerning approximation complexity, we additionally show that no Fully Polynomial-time Randomized Approximation Scheme (FPRAS) can exist for approximating $\zeta^k_{\mathcal{A}'}(G)$ under any of the aforementioned constraints unless $NP = RP$ (Theorem 3). Finally, we extend each of these hardness results to approximately computing parameters $pc^k_{opt}(G)$ and $pc^k_{opt'}(G)$ (Corollary 2).

We remark that our affine optimal k-proper connected edge colorings have direct application to the problem of minimizing interference between channels of communication in wireless networks. Briefly, in the well-known *frequency* *assignment problem* (traditionally abstracted as a vertex proper coloring problem on what are known as *interference graphs*) [1, 13, 15, 18, 22–24], one is tasked with assigning a sparse set of frequencies (colors) to a set of transmitters (vertices) embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$, such that closely spaced transmitters (adjacent vertices) have a threshold frequency separation (e.g., $\approx$ 50–100 kHz [13]) in order to avoid interference. In circa 2015, Li and Magnant [21] observed that the notion of proper connectivity could allow one to consider a frequency assignment model at the finer-grained level of message passing. More specifically, noting that transceivers in a wireless network (e.g., cellular towers mediating LTE communications) can receive and emit at different frequencies, Li and Magnant [21] considered the problem of coloring the edges of a network's interference graph to minimize interference by ensuring a minimum threshold frequency separation between incoming and outgoing signals at each transceiver.

Consider now that, in the context of Li and Magnant's [21] model, a k-proper connected network coloring using at most n colors correspondingly implies the existence of a plausible message passing scheme using at most n frequencies, with k vertex disjoint redundant paths between all pairs of nodes. Here, for a simple albeit realistic model [1, 13, 15, 18, 22–24] where we assume that different bands of the frequency spectrum (abstracted as color classes $C_1, C_2, \ldots, C_n$) have distinct costs or weights $(w_1, w_2, \ldots, w_n)$, and where we require at least k vertex disjoint redundant paths for passing messages between nodes, we can observe that finding an optimal frequency assignment becomes the problem of finding an affine optimal k-proper connected edge coloring.

2 Preliminaries

Concerning fundamental graph theoretic concepts and notation, we will generally follow Diestel [14] or, where appropriate, Bondy and Murty [5]. We clarify that all graphs in this work should be assumed to be simple (i.e., loop and multi-edge free) and undirected. Concerning complexity theoretic concepts and terminology, we call a problem Fixed-Parameter Tractable (FPT) for a specified parameter k if its time complexity can be expressed as $f(k) \cdot |x|^{\mathcal{O}(1)}$, where x is a string specifying a given problem instance and $f(k)$ is any computable function. Concerning approximation complexity and the concept of a Fully Polynomial-time Randomized Approximation Scheme (FPRAS), we will follow definitions and notation from Karp and Luby [20].

3 Treewidth fixed-parameter tractability

Recall that $\zeta^k_{\mathcal{A}}(G)$ is the affine optimal k-proper connection number of a graph G for an arbitrary affine function of the form $\mathcal{A} := \sum_{i=1}^{n} \left(w_i \cdot |C_i|\right)$. We begin by observing the following theorem:

Theorem 1 *For any fixed number of color classes n, determining $\zeta_{\mathcal{A}}^{k}(G)$ is treewidth FPT.*

Proof Let G be a graph with edge set E_G. To establish the theorem at hand, we will show that $\zeta_{\mathcal{A}}^{k}(G)$ admits a formulation as an extremum problem in Monadic Second-order (MS_2) logic, where the description size is bounded by the number of color classes n into which E_G can be decomposed. This will allow us to use an extension of Courcelle's well-known algorithmic metatheorem [7–10] to counting and optimization problems [4, 11, 12] to prove the existence of a treewidth FPT algorithm for determining $\zeta_{\mathcal{A}}^{k}(G)$.

To begin, we remark that the existence of a k-proper connected edge coloring for a graph G using at most n colors can be expressed in MS_2 logic. Here, we first write the sentence ψ_1 to express the decomposition of E_G into n distinct color classes, $(C_1, \dots C_n)$:

$$\psi_1 := \exists C_1, \dots, C_n \subseteq E_G\big(\forall e_i \in E_G\big(e_i \in C_1 \vee \dots \vee e_i \in C_n\big)\big) \wedge$$
$$\forall i, j \in \{1, \dots, n\}\big(i = j \vee C_i \cap C_j = \emptyset\big)$$

We next write a sentence ψ_2 to express the property of G having k vertex disjoint proper paths between all pairs of vertices $v_x, v_y \in V_G$. Here, we will use Courcelle's notion of a *quasipath*, where we have that an edge set X_i is a quasipath from a vertex v_x to a vertex v_y if and only if the following three first order conditions are met: (1) $v_x \neq v_y$; (2) both v_x and v_y are incident to a single unique edge in X_i; and (3) any vertex v_z not in the set $\{v_x, v_y\}$ is incident to exactly two edges in X_i. We can therefore use an auxiliary predicate $Quasipath(X_i, v_x, v_y)$ to check whether the edge set X_i corresponds to a quasipath representing a simple path between a pair of vertices v_x and v_y, and can moreover write the first order sentences "$X_1, \dots, X_k$ are pairwise disjoint" and "no vertex except v_x and v_y belongs to an edge of X_i and of X_j for $i \neq j$" to express that k quasipaths are vertex disjoint aside from having common endpoints at v_x and v_y (see, e.g., the section "*Disjoint paths in undirected graphs*" on "pg. 5" of Courcelle [9]).

Additionally, letting $inc(v_i, e_j)$ be a binary relation which expresses that a vertex v_i is adjacent to an edge e_j, we can use the following auxiliary predicate $PP(X_i)$ to express the property of a quasipath X_i being a proper path:

$$PP(X_i) := \forall e_a, e_b \in X_i, \forall v_z \in V_G(e_a \neq e_b \wedge$$
$$inc(e_a, z) \wedge inc(e_b, z) \implies \exists r \in \{1, \dots, n\}(e_a \in C_r \wedge e_b \notin C_r))$$

We can accordingly write ψ_2 as:

$$\psi_2 := \forall v_x, v_y \in V_G(\exists X_1, \dots, X_k \subseteq E_G(v_x = v_y \vee (Quasipath\big(X_1, v_x, v_y\big) \wedge$$
$$PP(X_1) \wedge \dots \wedge Quasipath\big(X_k, v_x, v_y\big) \wedge$$
$$PP(X_k) \wedge \text{“}X_1, \dots, X_k \textit{ are pairwise disjoint}\text{”} \wedge$$
$$\text{“}\textit{no vertex except } v_x \textit{ and } v_y \textit{ belongs to an edge of } X_i \textit{ and of } X_j \textit{ for } i \neq j\text{”})))$$

Putting everything together, we have that the sentence $\psi_1 \wedge \psi_2$ yields an MS_2 formula expressing the property of a graph possessing a k-proper connected edge coloring using at most n colors.

To now elaborate on our earlier remarks concerning MS_2-expressible extremum problems, by a result of Arnborg et al. [4]—see also "Theorem 7.12" on "pg. 184" of Cygan et al. [12] directly attributed to ref. [4]—for any fixed-size MS_2 formula ϕ with some number n of monadic free variables $X_1, \ldots, X_n$ (corresponding to sets of vertices or edges), we are able to formulate an MS_2 extremum problem of maximizing or minimizing any affine function over the cardinalities of the sets $X_1, \ldots, X_n$ for which ϕ is true. Furthermore, as long as the size of ϕ is fixed, we are guaranteed a treewidth FPT algorithm for the extremum problem [4, 12].

Here, letting $C_1, \ldots, C_n$ correspond to the set of free monadic variables, and letting $\phi = \psi_1 \wedge \psi_2$, we can formulate the MS_2 extremum problem of minimizing the earlier defined affine function $\mathcal{A}$. This correspondingly yields a treewidth FPT algorithm for determining $\zeta^k_{\mathcal{A}}(G)$ for any fixed n. □

Letting $\zeta^k_{\mathcal{A}^*}(G)$ be a modification of $\zeta^k_{\mathcal{A}}(G)$ for restricted affine functions of the form $\mathcal{A}^* := 0 \cdot |C_1| + \sum_{i=2}^{|E_G|} |C_i|$, we can observe the following proposition and corollary:

Proposition 1 *Determining $\zeta^k_{\mathcal{A}^*}(G)$ is treewidth FPT.*

Proof Let G be a graph with edge set E_G. Observe that any affine function of the form $\mathcal{A}^* := 0 \cdot |C_1| + \sum_{i=2}^{|E_G|} |C_i|$ will be minimized if the cardinality of the set $|C_1|$ is maximized. Accordingly, we can simplify the problem by specifying an affine function of the form $0 \cdot |C_1| + |C_2|$, and treating all edges in the color class C_2 as having a distinct coloration.

We remark that it is a trivial matter to modify the auxiliary predicate $PP(X_i)$ from Theorem 1 to check if a quasipath X_i constitutes a proper path in this context. Specifically, we can write down an MS_2 sentence for the modified auxiliary predicate, which we denote $PP'(X_i)$, as follows:

$$\begin{aligned} PP'(X_i) := \forall e_a, e_b \in X_i, \forall v_z \in V_G(e_a \neq e_b \ \wedge inc(e_a, z) \wedge inc(e_b, z) \\ \implies (e_a \in C_2 \vee e_b \in C_2)) \end{aligned}$$

Now, specifying only the two color classes (C_1, C_2) and everywhere substituting the auxiliary predicate $PP'(X_i)$ in place of $PP(X_i)$, we otherwise follow exactly the Theorem 1 proof argument to establish the existence of a treewidth FPT algorithm for computing $\zeta^k_{\mathcal{A}^*}(G)$. □

Corollary 1 *Determining the parameter $pc^k_{opt'}(G)$ for a graph G is treewidth FPT.*

Proof Recall that the parameter $pc^k_{opt'}(G)$ is a modification of the original optimal k-proper connection number $pc^k_{opt}(G)$, where, for an input graph G with monochromatic edge set E_G, one asks only for the total number of edges that must be recolored to ensure that G is k-proper connected. Here, as we are free to assume all recolored edges have distinct colorations, we can recast this optimization problem as one of bi-partitioning E_G into two distinct color classes $\left(C_1, C_2\right)$, and treating all edges in C_2 as having distinct colorations, maximizing $|C_1|$. As this is exactly our strategy in Proposition 1 for showing the existence of a treewidth FPT algorithm for $\zeta^k_{\mathcal{A}^*}(G)$, where $\mathcal{A}^* := 0 \cdot |C_1| + \sum_{i=2}^{|E_G|} |C_i|$, we have the corollary. □

4 Hardness results

Letting $\zeta^k_{\mathcal{A}'}(G)$ be a modification of $\zeta^k_{\mathcal{A}}(G)$ for highly restricted affine functions of the form $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$, we can observe the following hardness results:

Theorem 2 *It is NP-hard to determine $\zeta^1_{\mathcal{A}'}(G)$ for a 2-connected planar graph G.*

Proof We proceed via reduction from the *NP*-complete problem of deciding the existence of a Hamiltonian path on a cubic 2-connected planar graph (see Lemma 1 in the Appendix).

Letting G be an arbitrary cubic 2-connected planar graph with vertex set V_G and edge set E_G, our strategy will be to replace each vertex of G with a common subgraph γ, such that: (constraint 1) any 1-proper connected coloring for G must recolor at least 2 edges per γ subgraph, either with both edges internal to the same γ subgraph, or with one edge having both ends and two edges having one end in a common γ subgraph; and (constraint 2) if and only if G is traceable, it will suffice to recolor exactly two edges per γ subgraph, such that the resulting graph has exactly two types of edge colors. Together, (constraint 1) and (constraint 2) will ensure that $\zeta^1_{\mathcal{A}'}(H) \leq 2 \cdot |V_G|$ if and only if G is traceable.

To begin, we let γ correspond to the subgraph shown in Fig. 1a. Accordingly, we replace each vertex $v_i \in V_G$ with the subgraph given by the edge set $\{v_{(i,1)} \leftrightarrow v_{(i,2)}, v_{(i,1)} \leftrightarrow v_{(i,6)}, v_{(i,1)} \leftrightarrow v_{(i,7)}, v_{(i,2)} \leftrightarrow v_{(i,3)}, v_{(i,3)} \leftrightarrow v_{(i,4)}, v_{(i,3)} \leftrightarrow v_{(i,5)}, v_{(i,3)} \leftrightarrow v_{(i,6)}, v_{(i,3)} \leftrightarrow v_{(i,7)}, v_{(i,4)} \leftrightarrow v_{(i,5)}, v_{(i,4)} \leftrightarrow v_{(i,6)}, v_{(i,5)} \leftrightarrow v_{(i,6)}, v_{(i,5)} \leftrightarrow v_{(i,7)}\}$, reconnecting formerly adjacent vertices to vertices $v_{(i,3)}$, $v_{(i,4)}$, and $v_{(i,5)}$, respectively. Letting V_H and E_H be the vertex and edge sets for the graph H, respectively, we can observe that $|V_H| = 7 \cdot |V_G|$ and that $|E_H| = 12 \cdot |V_G| + |E_G| = \left(\frac{27}{2}\right) \cdot |V_G|$.

Next, brute-force enumeration of all possible manners of recoloring ≤ 3 edges in the neighborhood of each γ subgraph to ensure the resulting graph is 1-proper connected can be used to confirm that (constraint 1) holds for H. To elaborate on this enumeration, we refer the reader to Fig. 1 where we show the γ subgraph in all relevant local contexts in H, as well as an example minimum cost coloring for each instance. Here, vertices stylized as having a (hollow white) center are adjacent to but disjoint from the γ subgraph, (thin black) edges belong to color class C_1, (thick black) edges belong to color class C_2, and (curved dotted lines) correspond to paths of arbitrary length between the aforementioned (hollow white) vertices. Concerning the constraint that each graph in Fig. 1a–l is 1-proper connected, we allow proper

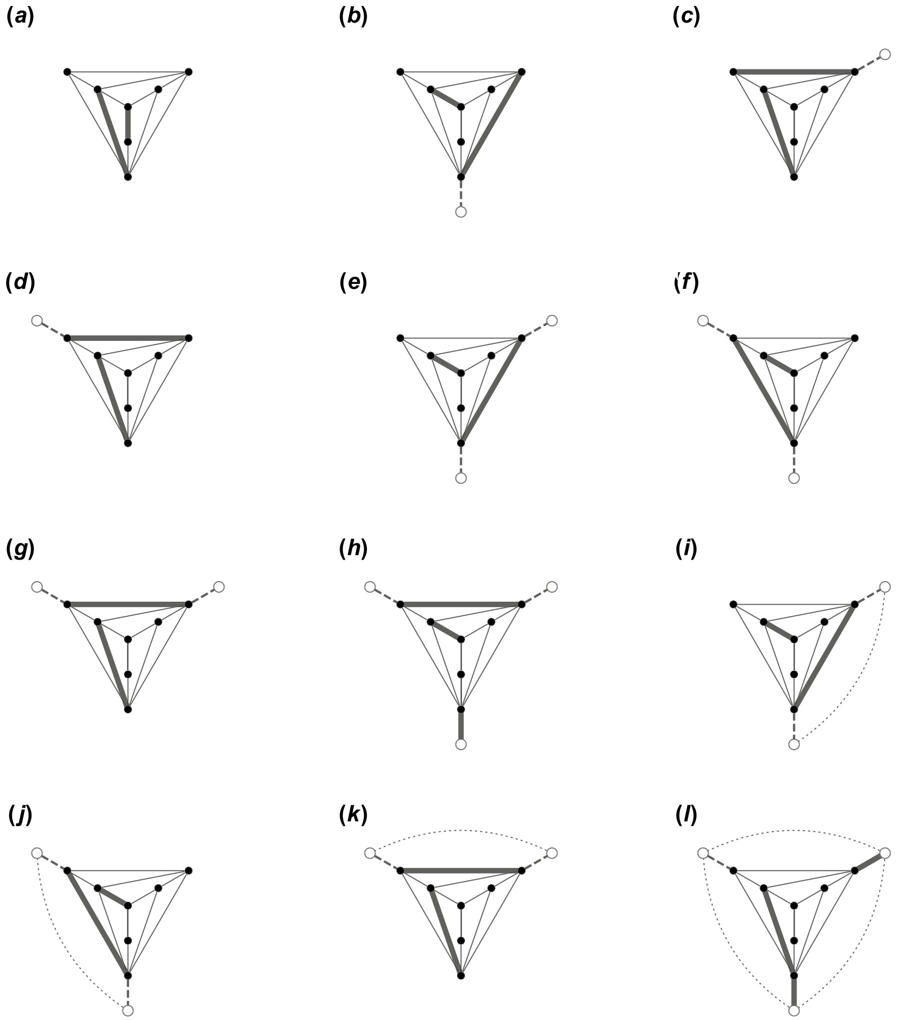

Fig. 1 Illustrations and edge colorings of the subgraph γ used in Theorem 2 to reduce the Hamiltonian path problem on a cubic 2-connected planar graph G, with vertex set V_G, to the problem of determining if the affine optimal $(k = 1)$-proper connection number $\zeta^1_{\mathcal{A}'}(H)$, for affine function $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$ and a 2-connected planar graph H, is $\leq 2 \cdot |V_G|$; **(a–l)** colorings minimizing $\zeta^1_{\mathcal{A}'}(G)$ while ensuring the existence of a proper path between all (solid black) vertices. See the proof argument of Theorem 2 for further details

paths to traverse paths between (hollow white) vertices (curved dotted lines), and to do so regardless of the coloration of the edge they traverse prior to ingressing or after egressing the path. Concerning the contribution to $\zeta^1_{\mathcal{A}'}(H)$ for each illustrated coloring in Fig. 1a–l, we sum the number of (thick black) edges between vertices internal to the same γ subgraph with a fraction $\frac{1}{2}$ (by the handshaking lemma) of the number of (thick black) edges connecting vertices not belonging to the same γ subgraph.

To now show that (constraint 2) holds, we can determine for the cases shown in Fig. 1a–g, i–l, which are consistent with a 1-proper connected coloring for H corresponding to a spanning tree T for G having minimum degree ≤ 2, that each γ subgraph incurs a cost of 2 to $\zeta^1_{\mathcal{A}'}(H)$ regardless of whether it corresponds to a leaf for T (as in the Fig. 1b–d cases) or whether G is an isolated vertex (yielding the Fig. 1a case). We can also determine that the remaining Fig. 1h instance, which would allow for a degree 3 vertex in the aforementioned spanning tree T, incurs a cost of $\frac{3}{2}$.

Putting everything together, we have that $\zeta^1_{\mathcal{A}'}(H) \leq 2 \cdot |V_G|$ if and only if G is traceable. As the Hamiltonian path decision problem for G is *NP*-complete by Lemma 1, this yields the theorem. □

Concerning cases where $k \geq 2$, we refer the reader to Theorem 4 (in the Appendix) for a proof that $\zeta^2_{\mathcal{A}'}(G)$ is *NP*-hard to determine for cubic 3-connected planar graphs, and Theorem 5 (in the Appendix) that, $\forall k \geq 3$, $\zeta^k_{\mathcal{A}'}(G)$ is *NP*-hard to determine for k-connected graphs. Together with Theorem 2, yields the following approximation hardness result:

Theorem 3 *No FPRAS can exist that approximates $\zeta^k_{\mathcal{A}'}(G)$ in any of the following cases*:

- (case 1) $k = 1$ and G is a 2-connected planar graph;
- (case 2) $k = 2$ and G is a cubic 3-connected planar graph;
- (case 3) $k \geq 3$ and G is a k-connected graph.

Proof From the proof arguments for Theorems 2, 4, and 5, recall that we are able to decide if a graph G with vertex set V_G has a Hamiltonian path (case 1), Hamiltonian cycle (case 2), or k-regular k-connected spanning subgraph with $\leq \frac{k}{2} \cdot |V_G|$ edges (case 3), by constructing a graph H from G in polynomial time and checking, for affine function $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$, that its affine optimal k-proper connection number, $\zeta^k_{\mathcal{A}'}(H)$, is equal to $\alpha \cdot |V_G|$ for some constant $\alpha \in \mathbb{N}$. Now, following the definition for an FPRAS given by Karp and Luby [20], let $\mathcal{Q}$ be an FPRAS for $\zeta^k_{\mathcal{A}'}(H)$ accepting an input string x, having an error parameter $\epsilon < \left(\frac{1}{2(\alpha \cdot |V_G|)}\right)$, and having an accuracy parameter $\delta = \frac{1}{3}$. We accordingly have that $\mathcal{Q}$ yields a *BPP* algorithm for checking if $\zeta^k_{\mathcal{A}'}(H) \leq \alpha \cdot |V_G|$, as we can simply round the output of $\mathcal{Q}$ to the nearest integer and make a correct guess with probability $1 - \delta = \frac{2}{3}$. However, as each of the aforementioned decision problems in (case 1) through (case 3) are *NP*-complete (as detailed in the proof arguments for Theorem 2, 4, and 5), we have that the existence of $\mathcal{Q}$ would necessarily imply in each of these cases that $NP \subseteq BPP$, and therefore, that $NP = RP$. □

Corollary 2 *The hardness results established in Theorems* 2 *through* 5 *hold for exactly and approximately computing the parameters* $pc^k_{opt}(G)$ *and* $pc^k_{opt'}(G)$.

Proof Recall that the proof arguments for Theorems 2, 4, and 5 proceed by reducing an *NP*-complete problem of deciding the existence of an object (e.g., a Hamiltonian path in Theorem 2) to the problem of deciding if, for affine function $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$, we have that $\zeta^k_{\mathcal{A}'}(G) \leq \mathcal{M}$ for some $\mathcal{M} \in \mathbb{N}_{>0}$. In each case we can also observe that $\mathcal{M}$ is the smallest possible value for $\zeta^k_{\mathcal{A}'}(G)$, and can check—by brute force enumeration in Theorems 2 and 4, and by using a simple induction proof in Theorem 5—that treating all edges in the color class C_2 as having a distinct coloration cannot yield a smaller value of $\zeta^k_{\mathcal{A}'}(G)$. Accordingly, we have that $\zeta^k_{\mathcal{A}'}(G)$ will be equivalent to the parameters $pc^k_{opt}(G)$ and $pc^k_{opt'}(G)$ in these instances. Finally, we can observe that the proof argument for Theorem 3 depends only on our using polynomial time reductions from *NP*-hard problems to deciding if $\zeta^k_{\mathcal{A}'}(G) \leq \mathcal{M}$ in Theorems 2, 4, and 5. Putting everything together, we therefore have that the proof arguments for Theorem 2 through Theorem 5 can be used to establish the same hardness results for parameters $pc^k_{opt}(G)$ and $pc^k_{opt'}(G)$. □

5 Concluding remarks and open problems

We have shown $\forall k \in \mathbb{N}_{>0}$, and for every fixed number of color classes $n \in \mathbb{N}_{>0}$ into which the graph's edge set can be decomposed, that there exists a treewidth FPT algorithm for computing the affine optimal k-proper connection number, $\zeta^k_{\mathcal{A}}(G)$. Here, while we have also shown that no such bound on the number of color classes is required if we restrict our consideration to affine functions to the form $\mathcal{A}^* := 0 \cdot |C_1| + \sum_{i=2}^{|E_G|} |C_i|$, we consider it an interesting open question as to whether there exist treewidth FPT algorithms for more general affine functions. We likewise pose the question as to whether treewidth FPT algorithms exist for the optimal k-proper connectivity parameter, $pc^k_{opt}(Q)$, without a bound for the number of color classes. Finally, we pose the question as to whether efficient dynamic programming algorithms exist for computing the parameters discussed in this work.

Appendix

This appendix contains a helper lemma that we refer to in the proof argument for Theorem 2 (Lemma 1), proof arguments for Theorem 4 and Theorem 5 referenced in the main text, and a helper lemma for Theorem 5 (Lemma 2). We remark that the full power of Lemma 1 is not utilized for the Theorem 2 proof argument.

Lemma 1 *The Hamiltonian path decision problem on cubic* 3-*connected planar graphs*, *where we additionally require that the shortest path between the endpoints of any embedded Hamiltonian path is* ≥ 4 *edges in length*, *is NP-complete*.

Proof We begin by remarking that there already exists a circa 1976 proof, due to Garey et al. [17], that the Hamiltonian path problem (referred to as the "Hamiltonian line" problem by the aforementioned authors) is *NP*-complete on cubic 3-connected planar graphs. Specifically, on "pg. 713" of ref. [17] the author's write: *"Finally, the undirected planar Hamiltonian line problem is NP-complete: convert the "or" linking edges* $\{v_{11}, w_{11}\}$ *and* $\{v_{n4}, w_{m6}\}$ *into an "exclusive or". A Hamiltonian line must either start at* v_{11} *and finish at* w_{11}*, or start at* v_{n4} *and finish at* w_{m6}*. Such a line will exist if and only if the original graph had a Hamiltonian circuit. Note that the construction preserves triple connectivity and degree threeness, as well as planarity."* Briefly, the aforementioned linking edges $\{v_{11}, w_{11}\}$ and $\{v_{n4}, w_{m6}\}$ correspond to the edges at the bottom and top of the author's "Fig. 7" example instance of their reduction construct.

Here, it is either the case that Garey et al. [17] inadvertently omitted further important details of their reduction, or that their proof is erroneous. In particular, consider that substituting the "exclusive or" (XOR) gadget as the authors specify does not prevent a Hamiltonian path from ingressing and egressing the gadget and having its endpoints embedded elsewhere. One can also construct an explicit example of the author's reduction construct starting with an unsatisfiable instance of 3*SAT*, then using Hamiltonian cycle or subgraph embedding enumeration algorithms—e.g., the Mathematica 12.0.0 'FindHamiltonianCycle[]' procedure [19] or the SageMath 8.3.0 'SubgraphSearch()' procedure [25]—find Hamiltonian paths that are difficult to meaningfully characterize.

To construct a specific counterexample, we performed the author's reduction for the trivially unsatisfiable 3*SAT* instance $(x \vee x \vee x) \wedge (\neg x \vee \neg x \vee \neg x)$, yielding the graph shown in Fig. 2 with 436 vertices and $436 \cdot \left(\frac{3}{2}\right) = 654$ edges. Our embedding for this graph is meant to resemble the one given by Garey et al. [18] in their "Fig. 7" illustration, and we use (dotted) boxes to highlight the sections of the reduction construct encoding the XOR gadget, clauses $(x \vee x \vee x)$ and $(\neg x \vee \neg x \vee \neg x)$, and the positive and negative literals x and $\neg x$, respectively. Here, despite the fact that the encoded formula is trivially unsatisfiable, the highlighted (thick black) edges trace a Hamiltonian path with both endpoints (indicated via black diamond markers) in the section of the graph encoding the $(x \vee x \vee x)$ clause.

Noting that there appears to be nothing wrong with the *NP*-completeness proof given by Garey et al. [17] for the Hamiltonian cycle decision problem on cubic 3-connected planar graphs, we will establish the current lemma via a different reduction to the Hamiltonian path decision problem.

Let M be an instance of a cubic 3-connected planar graph which is non-Hamiltonian but traceable, and create a graph R by deleting a vertex in M, doing so in such that the resulting graph remains traceable (observe that this will always be possible). Create two copies of R, which we'll denote R_1 and R_2, and create a reduction gadget Υ by: (step 1) adding an edge between one degree 2 vertex in R_1 and one degree 2 vertex in R_2; (step 2) creating a new vertex v_q, then adding an edge between v_q and a degree 2 vertex in both R_1 and R_2. Now, let G be an arbitrary cubic 3-connected planar graph with vertex set V_G, and let $v_i \in V_G$ be an arbitrary vertex of this graph adjacent to some set of vertices $v_x, v_y, v_z \in V_G$. Create a graph H from G by

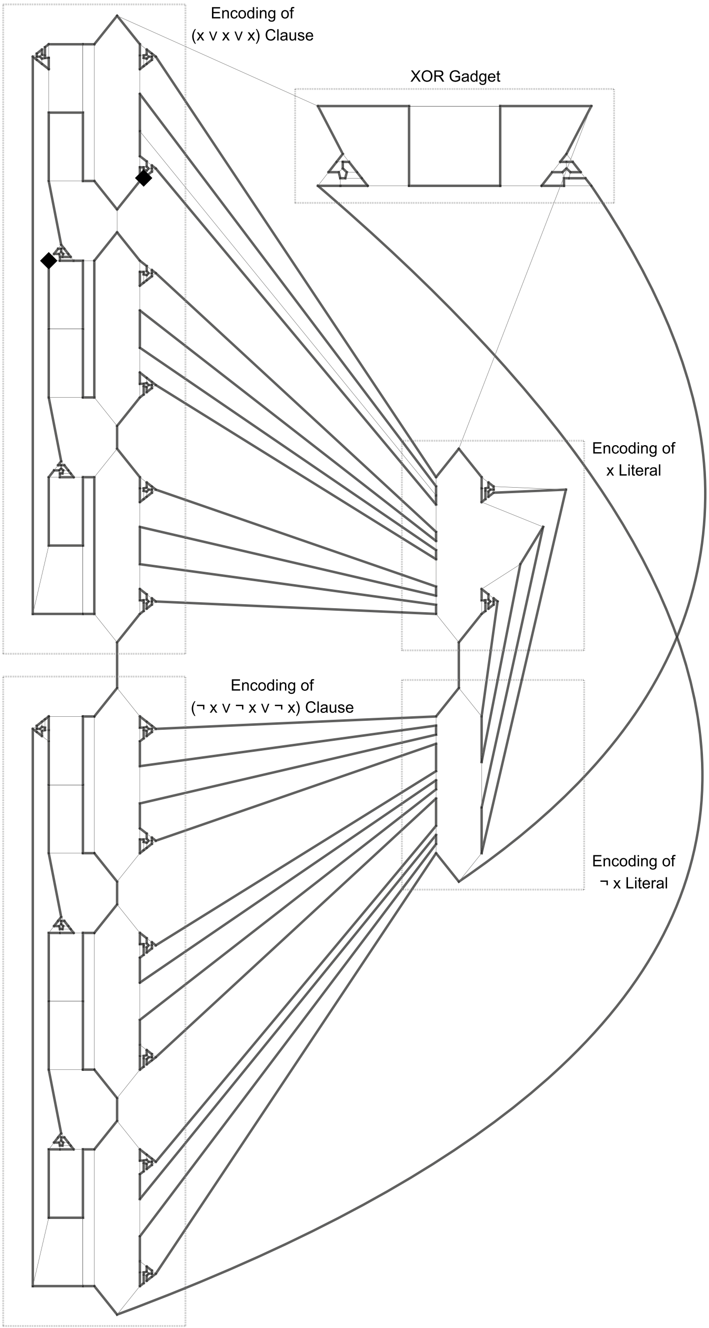

Fig. 2 Explicit counterexample the construction given in Garey et al. [17] to reduce arbitrary instances of 3*SAT* to the Hamiltonian path decision problem on cubic 3-connected planar graphs; see the proof argument for Lemma 1 for further details

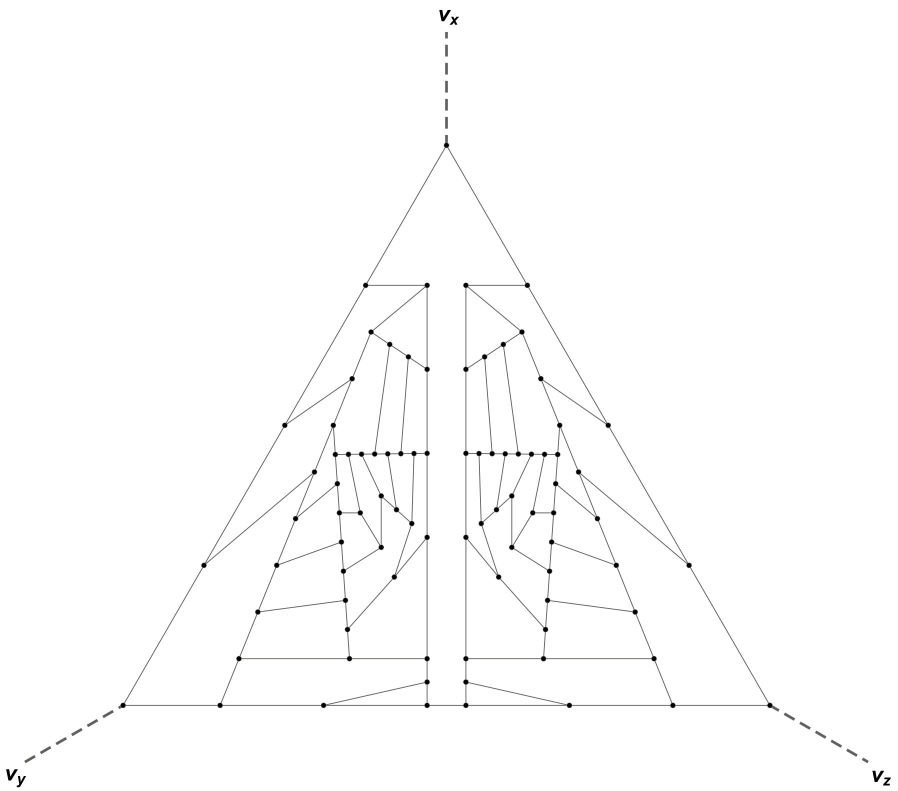

Fig. 3 Reduction gadget Υ, used in the proof argument for Lemma 1 to reduce the Hamiltonian cycle decision problem on cubic 3-connected planar graphs to the Hamiltonian path decision problem on the same class of graphs, where we additionally require that the shortest path between the endpoints of any embedded Hamiltonian path is ≥ 4 edges in length

substituting Υ in place of $v_i \in V_G$, in such a manner that the three degree 2 vertices for Υ are each connected to a distinct vertex in the set $v_x, v_y, v_z \subset V_G$. Observe that the result of this procedure will be a cubic 3-connected planar graph.

To see that H has a Hamiltonian path only if G has a Hamiltonian cycle, observe that any Hamiltonian path for H must have both its endpoints embedded internally to R_1 and R_2, respectively. More specifically, observe that R_1 and R_2 were derived by deleting a single vertex in a non-Hamiltonian graph M, and therefore, that any other type of embedding of a Hamiltonian path would imply the existence of a Hamiltonian cycle for M.

It now suffices to provide an explicit example of Υ where we can show that H has a Hamiltonian path if and only if G has a Hamiltonian cycle. Here, let M be a slightly modified variant of the graph "NH42.a" on 42 vertices shown in "Fig. 2" of Aldred et. al. [2], where we replace a select vertex with a Cyc_3 subgraph in such a manner that the graph remains cubic, 3-connected, and planar. Now let Υ be the subgraph on 87 vertices shown in Fig. 3—with 129 edges in addition to three outgoing edges to adjacent cubic graph vertices $\{v_x, v_y, v_z\}$—where the vertex v_q is adjacent to v_x, and the bilaterally symmetric embedding shows the 41 vertex R_1 and R_2 subgraphs on the left- and right-hand sides of the embedding, respectively. Using an algorithm based on either the Mathematica 12.0.0 'FindHamiltonianCycle[]' procedure [19] or the SageMath 8.3.0 'SubgraphSearch()' procedure [25] (with modifications in either case to decompose and individually treat subcomponents of the graph), we can determine that there are 3265280, 3265280, and 4431736 possible manners in which a Hamiltonian path can ingress and egress the Fig. 3 instance of Υ via the edges between the gadget and vertices $\{v_x, v_y\}$, $\{v_x, v_z\}$, and $\{v_y, v_z\}$, respectively. We therefore have that there exists at least one instance of the Υ gadget that can be used to guarantee that H has a Hamiltonian path if and only if G has a Hamiltonian cycle.

Finally, we can explicitly check that no graph containing the aforementioned example of the Υ gadget as an induced subgraph can embed a Hamiltonian path where the shortest path between the Hamiltonian path's endpoints is ≤ 3 edges in length, yielding the lemma. □

Theorem 4 *It is NP-hard to determine* $\zeta^2_{\mathcal{A}'}(G)$ *for a cubic* 3*-connected planar graph* G.

Proof To prove this theorem, we proceed via reduction from the *NP*-complete problem of deciding the existence of a Hamiltonian cycle on a cubic 3-connected planar graph [17]. To begin, let G be an arbitrary cubic 3-connected planar graph with vertex set V_G and edge set E_G. Generate a graph H from G by replacing each vertex $v_i \in V_G$ with a Cyc_3 subgraph given by the edge set $\{v_{(i,1)} \leftrightarrow v_{(i,2)}, v_{(i,1)} \leftrightarrow v_{(i,3)}, v_{(i,2)} \leftrightarrow v_{(i,3)}\}$, reconnecting formerly adjacent vertices v_a, v_b, and v_c to vertices $v_{(i,1)}$, $v_{(i,2)}$, and $v_{(i,3)}$, respectively. Letting V_H and E_H be the vertex and edge sets for the graph H, respectively, we can observe that $|V_H| = 3 \cdot |V_G|$ and that $|E_H| = 3 \cdot |V_G| + |E_G| = 6 \cdot |V_G|$.

It now suffices to observe that H will admit a $(k = 2)$-proper connected coloring if and only if $\zeta^2_{\mathcal{A}'}(H) \leq |V_G|$, where we specify $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$. As in the proof argument for Theorem 2, our strategy will be to show that, if G is Hamiltonian, exactly one edge per Cyc_3 subgraph must be placed in the color class C_2 (the remainder placed in the color class C_1) to obtain a 2-proper connected coloring for H, and that this is impossible if G is not Hamiltonian. We can do so by showing that the set of proper paths for any 2-proper connected coloring allowing for $\zeta^2_{\mathcal{A}'}(H) \leq |V_G|$ will, upon contraction of each Cyc_3 subgraph, be embedded along a fixed Hamiltonian cycle Q for G.

This leads us to the (very) simple caseology shown in Fig. 4, where we show the Cyc_3 subgraph in all relevant local contexts in H permitting the ingress and egress of two vertex disjoint proper paths—consistent with the requirement for a 2-proper connected coloring—with the same vertex and edge stylization as detailed in the Theorem 2 proof argument. Using brute force enumeration, we can determine that the only colorings allowing for $\zeta^2_{\mathcal{A}'}(H) \leq |V_G|$ partition (up to automorphism) the edges corresponding to the Fig. 4a illustrated (thick black lines) in color class C_2

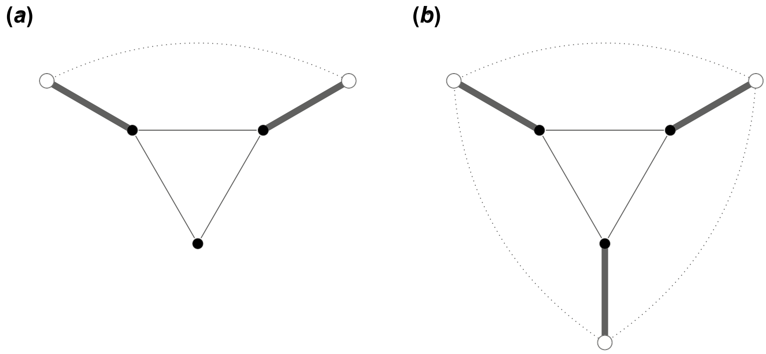

Fig. 4 Illustrations and edge colorings of the subgraph Cyc_3 used in Theorem 4 to reduce the Hamiltonian cycle decision problem for a cubic 3-connected planar graph G, with vertex set V_G, to the problem of determining if the affine optimal ($k = 2$)-proper connection number $\zeta^2_{\mathcal{A}'}(H)$, for affine function $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$ and a cubic 3-connected planar graph H, is $\leq |V_G|$; **(a)** coloring minimizing $\zeta^2_{\mathcal{A}'}(G)$ while ensuring the existence of a proper path between all (solid black) vertices; **(b)** coloring that must occur for some Cyc_3 subgraph if G is non-Hamiltonian. See the proof argument of Theorem 4 for further details

—one per Cyc_3 subgraph by the handshaking lemma—and the remainder in the color class C_1. We can also determine that, in the case where G is non-Hamiltonian, at least one instance of a subgraph colored in the manner of Fig. 4b will need to exist in H to allow for a 2-proper connected coloring.

Putting everything together, as the colorings shown in Fig. 4a, b contribute a factor of 1 and $\frac{3}{2}$ to $\zeta^2_{\mathcal{A}'}(H)$, respectively, we have that $\zeta^2_{\mathcal{A}'}(H) \leq |V_G|$ if and only if G is Hamiltonian. Accordingly, as the Hamiltonian cycle decision problem for G is NP-complete, we have that it is NP-hard to determine if $\zeta^2_{\mathcal{A}'}(H) \leq |V_G|$, yielding the theorem. $\square$

Lemma 2 *Deciding the existence of a k-regular k-connected spanning subgraph S for a given k-connected graph of maximum degree $k + 1$ is NP-complete $\forall k \geq 3$.*

Proof We proceed via reduction from the NP-complete problem [17] of deciding the existence of a Hamiltonian cycle (i.e., a spanning 2-connected subgraph) for a cubic 3-connected graph G with vertex set V_G, where we additionally have that all vertices are assigned a unique label from the interval $[1, |V_G|]$.

We begin by creating $k-2$ copies of the cycle graph $Cyc_{|V_G|}$, with labels $1, 2, \ldots, |V_G|$ assigned to vertices in the order that they occur in the unique cycle for the graph. We then add edges between vertices in the cycle graphs with the same integer label, yielding a graph Q isomorphic to $Cyc_{|V_G|} \square K_{k-2}$ (i.e., the Cartesian product of the cycle graph with $|V_G|$ vertices and the clique with $k-2$ vertices). Here, we can observe that Q will be a $(k-1)$-regular $(k-1)$-connected graph that is isomorphic to its own spanning $(k-1)$-regular subgraph. Finally, we create a graph H from G and Q by taking the union of the two graphs and adding edges between all non-adjacent vertices with identical integer labels. Letting V_H and E_H be the vertex and edge sets for H, respectively, we can note that $|V_H| = (k-1) \cdot |V_G|$ and $|E_H| = \frac{1}{2}(k \cdot (k-1) + 1) \cdot |V_G|$.

We can now observe that H will be k-connected, have maximum vertex degree $k+1$, and will possess a k-regular k-connected spanning subgraph with $|E_H| - \left(\frac{|V_G|}{2}\right) = \frac{1}{2}(k \cdot (k-1) + 1) \cdot |V_G| - 4$ edges if and only if G is Hamiltonian. Accordingly, we have that G is Hamiltonian if and only if a k-regular k-connected spanning subgraph S exists for a polynomial time constructable graph H with maximum vertex degree $k+1$, yielding the lemma. □

Theorem 5 *For all $k \geq 3$, it is NP-hard to determine $\zeta^k_{\mathcal{A}'}(G)$ for a k-connected graph G.*

Proof To prove this theorem, we proceed via reduction from the problem of deciding the existence of a k-regular k-connected spanning subgraph S for given k-connected graph G of maximum degree $k+1$, which is *NP*-complete as a consequence of Lemma 2.

To begin, let Λ_k and Λ'_k and be a pair of subgraphs generated from the cliques K_{2k} and K_{2k+1}, respectively, by adding a new vertex v_s and connecting it to k edges in the clique. Now let G be an arbitrary k-connected graph with minimum degree k, maximum degree $k+1$, vertex set V_G, and edge set E_G. Generate a graph H from G by replacing each vertex $v_i \in V_G$ of degree k and degree $k+1$ with subgraphs Λ and Λ', respectively, reconnecting each vertex formerly adjacent to v_i to a unique vertex in the subset of vertices in either Λ_k or Λ'_k that are non-adjacent to the vertex v_s. Letting V_H and E_H be the vertex and edge sets for the graph H, respectively, and letting n_k and n_{k+1} be the number of vertices in G of degree k and $k+1$, respectively, we can observe that $|V_H| = 2k \cdot n_k + (2k+1) \cdot n_{k+1}$ and that $|E_H| = 2k \cdot (k \cdot (n_k + n_{k+1}) + n_{k+1})$. For illustrative examples of Λ_k and Λ'_k for $k = 3, 4, 5$ we refer the reader to Fig. 5a–f, respectively.

It now suffices to observe that H will admit a k-proper connected coloring if and only if $\zeta^k_{\mathcal{A}'}(H) \leq k \cdot |V_G|$, where we specify $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$. Akin to the earlier Theorem 2 and Theorem 3 proof arguments, our strategy will be to show that, if G admits a k-regular k-connected spanning subgraph S, exactly k edges per Λ_k or Λ'_k subgraph must be placed in the color class C_2 (the remainder placed in the color class C_1) to obtain a k-proper connected coloring for H, and that this is impossible if G does not admit such a spanning subgraph S. We can do so by showing that the set of proper paths for any k-proper connected coloring allowing for $\zeta^k_{\mathcal{A}'}(H) \leq k \cdot |V_G|$ will, upon contraction of each Λ and Λ' subgraph, be embedded along the spanning subgraph S.

Here, letting v_a be a vertex adjacent to an "outgoing" edge with only one end in given subgraph Λ_k or Λ'_k, and letting v_b be a vertex adjacent to v_s, we can observe

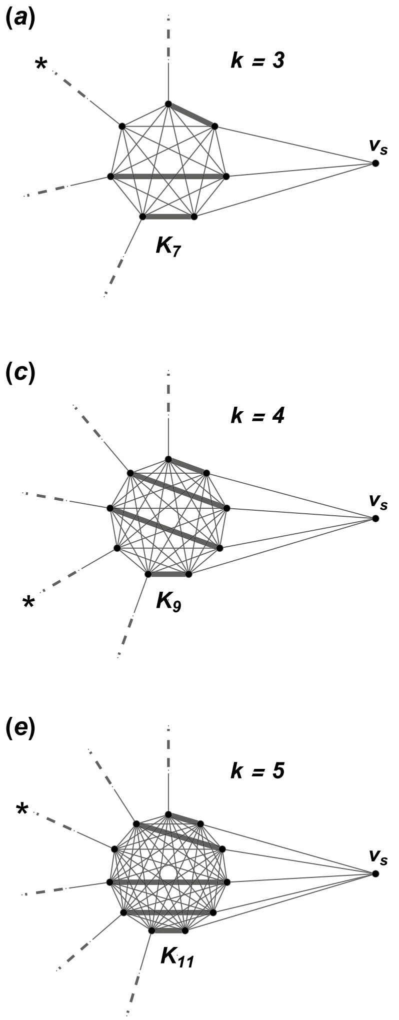

Fig. 5 Illustration and edge colorings of the subgraphs Λ' and Λ used in Theorem 5 to reduce the problem of deciding the existence of a k-regular k-connected spanning subgraph S for given k-connected graph G of maximum degree $k+1$, to the problem of determining if the affine optimal k-proper connection number $\zeta^k_{\mathcal{A}'}(H)$, for affine function $\mathcal{A}' := 0 \cdot |C_1| + |C_2|$ and a k-connected graph H, is $\leq k \cdot |V_G|$; **a–f** explicit examples of subgraphs Λ' **a**, **c**, **e** and Λ **b**, **d**, **f** for $3 \leq k \leq 5$, with colorings minimizing $\zeta^k_{\mathcal{A}'}(G)$ while ensuring the existence of k vertex disjoint proper path between all pairs of vertices

that – under the constraint $\zeta^k_{\mathcal{A}'}(H) \leq k \cdot |V_G|$ – the edge $v_a \leftrightarrow v_b$ must be assigned to color class C_2 to allow for the "outgoing" edge adjacent to v_a to belong to S in the aforementioned contraction of H. Proceeding with this argument, we can determine that exactly k such edges must be assigned to color class C_2 in the Λ_k or Λ'_k subgraphs. These minimal colorings are shown in Fig. 5a–f, where, in the case of the Λ_k' subgraph, we use the '$*$' character to indicate the edge in the contraction of H that does not participate in S.

Putting everything together, we have that $\zeta^k_{\mathcal{A}'}(H) \leq k \cdot |V_G|$ if and only if G admits a k-regular k-connected spanning subgraph S. As deciding if G admits S is NP-complete by Lemma 2, this yields the theorem. $\square$

Acknowledgements R. D. Barish would like to thank Shinya Fujita for introducing him to his concept of optimal proper graph connectivity, and for a number of interesting correspondences that directly inspired the current manuscript.

Funding Open Access funding provided by The University of Tokyo. This work was supported by a Grant-in-Aid for JSPS Research Fellow (18F18117 to R. D. Barish) from the Japan Society for the Promotion of Science (JSPS), as well as JSPS Kakenhi Grants {20K21827, 20H05967, 21H04871} and JST CREST Grant JPMJCR1402JST to T. Shibuya.

References

1. Aardal, K.I., van Hoesel, S.P.M., Koster, A.M.C.A., Mannino, C., Sassano, A.: Models and solution techniques for frequency assignment problems. Ann. Oper. Res. **153**(1), 79–129 (2007). https://doi.org/10.1007/s10479-007-0178-0
2. Aldred, R.E.L., Bau, S., Holton, D.A., McKay, B.D.: Nonhamiltonian 3-connected cubic planar graphs. SIAM J. Discrete Math. **13**(1), 25–32 (2000). https://doi.org/10.1137/S0895480198348665
3. Andrews, E., Lumduanhom, C., Laforge, E., Zhang, P.: On proper-path colorings in graphs. J. Comb. Math. Comb. Comput. **97**, 189–207 (2016)
4. Arnborg, S., Lagergren, J., Seese, D.: Easy problems for tree-decomposable graphs. J. Algorithms **12**(2), 308–340 (1991). https://doi.org/10.1016/0196-6774(91)90006-K
5. Bondy, J.A., Murty, U.S.R.: Graph Theory with Applications, 1st edn. Macmillan Press, New York (1976)
6. Borozan, V., Fujita, S., Gerek, A., Magnant, C., Manoussakis, Y., Montero, L., Tuza, Z.: Proper connection of graphs. Discrete Math. **312**(17), 2550–2560 (2012). https://doi.org/10.1016/j.disc.2011.09.003
7. Courcelle, B.: The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. Inform. Comput. **85**(1), 12–75 (1990). https://doi.org/10.1016/0890-5401(90)90043-H
8. Courcelle, B.: On the expression of graph properties in some fragments of monadic second-order logic. In: Immerman, N., Kolaitis, P.G. (eds.) Descriptive Complexity and Finite Models, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 31, pp. 33–62 (1997)
9. Courcelle, B.: The monadic second-order logic of graphs XII: planar graphs and planar maps. Theor. Comput. Sci. **237**(1–2), 1–32 (2000). https://doi.org/10.1016/S0304-3975(99)00305-9
10. Courcelle, B.: Graph structure and monadic second-order logic: language theoretical aspects. In: Proceedings of the 35th International Colloquium on Automata, Languages and Programming (ICALP), pp. 1–13 (2008). https://doi.org/10.1007/978-3-540-70575-8_1

11. Courcelle, B., Makowsky, J.A., Rotics, U.: On the fixed parameter complexity of graph enumeration problems definable in monadic second-order logic. Discrete Appl. Math. **108**(1–2), 23–52 (2001). https://doi.org/10.1016/S0166-218X(00)00221-3
12. Cygan, M., Fomin, F.V., Kowalik, L., Lokshtanov, D., Marx, D., Pilipczuk, M., Pilipczuk, M., Saurabh, S.: Parameterized Algorithms, 1st edn. Springer, Cham (2015)
13. de Werra, D., Gay, Y.: Chromatic scheduling and frequency assignment. Discrete Appl. Math. **49**(1–3), 165–174 (1994). https://doi.org/10.1016/0166-218X(94)90207-0
14. Diestel, R.: Graph Theory, 5th edn. Springer, Heidelberg (2016)
15. Eisenblätter, A., Grötschel, M., Koster, A.M.C.A.: Frequency planning and ramifications of coloring. Discuss. Math. Graph Theory **22**(1), 51–88 (2002). https://doi.org/10.7151/dmgt.1158
16. Fujita, S.: Optimal proper connection of graphs. Optim. Lett. **14**(6), 1371–1380 (2020). https://doi.org/10.1007/s11590-019-01442-9
17. Garey, M.R., Johnson, D.S., Tarjan, R.E.: The planar Hamiltonian circuit problem is NP-complete. SIAM J. Comput. **5**(4), 704–714 (1976). https://doi.org/10.1137/0205049
18. Hale, W.K.: Frequency assignment: theory and applications. Proc. IEEE **68**(12), 1497–1514 (1980). https://doi.org/10.1109/PROC.1980.11899
19. Inc., W.R.: Mathematica, Version 12.0.0. https://www.wolfram.com/mathematica. Champaign, IL (2019)
20. Karp, R.M., Luby, M.: Monte-Carlo algorithms for enumeration and reliability problems. In: Proceedings of the 24th annual symposium on Foundations Of Computer Science (FOCS), pp. 56–64 (1983). https://doi.org/10.1109/SFCS.1983.35
21. Li, X., Magnant, C.: Properly colored notions of connectivity—a dynamic survey. Theory Appl. Graphs (2015). https://doi.org/10.20429/tag.2015.000102
22. Metzger, B.H.: Spectrum management technique. Presentation at the 38th National ORSA meeting, Detroit, MI (1970)
23. Park, T., Lee, C.Y.: Application of the graph coloring algorithm to the frequency assignment problem. J. Oper. Res. Soc. Jpn. **39**(2), 258–265 (1996). https://doi.org/10.15807/jorsj.39.258
24. Roberts, F.S.: T-colorings of graphs: recent results and open problems. Discrete Math. **93**(2–3), 229–245 (1991). https://doi.org/10.1016/0012-365X(91)90258-4
25. The Sage Developers: SageMath, the Sage Mathematics Software System (Version 8.3.0) (2018). https://www.sagemath.org