Optimization Letters (2023) 17:1241–1259
https://doi.org/10.1007/s11590-022-01934-1

ORIGINAL PAPER

Two-machine flow shop scheduling with convex resource consumption functions

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Received: 7 August 2021 / Accepted: 8 September 2022 / Published online: 16 September 2022

Abstract
We consider a two-machine flow shop scheduling problem in which the processing time of each operation is inversely proportional to the power of the amount of resources consumed by it. The objective is to minimize the sum of the makespan and the total resource consumption cost. We show that the problem is NP-hard, and its constrained version remains so. Then, we develop 1.25- and 2-approximation algorithms for the problem and its constrained version, respectively.

Keywords Scheduling · Convex resource consumption · Computational complexity · Approximation algorithms

1 Introduction

Scheduling problems with controllable processing times have been extensively studied since Vickson [13]. Refer to [6, 10] for the comprehensive surveys. In most scheduling problems with controllable processing times, it is assumed that the processing time of job j is determined by a linear resource consumption function, which is described as

$$p_{i,j}(u_{i,j}) = \bar{p}_{i,j} - u_{i,j}, \quad 0 \le u_{i,j} \le \bar{u}_{i,j}, \tag{1}$$

where $u_{i,j}$ is the resource consumption amount of job j on machine i, and $\bar{p}_{i,j}$ and $\bar{u}_{i,j}$ are the initial processing time and upper bound on the resource consumption amount

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of job j on machine i, respectively. However, linear resource consumption functions in (1) cannot reflect the *law of diminishing marginal returns*, which can be found in many resource allocation problems in physical or economic systems. The law means that productivity increases at a decreasing rate with the resource consumption amount. In this paper, to reflect this, assume that each job processing time decreases at an increasing rate with the amount of resource consumption, which is described as

$$p_{i,j}(u_{i,j}) = \left(\frac{w_{i,j}}{u_{i,j}}\right)^k, \tag{2}$$

where $w_{i,j} > 0$ is the workload of operation $O_{i,j}$ and k is a positive constant. Thus, we mainly focus on introducing the previous research with convex resource consumption functions in (2). Table 1 shows a summary of complexity results for scheduling problems with convex resource consumption functions. Please refer to [10] for any missing definitions.

For a single-machine case, Kayan and Akturk [7] and Cheng and Janiak [2] introduced the application of convex resource consumption functions. In a CNC machine scheduling problem [7], the job processing time is determined by a convex function in (2) of the feed rate and spindle speed used for each operation. In a steel mill industry [2], the time to preheat each batch of ingots in each soaking pit to a certain temperature is determined by a convex decreasing function of the gas flow intensity. Shabtay and Kaspi [8] considered $1|conv, \sum_j u_j \le K| \sum_j v_j C_j$, where v_j is the weight of job j. Although the computational complexity remains open, they revealed a closed-form of the optimal resource allocation for a given job sequence, and presented polynomially

Table 1 Complexity results for scheduling with convex resource consumption functions

Problem	Complexity	Ref.
$1\|conv\|(\sum_j C_j, \sum_j u_j)$	$O(n \log n)$	[8]
$1\|conv\|(C_{\max}, \sum_j c_j u_j)$	$O(n)$	[10]
$1\|conv, d_j = d, \sum_j T_j \le T\| \sum_j u_j$	$O(n^2)$	[14]
$1\|conv, up, \sum_j c_j u_j \le K\|C_{\max}$	NP-hard, 1.5-approx, FPTAS	[12]
$1\|conv, up\|C_{\max} + \sum_j c_j u_j$	NP-hard, FPTAS	[4]
$Pm\|conv, pmtn, \sum_j u_j \le K\|C_{\max}$	$O(n^2)$	[9]
$Pm\|conv, \sum_j u_j \le K\|C_{\max}$	NP-hard	[9]
$Pm\|conv, \sum_j u_j \le K\| \sum_j C_j$	$O(n \log n)$	[9]
$Fm\|conv, prmu, \sum_{i,j} u_{i,j} \le K, \alpha_{i,j} \le u_{i,j} \le \beta_{i,j}\|C_{\max}$	NP-hard, m-approx (general convex)	[2]
$F2\|conv, nw, \sum_{i,j} u_{i,j} \le K\|C_{\max}$	Strongly NP-hard, $2^{1/k+1}$ -approx	[11]
$F2\|conv\|C_{\max} + \sum_{i,j} c_{i,j} u_{i,j}$	NP-hard, 1.25-approx	Theorem 1 & Theorem 3
$F2\|conv, \sum_{i,j} c_{i,j} u_{i,j} \le K\|C_{\max}$	NP-hard, 2-approx	Corollary 2 & Theorem 4

solvable cases. Xu et al. [14] showed that $1|conv, d_j = d, \sum_j T_j \leq T| \sum_j u_j$ is solvable in $O(n^2)$. Recently, Shabtay and Zofi [12] considered $1|conv, \sum_j u_j \leq K|C_{max}$ with one unavailability period, and showed that it is NP-hard and has a 1.5-approximation algorithm and an FPTAS. Choi and Park [4] extended the results into the case with multiple unavailability periods. For a parallel-machine case, Shabtay and Kaspi [9] showed that $Pm|conv, \sum_j u_j \leq K|C_{\max}$ is NP-hard while its preemption vesion and $Pm|conv, \sum_j u_j \leq K| \sum_j C_j$ are polynomially solvable.

For a flow shop case, Cheng and Janiak [2] considered $Fm|conv, \sum_{i,j} u_{i,j} \leq K|C_{\max}$ such that permutation schedules are only considered and the bound constraint for each operation, that is, $\alpha_{i,j} \leq u_{i,j} \leq \beta_{i,j}$, exists. They proved the NP-hardness of the case with $m = 2$, and developed a branch-and-bound algorithm and three m-approximation algorithms whose performances were effective through numerical experiments. Shabtay et al. [11] considered $F2|conv, \sum_{i,j} u_{i,j} \leq K|C_{\max}$ such that no-wait constraints exist, that is, no idle time exists between the first and the second operations of each job. They proved its strong NP-hardness, developed a $2^{1/k+1}$-approximation algorithm, and introduced three polynomially solvable cases. Furthermore, they developed two heuristics whose performance were effective through numerical experiments.

To the best of our knowledge, there has been no study on the complexity of $F2|conv, \sum_{i,j} u_{i,j} \leq K|C_{\max}$ without the bound and no-wait constraints. Note that the optimality property obtained from the bound constraints is to fully compress or not to compress each job under an optimal schedule, which had been used for the NP-hardness proof of [5]. Cheng and Janiak [2] also proved the NP-hardness of $F2|conv, \sum_{i,j} u_{i,j} \leq K|C_{\max}$ with $\alpha_{i,j} \leq u_{i,j} \leq \beta_{i,j}$ based on this optimality property. Since the convex resource consumption functions in (2) are not locally bounded, however, this optimality property cannot be used for the NP-hardness of $F2|conv, \sum_{i,j} u_{i,j} \leq K|C_{\max}$. Furthermore, two cases with and without no-wait constraint are completely different problems. Thus, it is not straightforwardly that the NP-hardness result of [2, 5, 11] holds in $F2|conv, \sum_{i,j} c_{i,j}u_{i,j} \leq K|C_{\max}$.

The contributions of this paper are twofold. First, we prove the NP-hardness of $F2|conv|C_{\max} + \sum_{i,j} c_{i,j}u_{i,j}$ and $F2|conv, \sum_{i,j} c_{i,j}u_{i,j} \leq K|C_{\max}$, which implies the NP-hardness of $F2|conv|(C_{\max}, \sum_{i,j} c_{i,j}u_{i,j})$ whose complexity remains open in [10]. Second, we develop 1.25- and 2-approximation algorithms for $F2|conv|C_{\max} + \sum_{i,j} c_{i,j}u_{i,j}$ and $F2|conv, \sum_{i,j} c_{i,j}u_{i,j} \leq K|C_{\max}$, respectively. Since an optimal schedule exists among the set of the permutation schedules in the two-machine flow shop scheduling problem with makespan criterion, an m-approximation algorithm of Cheng and Janiak [2] becomes a 2-approximation algorithm for $F2|conv, \sum_{i,j} c_{i,j}u_{i,j} \leq K|C_{\max}$. This fact is consistent with our 2-approximability result of $F2|conv, \sum_{i,j} c_{i,j}u_{i,j} \leq K|C_{\max}$.

The remainder of this paper is organized as follows: Sections 2 and 3 introduce the problem definition and some optimality properties, respectively. In Sects. 4 and 5, we prove the NP-hardness and develop approximation algorithms, respectively.

2 Problem definition

Our problem can be formally stated as follows. Let $\mathcal{J} = \{1, 2, ..., n\}$ and $\mathcal{M} = \{1, 2\}$ be the sets of jobs and machines, respectively. Let $\mathcal{O} = \{O_{i,j} \mid i \in \mathcal{M} \text{ and } j \in \mathcal{J}\}$ be the set of operations, where $O_{i,j}$ is the operation of job j on machine i. For $O_{i,j} \in \mathcal{O}$, let $w_{i,j}$ be the workload of $O_{i,j}$. Let $\sigma = (\pi; u)$ be a permutation schedule such that

- $\pi = (\pi(1), \pi(2), ..., \pi(n))$ is the job sequence on both machines, where $\pi(j)$ is the jth job to be processed on both machines in permutation π;
- $u = (u_{i,j})_{O_{i,j} \in \mathcal{O}}$, where $u_{i,j} > 0$ is the resource consumption amount of operation $O_{i,j} \in \mathcal{O}$.

Note that

- It is known from [1] that it suffices to consider only the permutation schedule with respect to any regular performance measures (e.g., makespan) in the two-machine flow shop model;
- For $O_{i,j} \in \mathcal{O}$, the resource consumption cost and the processing time of $O_{i,j}$ are calculated as $c_{i,j}u_{i,j}$ and $p_{i,j}(u_{i,j}) = (w_{i,j}/u_{i,j})^k$, respectively, where $c_{i,j} > 0$ is the unit consumption cost.

For simplicity, we will use $p_{i,j}$ instead of $p_{i,j}(u_{i,j})$ for $O_{i,j} \in \mathcal{O}$ when no confusion exists. For $O_{i,j} \in \mathcal{O}$, let $C_{i,j}(\sigma)$ and $S_{i,j}(\sigma)$ be the completion and start times of $O_{i,j}$ in σ, and $C_{\max}(\sigma) = C_{2,\pi(n)}(\sigma)$ be referred to as the makespan. The objective is to find a schedule σ with the minimum sum of the makespan and the total resource consumption cost, that is,

$$\min \quad z(\sigma) = C_{\max}(\sigma) + \sum_{O_{i,j} \in \mathcal{O}} c_{i,j}u_{i,j}.$$

Let the problem above be referred to as *Problem P*. Furthermore, let the constrained version of Problem P be stated as follows:

$$\begin{aligned} &\min C_{\max}(\sigma) \\ &\text{s.t.} \sum_{O_{i,j} \in \mathcal{O}} c_{i,j}u_{i,j} \le K, \end{aligned}$$

where K is a given threshold.

3 Optimal properties of an optimal schedule

In this section, we introduce some optimality properties of Problem P. First, we introduce a new terminology. Let a job j be referred to as a *pivot* in σ, if $C_{1,j}(\sigma) = S_{2,j}(\sigma)$. Then, we have the following optimality properties.

Proposition 1 *The first and last jobs are pivots in any optimal schedule.*

Proof Let f and l be the first and last jobs in the optimal schedule, respectively. Without increasing the makespan, we can decrease $u_{2,f}$ and $u_{1,l}$ until jobs f and l become pivots. Thus, Proposition 1 holds.

Proposition 2 *No idle time exists between consecutive jobs in any optimal schedule.*

By Propositions 1 and 2, henceforth, we consider only a schedule σ with

$$\sum_{j=2}^{n} p_{1,\pi(j)} = \sum_{j=1}^{n-1} p_{2,\pi(j)},$$

which implies that

$$C_{\max}(\sigma) = \sum_{j=1}^{n} p_{1,\pi(j)} + p_{2,\pi(n)} = p_{1,\pi(1)} + \sum_{j=1}^{n} p_{2,\pi(j)}. \tag{3}$$

Let $\sigma^* = \left(\pi^*;u^*\right)$ be an optimal schedule, and $p^*_{i,j} = p_{i,j}(u^*_{i,j})$ for $O_{i,j} \in \mathcal{O}$. For simplicity, let

$$t_{i,j}(u_{i,j}) = p_{i,j}(u_{i,j}) + c_{i,j}u_{i,j} \ \text{ and } \ \tau_{i,j} = \left(\frac{kw^k_{i,j}}{c_{i,j}}\right)^{\frac{1}{k+1}} \ \text{ for } \ O_{i,j} \in \mathcal{O}.$$

Since

$$t'_{i,j}(u_{i,j}) = p'_{i,j}(u_{i,j}) + c_{i,j} = -k\frac{w^k_{i,j}}{u^{k+1}_{i,j}} + c_{i,j},$$

we have

$$t'_{i,j}(\tau_{i,j}) = p'_{i,j}(\tau_{i,j}) + c_{i,j} = 0. \tag{4}$$

By Eq. (4) and the strict convexity of $t_{i,j}(u_{i,j})$, $t_{i,j}(u_{i,j})$ is minimized at $u_{i,j} = \tau_{i,j}$ and

$$t_{i,j}(u_{i,j}) > t_{i,j}(\tau_{i,j}) \ \text{ for } \ u_{i,j} \neq \tau_{i,j}, \tag{5}$$

and, furthermore, we have

$$c_{i,j}\tau_{i,j} = c_{i,j}\left(\frac{kw_{i,j}^k}{c_{i,j}}\right)^{\frac{1}{k+1}} = k\left(\frac{c_{i,j}w_{i,j}}{k}\right)^{\frac{k}{k+1}} = kp_{i,j}(\tau_{i,j}). \tag{6}$$

Lemma 1 *In* σ^*,

$$u^*_{i,j} \le \tau_{i,j} \text{ for } O_{i,j} \in \mathcal{O}.$$

Proof Suppose that $u^*_{a,b} > \tau_{a,b}$ for some $O_{a,b} \in \mathcal{O}$. Then, we can construct a schedule $\bar{\sigma}$ by letting $\bar{u}_{a,b} = \tau_{a,b}$ and $\bar{u}_{i,j} = u^*_{i,j}$ for $O_{i,j} \in \mathcal{O} \setminus \{O_{a,b}\}$. Note that by $u^*_{a,b} > \tau_{a,b}$, we have $p_{a,b}(u^*_{a,b}) < p_{a,b}(\tau_{a,b})$ and

$$C_{\max}(\bar{\sigma}) - C_{\max}(\sigma^*) \le p_{a,b}(\tau_{a,b}) - p_{a,b}(u^*_{a,b}). \tag{7}$$

Then, by inequalities (5) and (7),

$$z(\bar{\sigma}) - z(\sigma^*) \le p_{a,b}(\tau_{a,b}) - p_{a,b}(u^*_{a,b}) + c_{a,b}\left(\tau_{a,b} - u^*_{a,b}\right) = t_{a,b}(\tau_{a,b}) - t_{a,b}(u^*_{a,b}) < 0.$$

This is a contradiction.

Lemma 2 *In* σ^*,

$$u^*_{i,j} = \tau_{i,j} \text{ for } O_{i,j} \in \mathcal{P}^* := \{O_{1,\pi^*(1)}, O_{2,\pi^*(n)}\}.$$

Proof It is observed from Eq. (3) that two terms in

$$\left\{t_{i,j}(u_{i,j}) \mid O_{i,j} \in \mathcal{P}^*\right\}$$

are independent of $u_{i,j}$ for $O_{i,j} \in \mathcal{O} \setminus \mathcal{P}^*$. Thus, Lemma 2 holds from Eq. (5).

For simplicity, we introduce the following notations: When jobs $\pi(h)$ and $\pi(m)$ are the consecutive pivots in σ, let

- for $1 \le h < m \le n$,

$$\mathcal{A}_{h,m} = \{\pi(h+1), ..., \pi(m)\} \text{ and } \mathcal{B}_{h,m} = \{\pi(h), ..., \pi(m-1)\};$$

- $z_{h,m}(\sigma)$ be the objective value with respect to the operations in

$$\left\{O_{1,j} \mid j \in \mathcal{A}_{h,m}\right\} \cup \left\{O_{2,j} \mid j \in \mathcal{B}_{h,m}\right\}.$$

Then,

$$z_{h,m}(\sigma) = \sum_{j\in\mathcal{A}_{h,m}} p_{1,j}(u_{1,j}) + \sum_{j\in\mathcal{A}_{h,m}} c_{1,j}u_{1,j} + \sum_{j\in\mathcal{B}_{h,m}} c_{2,j}u_{2,j}, \tag{8}$$

where the first term is the total processing times on machines 1 and the second and third terms are the total resource consumption costs of these operations, respectively, and the following equation holds:

$$\sum_{j\in\mathcal{A}_{h,m}} p_{1,j}(u_{1,j}) = \sum_{j\in\mathcal{B}_{h,m}} p_{2,j}(u_{2,j}). \tag{9}$$

Lemma 3 *$z_{h,m}(\sigma)$ is minimized when two constants $\hat{\gamma}$ and $\tilde{\gamma}$ exist such that*

$$u_{1,j} = \hat{\gamma}\tau_{1,j} \text{ for } j \in \mathcal{A}_{h,m} \text{ and } u_{2,j} = \tilde{\gamma}\tau_{2,j} \text{ for } j \in \mathcal{B}_{h,m}.$$

Proof Suppose that $\sum_{j\in\mathcal{A}_{h,m}} p_{1,j}(u_{1,j}) = \delta$, where $\delta > 0$ is some value. Then, $z_{h,m}(\sigma)$ is minimized when σ is optimal for the following problem:

$$\begin{aligned} \min \sum_{j\in\mathcal{A}_{h,m}} c_{1,j}u_{1,j} \\ \text{s.t.} \sum_{j\in\mathcal{A}_{h,m}} p_{1,j}(u_{1,j}) \le \delta. \end{aligned}$$

Since Lagrangian $L(u;\lambda)$ for the above problem is expressed as follows:

$$L(u;\lambda) = \sum_{j\in\mathcal{A}_{h,m}} c_{1,j}u_{1,j} + \lambda\Big(\sum_{j\in\mathcal{A}_{h,m}} p_{1,j}(u_{1,j}) - \delta\Big),$$

By Karush-Kuhn-Tucker (KKT) necessary and sufficient conditions, there exists a constant λ such that

$$\frac{\partial}{\partial u_{1,j}} L(u;\lambda) := c_{1,j} + \lambda p'_{1,j}(u_{1,j}) = 0 \text{ for } j \in \mathcal{A}_{h,m}. \tag{10}$$

Note that by Eq. (4), $c_{1,j} = -p'_{1,j}(\tau_{1,j})$. Then, since $p'_{1,j}(u_{1,j}) \neq 0$, Eq. (10) can be rewritten as follows:

$$\lambda = -\frac{c_{1,j}}{p'_{1,j}(u_{1,j})} = \frac{p'_{1,j}(\tau_{1,j})}{p'_{1,j}(u_{1,j})} = \left(\frac{u_{1,j}}{\tau_{1,j}}\right)^{k+1},$$

which implies that for $j \in \mathcal{A}_{h,m}$,

$$u_{1,j} = \lambda^{\frac{1}{k+1}}\tau_{1,j}. \tag{11}$$

By Eq. (11) and setting $\hat{\gamma} = \lambda^{\frac{1}{k+1}}$,

$$u_{1,j} = \hat{\gamma}\tau_{1,j} \text{ for } j \in \mathcal{A}_{h,m}.$$

By Eqs. (8) and (9), $z_{h,m}(\sigma)$ is also minimized when σ is optimal for the following problem:

$$\min \sum_{j \in \mathcal{B}_{h,m}} c_{2,j} u_{2,j}$$
$$\text{s.t.} \sum_{j \in \mathcal{B}_{h,m}} p_{2,j}(u_{2,j}) \le \delta.$$

By a similar argument,

$$u_{2,j} = \tilde{\gamma} \tau_{2,j} \quad \text{for} \quad j \in \mathcal{B}_{h,m}.$$

□

For $1 \le h < m \le n$, let $A_{h,m}$ and $B_{h,m}$ be the total processing time of operations in $\mathcal{A}_{h,m}$ and $\mathcal{B}_{h,m}$ when $u_{i,j} = \tau_{i,j}$, respectively, which are calculated as follows:

$$A_{h,m} = \sum_{j \in \mathcal{A}_{h,m}} p_{1,j}(\tau_{1,j}) = \sum_{j \in \mathcal{A}_{h,m}} \left(\frac{c_{1,j} w_{1,j}}{k} \right)^{\frac{k}{k+1}} \tag{12}$$

and

$$B_{h,m} = \sum_{j \in \mathcal{B}_{h,m}} p_{2,j}(\tau_{2,j}) = \sum_{j \in \mathcal{B}_{h,m}} \left(\frac{c_{2,j} w_{2,j}}{k} \right)^{\frac{k}{k+1}}. \tag{13}$$

$$z_{h,m}(\sigma) \ge (k+1) \left(A_{h,m}^{\frac{k+1}{k}} + B_{h,m}^{\frac{k+1}{k}} \right)^{\frac{k}{k+1}},$$

Lemma 4

where the equality holds at

$$\hat{\gamma} = \left(\frac{A_{h,m}^{\frac{k+1}{k}}}{A_{h,m}^{\frac{k+1}{k}} + B_{h,m}^{\frac{k+1}{k}}} \right)^{\frac{1}{k+1}} \quad \text{and} \quad \tilde{\gamma} = \left(\frac{B_{h,m}^{\frac{k+1}{k}}}{A_{h,m}^{\frac{k+1}{k}} + B_{h,m}^{\frac{k+1}{k}}} \right)^{\frac{1}{k+1}}. \tag{14}$$

Proof By Lemma 3, set

$$u_{1,j} = \hat{\gamma} \tau_{1,j} \quad \text{for} \quad j \in \mathcal{A}_{h,m} \quad \text{and} \quad u_{2,j} = \tilde{\gamma} \tau_{2,j} \quad \text{for} \quad j \in \mathcal{B}_{h,m}.$$

Then, Eqs. (6), (8) and (9) can be modified as follows.

$$z_{h,m}(\sigma) = \frac{A_{h,m}}{\hat{\gamma}^k} + k A_{h,m} \hat{\gamma} + k B_{h,m} \tilde{\gamma} \tag{15}$$

and

$$\frac{1}{\hat{\gamma}^k}A_{h,m} = \frac{1}{\tilde{\gamma}^k}B_{h,m}. \tag{16}$$

By setting $\tilde{\gamma} = (B_{h,m}/A_{h,m})^{\frac{1}{k}}\hat{\gamma}$ from (16) and substituting it into (15),

$$z_{h,m}(\sigma) = \frac{A_{h,m}}{\hat{\gamma}^k} + kA_{h,m}\hat{\gamma} + kB_{h,m}\left(\frac{B_{h,m}}{A_{h,m}}\right)^{\frac{1}{k}}\hat{\gamma}.$$

Note that $z_{h,m}(\sigma)$ is minimized when

$$\frac{d}{d\hat{\gamma}}z_{h,m}(\sigma) = -k\frac{A_{h,m}}{\hat{\gamma}^{k+1}} + kA_{h,m} + kB_{h,m}\left(\frac{B_{h,m}}{A_{h,m}}\right)^{\frac{1}{k}} = 0,$$

that is, when two equalities in (14) hold. By substituting $\hat{\gamma}$ and $\tilde{\gamma}$ of (14) into (15), $z_{h,m}(\sigma)$ has the minimum of $(k+1)\left(A_{h,m}^{\frac{k+1}{k}} + B_{h,m}^{\frac{k+1}{k}}\right)^{\frac{k}{k+1}}$. □

It is observed from Lemma 4 that if jobs $\pi(h)$ and $\pi(m)$ are the consecutive pivots in σ and two equalities in (14) hold, then

$$P_{h,m}(\sigma) = \frac{A_{h,m}}{\hat{\gamma}^k} = \left(A_{h,m}^{\frac{k+1}{k}} + B_{h,m}^{\frac{k+1}{k}}\right)^{\frac{k}{k+1}} = \frac{1}{k+1}z_{h,m}(\sigma). \tag{17}$$

where $P_{h,m}(\sigma)$ is the total processing time of operations in $\mathcal{A}_{h,m}$.

Corollary 1 *In σ^*,*

$$z(\sigma^*) = (k+1)C_{\max}(\sigma^*).$$

Proof Let $\mathcal{Q}^*$ be the set of the pairs of the consecutive pivots in σ^*. Then, by Lemma 2 and relations (6) and (17), we have

$$\begin{aligned} z(\sigma^*) &= \textstyle\sum_{O_{i,j}\in\mathcal{P}^*} t_{i,j}(\tau_{i,j}) + \sum_{(h,m)\in\mathcal{Q}^*} z_{h,m}(\sigma^*) \\ &= (k+1)\big(\textstyle\sum_{O_{i,j}\in\mathcal{P}^*} p_{i,j}(\tau_{i,j}) + \sum_{(h,m)\in\mathcal{Q}^*} P_{h,m}(\sigma^*)\big) \\ &= (k+1)C_{\max}(\sigma^*). \end{aligned}$$

□

4 Computational complexity

In this section, we show that Problem P and its constrained version are NP-hard by using the optimality properties in Sect. 3.

Theorem 1 *Problem P is NP-hard, even when $k = 1$ and $w_{1,j} = w_{2,j}$ for $j \in \mathcal{J}$.*

Proof We prove it by reduction from the partition problem, which is known to be NP-complete: Given g integers in $\{a_1, a_2, ..., a_g\}$ with $\sum_{j=1}^{g} a_j = 2A$, is there a subset $\mathcal{A} \subset \{1, 2, ..., g\}$ such that

$$\sum_{j \in \mathcal{A}} a_j = A?$$

Given an instance of the partition problem, we can construct an instance of Problem P as follows: Let $\mathcal{J} = \{1, 2, ..., g+3\}$ and $\mathcal{O} = \left\{O_{i,j} \mid i = 1, 2,\ j \in \mathcal{J}\right\}$ such that

$$w_{i,j} = b_j \text{ for } O_{i,j} \in \mathcal{O} \text{ and } c_{i,j} = \begin{cases} b_j & \text{for } O_{i,j} \in \mathcal{O} \setminus \mathcal{P} \\ 1/M^2 & \text{for} O_{i,j} \in \mathcal{P}, \end{cases}$$

where

$$b_j = \begin{cases} a_j & \text{for } j = 1, 2, ..., g \\ 2A & \text{for } j = g+1 \\ A & \text{for j=g+2,g+3,} \end{cases}$$

$\mathcal{P} = \left\{O_{1,g+2}, O_{2,g+3}\right\}$, $M = 8\sqrt{A}/\delta$, $\delta = \min\{1, f(1) - f(0)\}$, and

$$f(x) = 2\sqrt{13A^2 - 10Ax + 2x^2} + 2\sqrt{13A^2 + 10Ax + 2x^2}.$$

By definition, we have

$$\tau_{i,j} = 1 \quad \text{for} \quad O_{i,j} \in \mathcal{O} \setminus \mathcal{P} \quad \text{and} \quad \tau_{i,j} = \sqrt{A}M \quad \text{for} \quad O_{i,j} \in \mathcal{P}. \tag{18}$$

Henceforth, we show that there exists a solution $\mathcal{A}$ to the partition problem if and only if the reduced instance of Problem P has a schedule σ with $z(\sigma) \le Z$, where $Z = 4\sqrt{13}A + \delta/2$.

(⇒) Suppose that there exists a solution $\bar{\mathcal{A}}$ to the partition problem. Let $\bar{\mathcal{L}}$ be the set of jobs in $\{1, 2, ..., g\}$ corresponding to the integers in $\bar{\mathcal{A}}$, and $\bar{\mathcal{R}} = \{1, 2, ..., g\} \setminus \bar{\mathcal{L}}$. Then, we can construct a schedule $\bar{\sigma} = \left(\bar{\pi}; \bar{u}\right)$ such that

- $\bar{\pi} = (g+2, \bar{\pi}_{\bar{\mathcal{L}}}, g+1, \bar{\pi}_{\bar{\mathcal{R}}}, g+3)$, where $\bar{\pi}_{\bar{\mathcal{L}}}$ and $\bar{\pi}_{\bar{\mathcal{R}}}$ are the sequences constructed by arbitrarily ordering the jobs in $\bar{\mathcal{L}}$ and $\bar{\mathcal{R}}$, respectively;

- $\bar{u} = (\bar{u}_{i,j})_{O_{i,j} \in \mathcal{O}}$, where

$$(\bar{u}_{1,j}, \bar{u}_{2,j}) = \begin{cases} (\tau_{1,j}, 2/\sqrt{13}) & \text{for } j = g+2 \\ (3/\sqrt{13}, 2/\sqrt{13}) & \text{for } j \in \bar{\mathcal{L}} \\ (3/\sqrt{13}, 3/\sqrt{13}) & \text{for } j = g+1 \\ (2/\sqrt{13}, 3/\sqrt{13}) & \text{for } j \in \bar{\mathcal{R}} \\ (2/\sqrt{13}, \tau_{2,j}) & \text{for } j = g+3. \end{cases}$$

Let $\bar{p}_{i,j} = w_{i,j}/\bar{u}_{i,j}$ for $O_{i,j} \in \mathcal{O}$. Then,

$$(\bar{p}_{1,j}, \bar{p}_{2,j}) = \begin{cases} \left(\frac{1}{M}\sqrt{A}, \frac{1}{2}\sqrt{13}A\right) & \text{for } j = g+2 \\ \left(\frac{1}{3}\sqrt{13}a_j, \frac{1}{2}\sqrt{13}a_j\right) & \text{for } j \in \bar{\mathcal{L}} \\ \left(\frac{2}{3}\sqrt{13}A, \frac{2}{3}\sqrt{13}A\right) & \text{for } j = g+1 \\ \left(\frac{1}{2}\sqrt{13}a_j, \frac{1}{3}\sqrt{13}a_j\right) & \text{for } j \in \bar{\mathcal{R}} \\ \left(\frac{1}{2}\sqrt{13}A, \frac{1}{M}\sqrt{A}\right) & \text{for j=g+3.} \end{cases}$$

It is observed that no idle time exists between the consecutive jobs in $\bar{\sigma}$ on both machines. Thus, since

$$C_{\max}(\bar{\sigma}) = \bar{p}_{1,\bar{\pi}(1)} + \sum_{j=1}^{g+3} \bar{p}_{2,\bar{\pi}(j)} = 2\sqrt{13}A + \frac{2}{M}\sqrt{A}$$

and

$$\sum_{O_{i,j} \in \mathcal{O}} c_{i,j}\bar{u}_{i,j} = 2\sqrt{13}A + \frac{2}{M}\sqrt{A},$$

we have

$$z(\bar{\sigma}) = 4\sqrt{13}A + \frac{4}{M}\sqrt{A} = 4\sqrt{13}A + \frac{\delta}{2} = Z.$$

($\Leftarrow$) Suppose that there exists an optimal schedule σ^* with $z(\sigma^*) \le Z$. Then, we can obtain $\tilde{\sigma} = \left(\pi^*;\hat{u}\right)$ by setting $\hat{u} = (\hat{u}_{i,j})_{O_{i,j} \in \mathcal{O}}$ with

$$\hat{u}_{i,j} = \begin{cases} \tau_{i,j} & \text{for } O_{i,j} \in \mathcal{P} \\ u^*_{i,j} & \text{for } O_{i,j} \in \mathcal{O} \setminus \mathcal{P}. \end{cases}$$

Since $\hat{u}_{i,j} \ge u^*_{i,j}$ for each $O_{i,j} \in \mathcal{O}$ by Lemma 1, $C_{\max}(\tilde{\sigma}) \le C_{\max}(\sigma^*)$. Let $\hat{\sigma} = \left(\hat{\pi};\hat{u}\right)$, where $\hat{\pi}$ is a sequence by Johnson's rule. Note that $C_{\max}(\hat{\sigma}) \le C_{\max}(\tilde{\sigma})$. Thus,

$$\begin{aligned}
z(\hat{\sigma}) =& C_{\max}(\hat{\sigma}) + \sum_{O_{i,j}\in\mathcal{O}} c_{i,j}\hat{u}_{i,j}\\
\le& C_{\max}(\tilde{\sigma}) + \sum_{O_{i,j}\in\mathcal{O}\setminus\mathcal{P}} c_{i,j}\hat{u}_{i,j} + \sum_{O_{i,j}\in\mathcal{P}} c_{i,j}\hat{u}_{i,j}\\
\le& C_{\max}(\sigma^*) + \sum_{O_{i,j}\in\mathcal{O}\setminus\mathcal{P}} c_{i,j}u^*_{i,j} + \sum_{O_{i,j}\in\mathcal{P}} c_{i,j}(u^*_{i,j} + \tau_{i,j})\\
=& C_{\max}(\sigma^*) + \sum_{O_{i,j}\in\mathcal{O}} c_{i,j}u^*_{i,j} + \sum_{j\in\mathcal{P}} c_{i,j}\tau_{i,j}\\
\le& Z + 2\sqrt{A}/M = 4\sqrt{13}A + \frac{3}{4}\delta.
\end{aligned} \tag{19}$$

Now, we will see the structure of $\hat{\sigma}$ in Claims 1 and 2.

Claim 1 Jobs $(g+2)$ and $(g+3)$ are the first and last jobs in $\hat{\sigma}$, respectively.

Proof Let $\hat{p}_{i,j} = w_{i,j}/\hat{u}_{i,j}$ for $O_{i,j} \in \mathcal{O}$. Then, by (18), we have

$$\hat{p}_{i,j} = \frac{w_{i,j}}{\tau_{i,j}} = \frac{A}{\sqrt{AM}} = \frac{\delta}{8} < 1 \quad \text{for} \quad O_{i,j} \in \mathcal{P}$$

and, by Lemma 1 and (18),

$$\hat{p}_{i,j} = \frac{w_{i,j}}{u^*_{i,j}} \ge \frac{w_{i,j}}{\tau_{i,j}} = w_{i,j} \ge 1 \quad \text{for} \quad O_{i,j} \in \mathcal{O} \setminus \mathcal{P}.$$

Hence, by Johnson's rule, jobs $(g+2)$ and $(g+3)$ are the first and last jobs in $\hat{\sigma}$, respectively. □ Like (12) and (13), for $1 \le h < m \le g+3$, let

$$\hat{A}_{h,m} = \sum_{j\in\hat{\mathcal{A}}_{h,m}} w_{1,j} \quad \text{and} \quad \hat{B}_{h,m} = \sum_{j\in\hat{\mathcal{B}}_{h,m}} w_{2,j},$$

where $\hat{\mathcal{A}}_{h,m} = \{\hat{\pi}(h+1), ..., \hat{\pi}(m)\}$ and $\hat{\mathcal{B}}_{h,m} = \{\hat{\pi}(h), ..., \hat{\pi}(m-1)\}$.

Claim 2 In $\hat{\sigma}$, job $(g+1)$ is a pivot.

Proof Suppose that, in $\hat{\sigma}$, job $(g+1)$ is not a pivot, it is the lth job, and it is sequenced between the consecutive pivots $\hat{\pi}(h)$ and $\hat{\pi}(m)$, where $h < l < m$. By (18) and Lemma 3,

$$\hat{u}_{1,j} = \hat{\gamma} \quad \text{for} \quad j \in \hat{\mathcal{A}}_{h,m} \quad \text{and} \quad \hat{u}_{2,j} = \tilde{\gamma} \quad \text{for} \quad j \in \hat{\mathcal{B}}_{h,m}.$$

i) $\hat{\gamma} \le \tilde{\gamma}$ Since $\hat{A}_{h,l} - \hat{B}_{h,l} = w_{1,\hat{\pi}(l)} - w_{2,\hat{\pi}(h)} \ge 2A - A > 0$, we have

$$\frac{1}{\hat{\gamma}}\hat{A}_{h,l} > \frac{1}{\tilde{\gamma}}\hat{B}_{h,l}.$$

This implies that job $\hat{\pi}(l)$ becomes a pivot, which is a contradiction.

ii) $\hat{\gamma} > \tilde{\gamma}$ Since $\hat{B}_{l,m} - \hat{A}_{l,m} = w_{2,\hat{\pi}(l)} - w_{1,\hat{\pi}(m)} \geq 2A - A > 0$, we have

$$\frac{1}{\hat{\gamma}}\hat{A}_{l,m} < \frac{1}{\tilde{\gamma}}\hat{B}_{l,m}.$$

This implies that jobs $\hat{\pi}(m)$ is not a pivot, which is a contradiction.

By cases i) and ii), Claim 2 holds. □

Now, we will derive a lower bound of $z(\hat{\sigma})$. Let $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ be the sets of jobs in $\{1, 2, ..., g\}$ before and after job $(g + 1)$ in $\hat{\sigma}$, respectively.

Claim 3 $f(x)$ is a lower bound of $z(\hat{\sigma})$, where $x = A - \sum_{j\in\hat{\mathcal{L}}} a_j$.

Proof Let $\{\hat{\pi}(\alpha_i) \mid i = 1, ..., v\}$ be the set of pivots in $\hat{\sigma}$. Note that by Proposition 1 and Claim 2, we may assume that $\hat{\pi}(\alpha_1) = g + 2$, $\hat{\pi}(\alpha_u) = g + 1$, and $\hat{\pi}(\alpha_v) = g + 3$, where $1 < u < v$. For simplicity, for $i \in \{2, 3, ..., v\}$, let $\hat{A}_i = \hat{A}_{\alpha_{i-1},\alpha_i}$, $\hat{B}_i = \hat{B}_{\alpha_{i-1},\alpha_i}$, and $z_i(\hat{\sigma}) = z_{\alpha_{i-1},\alpha_i}(\hat{\sigma})$. Since $\hat{\pi}(\alpha_{i-1})$ and $\hat{\pi}(\alpha_i)$ are consecutive pivots and by Lemma 4 for $k = 1$, we have

$$z_i(\hat{\sigma}) \geq 2\sqrt{\hat{A}_i^2 + \hat{B}_i^2} \quad \text{for } i \in \{2, 3, ..., v\}.$$

Note that by the Cauchy-Schwarz inequality, we have, for $i \in \{2, 3, ..., v - 1\}$,

$$\sqrt{\hat{A}_i^2 + \hat{B}_i^2} + \sqrt{\hat{A}_{i+1}^2 + \hat{B}_{i+1}^2} \geq \sqrt{(\hat{A}_i + \hat{A}_{i+1})^2 + (\hat{B}_i + \hat{B}_{i+1})^2}.$$

Since $\sum_{i=2}^{u} \hat{A}_i = \hat{A}_{\alpha_1,\alpha_u}$ and $\sum_{i=2}^{u} \hat{B}_i = \hat{B}_{\alpha_1,\alpha_u}$, we have

$$\sum_{i=2}^{u} z_i(\hat{\sigma}) \geq 2\sum_{i=2}^{u} \sqrt{\hat{A}_i^2 + \hat{B}_i^2} \geq 2\sqrt{\hat{A}_{\alpha_1,\alpha_u}^2 + \hat{B}_{\alpha_1,\alpha_u}^2}. \tag{20}$$

Similarly, since $\sum_{i=u+1}^{v} \hat{A}_i = \hat{A}_{\alpha_u,\alpha_v}$ and $\sum_{i=u+1}^{v} \hat{B}_i = \hat{B}_{\alpha_u,\alpha_v}$, we have

$$\sum_{i=u+1}^{v} z_i(\hat{\sigma}) \geq 2\sum_{i=u+1}^{v} \sqrt{\hat{A}_i^2 + \hat{B}_i^2} \geq 2\sqrt{\hat{A}_{\alpha_u,\alpha_v}^2 + \hat{B}_{\alpha_u,\alpha_v}^2}. \tag{21}$$

Then, by inequalities (20) and (21), we have

$$\begin{aligned} z(\hat{\sigma}) \geq & p_{1,\hat{\pi}(1)} + \sum_{i=2}^{v} z_i(\hat{\sigma}) + p_{2,\hat{\pi}(g+3)} \\ > & 2\sqrt{\hat{A}_{\alpha_1,\alpha_u}^2 + \hat{B}_{\alpha_1,\alpha_u}^2} + 2\sqrt{\hat{A}_{\alpha_u,\alpha_v}^2 + \hat{B}_{\alpha_u,\alpha_v}^2}. \end{aligned} \tag{22}$$

Since $\sum_{j\in\hat{\mathcal{R}}} a_j = 2A - \sum_{j\in\hat{\mathcal{L}}} a_j = A + x$, we obtain

$$\hat{A}_{\alpha_1,\alpha_u} = \sum_{j\in\hat{\mathcal{L}}} w_{1,j} + w_{1,g+1} = \sum_{j\in\hat{\mathcal{L}}} a_j + 2A = 3A - x,$$
$$\hat{B}_{\alpha_1,\alpha_u} = w_{2,g+2} + \sum_{j\in\hat{\mathcal{L}}} w_{2,j} = A + \sum_{j\in\hat{\mathcal{L}}} a_j = 2A - x,$$
$$\hat{A}_{\alpha_u,\alpha_v} = \sum_{j\in\hat{\mathcal{R}}} w_{1,j} + w_{1,g+3} = \sum_{j\in\hat{\mathcal{R}}} a_j + A = 2A + x, \text{ and}$$
$$\hat{B}_{\alpha_u,\alpha_v} = w_{2,g+1} + \sum_{j\in\hat{\mathcal{R}}} w_{2,j} = 2A + \sum_{j\in\hat{\mathcal{R}}} a_j = 3A + x.$$

Then, the inequality (22) can be rewritten as follows:

$$\begin{aligned} z(\hat{\sigma}) >& 2\sqrt{(3A-x)^2 + (2A-x)^2} + 2\sqrt{(3A+x)^2 + (2A+x)^2} \\ =& 2\sqrt{13A^2 - 10Ax + 2x^2} + 2\sqrt{13A^2 + 10Ax + 2x^2} = f(x). \end{aligned}$$

□

Now, we show that $\hat{\mathcal{L}}$ is a solution to the partition problem. By definition, $|x| \le A$. Suppose that $\sum_{j\in\hat{\mathcal{L}}} a_j \neq A$. Then, $1 \le |x| \le A$ holds by the integrality of a_j for $j \in \{1, 2, ..., g\}$. Note that $f(x)$ is an increasing function on $0 < x \le A$ since

$$f'(x) = \frac{10A + 4x}{\sqrt{13A^2 + 10Ax + 2x^2}} - \frac{10A - 4x}{\sqrt{13A^2 - 10Ax + 2x^2}} > 0.$$

By $f(-x) = f(x)$, we have $f(x) \ge f(1)$ on $1 \le |x| \le A$. Then, by Claim 3 and $f(0) = 4\sqrt{13}A$,

$$z(\hat{\sigma}) > f(x) \ge f(1) = 4\sqrt{13}A + f(1) - f(0) \ge 4\sqrt{13}A + \delta.$$

This is a contradiction to the inequality (19). Thus, $\sum_{j\in\hat{\mathcal{L}}} a_j = A$ holds and $\hat{\mathcal{L}}$ is a solution to the partition problem. □

Theorem 2 *Problem P is NP-hard, even when $k = 1$ and $c_{1,j} = c_{2,j}$ for $j \in \mathcal{J}$.*

Proof Given an instance of the partition problem, we can construct an instance of Problem P as follows: Let $\mathcal{J} = \{1, 2, ..., g+3\}$ and $\mathcal{O} = \{O_{i,j} \mid i = 1, 2,\ j \in \mathcal{J}\}$ such that

$$w_{i,j} = \begin{cases} b_j & \text{for } O_{i,j} \in \mathcal{O} \setminus \mathcal{P} \\ 1/M^2 & \text{for } O_{i,j} \in \mathcal{P} \end{cases} \quad \text{and} \quad c_{i,j} = b_j \ \text{ for } \ O_{i,j} \in \mathcal{O},$$

where b_j and M are the values defined in the proof of Theorem 1. By using the same argument in the proof of Theorem 1, we can show that there exists a solution $\mathcal{A}$ to the partition problem if and only if the reduced instance of Problem P has a schedule σ with $z(\sigma) \le Z$. We omit the details. □

Corollary 2 *The constrained version of Problem P is NP-hard, even when $k = 1$ and, for $j \in \mathcal{J}$, $w_{1,j} = w_{2,j}$ or $c_{1,j} = c_{2,j}$.*

Proof We can prove it by using the instances in the proofs of Theorems 1 and 2. We will show that there exists a solution $\mathcal{A}$ to the partition problem if and only if there exists a schedule σ with $C_{\max}(\sigma) \le Z/2$ and $\sum_{O_{i,j} \in \mathcal{O}} c_{i,j} u_{i,j} \le Z/2$. It is observed from the proofs of Theorems 1 and 2 that

- If the partition problem has a solution, then $\bar{\sigma}$ in the ($\Rightarrow$) part satisfies

$$C_{\max}(\bar{\sigma}) \le \frac{Z}{2} \text{ and } \sum_{O_{i,j} \in \mathcal{O}} c_{i,j} \bar{u}_{i,j} \le \frac{Z}{2};$$

- If there exists a schedule $\hat{\sigma}$ with $C_{\max}(\hat{\sigma}) \le Z/2$ and $\sum_{O_{i,j} \in \mathcal{O}} c_{i,j} \hat{u}_{i,j} \le Z/2$, then $z(\hat{\sigma}) \le Z$ and the partition problem has a solution by the ($\Leftarrow$) part.

By the observations above, Corollary 2 holds. □

5 Approximability

Since Problem P and its constrained version are proven to be NP-hard, it is reasonable to develop approximation algorithms instead of developing the exact algorithm. In this section, we develop approximation algorithms for Problem P and its constrained version by using the optimality properties in Sect. 3.

Let $\bar{\sigma} = \left(\pi^H; \tau\right)$ be a schedule, where $\tau = (\tau_{i,j})_{O_{i,j} \in \mathcal{O}}$ and π^H is a job sequence according to Johnson's rule. Note that $\bar{\sigma}$ can be obtained in $O(n \log n)$ time. For simplicity, let $\bar{p}_{i,j}$ be the processing time of $O_{i,j}$ when $u_{i,j} = \tau_{i,j}$, that is,

$$\bar{p}_{i,j} = p_{i,j}(\tau_{i,j}) = \left(\frac{w_{i,j}}{\tau_{i,j}}\right)^k \text{ for } O_{i,j} \in \mathcal{O}.$$

Theorem 3 *Problem P has $\left(1 + \frac{k}{(k+1)^2}\right)$-approximability.*

Proof First, we obtain the bound for $C_{\max}(\bar{\sigma})$. For an arbitrary schedule $\sigma = \left(\pi; u\right)$,

$$\begin{aligned}
z(\sigma) =& \max_{1\le h\le n}\left\{\sum_{j=1}^{h} p_{1,\pi(j)} + \sum_{j=h}^{n} p_{2,\pi(j)}\right\} + \sum_{O_{i,j}\in\mathcal{O}} c_{i,\pi(j)}u_{i,\pi(j)}\\
&\ge \max_{1\le h\le n}\left\{\sum_{j=1}^{h}\left(p_{1,\pi(j)} + c_{1,\pi(j)}u_{1,\pi(j)}\right) + \sum_{j=h}^{n}\left(p_{2,\pi(j)} + c_{2,\pi(j)}u_{2,\pi(j)}\right)\right\}\\
&\ge \max_{1\le h\le n}\left\{\sum_{j=1}^{h}\left(\bar{p}_{1,\pi(j)} + c_{1,\pi(j)}\tau_{1,\pi(j)}\right) + \sum_{j=h}^{n}\left(\bar{p}_{2,\pi(j)} + c_{2,\pi(j)}\tau_{2,\pi(j)}\right)\right\}\\
&=(k+1)\max_{1\le h\le n}\left\{\sum_{j=1}^{h}\bar{p}_{1,\pi(j)} + \sum_{j=h}^{n}\bar{p}_{2,\pi(j)}\right\}\\
&=(k+1)C_{\max}(\pi;\tau)\\
&\ge(k+1)C_{\max}(\bar{\sigma}),
\end{aligned} \tag{23}$$

where the second inequality and the second equality hold from Eqs. (5) and (6), respectively. Second, we obtain the bound for the total resource consumption cost of $\bar{\sigma}$. By Eqs. (5) and (6), furthermore, it is observed that

$$\begin{aligned}
\left(\frac{k+1}{k}\right)\sum_{O_{i,j}\in\mathcal{O}} c_{i,j}\tau_{i,j} &= \sum_{O_{i,j}\in\mathcal{O}}\left(\frac{1}{k}c_{i,j}\tau_{i,j} + c_{i,j}\tau_{i,j}\right)\\
&= \sum_{O_{i,j}\in\mathcal{O}}\left(\bar{p}_{i,j} + c_{i,j}\tau_{i,j}\right)\\
&\le \sum_{O_{i,j}\in\mathcal{O}}\left(p^*_{i,j} + c_{i,j}u^*_{i,j}\right)\\
&\le 2C_{\max}(\sigma^*) + \sum_{O_{i,j}\in\mathcal{O}} c_{i,j}u^*_{i,j}\\
&= z(\sigma^*) + C_{\max}(\sigma^*)\\
&= \left(1 + \frac{1}{k+1}\right)z(\sigma^*),
\end{aligned} \tag{24}$$

where the the last equality holds from Corollary 1. By inequalities (23) and (24),

$$z(\bar{\sigma}) = C_{\max}(\bar{\sigma}) + \sum_{O_{i,j}\in\mathcal{O}} c_{i,j}\tau_{i,j} \le \left(1 + \frac{k}{(k+1)^2}\right)z(\sigma^*).$$

□

Note that since

$$\frac{d}{dk}\left(1 + \frac{k}{(k+1)^2}\right) = \frac{1-k}{(k+1)^3},$$

$\left(1 + \frac{k}{(k+1)^2}\right)$ has the maximum of 1.25 at $k = 1$, and it converges to 1 when k goes to 0 or ∞. Henceforth, we will prove a 2-approximablity for the constrained version. Let $\hat{\sigma} = \left(\pi;\alpha\tau\right)$ be a schedule with an arbitrary sequence π, where

$$\alpha = \frac{K}{\sum_{O_{i,j} \in \mathcal{O}} c_{i,j}\tau_{i,j}}.$$

Note that $\hat{\sigma}$ can be obtained in $O(n)$ time. For simplicity, let $\hat{u}_{i,j} = \alpha\tau_{i,j}$ and $\hat{p}_{i,j} = p_{i,j}(\hat{u}_{i,j})$ for $O_{i,j} \in \mathcal{O}$.

Theorem 4 *The constrained version of Problem P has* 2*-approximability.*

Proof Consider the following problem.

$$\begin{aligned} \min \ & \sum_{O_{i,j} \in \mathcal{O}} \left(\frac{w_{i,j}}{u_{i,j}}\right)^k \\ \text{s.t.} \ & \sum_{O_{i,j} \in \mathcal{O}} c_{i,j}u_{i,j} \le K. \end{aligned} \tag{25}$$

Lagrangian $L(u;\lambda)$ for problem (25) is described as

$$L(u;\lambda) = \sum_{O_{i,j} \in \mathcal{O}} \left(\frac{w_{i,j}}{u_{i,j}}\right)^k + \lambda\Big(\sum_{O_{i,j} \in \mathcal{O}} c_{i,j}u_{i,j} - K\Big),$$

and KKT necessary and sufficient conditions are as follows:

$$\frac{\partial}{\partial u_{i,j}} L(u;\lambda) = -k\frac{w_{i,j}^k}{u_{i,j}^{k+1}} + \lambda c_{i,j} = 0 \ \text{ for } \ O_{i,j} \in \mathcal{O}, \tag{26}$$

and

$$\lambda\left(\sum_{O_{i,j} \in \mathcal{O}} c_{i,j}u_{i,j} - K\right) = 0. \tag{27}$$

Since $\lambda = \frac{1}{\alpha^{k+1}}$ and $u_{i,j} = \hat{u}_{i,j}$ for $O_{i,j} \in \mathcal{O}$ satisfy Eqs. (26) and (27), $(\hat{u}_{i,j})_{O_{i,j} \in \mathcal{O}}$ becomes an optimal solution to problem (25). Since $(u^*_{i,j})_{O_{i,j} \in \mathcal{O}}$ is a feasible solution of problem (25), we have

$$C_{\max}(\hat{\sigma}) \le \sum_{O_{i,j} \in \mathcal{O}} \hat{p}_{i,j} \le \sum_{O_{i,j} \in \mathcal{O}} p^*_{i,j} \le 2C_{\max}(\sigma^*).$$

$\square$

Note that it is still open that the above approximation factors $\left(1+\frac{k}{(k+1)^2}\right)$ and 2 are tight or not.

Funding This work was supported by the Ministry of Education of the Republic of Korea and the National Research Foundation of Korea (NRF-2021S1A5B8096365). This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (NRF-2021R1A2C1093960).

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References

1. Baker, K.R., Trietsch, D.: Principles of sequencing and scheeduling. Wiley, New Jersey (2009)
2. Cheng, T.C.E., Janiak, A.: A permutation flow-shop scheduling problem with convex models of operation processing times. Ann. Oper. Res. **96**, 39–60 (2000)
3. Cheng, T.C.E., Shakhlevich, N.: Proportionate flow shop with controllable processing times. J. Sched. **2**, 253–265 (1999)
4. Choi, B.C., Park, M.-J.: Single-machine scheduling with resource-dependent processing times and multiple unavailability periods. J. Sched. **25**, 191–202 (2022)
5. Janiak, A.: Minimization of resource consumption under a given deadline in the two-processor flop-shop scheduling problem. Inf. Process. Lett. **32**, 101–112 (1989)
6. Janiak, A., Janiak, W., Lichtenstein, M.: Resource management in machine scheduling problems: a survey. Decision Making Manuf. Serv. **1**, 59–89 (2007)
7. Kayan, R.K., Akturk, M.S.: A new bounding mechanism for the CNC machine scheduling problem with controllable processing times. Eur. J. Oper. Res. **167**, 624–643 (2005)
8. Shabtay, D., Kaspi, M.: Minimizing the total weighted flow time in a single machine with controllable processing times. Comput. Oper. Res. **31**, 2279–2289 (2004)
9. Shabtay, D., Kaspi, M.: Parallel machine scheduling with a convex resource consumption function. Eur. J. Oper. Res. **173**, 92–107 (2006)
10. Shabtay, D., Steiner, G.: A survey of scheduling with controllable processing times. Discret. Appl. Math. **155**, 1643–1666 (2007)
11. Shabtay, D., Kaspi, M., Steiner, G.: The no-wait two-machine flow shop scheduling problem with convex resource-dependent processing times. IIE Trans. **39**, 539–557 (2007)
12. Shabtay, D., Zofi, M.: Single machine scheduling with controllable processing times and an unavailablity period to minimize the makespan. Int. J. Prod. Econ. **198**, 191–200 (2018)
13. Vickson, R.G.: Choosing the job sequence and processing times to minimize total processing plus flow cost on a single machine. Oper. Res. **28**, 1155–1167 (1980)

14. Xu, K., Feng, Z., Ke, L.: Single machine scheduling with total tardiness criterion and convex controllable processing times. Ann. Oper. Res. **186**, 383–391 (2011)