Optimization Letters (2022) 16:1663–1673
https://doi.org/10.1007/s11590-021-01837-7

ORIGINAL PAPER

Convergence results for some piecewise linear solvers

Manuel Radons[1] · **Siegfried M. Rump**[2]

Received: 30 November 2020 / Accepted: 5 December 2021 / Published online: 25 December 2021

Abstract
Let A be a real $n \times n$ matrix and $z, b \in \mathbb{R}^n$. The piecewise linear equation system $z - A|z| = b$ is called an *absolute value equation*. In this note we consider two solvers for uniquely solvable instances of the latter problem, one direct, one semi-iterative. We slightly extend the existing correctness, resp. convergence, results for the latter algorithms and provide numerical tests.

Keywords Absolute value equation · Piecewise linear system · Sign-real spectral radius · Linear complementarity problems

1 Introduction

Denote by $\mathrm{M}_n(\mathbb{R})$ the space of $n \times n$ real matrices and let $A \in \mathrm{M}_n(\mathbb{R})$ and $z, b \in \mathbb{R}^n$. The piecewise linear equation system

$$z - A|z| = b \tag{1.1}$$

is called an *absolute value equation* (AVE) and was first introduced by Rohn in [16]. Mangasarian and Meyer proved its polynomial equivalence to the linear complementarity problem (LCP) [10]. In [11, P. 216-230] Neumaier authored a detailed survey about the AVEs intimate connection to the research field of *linear interval equations*. Especially closely related system types are equilibrium problems of the form

$$Bx + \max(0, x) = c, \tag{1.2}$$

where $B \in \mathrm{M}_n(\mathbb{R})$ and $x, c \in \mathbb{R}^n$. A prominent example is the first hydrodynamic model presented in [2]. Using the identity $\max(s, t) = (s + t + |s - t|)/2$, equality

The authors wish to thank Lutz Lehmann for his valuable input.

✉ Manuel Radons
radons@math.tuberlin.de

1 Chair of Discrete Mathematics/Geometry, Technische Universität Berlin, Berlin, Germany

2 Institute for Reliable Computing, Hamburg University of Technology, Hamburg, Germany

(1.2) can be reformulated as

$$Bx + \frac{x + |x|}{2} = c \quad \Longleftrightarrow \quad (2B + I)x + |x| = 2c, \tag{1.3}$$

and for nonsingular $(2B + I)$ this is equivalent to an AVE (1.1).

This position at the crossroads of several interesting problem areas has given rise to the development of efficient solvers for the AVE. Publications on the matter include approaches by linear programming [9], concave minimization [21], [6], split shifts [19], successive over-relaxation [22], as well as a variety of Newton and fixed point methods, cf. [2], [20], [5].

In this article we will present and further analyze two solvers for the AVE: the signed Gaussian elimination, which is a direct solver that was developed in [12], and a semi-iterative generalized Newton method which was developed for equivalent system types in [2,8] and adapted to the AVE in [4,18]. Previously, four correctness results for the signed Gaussian elimination have been proved, two of which were shown to hold for the semi-iterative generalized Newton method as well. We extend and unify these results. That is, after some preliminaries (Sect. 2), we slightly strengthen one of the correctness results for the signed Gaussian elimination, and then give a unifying proof that extends all correctness results for the signed Gaussian elimination to the generalized Newton method (Sects. 3–5). Further, we show that both algorithms are, nevertheless, not equivalent and provide some numerical results (Sect. 6).

2 Preliminaries

We denote by $[n]$ the set $\{1, \ldots, n\}$. For vectors and matrices *absolute values and comparisons are used entrywise*. A *signature matrix* S, or, briefly, *a signature*, is a diagonal matrix with entries $+1$ or -1, i.e., $|S| = I$. The set of n-dimensional signature matrices is denoted by $\mathcal{S}_n$. A single diagonal entry of a signature is a sign s_i, where $i \in [n]$. Let $z \in \mathbb{R}^n$. We write S_z for a signature, where $s_i = 1$ if $z_i \geq 0$ and -1 else. We then have $S_z z = |z|$. Using this convention, we can rewrite (1.1) as

$$(I - AS_z)z \;=\; b. \tag{2.1}$$

In this form it becomes apparent that the main difficulty of the AVE is to determine the proper signature S for z. That is, to determine in which of the 2^n orthants about the origin z lies. This is NP-hard in general [7].

An important statement from [17] that we will frequently use is

Theorem 2.1 *Let $A \in \mathrm{M}_n(\mathbb{R})$. Then the AVE (2.1) is uniquely solvable if $\|A\|_p < 1$, where $\|\cdot\|_p$ is any p-norm.*

3 The unifying theorem

Hereafter, sign denotes the signum function. The following simple observation is key to the subsequent discussion:

Proposition 3.1 *Let $A \in \mathrm{M}_n(\mathbb{R})$ and $z, b \in \mathbb{R}^n$ satisfy (2.1). If $\|A\|_\infty < 1$, then for at least one $i \in [n]$ we have $\mathrm{sign}(z_i) = \mathrm{sign}(b_i)$.*

Proof Let z_i be an entry of z s.t. $|z_i| \geq |z_j|$ for all $j \in [n]$. If $z_i = 0$, then $z = \mathbf{0}$ and thus $b \equiv z - A|z| = \mathbf{0}$, where $\mathbf{0}$ denotes the zero-vector in $\mathbb{R}^n$, and the statement holds trivially. If $|z_i| > 0$, then $\left|e_i^\mathsf{T} A|z|\right| < |z_i|$, due to the norm constraint on A. Thus, $b_i = z_i - e_i^\mathsf{T} A|z|$ will adopt the sign of z_i. □

We do not know though, for which indices the signs coincide. The theorem below states restrictions on A which guarantee the coincidence of the signs of z_i and b_i for all $i \in [n]$ where $|b_i| = vertbvert_\infty$ and thus provide the basis for the convergence proofs in Sects. 4 and 5. For $b \in \mathbb{R}^n$ we set

$$\mathcal{I}^b_{\max} \equiv \{\, i \in [n] \,:\, |b_i| = |b|_\infty \,\},$$

and define

$$\mathrm{Neq}(A, b, z) \equiv \{i \in \mathcal{I}^b_{\max} \,:\, \mathrm{sign}(b_i) \neq \mathrm{sign}(z_i)\}.$$

Theorem 3.2 *Let $A \in \mathrm{M}_n(\mathbb{R})$ and $b, z \in \mathbb{R}^n$ such that (2.1) is satisfied. Then we have*

$$\mathrm{Neq}(A, b, z) = \emptyset$$

if either of the following conditions is satisfied.

1. $\|A\|_\infty < \frac{1}{2}$.
2. *A is irreducible, and* $\|A\|_\infty \leq \frac{1}{2}$.
3. *A is strictly diagonally dominant and* $\|A\|_\infty \leq \frac{2}{3}$.
4. *$|A|$ is tridiagonal, symmetric,* $\|A\|_\infty < 1$, *and* $n \geq 2$.

The first three points are cited from [12, Thm. 3.1]. We will here prove the fourth point.

Proof (Theorem 3.2.4) The proof is performed by induction. The (2×2)-case can be verified by direct computation. Now assume the statement of the theorem holds for an $N \geq 2$, but the tuple (A, z, b) contradicts it in dimension $N + 1$. Since $\|A\|_\infty < 1$, it is $\mathrm{sign}(z_i) = \mathrm{sign}(b_i)$ for all $i \in [N + 1]$, if $|z_1| = \cdots = |z_{N+1}|$, in which case we would be done. Thus we will assume that not all entries of z have the same absolute value.

Let $z \in \mathbb{R}^{N+1}$ so that $z - A|z| = b$. Further, let $i \in [N + 1]$ so that $z_i = \|z\|_\infty$. Then $\sum_{j=1}^{N+1} |a_{ij} z_j| < |z_i|$, since $\|A\|_\infty < 1$, and we thus have $\mathrm{sign}(b_i) = \mathrm{sign}(z_i)$. Now let $k \in \mathrm{Neq}(A, b, z)$, which implies $\mathrm{sign}(z_k) \neq \mathrm{sign}(b_k)$. Then we cannot have $|z_k| = \|z\|_\infty$, and it must hold $|z_k| < \|z\|_\infty$ instead.

Let i be an index such that $|z_i| = \|z\|_\infty$. Then another row than the i-th must hold the contradiction because of the arguments made above. If this row has an index j so that $j \leq i-2$ or $j \geq i+2$, we are done, because then we can eliminate the i-th row and column from the system and the resulting $(n-1)$-dimensional system would still contain the row holhing the contradiction (the matrix is tridiagonal, hence eliminating a row/column whose index differs from its own index by two or more does not affect it).

Thus it remains to deal with the case $j = i \pm 1$. We assume that $i \in 2, \ldots, N$. If not, simply omit the corresponding half of the operations outlined below. Since $|z_i| = \|z\|_\infty$, there exist scalars $\zeta_1, \zeta_2 \in [0,1]$ such that $\zeta_1 \cdot |z_i| = |z_{i-1}|$ and $\zeta_2 \cdot |z_i| = |z_{i+1}|$. Multiplying both sides with $a_{i,i-1}$, resp. $a_{i,i+1}$, we get

$$\zeta_1 \cdot a_{i,i-1} \cdot |z_i| = a_{i,i-1} \cdot |z_{i-1}| \quad \text{and} \quad \zeta_2 \cdot a_{i,i+1} \cdot |z_i| = a_{i,i+1} \cdot |z_{i+1}| \,.$$

Let $A \in \mathrm{M}_{N+1}(\mathbb{R})$, and denote by $A_{i,i}$ the matrix in $M_N(\mathbb{R})$ that is derived from A by removing its i-th row and column. Further, let D be an real $(N \times N)$-matrix which is zero with the exception of two entries

$$d_{i-1,i-1} = \zeta_1 \cdot a_{i,i-1} \quad \text{and} \quad d_{i,i} = \zeta_2 \cdot a_{i,i+1}$$

(note that the i-th row/column of $A_{i,i}$ holds entries that were formerly contained in row/column $i+1$ of A). Then

$$\bar{A} \;\equiv\; A_{i,i} + D$$

is still tridiagonal with $\|A\|_\infty < 1$ and $|A|$ symmetric. Further, for

$$\bar{z} \equiv (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{N+1})^\mathsf{T} \quad \text{and} \quad \bar{b} \equiv (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_{N+1})^\mathsf{T}$$

we have

$$\bar{z} - \bar{A}|\bar{z}| \;=\; \bar{b}.$$

Hence, the tuple $(\bar{A}, \bar{z}, \bar{b})$ contradicts the induction hypothesis for dimension N, as either the index $i-1$ or i (formerly $i+1$) is contained in $\mathrm{Neq}(\bar{A}, \bar{z}, \bar{b})$. □

4 Signed Gaussian elimination

Let C be a nonsingular coefficient matrix of a linear system. The Gaussian elimination algorithm performs a sequence of rank-1-updates on C whose result is an upper triangular matrix. This shape allows to read off the solution of the system via backwards substitution. For the algorithmic principle of the backwards substituion to apply, it suffices to have a coefficient matrix that can be transformed to an upper triangular matrix via symmetric row/column-permutations. We will use this fact to reformulate

the signed Gaussian elimination introduced in [12] in such a way that it does not require excessive row/column-pivoting anymore.

If one is sure of the sign s_k of z_k one can remove this variable from the left-hand side of the AVE. Let A_{*k} denote the k-th column Ae_k and A_{j*} the j-th row $e_j^\mathsf{T} A$. Then the removal of the variable is reflected in the formula

$$(I - A_{*k}e_k^\mathsf{T} s_k)z = b + (A - A_{*k}e_k^\mathsf{T})|z|.$$

The inverses of rank-1-modifications are well-known to be (see, e.g. [1])

$$(I - uv^\mathsf{T})^{-1} = I + \frac{1}{1 - v^\mathsf{T} u} uv^\mathsf{T}.$$

Thus it is easy to remove the matrix factor on the left side. We then have

$$z = \bar{b} + \bar{A}|z|, \tag{4.1}$$

where

$$\bar{b} = b + \frac{1}{1 - A_{kk}s_k} s_k A_{*k}\, b_k \quad \text{and} \quad \bar{A} = A_{red} + \frac{1}{1 - A_{kk}s_k} s_k A_{*k}\, (A_{red})_{k*},$$

with

$$A_{red} = A - A_{*k}e_k^\mathsf{T} = A(I - e_k e_k^\mathsf{T}).$$

Now let El be the set of those indices for which the latter rank-1-update has already been performed. We can then formulate an elimination step for an arbitrary row/column as follows:

Function 1 Sign Controlled Elimination Step

```
def elim(A,b,k,sig_k,El):
sk = A[:,k]*sig_k;
A[:,k] = 0;
sk = sk/(1-sk[k])
for i in El: sk[i] = 0;
b = b + sk*b[k];
A = A + sk*A[k,:];
```

Performing this elimination step n times (for n pairwise different indices, and always updating El) transforms A into a matrix which can be transformed to an upper triangular matrix via symmetric row/column pivots. From this transformed matrix, z can be computed in a straightforward fashion via (an adapted) backwards substitution, since all sign-choices have been eliminated.

Now let $J \subseteq [n]$ be an index set and define

$$\text{J\_b} \equiv \{i \in J \ : \ |b_i| \geq |b_j| \ \forall j \in J\}.$$

Further, let $|J|$ be the number of elements in J. Using this convention, we can give the pseudocode of a slight modification of the algorithm that was introduced as *signed Gaussian elimination* (SGE) in [12]:

Algorithm 2 Signed Gaussian Elimination

```
sge(A,b):
set J = [n];
while (|J|  > 1) do:
determine J_b;
forall k in J_b set sig_k = sign(b_k);
forall k in J_b elim(A,b,k,sig_k,El);
update El;
J = J\ J_b;
endwhile
perform reverse substitution for (I-A)z = b;
return z
```

Theorem 4.1 *Let $A \in \mathrm{M}_n(\mathbb{R})$ and $z, b \in \mathbb{R}^n$ such that (1.1) is satisfied. If A conforms to any of the conditions listed in Theorem 3.2, then the signed Gaussian elimination computes the unique solution of the AVE (1.1) correctly.*

Proof Theorem 3.2 ensures the correctness of the sign-picks. Further, the conditions listed in Theorem 3.2 are invariant under the (sign controlled) elimination step: For criteria $(1)-(3)$, we refer to [12] and [4]. Concerning criterium (4): Assume without loss of generality that $1 \in \mathcal{I}^b_{\max}$. Let A' be the matrix into which A is transformed via the elimination step, and A'_{11} the matrix obtained from A' by eliminating its first row and column, and define A_{11} analogously. Since A is tridiagonal, A'_{11} differs from A_{11} only in a single entry, the upper left one, which we will denote a'_{11}. We denote the corresponding entry of A by a_{22}. Hence, the symmetry is preserved. Further, for any $S \in \mathcal{S}_n$, the matrix $I - AS$ is strictly diagonally dominant, since $\|A\|_\infty < 1$. So $a_{12} = \alpha \cdot (1 - a_{11}s_{11})$, where $\alpha \in [0, 1)$. Since $a_{12} = a_{21}$, this implies $a'_{11} = a_{22} - \alpha \cdot a_{12}$. Keeping in mind that $a_{23} = a'_{12}$, we have

$$\begin{aligned} |a'_{11}| + |a'_{12}| &\le |a_{22}| + \alpha \cdot |a_{12}| + |a_{23}| = |a_{22}| + \alpha \cdot |a_{21}| + |a_{23}| \\ &< |a_{22}| + |a_{21}| + |a_{23}| < 1\,. \end{aligned}$$

This proves the preservation of the norm constraint. Now apply the argument recursively down to the scalar level. This results in a nonsingular linear system with a coefficient matrix that is permutationally similar to an upper triangular matrix, which can be solved by backwards substitution. This solution is unique, since criteria (1)-(4) imply the unique solvability of the AVE, cf. Theorem 2.1. □

For dense A the SGE has a cubic computational cost. For A with band structure it was shown in [12] that the computation has the asymptotic cost of sorting n floating point numbers. Moreover, note that the SGE is numerically stable, since $I - AS$ is strictly diagonally dominant if $\|A\|_\infty < 1$.

For counterexamples which demonstrate the sharpness of the conditions (1)-(3) in Theorem 3.2 with respect to the SGE's correctness, see [12]. Concerning condition (4),

let $A \equiv I$. Then we have $\|A\|_\infty = 1$ and the SGE does not necessarily have a solution, due to the fact that another quantity which measures numbers of AVE solutions, the aligning radius of A, is not smaller than one, cf. [15]. E.g. if b is the vector of ones, there is no solution and any sign-pick is necessarily wrong.

5 Full step Newton method

In this section we analyze the full step Newton method (FN) which is defined by the recursion

$$z^{k+1} \;=\; (I - AS_k)^{-1}b, \tag{5.1}$$

where $S_k \equiv S_{z^k}$, and $z_0 \equiv b$. The iteration has the terminating criterion

$$z^k = z^{k+1}\,.$$

To the best knowledge of the authors it was developed developed simultaneously for two different equivalent system types in [8] and [2], and then later adapted to the present formulation of the AVE in [4]. A first, albeit rather restrictive, convergence result is [4, Prop. 7.2]:

Proposition 5.1 *If $\|A\|_p < 1/3$ for any p-norm, then the iteration (5.1) converges for all b in finitely many iterations from any z_0 to the unique solution of (2.1).*

Moreover, in [4, Prop. 7] convergence was proved for the first two restrictions on A in Theorem 3.2. The following extends this result to the criteria in Theorem 3.2.3-4.

Theorem 5.2 *Let $A \in \mathrm{M}_n(\mathbb{R})$ and $z, b \in \mathbb{R}^n$ such that (1.1) is satisfied. If A conforms to any of the conditions listed in Theorem 3.2, then for any initial vector $z^0 \in \mathbb{R}^n$ the full step Newton method (5.1) computes the unique solution of the AVE (1.1) correctly in at most $n+1$ iterations.*

Proof Note that all conditions listed in Theorem 3.2 are invariant under scalings of A by a signature matrix. Now assume that z satisfies the equality

$$z - Az \;=\; b$$

and set $S \equiv S_z$. Then, since $SS = I$, we have

$$b \;=\; SSz - ASSz \;=\; z - AS|z| \;=\; z - A'|z|,$$

where $A' \equiv AS$ is still strictly diagonally dominant with $\|A'\|_\infty < 1$. This implies $\mathrm{Neq}(A', b, z)$ is empty, by Theorem 3.2. Hence, the signs with index in $\mathcal{I}^b_{\max}$ are fixed throughout all iterations.

Now assume without loss of generality that $1 \in \mathcal{I}^b_{\max}$. Then for all $k \geq 1$ we will have $\mathrm{sign}(z^k_1) = \mathrm{sign}(b_1)$. Let z_k, where $k \geq 1$, be an iterate and let $\tilde{z}_k$ be the vector

obtained from z_k by eliminating its first entry. Then $\tilde{z}_k$ is the unique solution of the system $\tilde{z}_k - \tilde{A}|\tilde{z}_k| = \tilde{b}$, where $\tilde{A}$ and $\tilde{b}$ are obtained from $\bar{A}$ and $\bar{b}$ as defined in (4.1) by eliminating their first row and column, and first entry, respectively. (To avoid confusion, note that this system never appears anywhere as an intermediate step of the algorithm. We merely use the fact that $\tilde{z}_k$ is its solution.) The latter system equals a subsystem obtained by one step of Gaussian elimination. As mentioned in the proof of Theorem 4.1, all restrictions listed in Theorem 3.2.1-4 are invariant under the latter operation. That means for all $\tilde{z}_K$ with $K \geq 2$ all signs s_j so that $j \in \mathcal{I}^{\tilde{b}}_{\max}$ stay fixed to the sign of $\tilde{b}_j$. But that means the signs with index $j+1$ in the z_K stay fixed as well. Applying this argument recursively implies that all signs of z are fixed correctly in at most $n+1$ iterations. Again, we remark that the conditions in Theorem 3.2.1-4 imply the uniqueness of the solution at which we arrive via the aforedescribed procedure. □

6 Comparison of both solvers

In this subsection we will make theoretical and numerical comparisons of both solvers. We will show that they are not equivalent, despite similar correctness, resp. convergence, results. Further, we will test them on random data, as well as two systems from the literature.

6.1 The algorithms are not equivalent

Both solvers are not equivalent and neither solver has a range strictly larger range than the other: Let

$$A \equiv \begin{bmatrix} \frac{\varepsilon}{2} & \frac{1+\varepsilon}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad z \equiv \begin{bmatrix} \frac{\varepsilon}{2} \\ 1 \end{bmatrix},$$

where $\varepsilon > 0$ is arbitrarily small. Then, for $b \equiv z - A|z|$ we have $b = (-\frac{2+\varepsilon^2}{4}, \frac{1}{2})^\intercal$. And clearly $|b_1| > |b_2|$, but $\operatorname{sign}(b_1) \neq \operatorname{sign}(z_1)$. It was shown in [12, Prop.5.2] that for A and b as described, the SGE is lead astray – but we have $\|A\|_\infty = \frac{1}{2} + \varepsilon$. An elementary calculation shows that for $n \leq 2$ we have convergence of the FN method if $\|A\|_\infty < 1$ [13]. In [4, Sec. 7] it was shown that for a system with

$$A \equiv \begin{bmatrix} 0 & 0 & a \\ a & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \quad \text{and} \quad b \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where $a = 5/8$, the FN method cycles for all starting signatures that contain both positive and negative signs. It is straightforward to show that the SGE solves the corresponding AVE.

6.2 Random systems

500 tuples (A, b) of dimension 2.000 were generated at random as follows. Entries of A and b were picked uniformly at random in $[0, 1]$. Then signs were randomly flipped. Finally, A was scaled by $1/(\|A\|_\infty + 1/n)$ to achieve unique solvability of the system.

1. Signed Gaussian elimination: *All systems were solved.* We remark that this is no surprise, since, for random systems, since for large systems, which are constructed as above, we have equality of the sign-patterns of b and z coincide with high probability, cf. [12].
2. Full step Newton: *All systems were solved.* The average number of iterations, when started with S_b as the initial signature, was approx. 3. Moreover, the number of updated signs never exceeded 20 and was approximately 10 on average.

Remark 6.1 The results imply a practical advantage of the full step Newton of over the signed Gaussian elimination, since the former can be assembled from existing linear solvers, while the signed Gaussian elimination has to be implemented from scratch. For example, our test-implementation of FN utilizing the `numpy` library is roughly $1,000\times$ faster than our proof-of-concept SG implementation using the code presented above.

Theoretically, an efficient implementation of SG should outperform the FN algorithm on the given examples by the factor of number of iterations of FN. But to create an implementation from scratch whose eficciency rivals that of modern linear algebra libraries would be a research project in and of itself.

6.3 The system of Brugnano and Casulli

The text [2] is a standard reference. In it a hydrodynamic system of the form (1.2) is solved. We repeat it here:

$$Bz + \max(0, z) = c, \tag{6.1}$$

where $B \equiv \text{tridiag}(-1, 2, -1)$ and

$$z_i \equiv \exp\left(6\frac{i-1}{n-1} - 5\right) - 1 \qquad \text{for all} \quad i \in [n]\,.$$

Reformulation into an AVE via (1.3) yields $A \equiv -(2B + I)^{-1}$.

We tested both algorithms with system sizes of $n = 1000$ and $n = 10000$. It was already noted that the generalized Newton is equivalent to the algorithms presented in [2]. Hence the convergence proof developed in the latter reference also applies to our setting. The algorithm terminated after the third iteration for both system sizes. It is perhaps interesting that in both dimensions the majority of sign updates were performed during the first iteration of the algorithm (828, resp 8330). Only 3, resp. 4 more signs were corrected in the step after that. The signed Gaussian elimination solved both systems correctly.

6.4 The system of Wu and Li

In the aforecited reference [19], the following tridiagonal system is proposed:

$$B \equiv \text{tridiag}(-1, 4, -1) \in \mathrm{M}_n(\mathbb{R}), \quad z \equiv (-1, 1, -1, \ldots, 1, -1, 1)^T,$$

where $c = |z| - Bz$. To convert this into an AVE in the sense of this paper, one needs to invert B. This can be accomplished in linear time to working precision, e.g., via the algorithm in [14]. One readily checks, e.g. via the explicit inversion formulas in [3], that $\|B^{-1}\|_\infty < \frac{1}{2}$. Hence, both algorithms solve the problem correctly, cf. Thms. 4.1 and 5.2. Moreover, the generalized Newton method terminates after one iteration for any dimension. Proof: Let $b = B^{-1}c = z - B^{-1}|z|$. Since $\|B^{-1}\|_\infty < \frac{1}{2} < 1$, and $|z_1| = |z_2| = \cdots = |z_n|$, we have $\text{sign}(b_i) = \text{sign}(z_i)$ for all $i \in [n]$. And the generalized Newton method starts with S_b as its initial best guess signature-pick.

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Bartlett, M.S.: An inverse matrix adjustment arising in discriminant analysis. Ann. Math. Statist. **22**(1), 107–111 (1951)
2. Brugnano, L., Casulli, V.: Iterative solution of piecewise linear systems. SIAM J. Sci. Comput. **30**(1), 463–472 (2008)
3. Fonseca, C.M., Petronilho, J.: Explicit inverses of some tridiagonal matrices. Linear Algeb. Appl. **325**, 7–21 (2001)
4. Griewank, A., Bernt, J.U., Radons, M., Streubel, T.: Solving piecewise linear equations in abs-normal form. Linear Algeb. Appl. **471**, 500–530 (2015)
5. Hu, S.-L., Huang, Z.-H., Zhang, Q.: A generalized Newton method for absolute value equations associated with second order cones. J. Comput. Appl. Math. **235**(5), 1490–1501 (2011)
6. Mangasarian, O.L.: Absolute value equation solution via concave minimization. Optim. Lett. **1**(1), 3–8 (2007)
7. Mangasarian, O.L.: Absolute value programming. Comput. Optim. Appl. **36**(1), 43–53 (2007)
8. Mangasarian, O.L.: A generalized Newton method for absolute value equations. Optim. Lett. **3**, 101–108 (2009)
9. Mangasarian, O.L.: Absolute value equation solution via linear programming. J. Optim. Theory Appl. **161**(3), 870–876 (2014)
10. Mangasarian, O.L., Meyer, R.R.: Absolute value equations. Linear Algeb. Appl. **419**, 359–367 (2006)
11. Neumaier, A.: Interval Methods for Systems of Equations. Cambridge University Press (1990)
12. Radons, M.: Direct solution of piecewise linear systems. Theor. Comput. Sci. **626**, 97–109 (2016)
13. Radons, M.: Efficient solution of piecewise linear systems. Master-Thesis (2016)
14. Radons, M.: O(n) working precision inverses for symmetric tridiagonal toeplitz matrices with O(1) floating point calculations. Optim. Lett. **12**, 425–434 (2018)
15. Manuel, R.: Generalized perron roots and solvability of the absolute value equation (2021)

16. Rohn, J.: Systems of linear interval equations. Linear Algeb. Appl. **126**, 39–78 (1989)
17. Rump, S.M.: Theorems of Perron-Frobenius type for matrices without sign restrictions. Linear Algeb. Appl. **266**, 1–42 (1997)
18. Streubel, T., Griewank, A., Radons, M., Bernt, J.U.: Representation and analysis of piecewise linear functions in abs-normal form. Proc. IFIP TC **7**, 323–332 (2014)
19. ShiLiang, Wu., Li, CuiXia: A special shift splitting iteration method for absolute value equation. AIMS Math. **5**, 51–71 (2020)
20. Yuan, X.-T., Yan, S.: Nondegenerate piecewise linear systems: a finite newton algorithm and applications in machine learning. Neural Comput. **24**(4), 1047–1084 (2012)
21. Zamani, M., Hladík, M.A.: A new concave minimization algorithm for the absolute value equation solution. Optim. Lett. (2021)
22. Zheng, L.: The picard-hss-sor iteration method for absolute value equations. J. Inequal. Appl. (258) (2020)