A theoretical study of correlation between scaled energy and earthquake magnitude based on two source displacement models

Jeen-Hwa Wang

Abstract The correlation of the scaled energy, $\hat{e} = E_s/M_0$, versus earthquake magnitude, M_s, is studied based on two models: (1) Model 1 based on the use of the time function of the average displacements, with a ω^{-2} source spectrum, across a fault plane; and (2) Model 2 based on the use of the time function of the average displacements, with a ω^{-3} source spectrum, across a fault plane. For the second model, there are two cases: (a) As $\tau \doteqdot T$, where τ is the rise time and T the rupture time, $\lg(\hat{e}) \sim -M_s$; and (b) As $\tau \ll T$, $\lg(\hat{e}) \sim -(1/2)M_s$. The second model leads to a negative value of $\hat{e}$. This means that Model 2 cannot work for studying the present problem. The results obtained from Model 1 suggest that the source model is a factor, yet not a unique one, in controlling the correlation of $\hat{e}$ versus M_s.

Keywords Scaled energy · Earthquake magnitude · Seismic-wave energy · Seismic moment · Source displace model

1 Introduction

The seismic-wave energy, E_s, and seismic moment, M_0, are two main parameters quantifying earthquakes. The scaled energy, $\hat{e} = E_s/M_0$, can link dynamic properties (from E_s) and static behavior (from M_0) of earthquake sources, and, thus, the correlation of $\hat{e}$ versus M_s would reflect the source properties (cf. Kanamori and Heaton 2000; Kanamori and Rivera 2004). The measured values of $\hat{e}$ range from 10^{-6} to 10^{-3}

J.-H. Wang (✉)
Institute of Earth Sciences, Academia Sinica, Nangang,
P.O. Box 1-55, Taipei, China
e-mail: jhwang@earth.sinica.edu.tw

(Kanamori 1977; Vassiliou and Kanamori 1982; Kikuchi and Fukao 1988; Choy and Boatwright 1995; Brodsky and Kanamori 2001; Hwang et al. 2001; Huang et al. 2002, 2009; Kinoshita and Ohike 2002). A large range of $\hat{e}$ might be due to the uncertainty of measuring E_s, because the signals are often affected by several factors, including finite frequency bandwidth limitation as the major one (cf. Wang 2004). From the equation to express the effect due to finite frequency bandwidth limitation shown in Wang (2004), it is obvious that the under-estimate is larger for large earthquakes than for small ones. On the other hand, the estimate of M_0 from longer-period seismic data has lower uncertainty.

Brodsky and Kanamori (2001) first claimed an abrupt rise in $\hat{e}$ from 10^{-6} to 10^{-4} almost at $M_w = 5$, where M_w is the moment magnitude, that is, $\hat{e}$ is of size-dependence. On the contrary, Ide and Beroza (2001) stated that the values of E_s, especially for $M_w < 5$ events, used by Brodsky and Kanamori (2001) were underestimated due to finite frequency bandwidth limitation. They corrected the E_s, especially for intermediate-sized and small events, based on a ω^{-2} source model, and then stressed that $\hat{e}$ is almost a constant of about 3×10^{-5} in a large range of $\lg(M_0)$ from 4 to 21. Kinoshita and Ohike (2002) stated that $\hat{e}$ is weakly dependent on M_0.

From a 1-D elastodynamic earthquake fault model, Shaw (1998) stressed that the $\hat{e}$ is a function of M_0 and affected by friction. Kanamori and Rivera (2004) studied size-dependence of $\hat{e}$ on M_w through the formulation of the problem using variational calculus. They related the minimum of $\hat{e}$ (denoted as $\hat{e}_{min}$) to several source parameters. They stated that the $M_0 \sim f_c^{-3}$ scaling leads to independence of $\hat{e}_{min}$ on M_w, while the $M_0 \sim f_c^{-(3 + \varepsilon)}$ scaling, with $0 < \varepsilon \leq 1$, results in an increase in $\hat{e}_{min}$ with M_w. They also claimed that the predicted value of $\hat{e}_{min}$ for the largest crustal earthquakes is close to the average for large events, and $\varepsilon = 0.5$ is the optimum value. In order to

minimize the functional $J(u,v) = E_s - \lambda M_o$, where $u(t)$ is the radiation pattern-corrected far-field displacement in terms of time t, $v(t) = du(t)/dt$, and λ is the Lagrangian multiplier, Kanamori and Rivera (2004) considered $u(t)$ to be a one-sided pulse with a total duration of T, and thus, they took $u(0) = u(T) = 0$ to be the initial conditions for solving the equation of the displacement. The parameter T is the source rupture duration. They obtained the velocity: $v(t) = (6M_0/C_M T^3)(T-2t)$, where C_M is a factor in describing M_0 as explained below, and the displacement: $u(t) = (6M_0/C_M T^3)t(T-t)$. Based on $u(t)$ and $v(t)$, they obtained $\hat{e}_{min} = 12C_E M_0/C_M^2 T^3$, where C_E is a factor in describing E_s as explained below, in the extreme state of E_s/M_0. This gives a fact that $\hat{e}_{min}$ is independent upon M_s. Their results lead to a positive velocity of $6M_0/C_M T^2$ at $t = 0$ and a negative one of $-6M_0/C_M T^2$ at $t = T$.

Kanamori and Rivera (2004) obtained significant results for understanding the correlation between $\hat{e}_{min}$ and M_s. Nevertheless, it is still necessary to examine the correlation when different initial conditions are used, because the sum of the waveforms generated from the two planes of a fault recorded at a remote observational point is not always to form a one-sided pulse with $u(0) = u(T) = 0$ as used in Kanamori and Rivera (2004). In addition, they did not concern the type of the source spectra. The finite frequency bandwidth effect can affect the measure of seismic radiation energy (cf. Wang 2004). Under such an effect, different source spectra will lead to different measured values of seismic radiation energy, thus resulting in different estimated ones of scaled energy. Hence, it is also necessary to study dependence of $\hat{e}$ on M_s based on a ω^{-2} source model.

In this study, an attempt is made to theoretically investigate the correlation of $\hat{e}$ versus M_s using two models: (1) the use of the time function of the average displacements, with a ω^{-2} source spectrum, across a fault plane; and (2) the use of the time function of the average displacements, with a ω^{-3} source spectrum, across a fault plane. The time functions were proposed by Beresnev and Atkinson (1997). For the first way, like Kanamori and Rivera (2004) the source model is not prescribed, and only the extreme state of E_s/M_0 is taken into account. On the other hand, for the second one, a ω^{-2} source model is considered, and it is not necessary to minimize a functional. Since the observed seismograms are quite complex, Model 1 is just a special case of the problem. Comparison of the present results and those obtained by Kanamori and Rivera (2004) will be discussed below.

2 Theory and discussion

Following Kanamori and Rivera (2004), from the radiation pattern-corrected far-field displacement, $u(t)$, E_s, and M_0 are defined to be:

$$E_s = C_E v^2(t)dt, \tag{1}$$

$$M_0 = C_M u(t)dt. \tag{2}$$

In Eqs. (1) and (2), $v = du/dt$, $C_E = 8\pi\rho r^2\beta/5$, and $C_M = 4\pi\rho r\beta^3$, where r is the hypocentral distance between an event and a recording station and ρ and β are, respectively, the density and shear-wave velocity in between the source and the observational point, and the integration is performed from 0 to T. The radiation pattern-corrected far-field displacement, $u(t)$, is related to the velocity, $v(t)$, which is the first derivate of the average displacement, d(t), across a faut plane in the form: $u(t) \sim v(t-r/\beta)$.

The far-field time functions of the displacement and velocity are commonly complex, because there are numerous factors in affecting them. At a hard rock site, without substantial site effects, the modulus of shear-wave displacement spectrum can be written as

$$D(\omega) = 2S(\omega)P(\omega)\exp(-\omega r/2Q\beta), \tag{3}$$

where ω is the angular frequency, Q is the quality factor, $S(\omega)$ is the source spectrum, $P(\omega)$ accounts for the spectral cutoff above a certain frequency. $S(\omega)$ is modeled by multiplying a deterministic function by the Fourier spectrum of windowed Gaussian noise (cf. Beresnev and Atkinson 1997). There are different ways to select the deterministic function, for example, the ω^{-2} or ω^{-3} model (e.g., Aki 1967; Brune 1970). In this study, we apply the source time functions proposed by Beresnev and Atkinson (1997). An expression for the displacement from a point shear dislocation in a homogeneous elastic space (Aki and Richards 1980), together with a definition of the seismic moment, $M(t) = \mu u_o(t)A$, where μ, $u_o(t)$, and A are the rigidity, time function of source displacement, and fault area, gives the far-field shear wave in the form

$$u(x,t) = (R_{\theta\phi}/4\pi r\beta^3 r)v_o(t - r/\beta), \tag{4}$$

where $u(x,t)$ is the displacement at spatial point x, m is the shear modulus, $R_{\theta\phi}$ is the angular radiation pattern, $v_o(t)$ is the time derivative of the average displacement across the fault plane, and A is the fault area.

The studies of the correlation of $\hat{e}$ versus M_s using two models as mentioned above are described below.

2.1 Model 1

This model is based on the condition that the average source displacement, $d(t)$, shows a ω^{-2} source spectrum as proposed by Aki (1967). Beresnev and Atkinson (1997) proposed three time functions of the displacements across a fault plane, which start from zero and approach a certain level D_o, to satisfy the request that the spectra are a ω^{-n} form. In this study, we apply the time functions of the

source displacements with a ω^{-2} spectrum as described by the following expression:

$$d_2(t) = D_o\left[1 - (1 + t/\tau)\mathrm{e}^{-t/\tau}\right], \tag{5}$$

where $\tau = 1/\omega_c$, τ is the rise time, $\omega_c = f_c/2\pi$, and $D_o = d_2(t \to \infty)$. The modulus of the Fourier transform of $d_2(t)$ is $D_2(\omega) = D_o/[1 + (\omega/\omega_c)^2]$, which is a ω^{-2}-form function at high frequencies. The time function of the source velocity is:

$$v_2(t) = D_o(t/\tau)\mathrm{e}^{-t/\tau}/\tau. \tag{6}$$

Inserting Eqs (4) and (6), respectively, into Eqs (1) and (2) gives

$$E_s = (C_E D_o^2/\tau)\{1 - [1 + 2T/\tau + 2(T/\tau)^2]\mathrm{e}^{-2T/\tau}\}; \tag{7}$$

$$M_o = (C_M D_o \tau)\{(1 + \mathrm{e}^{-T/\tau})(T/\tau) + 2\mathrm{e}^{-T/\tau} - 2\}. \tag{8}$$

where $\eta = T/\tau$. Dividing Eq (7) by Eq (8) gives

$$\hat{e} = C_E D_o\{1 - [1 + 2T/\tau + 2(T/\tau)^2]\mathrm{e}^{-2T/\tau}\}/$$
$$C_M \tau^2\{(1 + \mathrm{e}^{-T/\tau})(T/\tau) + 2\mathrm{e}^{-T/\tau} - 2\}. \tag{9}$$

Eq (9) shows that $\hat{e}$ is a complex function of T/τ. In principle, for normal earthquakes τ cannot be larger than T (cf. Kanamori and Anderson 1975). There are two assumptions of relating τ to T: (1) $\tau \fallingdotseq T$ (from a conventional viewpoint); and (2) $\tau \ll T$, which is from the slip-pulse concept proposed by Heaton (1990). As $T/\tau \fallingdotseq 1$, $\hat{e} \fallingdotseq 3.12$ ($C_E D_o/C_M \tau^2$), thus, giving $\hat{e} \sim \tau^{-2} \sim f_c^2$. Since $f_c \sim M_0^{-1/3}$ (Brune 1970), we have $\hat{e} \sim M_0^{-2/3}$ or $\lg(\hat{e}) \sim -(2/3)\lg(M_0)$. This leads to $\lg(\hat{e}) \sim -M_s$ as considering the PB relationship of M_0 versus M_s obtained by Purcaru and Berckhemer (1978). Obviously, $\hat{e}$ negatively correlates to M_s, with a scaling exponent of -1.

As $\tau \ll T$, $1 - [1 + 2T/\tau + 2(T/\tau)^2]\mathrm{e}^{-2T/\tau} \fallingdotseq 1$ and $(T/\tau)(1 + \mathrm{e}^{-T/\tau}) + 2\mathrm{e}^{-T/\tau} - 2 \fallingdotseq T/\tau$. Hence, we have $\hat{e} \fallingdotseq C_E V/C_M \tau$, where $V = D_o/T$ is the average particle velocity. Obviously, this gives $\hat{e} \sim \tau^{-1} \sim f_c$. Since $f_c \sim M_0^{-1/3}$ (Brune 1970), we have $\hat{e} \sim M_0^{-1/3}$ or $\lg(\hat{e}) \sim -(1/3)\lg(M_0)$. This leads to $\lg(\hat{e}) \sim -(1/2)M_s$ as considering the PB relationship. Obviously, $\hat{e}$ negatively correlates to M_s, with a scaling exponent of $-1/2$.

2.2 Model 2

This model is based on the condition that the average source displacement, $d(t)$, shows a ω^{-3} source spectrum as first proposed by Haskell (1966), and then deeply studied by Aki (1967). Beresnev and Atkinson (1997) proposed three time functions of the displacements across a fault plane, which start from zero and approach a certain level D_o, to satisfy the request that the spectra are a ω^{-n} form. In this study, we

apply the time function of the source displacements with a ω^{-3} spectrum as described by the following expression:

$$d_3(t) = D_o\{2 - [1 + (1 + t/\tau)^2]\mathrm{e}^{-t/\tau}\}, \tag{10}$$

where $D_o = d_3(t \to \infty) = d_2(t \to \infty)$. The modulus of the Fourier transform of $d_3(t)$ is $D_3(\omega) = D_o/[1 + (\omega/\omega_c)^2]^{3/2}$, which is a ω^{-3}-form function at high frequencies. The time function of the source velocity is:

$$v_3(t) = D_o(t/\tau)^2\mathrm{e}^{-t/\tau}/\tau. \tag{11}$$

Inserting Eqs (11) and Eq. (10), respectively, into Eqs (1) and (2) gives

$$E_s = (C_E D_o^2/4\tau)\{3 - \mathrm{e}^{-2T/\tau}[3 - 6(T/\tau) - 6(T/\tau)^2$$
$$- 4(T/\tau)^3 + 2(T/\tau)^4]\}; \tag{12}$$

$$M_o = (C_M D_o \tau)[6 - 2(T/\tau) - 6\mathrm{e}^{-T/\tau} - 4(T/\tau)\mathrm{e}^{-T/\tau}$$
$$- 2(T/\tau)^2\mathrm{e}^{-T/\tau}. \tag{13}$$

Dividing Eqs (12) by (13) gives

$$\hat{e} = (C_E D_o/4C_M \tau^2)\{3 - \mathrm{e}^{-2T/\tau}[3 - 6(T/\tau) - 6(T/\tau)^2$$
$$- 4(T/\tau)^3 + 2(T/\tau)^4]\}/[6 - 2(T/\tau)$$
$$- 6\mathrm{e}^{-T/\tau}4(T/\tau)\mathrm{e}^{-T/\tau} - 2(T/\tau)^2\mathrm{e}^{-T/\tau}]. \tag{14}$$

Eq (14) shows that $\hat{e}$ is a complex function of T/τ. As mentioned above, there are two assumptions of relating τ to T: (1) $\tau \fallingdotseq T$ (from a conventional viewpoint); and (2) $\tau \ll T$, which is from the slip-pulse concept proposed by Heaton (1990). As $T/\tau \fallingdotseq 1$, the value of $\hat{e}$ from Eq (14) is negative. This is unreasonable, thus, meaning that Model 2 cannot work for the present problem.

3 Discussion

In Model 1, a time function of the average displacements, with a ω^{-2} source spectrum, across a fault plane as proposed by Beresnev and Atkinson (1997) is applied. Results show that as $\tau \fallingdotseq T$ from the conventional viewpoint, $\lg(\hat{e}) \sim -M_s$; while as $\tau \ll T$ from the slip-pulse concept, $\lg(\hat{e}) \sim -(1/2)M_s$. The results of Model 1 suggest that as the extreme state of E_s/M_0 is not taken into account, the source model, including the scaling law and the relation between τ and T, is a significant factor in controlling the correlation of $\hat{e}$ versus M_s.

In Model 2, a time function of the average displacements, with a ω^{-3} source spectrum, across a fault plane as proposed by Beresnev and Atkinson (1997) is applied. Results show that as $\tau \fallingdotseq T$, the value of scaled energy is negative. This means that this model is not acceptable for studying the present problem.

Kanamori and Rivera (2004) modified the M_0-f_c correlation from $M_0 \sim f_c^{-3}$ to $M_0 \sim f_c^{-(3 + \varepsilon)}$, with $0 < \varepsilon \le 1$,

to yield a positive correction of $\hat{e}$ versus M_s. It is interesting to ask if M_0 can relate to f_c in a form of $M_0 \sim f_c^{-3.5}$ or not. One of the major factors in controlling the generation of events is the ratio of the fault width, W, to the characteristic width, W_s, of the seismogenic zone. Small events happen when $W/W_s < 1$ and large ones do when W/W_s is equal to or slightly larger than 1. The W_s could change from area to area. From $M_0 = \mu\delta LW$, we can examine the scaling of M_0 versus L for different events. For larger events with $L > 10$ km, Wang and Ou (1998) stated that $M_0 \sim L^2$ due to $\delta \sim L$ and $W \doteqdot$ constant. Although they did not clearly report the scaling for smaller events with $L < 10$ km, from their figures, it can be seen that $M_0 \sim L^3$ due to $\delta \sim L$ and $W \sim L$. For both P- and S-waves, f_c is inversely proportional to the dimension of the earthquake source, i.e., $f_c \sim L^{-1}$, (cf. Aki and Richards 1980), and, thus, $M_0 \sim f_c^{-2}$ for large earthquake and $M_0 \sim f_c^{-3}$ for small events. Obviously, the exponents of the M_0–f_c scaling are -2 and -3, which are both not larger than 3, for large and small earthquakes, respectively.

4 Conclusions

The correlation of $\hat{e}$ versus M_s is studied using two models: (1) the first one is the application of the time function of the average displacements, with an ω^{-2} spectrum, across a fault plane; and (2) the second one is the application of the time function of the average displacements, with an ω^{-3} spectrum, across a fault plane. The two models are proposed by Beresnev and Atkinson (1997). Like Kanamori and Rivera (2004), Model 1 gives independence of $\hat{e}$ on M_s. This means that in the extreme state of E_s/M_0, the two different initial conditions, which associated with different source models, lead to a same conclusion that the scaled energy is of size-independence. For Model 2, the value of scaled energy is negative as $\tau \doteqdot T$. The results of this study suggest that the source model, including the scaling law and the relation between τ and T, is a factor, yet not a unique one, in controlling the correlation of $\hat{e}$ versus M_s. This will depend upon whether the extreme state of E_s/M_0 is taken into account or not. At present, it is not yet known which model is the most appropriate one to explain the correlation of $\hat{e}$ versus M_s, because the observed correlation is still questionable due to high uncertainties in the estimates of E_s, especially for large earthquakes.

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