Doi: 10.1007/s11589-010-0738-2

Equipartition and retrieval of Green's function*

Richard L. Weaver †
Department of Physics, *University of Illinois*, *Urbana, IL* 61801, *USA*

Abstract Physical and mathematical arguments are presented for equipartition as the statistical state achieved by a random field, independent of its sources, in the limit of enough scattering. The arguments are simplest for the case of thermally excited fields, but are shown to apply also, with caveats, in non-equilibrium acoustics and seismology. Practical implications are discussed.

Key words: **noise correlation; equipartition; elastic waves; information entropy**
CLC number: P315.01 **Document code**: A

1 Introduction

The term equipartition signifies that appropriately normalized average energy densities in different places, or different modes, of a diffuse wave field are equal (Nyquist, 1928; Kittel and Kroemer, 1980; Weaver, 1982). This statistical concept is venerable and useful, and sometimes problematic. Its utility springs from the enormous simplifications it engenders. It is striking that one can say anything general about the statistics of a wave field. Surely there are many degrees of freedom; why should the statistics of the field be so simple, and so independent of its sources? And yet it appears that under relatively simple conditions these fields are equipartitioned and their statistics are simple. The properties of a random wavefield, in particular its correlations, are universal and independent of its source.

There are several aspects to the concept. In seismology the term has come to signify the state of a diffuse field in which wave energy is partitioned between shear and longitudinal waves in a precise ratio (see below). It is less well appreciated that full equipartition implies corresponding ratios for other kinds of waves as well, e.g., Rayleigh and Love. The term also signifies that waves are equally intense from all directions of incidence.

This paper is mostly pedagogic. It is intended to review the concept, including its origins in thermodynamics and its consequences for energy ratios and retrieval of the Green's function, and to argue for its applicability to media with non-thermal sources such as those of seismology. It is intended to introduce the concept of *local* equipartition in unbounded media, and to outline conditions under which one expects to find equipartition in practice. It is also argued that equipartition, being so universal, is an important state against which to compare any random seismic field.

2 Thermal equipartition

Classical statistical thermodynamics shows that a system, at specified total energy, achieves maximal thermodynamic entropy if the natural (eigen) modes have equal expected energies (Nyquist, 1928; Kittel and Kroemer, 1980). For the case of acoustics, for example, thermally excited phonons in a finite solid (Kittel and Kroemer, 1980), we write the amplitudes a_n of the natural modes as uncorrelated Gaussian random variables with the following correlation function:

$$\begin{cases} < a_n a_m^* > = \delta_{nm} 2\varepsilon_n / \omega_n^2 \\ < a_n a_m > = 0 \end{cases}, \quad (1)$$

where ε_n is the average energy in the nth natural mode and ω_n is its frequency.

The thermal case is special in that energy re-distributing scatterings permit flows of energy between different frequencies. This implies that average energy is the same for all natural modes; the quantity ε_n is then interpretable as Boltzmann constant times absolute temperature, $k_B T$ (Quantum mechanics provides a correction which here we neglect). They also differ in

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* **Received** 2 June 2010; **accepted in revised form** 3 August 2010;
published 10 October 2010.
† **Corresponding author. e-mail**: r-weaver@illinois.edu

that sources are distributed (the fluctuation dissipation theorem (Callen and Welton, 1951)) precisely in proportion to dissipation. For acoustics above about 100 kHz, noise fields are usually dominated by thermal fluctuations because man-made and natural sources are weak or rapidly dissipated. Due to the robustness of equipartition in this regime, and despite the weakness of thermal fluctuations, it is here that some of the more striking demonstrations of Green's function retrieval have been made (Marvin et al., 1980; Weaver and Lobkis, 2001).

Noise in acoustics and seismology and ocean acoustics below 100 kHz is dominated by non-thermal sources. Such fields are not in equilibrium with the ambient temperature, and source strengths are not distributed in the same way as the dissipation. The thermodynamic arguments do not apply and equipartition does not necessarily follow. The lack of significant inelastic scattering means that ε_n (even if (1) applies) is in principle a function of frequency. As a consequence, a random field's statistics can be dependent on details of its source, in particular the source's distribution in time and space and frequency. Nevertheless, a version of the maximum entropy concept often continues to apply and a variation on thermal equipartition is sometimes retained.

3 Consequences of equipartition

Below we review derivations of the principle of equipartition for the non-thermal case. Before doing that, though, it is instructive to first demonstrate two of its more striking consequences.

The usual modal expansion for the free linear vibrations of a finite structure is

$$\psi(\boldsymbol{x}, t) = \mathrm{Re}\sum_n a_n u^{(n)}(\boldsymbol{x})\exp(\mathrm{i}\omega_n t), \tag{2}$$

where $u^{(n)}$ is the nth natural mode, ω_n is its frequency, and a_n is its complex amplitude. Thus (the time-derivative of) the field-field correlation is

$$\begin{aligned} C'(\boldsymbol{x}, \boldsymbol{x}', \tau) = \frac{\mathrm{d}}{\mathrm{d}\tau}\langle\psi(\boldsymbol{x},t)\psi(\boldsymbol{x}', t+\tau)\rangle = \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg\langle\Bigg[\sum_n a_n u^{(n)}(\boldsymbol{x})\exp(\mathrm{i}\omega_n t) + \sum_n a_n^* u^{(n)}(\boldsymbol{x})\exp(-\mathrm{i}\omega_n t)\Bigg]\times \\ \Bigg\{\sum_m a_m u^{(m)}(\boldsymbol{x}')\exp[\mathrm{i}\omega_m(t+\tau)] + \sum_m a_m^* u^{(m)}(\boldsymbol{x}')\exp[-\mathrm{i}\omega_m(t+\tau)]\Bigg\}\Bigg\rangle = \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\sum_n\left\{\frac{\varepsilon_n}{\omega_n^2}\right\}u^{(n)}(\boldsymbol{x})\,u^{(n)}(\boldsymbol{x}')\cos(\omega_n\tau) = -\sum_n\{\varepsilon_n\}\,u^{(n)}(\boldsymbol{x})\,u^{(n)}(\boldsymbol{x}')\,\frac{\sin(\omega_n\tau)}{\omega_n}, \end{aligned} \tag{3}$$

the last line of which was derived using the assumed correlations (1). Except for the minus sign and the factor of ε_n and the support at negative τ, this is the familiar natural mode expansion of Green's function (Morse and Feshbach, 1953; Economou, 1983). Hence one draws the conclusion that an equipartitioned field has correlations equal to Green's function. In the event that ε has a weak frequency dependence, one concludes that C' corresponds to a band-pass filtered version of G. In the event that ε has a rapid frequency dependence, the filter has long and possibly irregular duration and C' resembles G more poorly.

Another striking consequence of equipartition is that a diffuse elastic wave field will, in the bulk, partition its energy between shear and longitudinal waves in a proportion

$$E_\mathrm{S}/E_\mathrm{P} = 2c_\mathrm{P}^3/c_\mathrm{S}^3. \tag{4}$$

This number depends on Poisson's ratio but is typically about 10 or more. The result follows from simple mode counting procedures (Kittel and Kroemer, 1980; Kinsler et al., 1982; Weaver, 1982) in which modal density is seen to be inversely proportional to the cube of wavelength (and hence the cube of velocity c). An additional factor of two arises above from the two independent polarizations of shear waves. This ratio was derived independently by Papanicolaou et al. (1996) in a very different context as the steady state ratio of elastic energies achieved by a multiply scattered wave field. The expression is more complex at a free surface (Weaver, 1985; Hennino et al., 2001) where the energy density of Rayleigh waves is also of interest. One also recognizes that there is a characteristic ratio of horizontal to vertical kinetic energy. Characteristic energy ratios are indeed observed in multiply scattered seismic waves (Hennino et al., 2001). This can be used as a signature of equipartition.

Equipartition in non-thermal systems can be understood, and derived, in at least two distinct ways, each with the potential to give insight into how and whether equipartition may or may not occur in practice. One method posits that there should be some sort of dynamic balance between different kinds of energy, and furthermore that reciprocity demands that that balance be in a certain universal proportion. The other method posits that the statistics of a wave field ought to be such as to maximize information entropy. The following sections present the two approaches.

4 Dynamic balance of diffuse energy flow

That there should be some characteristic ratio of energies in a steady-state diffuse field is not difficult to appreciate. Even if a source is such that wave energy is produced in certain modes and not others, mode conversion will mix the energies. A steady state dynamic balance is then to be expected as energy in mode '*i*' is augmented due to conversion from modes '*j*' and diminished by conversion from mode '*i*' to modes '*j*'. On taking the average power flows to be proportional to the average energy densities we write

$$\begin{cases} \dfrac{\mathrm{d}E_i}{\mathrm{d}t} = \sum\limits_{j\neq i} \alpha_{ij} E_j - (\sum\limits_{j\neq i} \alpha_{ji}) E_i \\ \dfrac{\mathrm{d}\{E\}}{\mathrm{d}t} = \boldsymbol{A}\{E\} \end{cases}, \tag{5}$$

where the α are energy mode conversion coefficients, all non-negative. $\boldsymbol{A}$ is a positive semi-definite matrix of coefficients and $\{E\}$ is an array of mode energies. Here we may imagine that indices '*i*' represent shear or longitudinal waves, or different substructures, or different volumes in room acoustics. This first-order constant coefficient set of equations has exponential solutions $\exp(-\gamma t)$ with characteristic rates γ, all nonnegative. Conservation of total energy $E=\sum E_i$ (implicit in the above equations) demands that one of those rates is $\gamma=0$. The eigenvector $\{e\}$ associated with that eigenvalue satisfies $\boldsymbol{A}\{e\}=0$ and corresponds to some characteristic ratio of energies in the different substructures. As long as $\boldsymbol{A}$ is not block diagonal (i.e. when all modes are coupled to all others, directly or indirectly) there is only one such eigenvalue $\gamma = 0$ and eigenvector $\{e\}$. This is the only steady state solution of equation (5); it is the differential equation's attractor. The energy distribution achieved at late time is proportional to $\{e\}$ and independent of initial conditions.

Equation (5) is a limit of a more realistic case in which there are dissipations and sources of energy as well as mode conversions. In the presence of dissipation σ and steady input powers $\{P\}$, one modifies (5) as

$$\begin{cases} \dfrac{\mathrm{d}E_i}{\mathrm{d}t} = \sum\limits_{j\neq i} \alpha_{ij} E_j - (\sum\limits_{j\neq i} \alpha_{ji}) E_i - \sigma_i E_i + P_i \\ \dfrac{\mathrm{d}\{E\}}{\mathrm{d}t} = (\boldsymbol{A}-\boldsymbol{\sigma})\{E\} + \{P\} \end{cases}. \tag{6}$$

The steady state solution of (6) such that $\mathrm{d}\{E\}/\mathrm{d}t=0$ is $\{E\}=-(\boldsymbol{A}-\boldsymbol{\sigma})^{-1}\{P\}$. It is in general now not proportional to $\{e\}$. The presence of dissipation and steady power input may distort the equilibrium distribution of energy away from $\{e\}$. If, however, P happens to be proportional to the damping and to the natural distribution $\{e\}$ of energy: $P_i = p\sigma_i e_i$ with some proportionality p, then the steady state distribution is $\{E\} = -[\boldsymbol{A}-\boldsymbol{\sigma}]^{-1}\{P\}= -[\boldsymbol{A}-\boldsymbol{\sigma}]^{-1}p\boldsymbol{\sigma}\{e\}=p\{e\}$. The distribution is unchanged from the special limiting case $\propto\{e\}$ obtained at late times in the absence of dissipation. While the proportionality $P_i=p\sigma_i e_i$ may seem arbitrary, it is obeyed precisely in the thermal case (Callen and Welton, 1951).

That the special distribution $\{e\}$ is *equipartitioned*, i.e., that this eigenvector corresponds to each natural mode having equal average energy (independent of the details of the α), is not as obvious as the conclusion that there must be some characteristic steady state ratio $\{e\}$. Such a conclusion requires further argument.

5 Equipartition in finite structures

In a finite structure proofs of equipartition can be composed in various ways. Perhaps the simplest are those based on maximum information entropy (Jaynes, 1982). Given a random wavefield described by its modal expansion as in equation (2), we inquire as to the joint probability function $p(a_1, a_2, a_3, \cdots)$ for the complex modal amplitudes a. The information entropy associated with such a probability density function is

$$I = \int p(a_1, a_2, \cdots) \ln[p(a_1, a_2, \cdots)]\, \Pi_n \mathrm{d}\,\mathrm{Re}\, a_n\, \mathrm{d}\,\mathrm{Im}\, a_n, \tag{7}$$

which, in accord with the usual arguments (Jaynes, 1982) must be maximized subject to the condition that sundry additional pieces of information are known. For example, we might impose two conditions: the normalization of the probability density function, and the total energy.

$$\begin{cases} \int p(a_1, a_2, \cdots)\, \Pi_n \mathrm{d}\,\mathrm{Re}\, a_n\, \mathrm{d}\,\mathrm{Im}\, a_n = 1 \\ \int \left(\sum_n \omega_n^2 \frac{|a_n|^2}{2} \right) p(a_1, a_2, \cdots)\, \Pi_n \mathrm{d}\,\mathrm{Re}\, a_n \mathrm{d}\,\mathrm{Im}\, a_n = E \end{cases} \quad (8)$$

On maximizing I while imposing constraints by means of Lagrange multipliers, one concludes that the probability density function must be

$$p(a_1, a_2, \cdots) = F \exp\left(-\beta \sum_n \omega_n^2 \frac{|a_n|^2}{2} \right) \quad (9)$$

with F being a normalization constant and β being one of the Lagrange multipliers. One immediately concludes that the field has Gaussian statistics with correlations indicated by (1) with constant $\varepsilon_n = E/$ number of modes $= 1/\beta$. The correlation function is exceptionally simple in terms of the modal amplitudes (called normal coordinates in vibrations) a. In terms of more immediately useful field variables such as displacements or strains at sundry positions, the correlation function is just G. Gaussian statistics follow from maximal information entropy and in turn imply both equipartition *and* retrieval of G. Equipartition, and retrieval $C'=G$, are two ways of stating the same thing, merely in different coordinates. A similar point is emphasized by Sánchez-Sesma et al. (2008).

These arguments are slightly modified if the constraint is on the smooth spectral density of energy $E(\omega)$ rather than the total energy E. In non-thermal acoustics the field is excited by sources with specified spectral densities, and energy does not migrate in frequency. In this case the statistics of p are constrained by the (smooth) frequency-dependent spectral density of energy $E(\omega)$, not by total energy. In that case β has a (smooth) dependence on ω_n.

It may be that the system is composed of two or more independent subsystems that do not exchange energy, for example two uncoupled volumes in room acoustics, or shear and longitudinal waves in an elastic body with rigid-smooth boundaries. In this case the energy in each subsystem is determined only by its own source, and the modes of the two subsystems will have separate probability density functions with different β.

If, however, those two kinds of modes do exchange energy, then one concludes that the amount of energy of each kind is merely proportional to the number of modes of that kind. This is the origin of the conclusion equation (4) (Weaver, 1982). A related consequence is that two coupled acoustic volumes, e.g., coupled reverberation rooms, have energies proportional to their respective number of modes, in turn proportional to their volumes if they have the same wave speeds. After equilibrium is reached, the two coupled rooms will have identical energy densities, even if the sources are only in one room. Similarly, a source of purely longitudinal waves in a finite elastic structure will generate, after sufficient mode conversion, a field that is predominantly composed of shear waves in accord with equation (4). Another consequence is that seismic codas are dominated by shear waves.

Equipartition may also be established by a detailed consideration of mode conversion rates α. Mode conversion reflection coefficients from longitudinal to shear are typically much greater than the reverse. Thus a steady state balance requires a much higher energy density in shear waves than in longitudinal waves. Egle (1981) used this to derive the ratio [equation (4)] numerically for ultrasonics in a finite body with a traction free surface. The same inequalities, and consequences, apply to scattering cross sections, as discussed analytically by Papanicolaou et al. (1996) in an argument based on reciprocity. Weaver (1984) showed that the reciprocity of the S-matrix governing edge reflections implies that the steady state balance of Lamb wave energy densities in a thick plate must correspond to equipartition, regardless of dispersion or anisotropy or boundary conditions. His argument applies equally well to the simpler case of bulk waves in a 3-D solid. Lyon and Maidanik (1962) showed by means of a detailed discussion of the equations of motion, that a single mechanical oscillator in contact with a large system must have an average energy equal to that of the average mode in the large system.

An additional caveat of only rare practical importance, but fascinating to many, may be noted. If the coupling is very weak, and energy flow is slow on the time scale of the modal density (called the Heisenberg time), then energy does not become equipartitioned after an arbitrary source. This is the physical basis for Anderson localization (Weaver and Lobkis, 2000; Hu et al., 2008).

The upshot is that as long as mode conversion rates α are fast compared to a) differences of dissipation σ and b) inverses of Heisenberg times, then an arbitrary source yields, perhaps after a short time, a field whose statistics are well characterized as equipartitioned. Not only may one conclude $\{E\} \propto \{e\} \propto$ number of modes, but also the more striking retrieval equation, $C'=G$ equivalent to (1) and to (3).

6 Open systems and local equipartition

In an open or very large system, however (consider for example the entire Earth) the notion of non-thermal equipartition *per se* can be nonsense. Seismic energy is not distributed equally over all the Earth. In the context of the present discussion we may understand this as owing to dissipations, σ, of seismic waves being comparable to or faster than scattering and re-distributions, α, across the Earth. Nevertheless open systems can achieve a kind of *local* equipartition, in which all the modes of a region and/or of a certain type, are equally energetic on the average. In this case the correlation function C' only resembles a local version of G, a G that lacks the influence of the missing types of waves or the influence of distant scatterers.

Much of the early work on retrieval of G in open systems considered homogeneous media with sources at infinity. If these sources are distributed uniformly in angle and if there is no dissipation, one promptly derives the relation $C'=G$, i.e. equipartition (Roux and Kuperman, 2004; Snieder, 2004). Those discussions are perhaps particularly accessible and intuitive. However, the theorem is more general; it applies also to inhomogeneous media and even to media with dissipation if the sources are distributed in proportion to the dissipation. Wapenaar (2004) in a series of papers shows how a specific set of random sources on a surface surrounding a non-dissipative heterogeneous region, gives rise to a random field in the interior with the required correlations. Weaver and Lobkis (2004) showed that a random field incident upon a finite dissipationless heterogeneous region will give rise to the required correlations if the incident random field itself has the required correlations. Thus a field locally equipartitioned in one place will, if in equilibrium with another place, generate a field in the other place that is also locally equipartitioned.

7 What do we expect in practice?

Much of the current excitement in seismic Green's function retrieval has arisen from circumstances in which noise fields are diffuse (i.e. random with smooth variations in intensity) but not equipartitioned. Seismic correlation functions tend to be dominated by Rayleigh waves, indicating perhaps that the ambient noise field contains little contribution from bulk waves. That the amplitude of the retrieved waveform depends upon direction indicates that the propagation directions in the noise field are not isotropically distributed either.

This means that the wave field is not equipartitioned and thus is not very multiply-scattered; little mode conversion α has transpired. The wide success of time-of flight retrieval from ambient seismic noise is owed more to asymptotic arguments for ray propagation in a continuous distribution of intensity versus direction, arguments like those of references (Roux and Kuperman, 2004; Snieder, 2004; Weaver et al., 2009; Froment et al., 2010), rather than to the principle of equipartition or equation (3). The researcher interested in time-of flight retrieval perhaps need not consider the concept of equipartition. Nevertheless partition ratios are in principle important parameters. In those cases in which partition is equal, Green's functions should be recoverable with relative amplitudes that are robust. It appears to be a somewhat neglected issue: to what extent is a given ambient seismic noise field (locally) equipartitioned? Hennino et al. (2001) measured the late time coda at a station in Mexico. Their array was configured so that both seismic displacements and strains could be recovered. In consequence they could assess the energy in various modes. They showed that, within the late coda of earthquake waveforms, the field was equipartitioned. The correct energy ratios were observed, for example shear to longitudinal, or horizontal to vertical, independent of the earthquake source. The conclusion was that their late time coda was multiply scattered; enough mode conversion had taken place by scattering for the equilibrium ratios to be established. Thus coda, if attention is confined to late times, should be superior to ambient seisms for the retrieval of Green's functions. (The chief reason for preferring ambient noise is that there is so much more of it; averages converge more quickly; one need not wait for earthquakes). Campillo and Paul (2003) successfully retrieved Green's functions from codas in Mexico. Paul et al. (2005) using coda waveforms from Alaska, recovered non-time-symmetric Rayleigh wave arrivals, thus indicating that the propagation directions in their codas were not isotropically distributed. They noted, however, that the time symmetry increased as the coda aged. Multiple scattering was tending to equilibrate the coda partition. They also noted, in numerical simulations, that the characteristic ratio (4) of propagation mode energies E_S/E_P was achieved long before the distribution of propagation directions became isotropic. This suggests that as waves multiply-scatter, isotropy is achieved slowly but other signatures of equipartition are achieved quickly.

8 Conclusions

Equipartition is a general principle describing the statistics of a diffuse field as independent of initial conditions or sources. Equipartition obtains in the limit of a large number of mode converting scatterings or reflections. It is observed in the laboratory and in late seismic coda. As such it is an important state against which to compare any random wave field. Ambient seismic noise appears to be rarely equipartitioned, even locally. Fortunately, practical retrievals of surface wave propagation time from ambient seismic noise have not depended on having full equipartition. Nevertheless it may be useful to ask of certain seismic fields the degree to which they do approximate equipartition. Constraints on mode conversion rates versus losses could be obtained from such comparisons. In those cases in which the fields *are* equipartitioned, correlations will provide highly robust measures of Green's functions.

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