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Numerical modeling of wave equation by a truncated high-order finite-difference method*

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Abstract Finite-difference methods with high-order accuracy have been utilized to improve the precision of numerical solution for partial differential equations. However, the computation cost generally increases linearly with increased order of accuracy. Upon examination of the finite-difference formulas for the first-order and second-order derivatives, and the staggered finite-difference formulas for the first-order derivative, we examine the variation of finite-difference coefficients with accuracy order and note that there exist some very small coefficients. With the order increasing, the number of these small coefficients increases, however, the values decrease sharply. An error analysis demonstrates that omitting these small coefficients not only maintain approximately the same level of accuracy of finite difference but also reduce computational cost significantly. Moreover, it is easier to truncate for the high-order finite-difference formulas than for the pseudospectral formulas. Thus this study proposes a truncated high-order finite-difference method, and then demonstrates the efficiency and applicability of the method with some numerical examples.

Key words: **finite difference; high-order accuracy; truncation; efficiency; numerical modeling**
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1 Introduction

As one of the most popular methods of numerical solution for partial differential equations, the finite-difference method (FDM) has been widely utilized in seismic modeling (Kelly et al, 1976; Dablain, 1986; Virieux, 1986; Igel et al, 1995; Etgen and O'Brien, 2007; Bansal and Sen, 2008) and migration (Claerbout, 1985; Larner and Beasley, 1987; Li, 1991; Ristow and Ruhl, 1994; Zhang et al, 2000; Fei and Liner, 2008). In order to improve the accuracy and stability of a FDM in numerical modeling, many methods have been developed, including difference schemes of variable grid (Wang and Schuster, 1996; Hayashi and Burns, 1999), irregular grid (Opršal and Zahradník, 1999), standard staggered grid (Virieux, 1986; Levander, 1988; Robertsson et al, 1994; Graves, 1996), rotated staggered grid (Gold et al, 1997; Saenger and Bohlen, 2004; Bohlen and Saenger, 2006), variable time step (Tessmer, 2000), and implicit methods (Emerman et al, 1982; Zhang and Zhang, 2007).

Since the efficiency of an algorithm has to take the CPU demands and the memory demands into consideration, a desirable algorithm must balance input/output and memory needs. Generally speaking, a lower-order finite-difference algorithm uses a shorter operator but requires more grid points; while a higher-order finite-difference algorithm uses a longer operator but requires fewer grids. Dablain (1986) demonstrated the advantages of high-order difference schemes in application to the scalar wave equation. One is the economy achieved in storage requirements at no extra cost of total computational time, which make it possible to compute larger models or higher frequency solutions within the available core memory on the computer. Fornberg (1987) derived the finite-difference formulas with any-order accuracy for the first-order derivative and compared high-order finite-difference method with a pseudospectral method for the elastic wave equation. To increase efficiency and accuracy of such modeling, Crase (1990)

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presented an elastic finite-difference scheme with arbitrary-order accuracy in both space and time. Liu et al (1998) developed finite-difference formulas with any even-order accuracy for any order derivative, which were used to simulate elastic wave propagation in two-phase anisotropic media (Liu et al, 2008). Chen (2007) stated some high-order time discretizations in seismic modeling. Arbitrary staggered finite-difference formulas have also been developed for simulation of elastic waves (Dong et al, 2000; Pei, 2004).

In this paper, we study the characteristics of high-order finite-difference coefficients and the errors caused by truncation of these coefficients. Then we develop a truncated finite-difference method and demonstrate its efficiency and applicability with some numerical results.

2 Characteristics of finite-difference coefficients and error analysis on the truncation of the coefficients

2.1 Even-order finite-difference formulas for the first- and second-order derivatives

Centered finite-difference formulas are usually used to numerically solve partial differential equations, which generally involve the first- and second-order derivatives. In the following, we introduce centered finite-difference formulas with any even-order accuracy for the first- and second-order derivatives. And similar formulas for the staggered finite difference are also given.

A $(2N)$th-order finite-difference formula for the first-order derivative can be expressed as (Dablain, 1986)

$$\frac{\partial p}{\partial x} \approx \frac{\delta p}{\delta x} = \frac{1}{h}\sum_{n=-N}^{N} c_n p(x+nh), \tag{1}$$

where x is a real variable, $p(x)$ is a function, h is a small value, N is a positive integer, the finite-difference coefficient c_n is an odd discrete sequence and can be computed by the following formula (Fornberg, 1987; Liu et al, 1998, 2008):

$$c_n = \frac{(-1)^{n+1}}{2n} \prod_{1\le i\le N,\, i\ne n} \left| \frac{i^2}{n^2 - i^2} \right| \qquad n = 1, 2, \cdots, N. \tag{2}$$

When $N \to \infty$,

$$c_n = \frac{(-1)^{n+1}}{n} \qquad n = 1, 2, \cdots, \infty. \tag{3}$$

Thus, there exists a limit at which we can attain infinite accuracy theoretically and arrive at the pseudospectral method (Fornberg, 1987).

A $(2N)$th-order finite-difference formula for the second-order derivative can be expressed as

$$\frac{\partial^2 p}{\partial x^2} \approx \frac{\delta^2 p}{\delta x^2} = \frac{1}{h^2}\sum_{n=-N}^{N} c_n p(x+nh), \tag{4}$$

where the difference coefficient c_n, an even discrete sequence, can be computed by the following (Liu et al, 1998, 2008):

$$c_n = \begin{cases} \dfrac{(-1)^{n+1}}{n^2} \displaystyle\prod_{1\le i\le N,\, i\ne n} \left| \frac{i^2}{n^2 - i^2} \right| & n = 1, 2, \cdots, N \\ -2\displaystyle\sum_{i=1}^{N} c_i & n = 0 \end{cases} \tag{5}$$

When $N \to \infty$,

$$c_n = \frac{2(-1)^{n+1}}{n^2} \qquad n = 1, 2, \cdots, \infty. \tag{6}$$

A staggered-grid finite-difference formula is defined as follows (Kindelan et al, 1990):

$$\frac{\partial p}{\partial x} \approx \frac{\delta p}{\delta x} = \frac{1}{h}\sum_{n=-N}^{N} c_n [p(x+nh-0.5h) - p(x-nh+0.5h)], \tag{7}$$

where the difference coefficients c_n form an odd discrete sequence and can be computed by the following formula (Pei, 2004):

$$c_n = \frac{(-1)^{n+1}}{2n-1} \prod_{1\le i\le N,\, i\ne n} \left| \frac{(2i-1)^2}{(2n-1)^2 - (2i-1)^2} \right|. \tag{8}$$

When $N \to \infty$,

$$c_n = \frac{4(-1)^{n+1}}{\pi (2n-1)^2} \qquad n = 1, 2, \cdots, \infty. \tag{9}$$

All the expressions for c_n given by formulas (2), (5), and (8) can be derived from a Taylor series expansion. The formulas (1), (4), and (7) indicate that more points are required progressively in higher-order finite difference schemes.

2.2 Accuracy of finite-difference formula

To analyze the accuracy of finite-difference formulas, we let

$$p = p_0 \mathrm{e}^{\mathrm{i}kx} \tag{10a}$$

and

$$\alpha = \frac{kh}{2}, \tag{10b}$$

where, k is wavenumber, $\mathrm{i} = \sqrt{-1}$, p_0 is a constant. Substituting equations (10a) and (10b) into equations (1), (4), and (7), we have

$$\alpha \approx \sum_{n=1}^{N} c_n \sin(2n\alpha), \tag{11}$$

$$\alpha^2 \approx -\frac{1}{4}c_0 - \frac{1}{2}\sum_{n=1}^{N} c_n \cos(2n\alpha), \tag{12}$$

and

$$\alpha \approx \sum_{n=1}^{N} c_n \sin[(2n-1)\alpha]. \tag{13}$$

Since the requirement of finite-difference coefficients is to satisfy equations (11), (12) and (13), we examine the following functions to test the accuracy of these formulas:

$$f_1(\alpha) = \sum_{n=1}^{N} c_n \sin(2n\alpha), \tag{14}$$

$$f_2(\alpha) = \sqrt{-\frac{1}{4}c_0 - \frac{1}{2}\sum_{n=1}^{N} c_n \cos(2n\alpha)}, \tag{15}$$

and

$$f_s(\alpha) = \sum_{n=1}^{N} c_n \sin[(2n-1)\alpha]. \tag{16}$$

The finite-difference coefficients c_n are obtained from equations (2), (5), and (8). The calculated $f_1(\alpha)$, $f_2(\alpha)$, and $f_s(\alpha)$ are compared with α for different orders (Figure 1), which clearly reveals that the accuracy increases with increase of order.

2.3 Characteristics of finite-difference coefficients

Since more points are involved in the higher order finite difference, computation cost increases. Next, we will study the characteristics of finite-difference coefficients and investigate ways to save computation time.

The variation of finite-difference (FD) coefficients with different accuracy orders (from 4 to 50) is shown in Figure 2, where the FD coefficients are calculated using equations (2), (5), and (8). It can be seen from the figure that the absolute values of the coefficients c_n decrease with increasing n. The degree of decrease is larger for higher-order FD, for example, c_n approaches zero when n>10 for a 50th-order FD.

2.4 Accuracy analysis on variation of error with truncation number

As mentioned above, c_n approaches zero for large n in higher order FD, therefore, these c_n may be omitted. Next we will quantify the error caused by the truncation of c_n. As an example, the analysis is performed for the staggered grid FD formulas.

According to equation (16), the truncation function for testing the accuracy is defined as follows:

$$f_M(\alpha) = \sum_{n=1}^{M} c_n \sin[(2n-1)\alpha] \qquad M < N. \tag{17}$$

This formula reduces to the conventional function (16) when M=N.

To quantitatively evaluate the accuracy of equation (17), the following error function is introduced,

$$\varepsilon_M = \frac{1}{K+1} \sum_{\alpha=0,\,\Delta\alpha,\,2\Delta\alpha,\,\cdots,\,K\Delta\alpha} \left|\alpha - f_M(\alpha)\right|. \tag{18}$$

ε_N is the error of the conventional finite difference and less than ε_M for M<N. K=int[π/(2Δα)], is a function to get the integer part of a value. To magnify the error ε_M, we calculate the relative error $\lg|(\varepsilon_M-\varepsilon_N)/\varepsilon_N|$ for different M. In the following calculation, N=21, $\Delta\alpha$=0.001 and K=1 570. Figure 3a shows the variation of the relative error $\lg|(\varepsilon_M-\varepsilon_N)/\varepsilon_N|$ with the truncation number M, which demonstrates that the relative error between ε_M and ε_N sharply decreases with increasing M. For instance, the relative error is around 10^{-6} when M=14, and only 10^{-10} when M=18. Therefore, it is possible to omit some c_n for large n.

To support the truncated method further, we calculate the variation of $f_M(\alpha)$ for different truncation number M with α, shown in Figure 3b. The curves from the truncated method (M=6, 8, 10) are very close to the conventional method (M=N=21). The variation of the error $f_M(\alpha)-\alpha$ for different truncation number M with α is shown in Figure 3c. We can see that the error is small when M is taken as 10, which is closer to the conventional error (M=N=21). If this error can be accepted, we can choose M=10 to replace M=21 and thus to save the computational time. Figure 3d illustrates the variation of $\lg|c_n/c_1|$ with n. c_{10} is approximately equal to $10^{-5}c_1$, while c_n (n>10) is very small and has a little effect on the calculation of finite difference.

2.5 Comparison of coefficients of the finite-difference and pseudospectral methods

Since some of the high-order finite-difference coefficients can be omitted, the truncation may also be proposed to the pseudospectral methods whose coefficients are given by equations (3), (6), and (9). Figure 4 shows the variation of $\lg|c_n/c_1|$ with n for the 60th-order finite-difference method. Figure 5 illustrates the variation of $\lg|c_n/c_1|$ with n for the pseudospectral method. These figures demonstrate that the coefficients in the high-order finite-difference methods decrease sharply, but those in the pseudospectral methods decrease smoothly. Therefore, it is easier to truncate the coefficients in the high-order finite-difference methods than those used in pseudospectral method. If the truncation is employed in the pseudospectral method at all, a window function will be necessary.

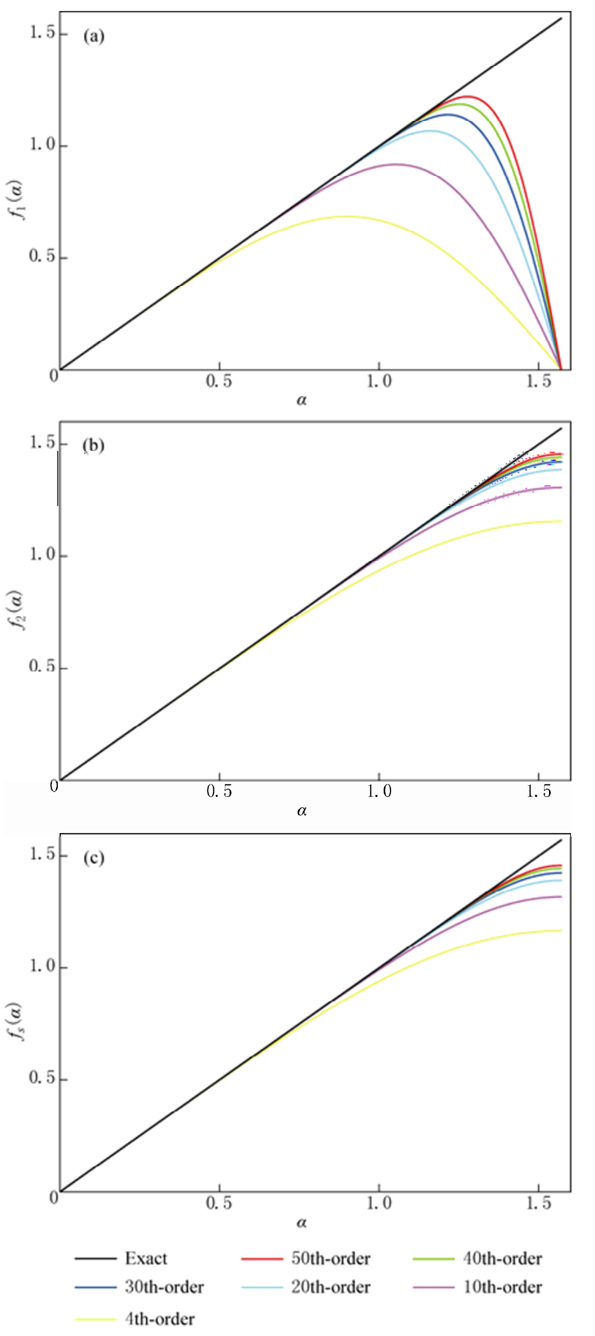

Figure 1 Plot of the exact α (α=$kh/2$), $f_1(\alpha)$, $f_2(\alpha)$, $f_s(\alpha)$ for different accuracy orders from 4 to 50. (a) Finite-difference for the first-order derivative; (b) Finite-difference for the second-order derivative; (c) Staggered finite-difference for the first-order derivative.

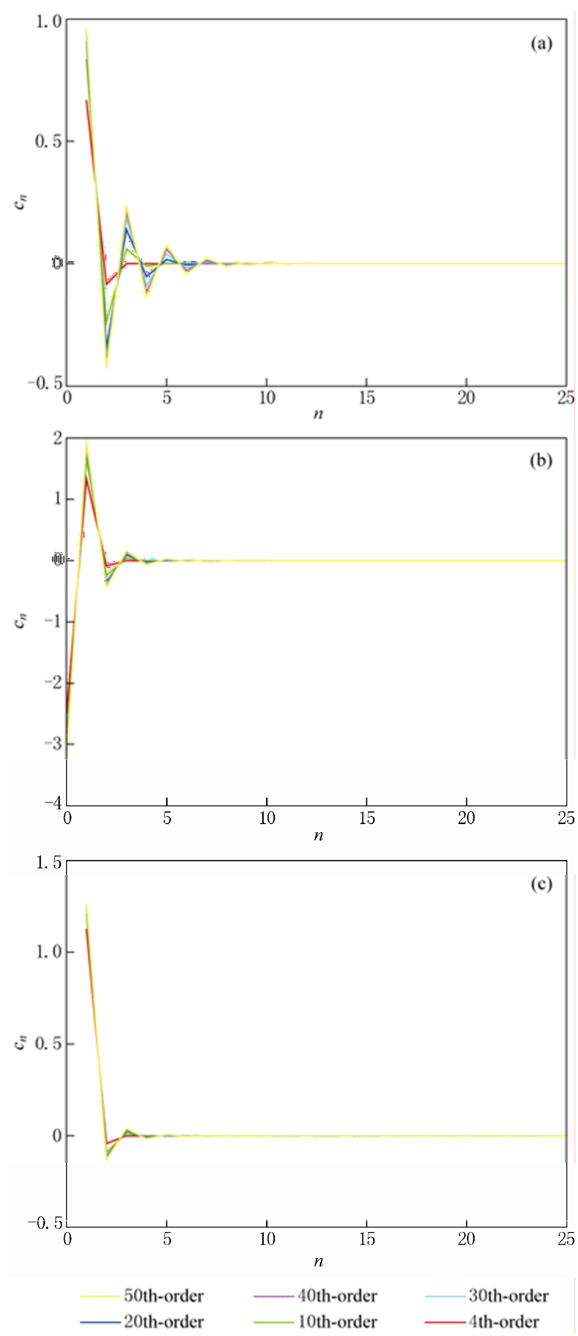

Figure 2 Variation of finite-difference coefficient c_n with different accuracy orders from 4 to 50. (a) Finite-difference coefficients for the first-order derivative; (b) Finite-difference coefficients for the second-order derivative; (c) Staggered finite-difference coefficients for the first-order derivative.

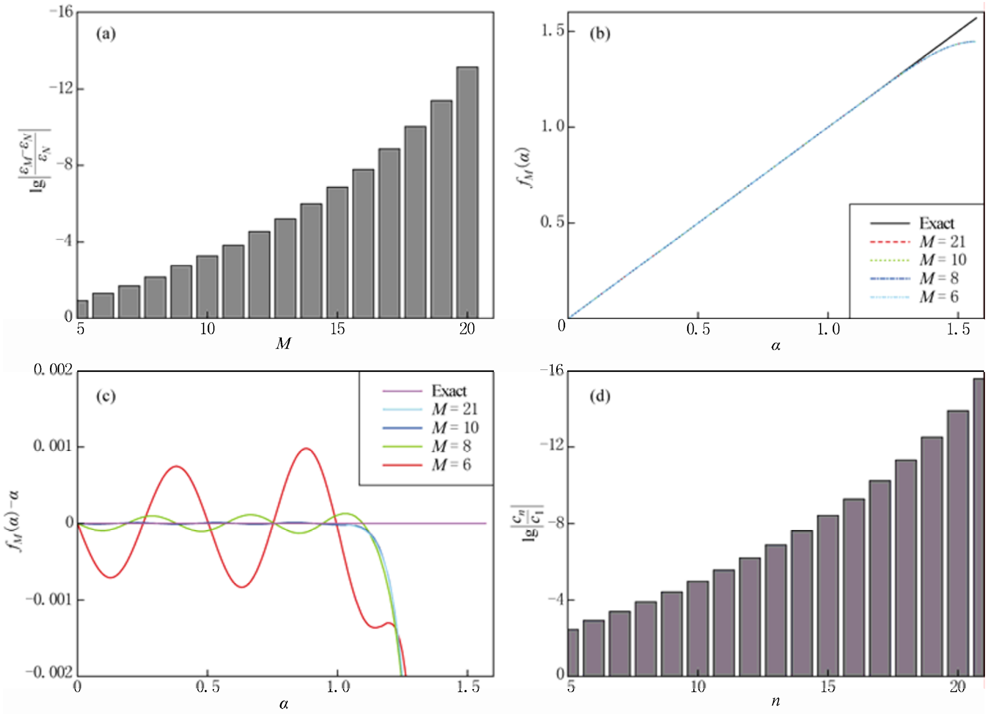

Figure 3 Error analysis of the truncated (2N)th-order staggered finite-difference formula (M is the truncation number, and 2M points are used to perform (2N)th-order staggered finite-difference, $M<N$, N=21). (a) Variation of the relative error $\lg|(\varepsilon_M-\varepsilon_N)/\varepsilon_N|$ with the truncation number M; (b) Variation of $f_M(\alpha)$with α ($\alpha=kh/2$); (c) Variation of the error $f_M(\alpha)-\alpha$ with α ($\alpha=kh/2$); (d) Variation of $\lg|c_n/c_1|$ with n.

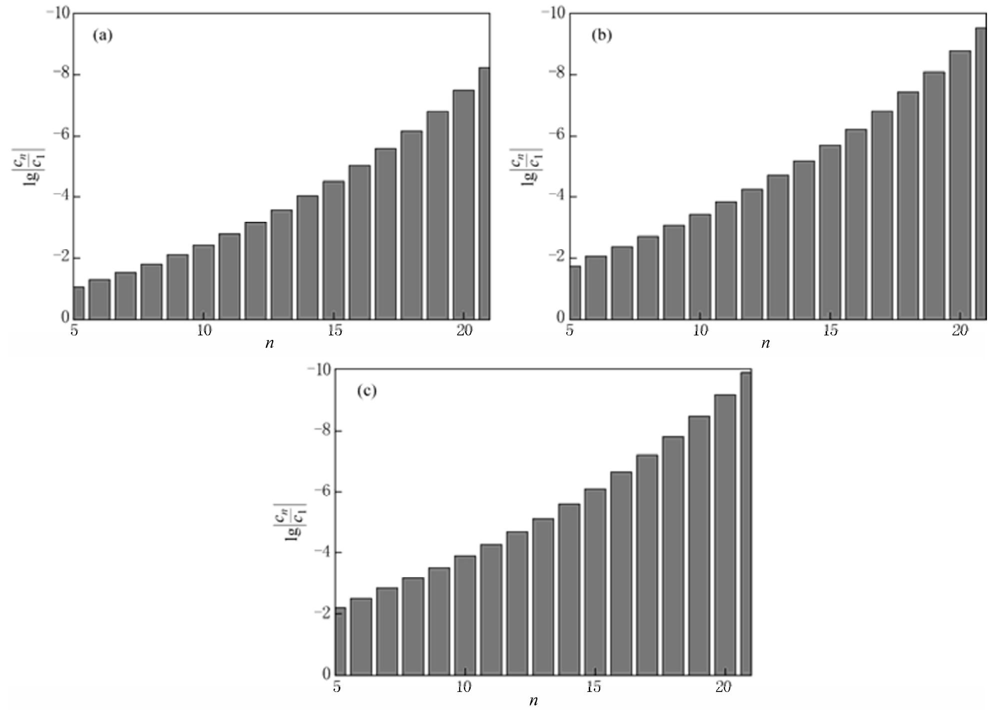

Figure 4 Variation of $\lg|c_n/c_1|$ with n for the 60th-order finite-difference method. (a) For the first-order derivative; (b) For the second-order derivative; (c) For the first-order derivative by staggered method.

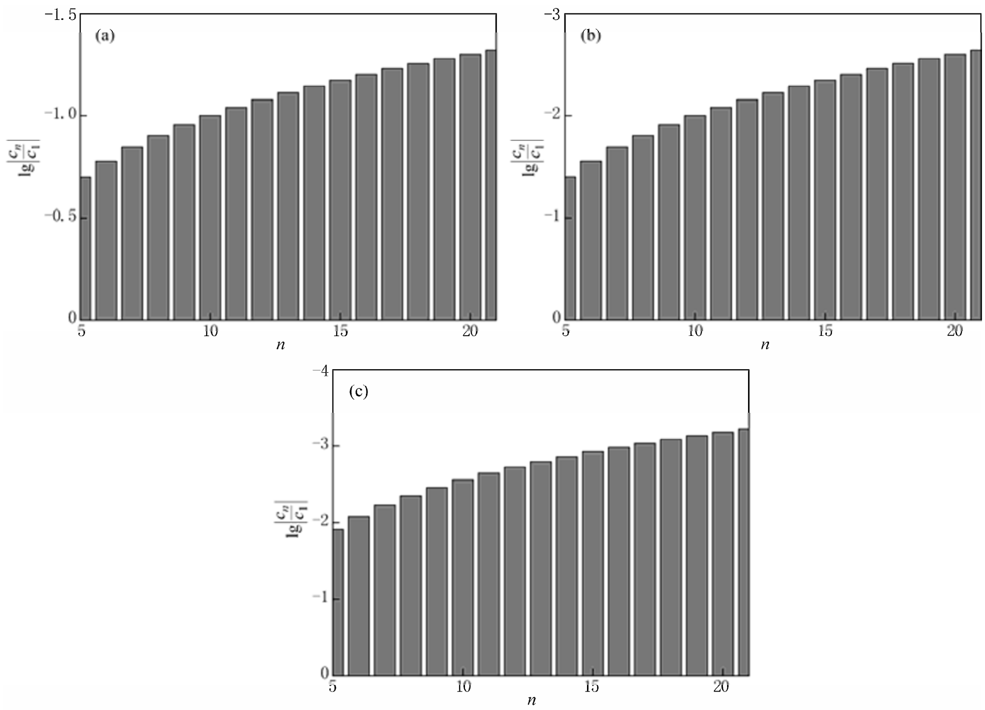

Figure 5 Variation of lg$|c_n/c_1|$ with n for the limit finite-difference method, which is equivalent to the pseudospectral method. (a) For the first-order derivative; (b) For the second-order derivative; (c) For the first-order derivative by staggered method.

3 Truncated high-order finite-difference method

According to the characteristics of finite-difference coefficients and error analysis of truncated finite-difference formulas, it is feasible to omit some small finite-difference coefficients. We introduce the following formula as the truncation rule for the (2N)th-order finite-difference method

$$|c_m| \geq r \max_i |c_i| \qquad m \leq M, \tag{19}$$

where r is a ratio, the coefficient whose absolute value is less than $r \max_i |c_i|$ will be omitted in the truncated finite difference, $\max_i |c_i|$ is equal to $|c(1)|$ for the first-order derivative and $|c(0)|$ for the second-order derivative. If r is zero, there is no truncation, 2N points for the first-order derivative and 2N+1 points for the second-order derivative will be involved in the conventional finite difference. If r is greater than zero, fewer points may be used in the truncated finite difference.

We obtain a value of M when r=10^{-5} for the (2N)th-order finite-difference formulas. Tables 1 and 2 list the relationship between 2M or 2M+1 and 2N for the first-, second-order derivatives, respectively. Table 3

Table 1 The used point number 2M in the truncated FDM for the first-order derivative

Order 2N	2M	Order 2N	2M
18	16	40–46	26
20–22	18	48–52	28
24–28	20	54–60	30
30–32	22	62–68	32
34–38	24		

Note: For (2N)th-order FDM, the conventional uses 2N points and the truncated uses 2M points when r=10^{-5}.

Table 2 The used point number 2M+1 in the truncated FDM for the second-order derivative

Order 2N	2M+1	Order 2N	2M+1
16–18	15	40–48	23
20–24	17	50–58	25
26–32	19	60–68	27
34–38	21		

Note: For (2N)th-order FDM, the conventional uses 2N+1 points and the truncated uses 2M+1 points when r=10^{-5}.

Table 3 The used point number 2M in the truncated SFDM for the first-order derivative

Order 2N	2M	Order 2N	2M
16–20	14	44–52	22
22–26	16	54–62	24
28–32	18	64–74	26
34–42	20		

Note: For (2N)th-order SFDM, the conventional uses 2N points and the truncated uses 2M points when r=10^{-5}.

lists the relationship for the staggered finite difference. These tables suggest that computation will be significantly decreased by the truncated method for the high-order finite difference. For example, the truncated 62nd-order staggered finite-difference method (SFDM) only cost the 40% calculation amount of the conventional 62nd-order SFDM.

4 Examples of numerical modeling

The conventional SFDM and the truncated SFDM are used to perform numerical modeling of the following elastic wave equations,

$$\frac{\partial v_x}{\partial t} = b\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}\right), \tag{20a}$$

$$\frac{\partial v_z}{\partial t} = b\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{zz}}{\partial z}\right), \tag{20b}$$

$$\frac{\partial \tau_{xx}}{\partial t} = (\lambda + 2\mu)\frac{\partial v_x}{\partial x} + \lambda\frac{\partial v_z}{\partial z}, \tag{20c}$$

$$\frac{\partial \tau_{zz}}{\partial t} = \lambda\frac{\partial v_x}{\partial x} + (\lambda + 2\mu)\frac{\partial v_z}{\partial z}, \tag{20d}$$

and

$$\frac{\partial \tau_{zx}}{\partial t} = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right). \tag{20e}$$

In these equations, (v_x, v_z) is the velocity vector, (τ_{xx}, τ_{zz}, τ_{xz}) is the stress tensor, $\lambda(x, z)$ and $\mu(x, z)$ are Lamé coefficients, $b(x, z)$ is the inverse of density. We perform the modeling on staggered grids (Virieux, 1986). In the modeling, the first-order space derivatives are calculated by the conventional SFDM and the truncated SFDM, respectively; the second time derivatives are computed by the second-order finite difference.

4.1 Numerical modeling for a homogeneous elastic model

A model is used in numerical modeling here and the parameters are listed in Table 4. Snapshots at 200 ms computed by the conventional SFDM and the truncated SFDM are shown in Figure 6. CPU time needed to compute Figures 6a and 6b is 26.1 s and 20.2 s on the same computational condition by a conventional 42nd-order SFDM and a truncated 42nd-order SFDM, respectively. The two figures are nearly the same, demonstrating that the truncation is indeed feasible for numerical modeling. The truncated 42nd-order SFDM provides higher precision than a conventional 20th-order SFDM (Figures 6b and 6c), and costs the same computation time as a conventional 20th-order SFDM because both the methods use 20 points to calculate space derivatives. Therefore, a truncated high-order SFDM is effective and may be used to reduce computation cost.

Table 4 Homogeneous elastic model and its modeling parameters

Parameter	Value
Model size/m^2	1 000×1 000
Model P wave velocity/km·s^{-1}	2
Model S wave velocity/km·s^{-1}	1
Model density/kg·m^{-3}	1 000
Grid size/m^2	10×10
Time step/ms	1
P-wave source function	Sine signal of 50 Hz with one period length
Source location	In the middle of the model

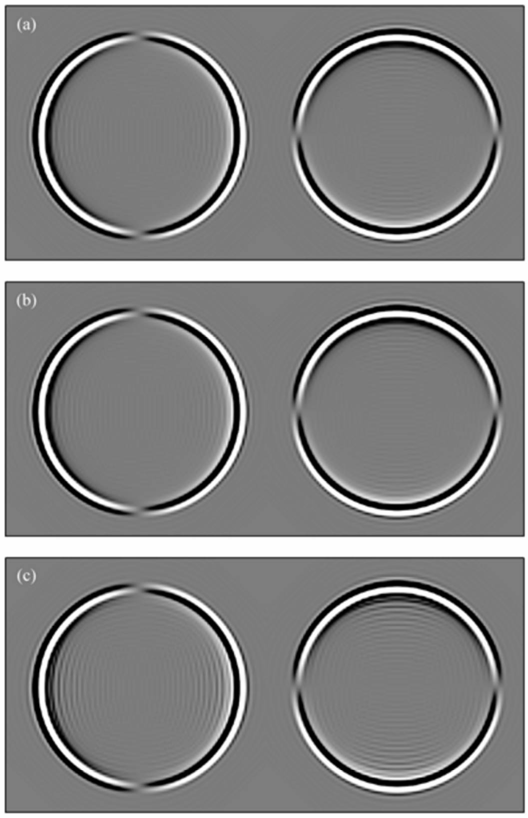

Figure 6 Snapshot comparisons of the conventional SFDM and the truncated SFDM. (a) The conventional 42nd-order SFDM with 42 points ($2M$=42); (b) The truncated 42nd-order SFDM with 20 points (r=10^{-5}, $2M$=20); (c) The conventional 20th-order SFDM with 20 points ($2M$=20).

4.2 Numerical modeling for SEG/EAEG salt model

Here, we will show results from numerical modeling of the elastic wave equations, by the conventional SFDM and the truncated SFDM, basing on the SEG/ EAEG salt model. For the two methods with accuracy orders of 20 and 42 respectively, 20 points are used to calculate

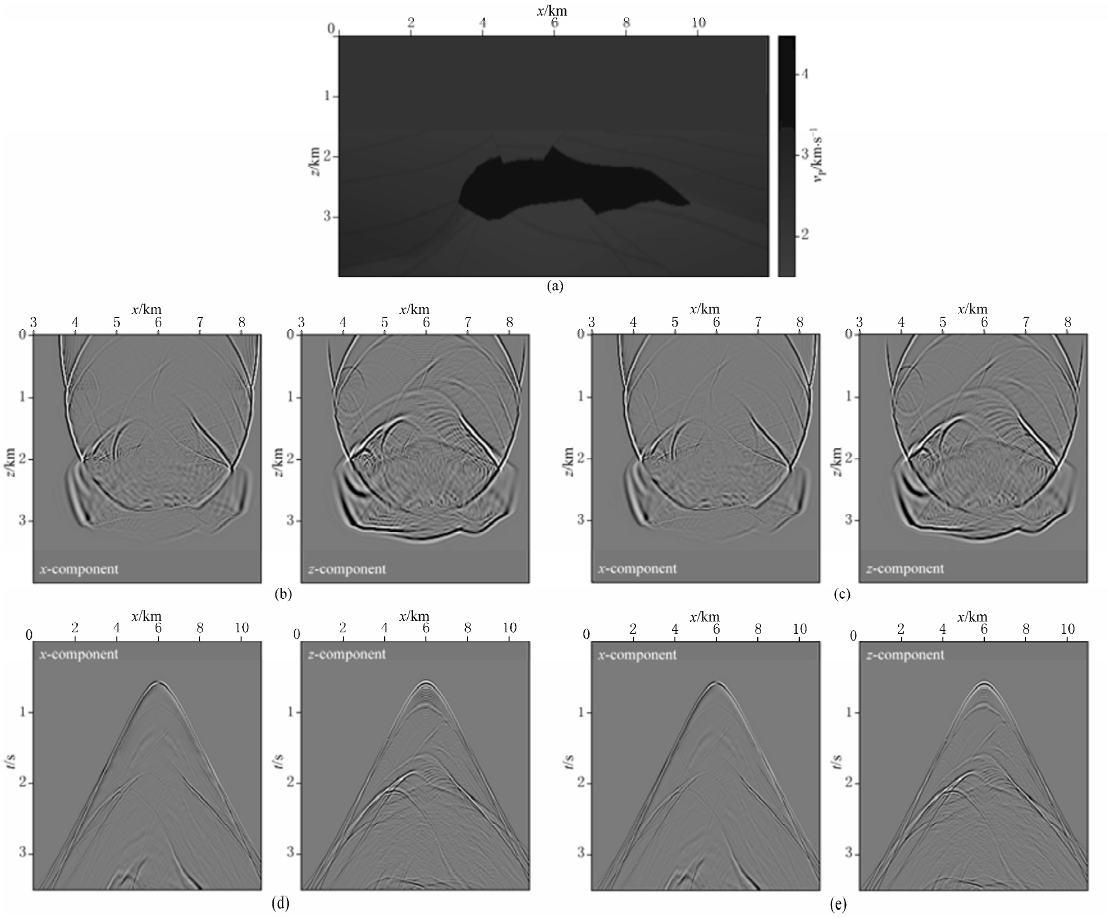

Figure 7 Numerical modeling results of SEG/EAEG salt model respectively by the conventional 20th-order SFDM and the truncated 42nd-order SFDM, both methods use 20 points to calculate derivatives and cost the same computation time. (a) SEG/EAEG salt model of P-wave velocity (S-wave velocity and density, having the similar variation characteristics to P-wave velocity, are not shown); (b) Snapshots of *x*- and *z*-components at $t = 1\ 600$ ms by the conventional SFDM; (c) Snapshots of *x*- and *z*-components at t=1 600 ms by the truncated SFDM; (d) OBC gather of *x*- and *z*-components by the conventional SFDM; (e) OBC gather of *x*- and *z*-components by the truncated SFDM.

Table 5 SEG/EAEG salt model and its simulating parameters

Parameters	Values
Model size/m^2	12 000×4 000
Model first layer	Ocean water
Grid size/m^2	20×20
Time step/ms	1
P-wave source function	Sine signal of 25Hz with one period length
Source location/m	x_0=6000, z_0=20
Receiver location	Ocean bottom

the derivatives and the same computation time are required. Figure 7a shows the model and Table 5 lists its simulation parameters. Snapshots and shot gathers are shown in Figures 7b, 7c, 7d and 7e, and Figure 8 shows a zoomed section of *x*-component of Figure 7. From these figures, we can see that the dispersion of the truncated

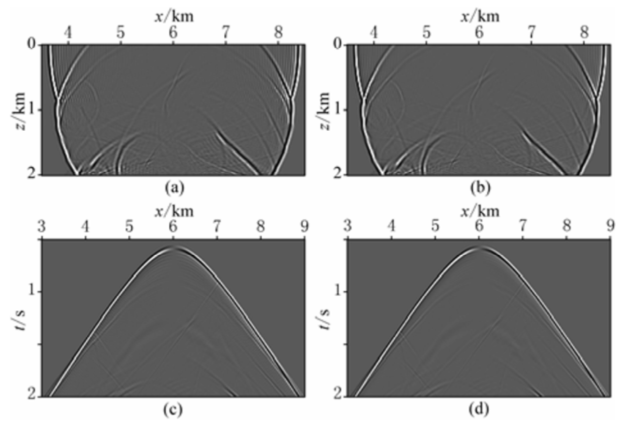

Figure 8 Zoom of *x* components in Figure 7. Subfigure a, b, c and d are zoom of Figure 7b, 7c, 7d and 7e, respectively.

method is obviously less than that of the conventional method. Therefore, the accuracy of the truncated method is higher than that of the conventional method under the same level of computation.

5 Conclusions

A conventional high-order finite-difference method provides higher accuracy but costs more computation time. Our analysis shows that the high-order finite-difference coefficients c_n decrease sharply with an increase of n, and omitting some small coefficients has a little effect on the accuracy of finite difference but it can reduce the computation amount significantly. Numerical modeling results demonstrate the validity of a truncated finite-difference method.

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