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Group identities on symmetric units under oriented involutions in group algebras

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Abstract
Let $\mathbb{F}G$ denote the group algebra of a locally finite group G over the infinite field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$, and let $\circledast : \mathbb{F}G \to \mathbb{F}G$ denote the involution defined by $\alpha = \Sigma\alpha_g g \mapsto \alpha^{\circledast} = \Sigma\alpha_g \sigma(g) g^*$, where $\sigma : G \to \{\pm 1\}$ is a group homomorphism (called an orientation) and $*$ is an involution of the group G. In this paper we prove, under some assumptions, that if the $\circledast$-symmetric units of $\mathbb{F}G$ satisfies a group identity then $\mathbb{F}G$ satisfies a polynomial identity, i.e., we give an affirmative answer to a Conjecture of B. Hartley in this setting. Moreover, in the case when the prime radical $\eta(\mathbb{F}G)$ of $\mathbb{F}G$ is nilpotent we characterize the groups for which the symmetric units $\mathcal{U}^+(\mathbb{F}G)$ do satisfy a group identity.

Keywords Group algebras · Group identity · Involutions · Symmetric units · Unit group

Mathematics Subject Classification 16U60 · 16W10 · 16S34 · 16R50

1 Introduction

Let $\mathbb{F}G$ denote the group algebra of the group G over the field $\mathbb{F}$. Any involution $* : G \to G$ can be extended $\mathbb{F}$-linearly to an algebra involution of $\mathbb{F}G$. Such a map is

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called a group involution of $\mathbb{F}G$. A natural example is the so-called *classical involution*, which is induced from the map $g \mapsto g^* = g^{-1}$, for all $g \in G$.

Let $\sigma : G \to \{\pm 1\}$ be a non-trivial homomorphism (called an *orientation* of G). If $* : G \to G$ is a group involution, an *oriented group involution* of $\mathbb{F}G$ is defined by

$$\alpha = \sum_{g \in G} \alpha_g g \mapsto \alpha^{\circledast} = \sum_{g \in G} \alpha_g \sigma(g) g^*. \tag{1}$$

Notice that, as σ is non-trivial, char($\mathbb{F}$) must be different from 2. It is clear that, $\alpha \mapsto \alpha^{\circledast}$ is an involution in $\mathbb{F}G$ if and only if $gg^* \in N = ker(\sigma)$ for all $g \in G$.

In the case when the involution on G is the classical involution, the map $\circledast$ is precisely the oriented involution introduced by S. P. Novikov, [1], in the context of K-theory.

We denote with $\mathbb{F}G^+ = \{\alpha \in \mathbb{F}G : \alpha^{\circledast} = \alpha\}$ and $\mathbb{F}G^- = \{\alpha \in \mathbb{F}G : \alpha^{\circledast} = -\alpha\}$ the sets of symmetric and skew-symmetric elements of $\mathbb{F}G$ under $\circledast$ and, writing $\mathcal{U}(\mathbb{F}G)$ for the group of units of $\mathbb{F}G$, we let $\mathcal{U}^+(\mathbb{F}G)$ denote the set of symmetric units, i.e., $\mathcal{U}^+(\mathbb{F}G) = \{\alpha \in \mathcal{U}(\mathbb{F}G) : \alpha^{\circledast} = \alpha\}$.

Let $\langle x_1, x_2, \ldots \rangle$ be the free group on a countable set of generators. If H is any subset of a group G, we say that H satisfies a group identity ($H \in$ GI or H is GI for short) if there exists a non-trivial reduced word $\omega(x_1, x_2, \ldots, x_n) \in \langle x_1, x_2, \ldots \rangle$ such that $\omega(h_1, h_2, \ldots, h_n) = 1$ for all $h_i \in H$. For instance, if we write $(x_1, x_2) = x_1^{-1} x_2^{-1} x_1 x_2$ and $(x_1, x_2, \ldots, x_n, x_{n+1}) = ((x_1, x_2, \ldots, x_n), x_{n+1})$, for all $n \geq 2$, then $\langle H \rangle$ is abelian if it satisfies the group identity (x_1, x_2), nilpotent if it satisfies $(x_1, x_2, \ldots, x_n)$, for some n and n-Engel if it satisfies $(x_1, \underbrace{x_2, x_2, \ldots, x_2}_{n \text{ times}})$ for some n.

Some time ago and with the idea of establishing a connection between the additive and multiplicative structure of a group algebra $\mathbb{F}G$, Brian Hartley made the following famous conjecture:

Conjecture 1.1 (Hartley's Conjecture) *Let G be a torsion group and $\mathbb{F}$ a field. If the unit group $\mathcal{U}(\mathbb{F}G)$ of $\mathbb{F}G$ satisfies a group identity, then $\mathbb{F}G$ satisfies a polynomial identity.*

Let R be an $\mathbb{F}$-algebra. Recall that a subset S of R satisfies a polynomial identity ($S \in$ PI or S is PI for short) if there exists a non-zero polynomial $f(x_1, x_2, \ldots, x_n)$ in the free associative algebra $\mathbb{F}\{X\}$ on the countable infinite set of non-commuting variables $X = \{x_1, x_2, \ldots\}$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_i \in S$. For instance, R is commutative if it satisfies the polynomial identity $f(x_1, x_2) = x_1x_2 - x_2x_1$ and, any finite dimensional associative algebra satisfies the *standard polynomial identity* of degree $n + 1$, where $n = \dim_{\mathbb{F}} R$ [2, Lemma 5.1.6, p. 173],

$$St_{n+1}(x_1, x_2, \ldots, x_{n+1}) = \sum_{\rho \in \mathcal{S}_{n+1}} (sgn \rho) x_{\rho(1)} x_{\rho(2)} \cdots x_{\rho(n+1)}.$$

Group algebras $\mathbb{F}G$ satisfying a PI were classified in two subsequent papers of Passman and Isaacs-Passman, see [2, Corollaries 5.3.8 and 5.3.10, p. 196-197].

Giambruno et al. [3] solved the Hartley's conjecture for semiprime group rings, and Giambruno et al. [4] solved it in general for group algebras over infinite fields. By using the results of [4], Passman [5] gave necessary and sufficient conditions for $\mathcal{U}(\mathbb{F}G)$ to satisfy a group identity, when $\mathbb{F}$ is infinite. Subsequently, Liu [6] confirmed that the conjecture also holds for finite fields and Liu and Passman in [7] extended the results of [5] to this case. The same question for groups with elements of infinite order was studied by Giambruno et al. in [8]. For further details about these results see Lee [9, Chapter 1] and the references quoted therein.

Let $*$ be an involution of a group algebra $\mathbb{F}G$ induced by an involution of the group G, the so-called group involution. When $*$ is the classical involution induced from $g \mapsto g^{-1}$, $g \in G$, Giambruno et al. [10] showed that if G is a torsion group, $\mathbb{F}$ is infinite with $\mathrm{char}(\mathbb{F}) \neq 2$, and $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity then $\mathbb{F}G$ satisfies a polynomial identity. They also classified groups G such that $\mathcal{U}^+(\mathbb{F}G)$ satisfies a group identity. Sehgal and Valenti [11] extended the results of [10] to non-torsion groups under the usual restriction for the only if part related to Kaplansky's Conjecture (the units of $\mathbb{F}G$ are trivial if G is a torsion-free group and $\mathbb{F}$ is a field).

Considering group involutions $*$, i.e., $*$ is an involution on G extended $\mathbb{F}$-linearly to the group algebra $\mathbb{F}G$, Dooms and Ruiz [12] proved the following.

Theorem 1.1 ([12, Theorem 3.1]) *Let $\mathbb{F}$ be an infinite field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let G be a non-abelian group such that $\mathbb{F}G$ is regular. Let $*$ be an involution on G. Suppose one of the following conditions holds:*

(i) *$\mathbb{F}$ is uncountable,*
(ii) *all finite non-abelian subgroups of G which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in GI$ if and only if G is an SLC-group with canonical involution given by the expression (2) below. Moreover, in this case $\mathbb{F}G^+$ is a ring contained in $\zeta(\mathbb{F}G)$.

Using the last result and under some assumptions, Dooms and Ruiz proved that if $\mathcal{U}^+(\mathbb{F}G)$ is GI then $\mathbb{F}G$ is PI, giving an affirmative answer to the Hartley's Conjecture in this setting. They also characterized, with mild restrictions, the locally finite groups for which the symmetric units $\mathcal{U}^+(\mathbb{F}G)$ satisfy a group identity, when the prime radical $\eta(\mathbb{F}G)$ of $\mathbb{F}G$ is nilpotent. Giambruno et al. [13] completely solved the question for group algebras of torsion groups, with group involutions such that $\mathcal{U}^+(\mathbb{F}G)$ is GI.

In the classification results on group algebras whose symmetric units with respect to the classical involution satisfy a group identity in some sense the exceptional cases turned out to involve Hamiltonian 2-groups, [10, Theorem 7, p. 459], because they are non-abelian groups such that the symmetric elements in the group algebras commute, [10, Remark 3, p. 451]. We recall that a non-abelian group G is a Hamiltonian group if every subgroup of G is normal. It is well-known that in this case $G \cong \mathcal{Q}_8 \times E \times O$, [14, Theorem 1.8.5, p. 63], where $\mathcal{Q}_8 = \langle x, y : x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle$ is the quaternion group of order 8, E is an elementary abelian 2-group and O is an abelian group with every element of odd order. When $O = \{1\}$, G is called a Hamiltonian 2-group.

When one works with linear extensions of arbitrary involutions of the base group of the group algebra, $* : G \to G$, one finds a larger class of groups such that the symmetric elements also commute. We recall that a group G is said to be an LC-group (a group with "limited commutativity" property) if it is non-abelian and for any pair of elements $g, h \in G$, we have that $gh = hg$ if and only if at least one element of $\{g, h, gh\}$ lies in $\zeta(G)$, where $\zeta(G)$ denotes the center of G. This family of groups was introduced by Goodaire. From [15, Proposition III.3.6, p. 98], a group G is an LC-group with a unique non-trivial commutator s (which must have order 2 and be central) if and only if $G/\zeta(G) \cong C_2 \times C_2$, where C_2 is the cyclic group of order 2. If G is endowed with an involution $*$, then we say that G is a special LC-group, or SLC-group, if it is an LC-group, it has a unique non-trivial commutator s and on such a group, the map $*$ is defined by

$$g^* = \begin{cases} g, & \text{if g is central;} \\ sg, & \text{otherwise.} \end{cases} \tag{2}$$

We refer to this as the *canonical* involution on an SLC-group. For instance, if $*$ is the classical involution, then from expression (2) all elements have order 1, 2, or 4. Furthermore, if g is a non-central element, then $g^2 = s$ and we obtain that every cyclic subgroup of G is normal, and thus in this case the SLC-groups are precisely the Hamiltonian 2-groups.

When we consider on $\mathbb{F}G$ the *oriented group involution* $\left(\sum_{g\in G} \alpha_g g\right)^{\circledast} = \sum_{g\in G} \alpha_g \sigma(g) g^*$, where G is a group with a non-trivial homomorphism $\sigma : G \to \{\pm 1\}$ and an involution $*$, the kernel of σ is a subgroup N in G of index 2. It is clear that the involution $\circledast$ coincides on the subalgebra $\mathbb{F}N$ with the group involution $*$. Also, we have that the symmetric elements in G, under $\circledast$, are the symmetric elements in N regarding $*$. If we denote the sets of symmetric elements in G, under the involutions $\circledast$ and $*$, by N^+ and G^+, respectively, then we can write $N^+ = N \cap G^+$. In recent years, this type of involution has been of interest and some results were obtained in the study of properties of $\mathbb{F}G^+$, $\mathbb{F}G^-$ and $\mathcal{U}^+(\mathbb{F}G)$, see [16–18]. For instance, the authors in [18] proved that $\mathbb{F}G$ satisfies the $\circledast$-PI, i.e., $\mathbb{F}G$ satisfies a PI where $x_i^{\circledast}$ for some i's appear, $\alpha^{\circledast}\alpha = \alpha\alpha^{\circledast}$ if and only if the set $\mathbb{F}G^+$ of symmetric elements in regard to $\circledast$ is commutative. Since $[G : N] = 2$, the structure of the group N and the action of $*$ on N are both known, see [19, Theorem 2.4, p.730] and [18, Theorem 3.1, p. 4395], then this classification depend on whether $N = ker(\sigma)$ is either abelian or an SLC-group. However, this result does not provide a complete description of G and the action of $\circledast$ on G. Therefore this is the principal aim in this kind of research.

In this paper, we extend the results obtained by Dooms and Ruiz [12] to the case of the oriented group involution (1). More precisely, we classify under middle hypothesis, the groups with a regular group algebra over an infinite field $\mathbb{F}$ of char$(\mathbb{F}) \neq 2$ for which the $\circledast$-symmetric units satisfy a GI. Further, we prove that if the $\circledast$-symmetric units of $\mathbb{F}G$, where G is a locally finite group, satisfies a group identity then $\mathbb{F}G$ satisfies a polynomial identity, see Theorems 3.1 and 3.2. Moreover, in the case when the prime radical $\eta(\mathbb{F}G)$ of $\mathbb{F}G$ is nilpotent we characterize the groups for which the symmetric units $\mathcal{U}^+(\mathbb{F}G)$ do satisfy a group identity, see Theorem 3.3.

Throughout this paper $\mathbb{F}$ will always denote an infinite field with $\text{char}(\mathbb{F}) \neq 2$, G a group, $*$ and σ an involution and a non-trivial orientation of G, respectively. We will denote with $\circledast$ an oriented group involution of $\mathbb{F}G$ given by expression (1), which is linear extension of the involution $*$ of G, twisted by the homomorphism σ.

2 Preliminaries and notations

Let R be a ring with involution $\star$. Let $\mathcal{U}(R)$ its group of units and $\mathcal{U}^+(R) = \mathcal{U}(R) \cap R^+$ the set of symmetric units. It is well-known that central idempotents are very important in the study of group identities. Moreover the following fact is proved.

Lemma 2.1 ([10, Theorem 2]) *Let R be a semiprime ring with involution $\star$ such that $\mathcal{U}^+(R)$ is GI. Then every symmetric idempotent of R is central.*

For a given prime p, an element $x \in G$ will be called a p-element if its order is a power of p and it is called p'-element if its order is finite and, not divisible by p. Moreover, a torsion subgroup H of G is a p'-subgroup if every element $h \in H$ is a p'-element. We agree that if $p = 0$ every torsion subgroup is a p'-subgroup.

An immediate consequence in the setting of group algebras of Lemma 2.1 is the following: Let $\mathbb{F}$ be a field with $\text{char}(\mathbb{F}) \geq 0$ and G a group such that $\mathbb{F}G$ is semiprime. If $\mathcal{U}^+(\mathbb{F}G)$ is GI under the classical involution, then every torsion p'-subgroup of G is normal in G. Indeed to prove this result is important the use of the idempotent element

$$\frac{1}{o(g)}\widehat{g} = \frac{1}{o(g)}(1 + g + \cdots + g^{o(g)}),$$

where g is a p'-element, see the proof in [10, Corollary 2, p. 451]. Note that this element is symmetric under the classical involution, but it is not when $*$ is a group involution. This fact, in the former case, gives important information on cyclic subgroups. Unfortunately, this property is lost in the case of a group involution $*$.

Semisimple algebras whose units satisfy a GI were widely studied, see for instance [20, Theorem 1, p. 197], [12, Theorem 2.2, p. 743], [21, Theorem 3.1, p. 1732] and [13, Lemma 2.1, p. 2803]. The following three lemmas will be needed in the sequel.

Lemma 2.2 ([13, Lemma 2.1]) *Let R be a finite dimensional semisimple algebra with involution $\star$ over an infinite field K,* $\text{char}(K) \neq 2$. *Suppose that $\mathcal{U}^+(R)$ is GI. Then R is a direct sum of simple algebras of dimension at most four over their centers and the symmetric elements R^+ are central in R, i.e.,*

$$R \cong D_1 \oplus D_2 \oplus \cdots \oplus D_k \oplus M_2(\mathbb{F}_1) \oplus M_2(\mathbb{F}_2) \oplus \cdots \oplus M_2(\mathbb{F}_l) \;\; \textit{and} \;\; R^+ \subseteq \zeta(R).$$

Lemma 2.3 ([12, Theorem 2.2]) *Let R be a semisimple K-algebra with involution $\star$, where K is an infinite field with* $\text{char}(K) \neq 2$. *Suppose one of the following conditions holds:*

(i) *K is uncountable,*

(ii) *R has no simple components that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(R) \in$ *GI if and only if* R^+ *is central in* R.

Lemma 2.4 ([9, Lemma 2.3.5]) *Suppose that* R *is an* K*-algebra with involution* $\star$*, where* $\mathrm{char}(K) \neq 2$. *Let* I *be a* $\star$*-invariant nil ideal. If* $\mathcal{U}^+(R)$ *satisfies the group identity* $\omega(x_1, \ldots, x_n) = 1$*, then so does* $\mathcal{U}^+(R/I)$*. Conversely, if* $p > 0$*,* I *is nil of bounded exponent and* $\mathcal{U}^+(R/I) \in$ *GI, then* $\mathcal{U}^+(R) \in$ *GI.*

Remark 1 By [22, Lemma 2.4, p. 891], under the assumptions of Lemma 2.3, it follows that $\mathcal{U}^+(R) \in$ GI is equivalent to R^+ is Lie n-Engel, for some n.

We conclude this section with a result due to Jespers and Ruiz Marín [19], where the SLC groups arise naturally.

Lemma 2.5 ([19, Theorem 2.4]) *Let* $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let* G *be a group with an involution* $*$ *extended* $\mathbb{F}$*-linearly to* $\mathbb{F}G$*. Then* $\mathbb{F}G^+$ *is commutative if and only if* G *is abelian or an SLC group. In this case,* $\mathbb{F}G^+ = \zeta(\mathbb{F}G)$.

3 Results

3.1 Regular group algebras

We need the following results about group algebras endowed with an oriented group involution. We recall that for a fixed orientation σ of G, we denote with $N = \ker(\sigma)$.

Lemma 3.1 ([16, Lemma 1.1]) *Let* R *be a commutative ring with unity of characteristic different from* 2 *and let* G *be a group with a non-trivial orientation* σ *and an involution* $*$*. Suppose that* RG^+ *is commutative under oriented group involution and let* $g \in (G\backslash N)\backslash G^+$*,* $h \in G$*. Then one of the following holds:*

(i) $gh = hg$*; or*
(ii) $\mathrm{char}(R) = 4$ *and* $gh = g^*h^* = hg^* = h^*g$.

Furthermore, $gg^* = g^*g$.

Lemma 3.2 ([16, Theorem 2.2]) *Let* R *be a commutative ring with unity of characteristic different from* 2 *and let* G *be a non-abelian group with a non-trivial orientation* σ *and an involution* $*$*. Then,* RG^+ *is a commutative ring if and only if one of the following conditions holds:*

(i) $N = ker(\sigma)$ *is an abelian group and* $(G \setminus N) \subset G^+$*;*
(ii) G *and* N *have the* LC*-property, and there exists a unique non-trivial commutator* s *such that the involution* $*$ *is given by*

$$g^* = \begin{cases} g, & \text{if } g \in N \cap \zeta(G) \text{ or } g \in (G \setminus N) \setminus \zeta(G); \\ sg, & \text{otherwise.} \end{cases}$$

(iii) char$(R) = 4$, *G has the* LC*-property, and there exists a unique non-trivial commutator s such that the involution $*$ is the canonical involution.*

Recall that a ring R with identity is said to be (von Neumann) regular if for any $x \in R$ there exists an $y \in R$ such that $xyx = x$. Villamayor [2, Theorem 3.1.5, p. 69] showed that the group algebra $\mathbb{F}G$ is regular if and only if G is locally finite and has no elements of order p in case $\mathbb{F}$ has characteristic $p > 0$. Note that in this case the set of p-elements P is trivial and thus $\mathbb{F}G$ is semiprime, [2, Theorem 4.2.13, p. 131] (in case char$(\mathbb{F}) = 0$, we agree that $P = \{1\}$).

We are now able to classify the groups with a regular group algebra over an infinite field $\mathbb{F}$ of char$(\mathbb{F}) \neq 2$ for which the $\circledast$-symmetric units satisfy a GI, result which is the oriented version of Dooms and Ruiz, see Theorem 1.1. Note that in this context the third condition in Lemma 3.2 will not be considered.

Theorem 3.1 *Let $\mathbb{F}$ be an infinite field with* char$(\mathbb{F}) \neq 2$ *and let G be a non-abelian group such that $\mathbb{F}G$ is regular. Let $\sigma : G \to \{\pm 1\}$ be a non-trivial orientation and an involution $*$ on G. Suppose one of the following conditions holds:*

(i) *$\mathbb{F}$ is uncountable,*
(ii) *all finite non-abelian subgroups of G which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in GI$ if and only if one of the following conditions holds:

(1) *$N = ker(\sigma)$ is an abelian group and $(G \setminus N) \subset G^+$;*
(2) *G and N have the* LC*-property, and there exists a unique non-trivial commutator s such that the involution $*$ is given by*

$$g^* = \begin{cases} g, & \text{if } g \in N \cap \zeta(G) \text{ or } g \in (G \setminus N) \setminus \zeta(G); \\ sg, & \text{otherwise.} \end{cases} \tag{3}$$

Consequently, $\mathcal{U}^+(\mathbb{F}G) \in GI$ if and only if $\mathcal{U}^+(\mathbb{F}G)$ is an abelian group.

Proof Assume that $\mathcal{U}^+(\mathbb{F}G) \in$ GI and let $N = ker(\sigma)$. Then $\mathcal{U}^+(\mathbb{F}N) \in$ GI. Hence, by Theorem 1.1 and Lemma 2.5, we have two possibilities for N; either

(A) N is an abelian group; or
(B) N has the LC-property, and there exists a unique non-trivial commutator s such that the involution $*$ in N, is the canonical involution.

Now, let $g, h \in G$ such that $gh \neq hg$. Consider the collection $\{H_i\}_{i \geq 1}$ (possibly infinite) of all finite subgroups of G which are $*$-invariant and contain g and h. It is clear that $G = \bigcup_i H_i$ and that $\mathcal{U}^+(\mathbb{F}H_i) \in$ GI. Since $\mathbb{F}G$ is regular, we have that all $\mathbb{F}H_i$ are semisimple.

Set $\sigma_i = \sigma|_{H_i}$ and let $N_i = ker(\sigma_i)$. By Lemma 2.3, $\mathbb{F}H_i^+$ is central in $\mathbb{F}H_i$ for all i, and applying the Lemma 3.2, one of the following conditions holds:

(a) N_i is an abelian group and $(H_i \setminus N_i) \subset H_i^+$; or

(b) H_i and N_i have the LC-property, and there exists a unique non-trivial commutator s such that the involution $*$ is as given in the expression (3).

It is easy to see that, $N = ker(\sigma) = \bigcup_i N_i = \bigcup_i ker(\sigma_i)$ and $\bigcup_i (H_i \setminus N_i) = G \setminus N$. Suppose that (A) is true. Then (a) holds for all i and thus (1) follows.

Assume that (B) is true. If there exists j such that (a) holds, then since N_j is abelian and $g, h \in H_j$, at least one of them belong to $H_j \setminus N_j$. Without loss of generality, if $g \in H_j \setminus N_j$ follows that g is symmetric, and as $\mathbb{F}H_j^+$ is central in $\mathbb{F}H_j$, we obtain $gh = hg$, which is a contradiction. So (b) holds for all i.

If s is not a unique non-trivial commutator of G, then there exist $x, y \in G$ such that $(x, y) \neq s$. We know that $x, y \in H_i$, for some i, for instance, $H_i = \langle x, y, g, h, x^*, y^*, g^*, h^* \rangle$. Therefore $(x, y) = (g, h) = s$, a contradiction.

Claim: For all i, $\zeta(N_i) = N_i \cap \zeta(H_i)$. In fact, let $x \in \zeta(N_i) \setminus \zeta(H_i)$. Then, there exists $y \in H_i \setminus N_i$ such that $xy \neq yx$ and by the behavior of $*$ on H_i given by (3), $y^* = y$. Since $\mathbb{F}N_i^+$ is commutative, we have that $x^* = x$. Thus $xy \in (H_i \setminus N_i) \setminus \zeta(H_i)$ and by (b) $xy \in H_i^+$. Therefore $xy = (xy)^* = y^* x^* = yx$, a contradiction. Hence $\zeta(N_i) = N_i \cap \zeta(H_i)$.

Thus for all i, we obtain that $H_i^+ = \zeta(N_i) \cup [(H_i \setminus N_i) \setminus \zeta(H_i)]$. Now, it is clear that $\zeta(G) \subseteq \bigcup_i \zeta(H_i)$. When $x \in \bigcup_i \zeta(H_i)$, then $x \in \zeta(H_j)$ for some j. By the construction of H_j, $g \in H_j$ and hence $xg = gx$. Since g is an arbitrary element of $G \setminus \zeta(G)$, we conclude that $x \in \zeta(G)$ and $\zeta(G) = \bigcup_i \zeta(H_i)$. Therefore $* : G \to G$ is given as in the statement, and (2) holds.

The converse is clear, because conditions (1) and (2) by Lemma 3.2 imply that $\mathbb{F}G^+$ is commutative and hence, $(u, v) = 1$ is a GI for $\mathcal{U}^+(\mathbb{F}G)$.

The last assertion is now clear. □

We can find group algebras that fulfil the condition (2) in Theorem 3.1, see Remark in [12, p. 746] and the references quoted therein.

3.2 Non-regular group algebras

Dooms and Ruiz, in [12, Lemma 3.3, p. 747], assuming that $\mathbb{F}$ is an infinite field with char$(\mathbb{F}) \neq 2$ and G a locally finite group such that $\mathcal{U}^+(\mathbb{F}G) \in$ GI under a group involution $*$, demonstrated that the set of p-elements of G is a normal subgroup of G. They obtained a similar result to Theorem 3.1 for non-regular group algebras, see [12, Theorem 3.4, p. 748].

To handle group algebras which are not necessarily regular, we need the following two lemmas which are the natural extensions of known results. As usual, for a normal subgroup H of G we denote by $\Delta(G, H)$ the kernel of the map $\mathbb{F}G \overset{\Psi}{\to} \mathbb{F}(G/H)$ defined by

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \alpha_g g H$$

and $\Delta(G, G) = \Delta(G)$ is the augmentation ideal.

Lemma 3.3 *Let G be a locally finite group and* $\mathrm{char}(\mathbb{F}) = p \neq 2$. *If* $\mathcal{U}^+(\mathbb{F}G) \in GI$, *then the set P of p-elements of G is a subgroup.*

Proof Let $g, h \in P$ and let $H = \langle g, h, g^*, h^* \rangle$. Since G is locally finite, then H is finite. Moreover, H is $*$-invariant and $H \subset N = ker(\sigma)$ (every element $x \in H$ has odd order). Since $\mathbb{F}H$ is a finite dimensional algebra the Jacobson radical $\mathcal{J}$ is nilpotent. Let $R = \mathbb{F}H/\mathcal{J}$. Then R is semisimple and, by Lemma 2.4, $\mathcal{U}^+(R)$ satisfies a group identity. Hence by Lemma 2.2, R is a direct sum of simple algebras of dimension at most four over their centers. Finally, by [22, Lemma 2.6, p. 892] we get that P is a subgroup. □

Lemma 3.4 *Let $\mathbb{F}$ be a field with* $\mathrm{char}(\mathbb{F}) = p > 2$ *and G a group such that* $\mathcal{U}^+(\mathbb{F}G)$ *satisfies a GI* $\omega(x_1, ..., x_n) = 1$, *under an oriented group involution* $\circledast$. *If H is a normal $*$-invariant p-subgroup of G, and either H is finite or G is locally finite, then* $\mathcal{U}^+(\mathbb{F}(G/H))$ *satisfies* $\omega(x_1, ..., x_n) = 1$.

Proof Since H is a p-subgroup, then $H \subset N$ and hence H is $\circledast$-invariant.
If H is finite, then by [9, Lemma 1.1.1, p. 1], $\Delta(G, H)$ is nilpotent and the statement follows from Lemma 2.4.

Now assume that G is locally finite. Let $\overline{G} = G/H$ and take $\overline{\alpha_1}, \ldots, \overline{\alpha_n} \in \mathcal{U}^+(\mathbb{F}\overline{G})$. Since the map $\mathbb{F}G \to \mathbb{F}\overline{G}$ is an epimorphism we have that $\mathbb{F}\overline{G}^+$ is the image of $\mathbb{F}G^+$, thus we may lift these elements up to $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}G^+$ and similarly for their inverses. Let G_1 the subgroup of G generated by the supports of all of these elements. As G is locally finite, G_1 is finite. Taking $H_1 = G_1 \cap H$, we have by the finite case that $\mathcal{U}^+(\mathbb{F}(G_1/H_1))$ satisfies $\omega(x_1, ..., x_n) = 1$. Replacing G with G_1 and H with H_1, we get the statement. □

Consider the group algebra $\mathbb{F}G$, where G is a locally finite group with an oriented group involution $\circledast$, P the set of p-elements of G and $\mathbb{F}$ an infinite field with $\mathrm{char}(\mathbb{F}) = p \neq 2$. Suppose that $\mathcal{U}^+(\mathbb{F}G) \in$ GI and let $\overline{G} = G/P$. If $\mathbb{F}\overline{G}$ satisfies conditions (i) and (ii) of Theorem 3.1, then by Lemma 3.3, we have that P is a normal subgroup of G and by Lemma 3.4 $\mathcal{U}^+(\mathbb{F}\overline{G})$ is GI. Since $\mathbb{F}\overline{G}$ is regular, it follows that either $\overline{G}$ is abelian, or $\overline{N}$ and $\overline{G}$ satisfy one of the conclusions of Theorem 3.1 and the involution $\overline{*} : \overline{G} \to \overline{G}$ is as given in the expression (3). Moreover, $\mathbb{F}\overline{G}^+$ is central in $\mathbb{F}\overline{G}$ and thus $\mathbb{F}\overline{G}$ is PI. Since $\mathbb{F}\overline{G} \cong \mathbb{F}G/\Delta(G, P)$ and $\Delta(G, P)$ is a nil subring of $\mathbb{F}G$ $\circledast$-invariant, by [10, Remark 2, p. 450] we have that $\Delta(G, P)$ is PI and as being PI is closed under ideal extensions, we get that $\mathbb{F}G$ is also PI. Therefore, we obtain the next result for non-regular group algebras.

Theorem 3.2 *Let $g \mapsto g^*$ be an involution on a locally finite group G, $\sigma : G \to \{\pm 1\}$ a non-trivial orientation with $N = ker(\sigma)$, P the set of p-elements of G and $\mathbb{F}$ an infinite field with* $\mathrm{char}(\mathbb{F}) = p \neq 2$. *Suppose that* $\mathcal{U}^+(\mathbb{F}G) \in GI$ *and that one of the following conditions holds:*

(i) $\mathbb{F}$ *is uncountable,*
(ii) *all finite non-abelian subgroups of G/P which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras,*

then we have that

(1) $\overline{G} = G/P$ *is abelian, or*
(2) $\overline{N} = N/P = ker(\overline{\sigma})$ *is abelian and* $(\overline{G}\backslash\overline{N}) \subset \overline{G^+}$*, or*
(3) $\overline{G}$ *and* $\overline{N}$ *have the* LC-*property and there exists a unique non-trivial commutator* $\overline{s}$ *such that the involution* $\overline{*}$ *in* $\overline{G}$ *is given by*

$$\overline{g^*} = \begin{cases} \overline{g}, & \text{if } \overline{g} \in \overline{N} \cap \zeta(\overline{G}) \text{ or } \overline{g} \in (\overline{G} \setminus \overline{N}) \setminus \zeta(\overline{G}); \\ \overline{sg}, & \text{otherwise.} \end{cases} \quad (4)$$

Moreover, $\mathbb{F}G \in PI$.

To obtain sufficient conditions for locally finite groups G such that $\mathcal{U}^+(\mathbb{F}G) \in$ GI, we need the following lemma:

Lemma 3.5 *Let* $\mathbb{F}$ *be a field with* $\text{char}(\mathbb{F}) = p \neq 2$. *Let* G *be a locally finite group with an involution* $*$ *and a non-trivial orientation* σ. *If* P *is a subgroup of bounded exponent, and either* G/P *is abelian, or* G/P *and* N/P *are as in Theorem 3.1, then* $\mathcal{U}^+(\mathbb{F}G) \in GI$.

Proof Suppose P is a subgroup of bounded exponent and that N/P, G/P, $*$ and σ are as in the statement. Then by Lemma 3.2, $\mathcal{U}^+(\mathbb{F}\overline{G})$ is abelian. Hence $(\mathcal{U}^+(\mathbb{F}G), \mathcal{U}^+(\mathbb{F}G)) \subset 1 + \Delta(G, P)$. Now $\Delta(G, P)$ is nil of bounded exponent and thus $(\mathcal{U}^+(\mathbb{F}G), \mathcal{U}^+(\mathbb{F}G))^{p^n} = 1$ for some $n \geq 0$. Hence $\mathcal{U}^+(\mathbb{F}G) \in$ GI. □

Remark 2 Note that, under the assumptions of Lemma 3.5, in case $\mathbb{F}G$ is PI and G/P is abelian, we obtain that $G' \subseteq P$ is of bounded exponent. Hence by [5, Theorem 1.1, p. 657] even $\mathcal{U}(\mathbb{F}G)$ satisfies a GI. Now, if G/P is an SLC-group one easily deduces that also in this case G' is of bounded exponent, but not necessarily a p-group. Finally, if $\overline{N} = N/P$ is abelian and $(\overline{G} \setminus \overline{N}) \subset \overline{G^+}$ we can not assure that G' is neither of bounded exponent nor a p-group.

In the sequel, we characterize the groups for which the symmetric units $\mathcal{U}^+(\mathbb{F}G)$ under ⊛ do satisfy a group identity. For $g \in G$, let $C_G(g) = \{h \in G : hg = gh\}$ be the centralizer of g in G. Set $\Phi(G) = \{g \in G : [G : C_G(g)] < \infty\}$ the finite conjugacy subgroup of G and $\Phi_p(G) = \langle P \cap \Phi(G) \rangle$.

Theorem 3.3 *Let* $g \mapsto g^*$ *be an involution on a locally finite group* G*,* $\sigma : G \to \{\pm 1\}$ *a non-trivial orientation and* $\mathbb{F}$ *an infinite field with* $\text{char}(\mathbb{F}) = p \neq 2$. *Suppose that the prime radical* $\eta(\mathbb{F}G)$ *of* $\mathbb{F}G$ *is a nilpotent ideal and that one of the following conditions holds:*

(i) $\mathbb{F}$ *is uncountable,*
(ii) *all finite non-abelian subgroups of* G/P *which are* $*$*-invariant have no simple components in their group algebra over* $\mathbb{F}$ *that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in GI$ *if and only if* P *is a finite normal subgroup and* G/P *is abelian or* G/P *and* N/P *are as in Theorem 3.1.*

Proof Suppose that $\mathcal{U}^+(\mathbb{F}G) \in$ GI, then by Lemma 3.3 we have that P is a normal subgroup. Now, by Theorem 3.2, either G/P is abelian or G/P and N/P are as in Theorem 3.1 and hence, $\mathbb{F}G \in$ PI. Thus by [2, Theorem 5.2.14, p. 189], $[G : \Phi(G)] < \infty$ and $\left|\Phi'(G)\right| < \infty$. Since $\eta(\mathbb{F}G)$ is nilpotent [2, Theorem 8.1.12, p. 311] gives that $\Phi_p(G) = P \cap \Phi(G)$ is a finite normal p-subgroup. As $P\Phi(G)/\Phi(G) \cong P/P \cap \Phi(G)$ is finite, then P is finite.

Now, the converse is clear by Lemma 3.5. □

As a corollary following the arguments in Dooms and Ruiz [12, Corollary 3.7, p. 749], we obtain a characterization of the locally finite groups with semiprime group algebras such that the set of ⊛-symmetric units is GI.

Corollary 3.1 *Let $g \mapsto g^*$ be an involution on a locally finite group G, $\sigma : G \to \{\pm 1\}$ a non-trivial orientation and $\mathbb{F}$ an infinite field with* $\mathrm{char}(\mathbb{F}) = p \neq 2$ *such that $\mathbb{F}G$ is semiprime. Suppose one of the following conditions holds:*

(i) *$\mathbb{F}$ is uncountable,*
(ii) *all finite non-abelian subgroups of G/P which are $*$-invariant have no simple components in their group algebra over $\mathbb{F}$ that are non-commutative division algebras other than quaternion algebras.*

Then $\mathcal{U}^+(\mathbb{F}G) \in GI$ if and only if $\mathcal{U}^+(\mathbb{F}G)$ is an abelian group.

Proof Suppose $\mathcal{U}^+(\mathbb{F}G) \in$ GI, then following the lines of the proof of Theorem 3.3, $\mathbb{F}G$ is semiprime PI. Hence by [2, Theorem 4.2.13, p. 131] $\Phi_p(G) = \{1\}$ and by the previous proof, we get that $P = \{1\}$. Thus $\mathbb{F}G$ is regular, and the result follows from Theorem 3.1. □

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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