Ricerche di Matematica
https://doi.org/10.1007/s11587-022-00705-5

Stability of plane shear flows in a layer with rigid and stress-free boundary conditions

Paolo Falsaperla[1] · **Giuseppe Mulone**[1] · **Carla Perrone**[1]

Received: 2 October 2021 / Accepted: 3 May 2022

Abstract
We study the stability of shear flows of an incompressible fluid contained in a horizontal layer. We consider rigid–rigid, rigid—stress-free and stress-free—stress-free boundary conditions. We study (and recall some known results) linear stability/instability of the basic Couette, Poiseuille and a laminar parabolic flow with the spectral analysis by using the Chebyshev collocation method. We then use an L_2-energy with Lyapunov second method to obtain nonlinear critical Reynolds numbers, by solving a maximum problem arising from the Reynolds energy equation. We obtain this maximum (which gives the minimum Reynolds number) for streamwise perturbations $\mathrm{Re}_c = \mathrm{Re}^y$. However, this contradicts a theorem which proves that streamwise perturbations are always stabilizing, $\mathrm{Re}^y = +\infty$. We solve this contradiction with a conjecture and prove that the critical nonlinear Reynolds numbers are obtained for two-dimensional perturbations, the spanwise perturbations, $\mathrm{Re}_c = \mathrm{Re}^x$, as Orr had supposed in the classic case of Couette flow between rigid planes.

Keywords Couette flow · Poiseuille · Stress-free boundary conditions · Nonlinear stability

Mathematics Subject Classification 76D05 · 76E05

1 Introduction

The stability of parallel shear flows has been studied by many authors analytically, numerically and with experiments, see Refs. [1–16]. Shear flows have application in many physical situations, for example in astrophysics, meteorology, oceanography, geophysics, and engineering, The stability/instability problem is nowadays object of study because the transition from laminar flows to instability, turbulence and chaos

✉ Giuseppe Mulone
giuseppe.mulone@unict.it

1 Department of Mathematics and Computer Science, University of Catania, Catania, Italy

is not completely understood and there are some discrepancies between the critical Reynolds numbers of linear and nonlinear analysis and the experiments at least in the case of rigid boundary conditions (the so called *Couette-Sommerfeld paradox*).

Recently [17], we have studied the nonlinear stability of plane Couette and Poiseuille flows with rigid (and periodic) boundary conditions, with the Lyapunov second method by using the classical L_2-energy. We proved that the streamwise perturbations are L_2-energy stable for any Reynolds number. This contradicts the results of Joseph [18], Joseph and Carmi [19] and Busse [8], and led us to make a guess to solve that problem. With this conjecture, for plane Couette and Poiseuille flows with rigid boundaries, we proved that the critical nonlinear Reynolds numbers are obtained along two-dimensional perturbations, the spanwise perturbations, as Orr [4] had supposed. This conclusion combined with some results by Falsaperla et al. [15] on the stability with respect to tilted rolls, provides a possible solution to the Couette-Sommerfeld paradox.

Here we investigate the stability for the classical shear flows, Couette and Poiseuille, in the cases of one rigid an the other stress-free (also called slip-free) boundaries, RF, and both stress-free FF boundaries, for completeness we also add the results for rigid RR boundaries. We also consider a case of a parabolic laminar flow where we assign the velocity at the bottom and the third component of the velocity and the tangential stress at top boundary, this problem is more appropriate for studying the flow of water in a river [20].

Although the case of stress-free planes is an ideal case (which is more appropriate in astrophysics and meteorology) in some applications it is useful to consider such boundary conditions. For example at the interface of a multiphase flow FF boundary conditions can occur in situations including micro-and nano-fluidic flow, flow over hydrophobic surfaces, rising bubbles in quiescent liquid, and polymer extrusion processes, see [21] where the free-slip boundary condition with an adaptive Cartesian grid method has been implemented. In numerical applications, the possibility of using free-slip conditions within the context of the particle finite element method (PFEM) has been investigated [22] “for high Reynolds number engineering applications in which tangential effects at the fluid-solid boundaries are not of primary interest, the use of free-slip conditions can alleviate the need for very fine boundary layer meshes”.

In Rao and Rajagopal [23] a history of slip and no-slip boundary conditions and a list of references can be found. The authors in particular observe that “it has been found that a large class of polymeric materials slip or stick-slip on solid boundaries. For instance, when polymeric melts flow due to an applied pressure gradient, there is a sudden increase in the throughput at a critical", see Rf. [23] p. 113.

The existence of slip between the velocity of the fluid at the wall and the speed of the wall sometimes is considered [24], the relative velocity is assumed to be proportional to the shear rate at the wall with a suitable slip coefficient (here we do not consider this interesting case that will be object of future study).

We recall that the classical results for Couette and Poiseuille for RR boundary conditions are the following: the plane Poiseuille flow is *unstable* for $\text{Re} > 5772$ (Orszag [10]); the plane Couette flow is *linearly stable for any* Reynolds numbers, Romanov [9]; for laboratory experiments and numerical simulations see, for instance [13, 14]. The nonlinear asymptotic L_2-*energy stability* has been proved for Reynolds numbers Re below some critical nonlinear value Re_c which is of the order 10^2. In particular Joseph [7, 18] proved that $\mathrm{Re}_c = \mathrm{Re}^y = 20.65$ (and $\mathrm{Re}^x = 44.3$) for plane Couette flow, and Joseph and Carmi [19] proved that $\mathrm{Re}_c = \mathrm{Re}^y = 49.55$ (and $\mathrm{Re}^x = 87.6$) for plane Poiseuille flow. Here and in what follows Re^y refers to critical value for streamwise (or longitudinal) perturbations, Re^x refers to the critical value for spanwise (or transverse) perturbations.

The use of *weighted* L_2*-energy* has been fruitful for studying nonlinear stability in fluid mechanics (see Straughan [25]).

Rionero and Mulone [26] studied the nonlinear stability of Couette and Poiseuille flows with the Lyapunov second method in the case of stress-free boundary conditions. By using a weighted energy they proved that plane Couette flows and plane Poiseuille flows are conditionally asymptotically nonlinear stable for any Reynolds numbers. They observed that, by applying the classical L_2-energy method, it is possible to obtain global nonlinear stability. However, they have not studied the maximum problem obtained from the Reynolds-Orr equation, and by introducing a suitable kinematically admissible velocity field, they proved that the nonlinear critical Reynolds numbers are less than 80 and 40 in the FF case of Couette and Poiseuille flows, respectively.

The problem of finding the best conditions for global nonlinear energy stability with respect to three-dimensional perturbations is still an open problem (for RR, RF and FF boundary conditions). This problem is equivalent to finding the maximum of a functional ratio that arises from the Reynolds-Orr energy equation [3].

In the nonlinear case it is often assumed that the least stabilizing perturbations, as in the linear case, are the two-dimensional spanwise perturbations, see Orr [4]. However, Joseph in his paper on Couette flow [18] proved that the least stabilizing perturbations are the streamwise perturbations and concluded that the Orr result is wrong. Joseph and Carmi [19] and Busse [8] obtained a result similar to Joseph [18] for Poiseuille flow. Moreover, our numerical calculations (for rigid and stress-free boundary conditions) with the Chebyshev collocation method (see Figs. 6, 7 and 8), show that the minimum Reynolds number for the energy method is obtained with respect to streamwise perturbations. In the RR case we obtained (see Ref. [17]) the same numerical results of Joseph [18] for the Couette case, and Joseph and Carmi [19] and Busse [8] for the Poiseuille case ($\mathrm{Re}_c = \mathrm{Re}^y = 20.6$ in Couette case, and $\mathrm{Re}_c = \mathrm{Re}^y = 49.55$ in Poiseuille case).

Despite of the results of Joseph [7, 18], Joseph and Carmi [19], and Busse [8] and our numerical calculations with the Chebyshev collocation method, we shall prove, also in the case of a stress-free boundary, that these results lead to a contradiction. In fact, in Sect. 3, we prove that streamwise perturbations are nonlinear L_2-energy exponentially stable for any Reynolds number (i.e. $\mathrm{Re}^y = +\infty$). Although some authors [27] assume that streamwise perturbations are always stable, as far as we know, in the literature there is no precise proof apart from Moffatt's proof in a semi-space [28], for rigid boundary conditions and a vanishing condition of the product of perturbations of pressure and a component of the velocity field at infinite. We then observe that *there is a contradiction* between this result ($\mathrm{Re}^y = +\infty$) and the numerical ones, i.e. the critical Reynolds number is obtained from the maximum problem (see Sect. 3) on the streamwise perturbations: Re^y is a finite real value (for instance in the RR case, Re^y= 20.6 for Couette and Re^y=49.5 for Poiseuille). In Sect. 3 we *suggest how to* *solve this contradiction through a conjecture*: the maximum of the functional ratio that comes from the Reynolds-Orr energy equation is obtained in a subspace of the space of kinematically admissible perturbations, the space of *physically admissible perturbations* competing for the maximum. In this way, we are able to prove that the maximum is reached on two-dimensional perturbations, the spanwise perturbations, and that the streamwise perturbations are stable for any Reynolds number.

This result (i.e. the least stabilizing perturbations are two-dimensional perturbations) justifies the study of the stability with respect to two-dimensional rolls done by Falsaperla et al. [15]. These rolls appear at the onset of turbulence in the experiments and so we have a possible explanation of the Couette-Sommerfeld paradox.

The plan of the paper is the following.

In Sect. 2 we write the non-dimensional perturbation equations of laminar flows: plane Couette, plane Poiseuille and “parabolic" flow, and we recall the classical linear stability/instability results.

In Sect. 3 we recall the classical energy stability results of Orr [4], Joseph [18], Joseph and Carmi [19], Busse [8] (see Drazin and Reid [29]). Then, we prove non-linear exponential stability of streamwise perturbations for any Reynolds number and any boundary conditions, i.e. $\mathrm{Re}^y = +\infty$ with respect to the L_2-energy norm. Furthermore, we study the maximum problem arising from the Reynolds energy [3], we arrive at a contradiction and make a conjecture to solve it by introducing the space of physical admissible perturbations competing for the maximum problem. In this space we find the optimal perturbations which give the critical Reynolds number: the spanwise perturbations, as Orr [4] had supposed in the case RR.

In Sect.4 we make some final comments.

2 Laminar flows between two parallel planes

Given a reference frame $Oxyz$, with unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$, consider the layer $\mathcal{D} = \mathbb{R}^2 \times [-1, 1]$ of thickness 2 with horizontal coordinates x, y and vertical coordinate z.

Plane parallel shear flows are solutions of the stationary Navier-Stokes equations

$$\begin{cases} \mathbf{U}\cdot\nabla\mathbf{U} = \mathrm{Re}^{-1}\Delta\mathbf{U} - \nabla P \\ \nabla\cdot\mathbf{U} = 0, \end{cases} \tag{1}$$

characterized by the functional form

$$\mathbf{U} = \begin{pmatrix} f(z) \\ 0 \\ 0 \end{pmatrix} = f(z)\mathbf{i}, \tag{2}$$

where $\mathbf{U}$ is the velocity field and P the pressure field, and Re is the Reynolds number. The function $f(z) : [-1, 1] \to \mathbb{R}$ is assumed to be sufficiently smooth and is called the shear profile. All the variables are written in a non-dimensional form. To nondimensionalize the equations and the gap of the layer we use a Reynolds number based on the maximum velocity and half gap d (see [20]).

In particular, for fixed velocity at the boundaries $z = \pm 1$, we have the well known profiles:

(a) *Couette* $f(z) = z$,
(b) *Poiseuille* $f(z) = 1 - z^2$.
If the velocity vanishes at $z = -1$ and its derivative with respect to z vanishes at $z = 1$, we have the
(c) *laminar parabolic flow* $f(z) = \frac{1}{4}[-z^2 + 2z + 3]$.

2.1 Perturbation equations

The perturbation equations to the plane parallel shear flows, in non-dimensional form, are

$$\begin{cases} u_t = -\mathbf{u}\cdot\nabla u + \mathrm{Re}^{-1}\Delta u - (f u_x + f' w) - \dfrac{\partial p}{\partial x} \\ v_t = -\mathbf{u}\cdot\nabla v + \mathrm{Re}^{-1}\Delta v - f v_x - \dfrac{\partial p}{\partial y} \\ w_t = -\mathbf{u}\cdot\nabla w + \mathrm{Re}^{-1}\Delta w - f w_x - \dfrac{\partial p}{\partial z} \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \tag{3}$$

where $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is the perturbation to the velocity field, p is the perturbation to the pressure field.

Throughout the paper, we use the symbol h_x as $\frac{\partial h}{\partial x}$, h_t as $\frac{\partial h}{\partial t}$, etc., for any function h, $f' = \dfrac{df}{dz}$.

To system (3) we append the
rigid (R) boundary conditions

$$\mathbf{u}(x, y, \pm 1, t) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, +\infty),$$

stress-free (F) boundary conditions

$$u_z(x, y, \pm 1, t) = v_z(x, y, \pm 1, t) = w(x, y, \pm 1, t) = 0,$$

$(x, y, t) \in \mathbb{R}^2 \times (0, +\infty)$,
in this case, in order to guarantee the uniqueness we must add the average conditions

$$\int_\Omega u \, dz \, dy \, dz = \int_\Omega v \, dz \, dy \, dz = 0$$

(Ω is a cell of periodicity, see next section), or *mixed (RF, rigid - stress-free)* boundary conditions

$$\mathbf{u}(x, y, -1, t) = 0, \quad (x, y, t) \in \mathbb{R}^2 \times (0, +\infty),$$
$$u_z(x, y, 1, t) = v_z(x, y, 1, t) = w(x, y, 1, t) = 0,$$

$(x, y, t) \in \mathbb{R}^2 \times (0, +\infty)$.

We also give the initial condition

$$\mathbf{u}(x, y, z, 0) = \mathbf{u_0}(x, y, z), \quad \text{in} \quad \mathcal{D},$$

with $\mathbf{u_0}(x, y, z)$ solenoidal vector which vanishes at the boundaries.

Definition 2.1 We define streamwise (or longitudinal) perturbations the perturbations $\mathbf{u}$, p which do not depend on x.

Definition 2.2 We define spanwise (or transverse) perturbations the perturbations $\mathbf{u}$, p which do not depend on y.

2.2 Linear stability/instability

The linear stability/instability is obtained by studying the linearised system by neglecting the terms $\mathbf{u}\cdot\nabla u$, $\mathbf{u}\cdot\nabla v$, $\mathbf{u}\cdot\nabla w$ in (3).

The linear perturbation equations become

$$\begin{cases} u_t = \mathrm{Re}^{-1}\Delta u - (f u_x + f' w) - \dfrac{\partial p}{\partial x} \\ v_t = \mathrm{Re}^{-1}\Delta v - f v_x - \dfrac{\partial p}{\partial y} \\ w_t = \mathrm{Re}^{-1}\Delta w - f w_x - \dfrac{\partial p}{\partial z} \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \tag{4}$$

Since the system is autonomous, we consider solutions of the form (cf. [25, 29]):

$$f(x, y, z, t) = f(z) e^{i(ax+by)+ct}, \tag{5}$$

with $f = u, v, w$ or p, in the domain $\mathbb{R}^2 \times (-1, 1) \times (0, +\infty)$, $a \geq 0$, $b \geq 0$, $a^2 + b^2 > 0$, and c is a complex number. By substituting (5) in (4), we have the system

$$\begin{cases} cu + iafu + f'w = \mathrm{Re}^{-1}(D^2 - (a^2 + b^2))u - iap \\ cv + iafv = \mathrm{Re}^{-1}(D^2 - (a^2 + b^2))v - ibp \\ cw + iafw = \mathrm{Re}^{-1}(D^2 - (a^2 + b^2))w - Dp \\ iau + ibv + Dw = 0\,, \end{cases} \tag{6}$$

where D and D^2 indicate first and second derivatives with respect to z.

We recall that, for rigid boundary conditions, the classical result of Romanov [9] proves that Couette flow is *linearly stable* for *any Reynolds* number. Instead, Poiseuille flow is unstable for any Reynolds number bigger that 5772 (Orszag [10]).

We observe that, in the linear case, the Squire theorem [6] holds, and the most destabilizing perturbations are two-dimensional, in particular the spanwise perturbations (see Drazin and Reid [29], p. 155). The critical Reynolds value, for Poiseuille flow, can be obtained by solving the celebrated Orr-Sommerfeld equation (see Drazin and Reid [29]).

Applying the Squire transformation [6], we are led to study the system of the spanwise perturbations:

$$\begin{cases} cu + iafu + f'w = \mathrm{Re}^{-1}(D^2 - a^2)u - iap \\ cw + iafw = \mathrm{Re}^{-1}(D^2 - a^2)w - Dp \\ iau + Dw = 0\,, \end{cases} \tag{7}$$

We differentiate the first and second member of $(7)_1$ with respect to z, then multiply by ia; multiply both sides of $(7)_1$ by a^2. Taking into account $(7)_3$, adding term to term the equations so obtained, we get the *Orr-Sommerfeld equation* [5] (cf. [29], p. 156)

$$c(D^2 - a^2)w + iaf(D^2 - a^2)w - f''iaw = \mathrm{Re}^{-1}(D^2 - a^2)^2 w. \tag{8}$$

This equation can also be obtained by taking the third component of the double curl of the equation

$$c\mathbf{u} + f\mathbf{u}_x + f'w\mathbf{i} = \mathrm{Re}^{-1}\Delta\mathbf{u} - \nabla p \tag{9}$$

and applying the solenoidality of the velocity field and the Squire transformation.

It is easy to see that both Couette and Poiseuille flows under stress-free boundary conditions are *linearly stable* for any Reynolds number. This is in agreement with the result of Rionero and Mulone [26]. Moreover, plane Couette flow, with rigid boundary conditions is *linearly stable for any* Reynolds numbers (Romanov [9]).

By solving this equation with the Chebyshev collocation method, we obtain the following results for linear stability/instability:

Couette

1. In the stress-free—stress-free case we have stability for any Reynolds number [26] (see Fig. 1).
2. In the rigid—stress-free case we have stability for any Reynolds number (in this case we obtain a graph very similar to the FF case).
3. In the rigid–rigid case we have stability for any Reynolds number [9] (in this case we obtain a graph very similar to the FF case).

Poiseuille

1. In the stress-free—stress-free case we have stability for any Reynolds number [26], (see Fig. 2).

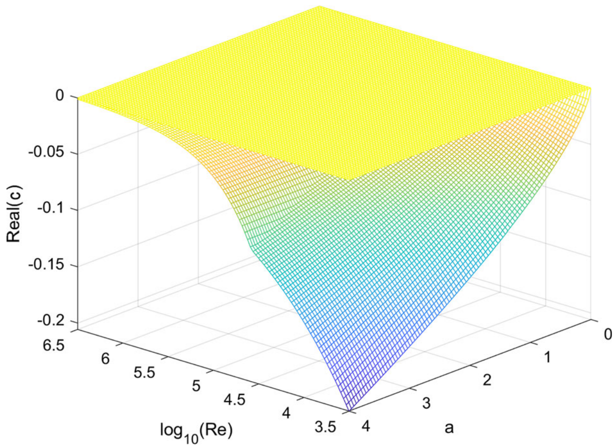

Fig. 1 Plane Couette (linear) stability. The surface gives the maximum real part of the time decay coefficient $Real(c)$, when a runs from 0 to 4, and the Reynolds number $\Re$ is in the interval $[10^{3.5}, 10^{6.5}]$ for Orr-Sommerfeld equation (8) with FF boundary conditions (for RF and RR boundaries the graphics are very similar). The horizontal plane corresponds to $Real(c) = 0$. Other computations in larger ranges of $\Re$ and a confirm these stability results

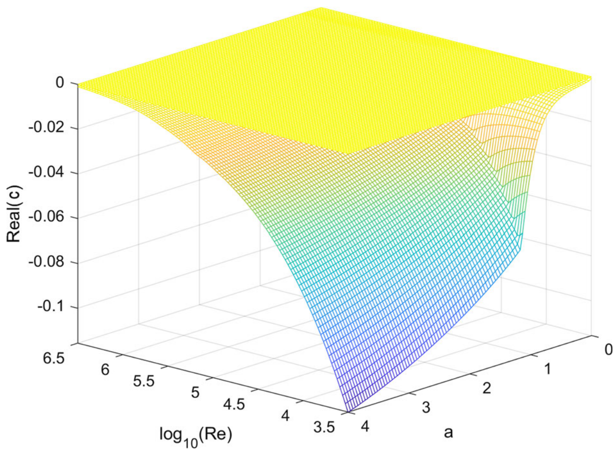

Fig. 2 Plane Poiseuille (linear) stability. The surface gives the maximum real part of the time decay coefficient $Real(c)$, when a runs from 0 to 4, and the Reynolds number $\Re$ is in the interval $[10^{3.5}, 10^{6.5}]$ for Orr-Sommerfeld Eq. (8) with FF boundary conditions. The horizontal planes correspond to $Real(c) = 0$

2. In the rigid—stress-free case we have instability for Re $>$ 169785 (see Fig. 3, left panel).
3. In the rigid–rigid case we have instability for Re $>$ 5772 (Orszag [10]), (see Fig. 4, right panel).

Laminar parabolic flow

1. In the stress-free—stress-free case we have stability for any Reynolds number (see Fig. 5, top panel)

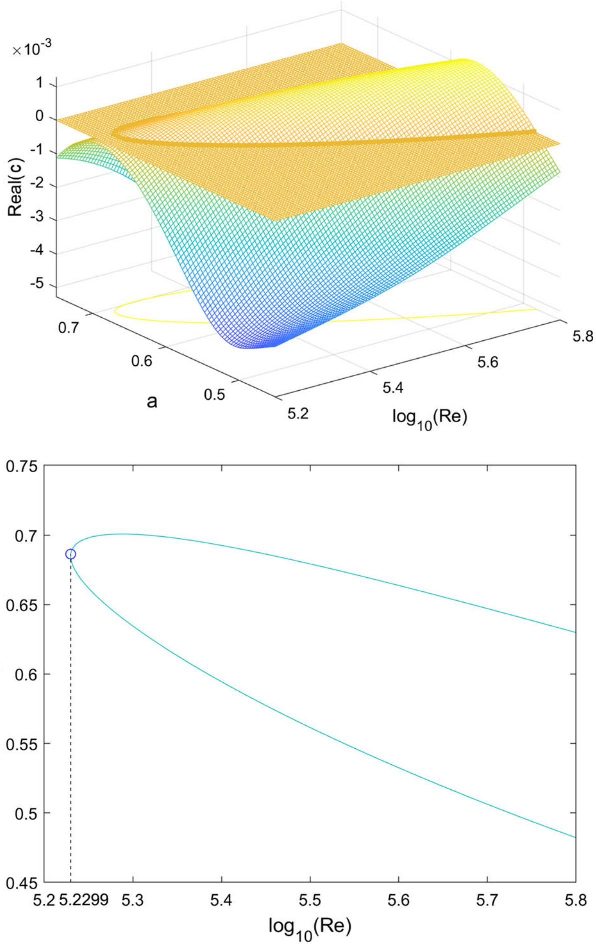

Fig. 3 On the top: the surface gives the real part of the time decay coefficient $Real(\sigma)$ for Orr-Sommerfeld Eq. (8) with RF boundary conditions. a runs from 0.45 to 0.75, and the Reynolds number $\Re$ is in the interval $[10^{5.2}, 10^{5.8}]$. The meshed plane corresponds to $Real(c) = 0$, the surface corresponds to the set of points $(a, log_{10}(\Re), Real(c))$. On the bottom: the correspondent critical curve in the a-Re plane is plotted; the minimum is highlighted by a circle and its numerical value is marked on the horizontal axis. The minimum is equal to $\Re = 169785$

2. In the rigid—stress-free case we have stability of any Reynolds number, see [20] (see Fig. 5, middle panel).
3. In the rigid–rigid case we have stability for any Reynolds number (see Fig. 5, bottom panel).

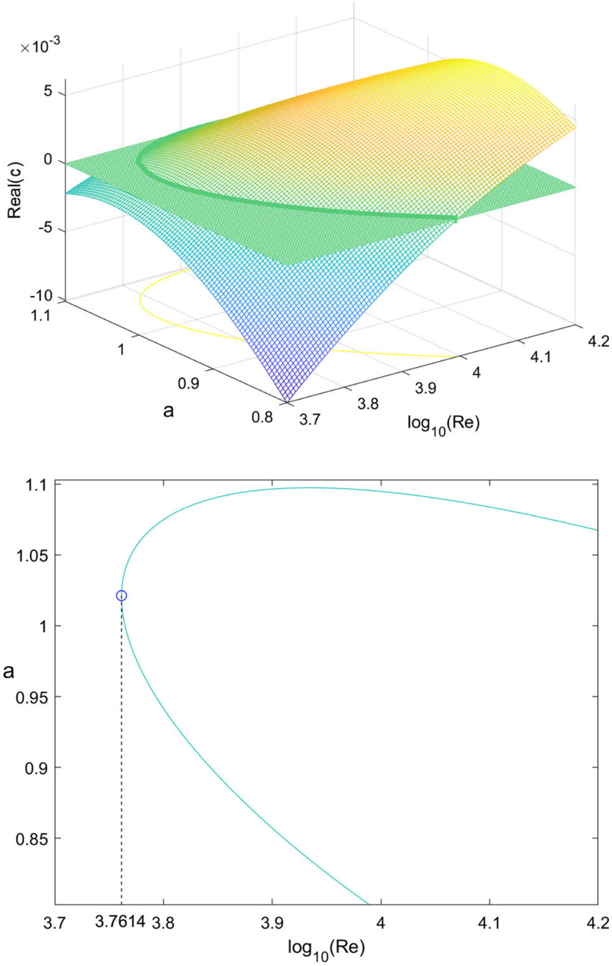

Fig. 4 On the top: the surface gives the real part of the time decay coefficient $Real(\sigma)$ for Orr-Sommerfeld Eq. (8) with RR boundary conditions. a runs from 0.45 to 0.75, and the Reynolds number $\Re$ is in the interval $[10^{3.7}, 10^{4.2}]$. The meshed plane corresponds to $Real(c) = 0$, the surface corresponds to the set of points $(a, log_{10}(\Re), Real(c))$. On the bottom: the correspondent critical curves in the a-Re plane are plotted; the minimum is highlighted by a circle and its numerical value is marked on the horizontal axis.The minimum is equal to $\Re = 5772$, the Orszag [10] result

3 Nonlinear energy stability

Assume that both $\mathbf{u}$ and ∇p are x, y-periodic with periods a and b in the x and y directions, respectively, with wave numbers $(a, b) \in \mathbb{R}^2_+$. In the following it suffices therefore to consider functions over the periodicity cell

$$\Omega = [0, \frac{2\pi}{a}] \times [0, \frac{2\pi}{b}] \times [-1, 1].$$

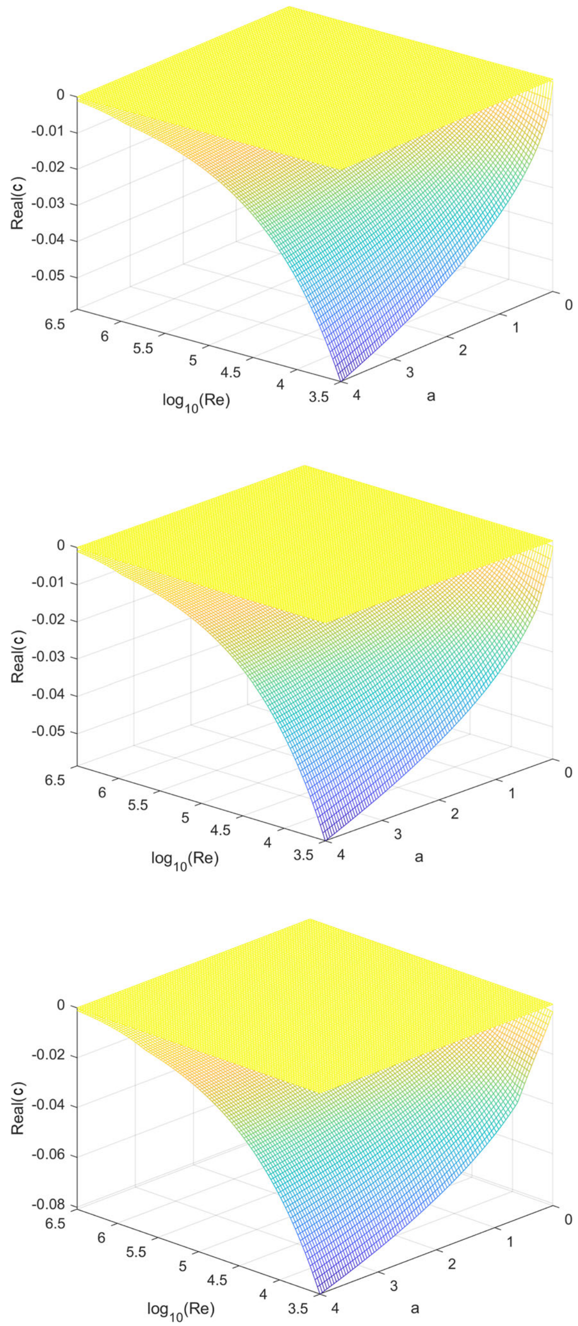

Fig. 5 Plane parabolic (linear) stability. Each surface gives the real part of the time decay coefficient $Real(c)$, when a runs from 0 to 4, and the Reynolds number $\Re$ is in the interval $[10^{3.5}, 10^{6.5}]$ for Orr-Sommerfeld Eq. (8) with FF, RF and RR boundary conditions (from top to bottom). Each meshed plane corresponds to $\Re(c) = 0$, the surface corresponds to the set of points $(a, log_{10}(\Re), Real(c))$

As the basic function space, we take $L_2(\Omega)$, which is the space of square-summable functions in Ω with the scalar product denoted by

$$(g, h) = \int_0^{\frac{2\pi}{a}} \int_0^{\frac{2\pi}{b}} \int_{-1}^{1} g(x, y, z)h(x, y, z)dxdydz,$$

and the norm given by

$$\|g\| = \Big[\int_0^{\frac{2\pi}{a}} \int_0^{\frac{2\pi}{b}} \int_{-1}^{1} g^2(x, y, z)dxdydz\Big]^{\frac{1}{2}}.$$

Here we study *the nonlinear energy stability with the Lyapunov method*, by using the classical energy

$$V(t) = \frac{1}{2}[\|u\|^2 + \|v\|^2 + \|w\|^2].$$

We obtain *sufficient conditions of global nonlinear stability*.

Taking into account the solenoidality of $\mathbf{u}$ and the boundary condition, we write the Reynolds-Orr energy identity [3]

$$\dot{V} = -(f'w, u) - \mathrm{Re}^{-1}[\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2], \tag{10}$$

and we have

$$\begin{aligned} \dot{V} &= -(f'w, u) - \mathrm{Re}^{-1}[\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2] \\ &= \left(\frac{-(f'w, u)}{\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2} - \frac{1}{\mathrm{Re}}\right) \|\nabla \mathbf{u}\|^2 \\ &\le \left(\frac{1}{\mathrm{Re}_c} - \frac{1}{\mathrm{Re}}\right) \|\nabla \mathbf{u}\|^2, \end{aligned} \tag{11}$$

where

$$\frac{1}{\mathrm{Re}_c} = m = \max_{\mathcal{S}} \frac{-(f'w, u)}{\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2}, \tag{12}$$

$\mathcal{S}$ is the space of the *kinematically admissible fields*

$$\begin{aligned} &\mathcal{S} = \{u, v, w \in H^2(\Omega),\ \text{satisfying the boundary} \\ &\text{conditions } RR,\ RF,\ FF,\ \text{periodic in} x,\ \text{and } y, \\ &u_x + v_y + w_z = 0, \quad \|\nabla \mathbf{u}\| > 0\}, \end{aligned} \tag{13}$$

and $H^2(\Omega)$ is the Sobolev space of the functions which are in $L_2(\Omega)$ together with their first and second generalized derivatives.

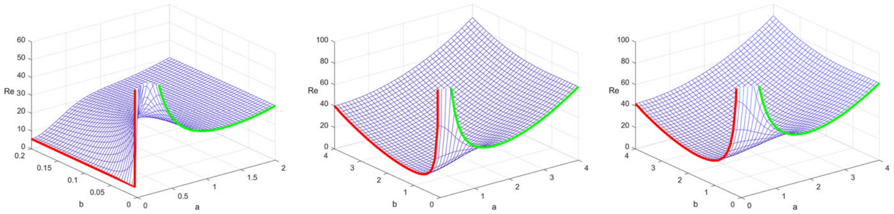

Fig. 6 Plane Couette energy Orr-Reynolds number $\Re = \Re_c$ as function of the wave numbers a and b, for system (17) with FF, RF and RR boundary conditions (from top to bottom). The absolute minimum of each surface is achieved on the streamwise perturbations ($a = 0$)

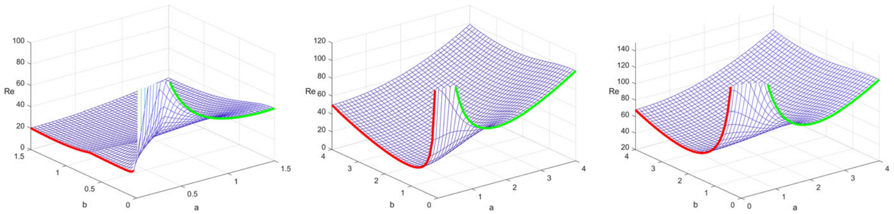

Fig. 7 Plane Poiseuille energy Orr-Reynolds number $\Re = \Re_c$ as function of the wave numbers a and b, for system (17) with FF, RF and RR boundary conditions (from top to bottom). The absolute minimum of each surface is achieved on the streamwise perturbations ($a = 0$)

The Euler-Lagrange equations of this maximum problem are given by

$$- f'w\mathbf{i} - f'u\mathbf{k} + 2m\Delta\mathbf{u} = \nabla\lambda, \tag{14}$$

where λ is a Lagrange multiplier in the cases RR and RF.

In the case FF, the Euler-Lagrange equations of this maximum problem are given by

$$- f'w\mathbf{i} - f'u\mathbf{k} + 2m\Delta\mathbf{u} = \nabla\lambda + \mathbf{h}, \tag{15}$$

where the function λ and the constant vector $\mathbf{h} = (h_1, h_2, 0)^T$ are Lagrange multipliers which come from the zero divergence of the velocity vector $\mathbf{u}$, and the zero mean conditions of u and v, respectively.

It is easy to prove that $\mathbf{h} = 0$. For this we write equation (15) in components

$$\begin{cases} -f'w + 2m\Delta u = \lambda_x + h_1 \\ 2m\Delta v = \lambda_y + h_2 \\ -f'u + 2m\Delta w = \lambda_z\,. \end{cases} \tag{16}$$

Integrating $(16)_2$ over Ω we have

$$2m\int_\Omega \Delta v d\,\Omega = \int_\Omega \lambda_y d\,\Omega + \int_\Omega h_2 d\,\Omega.$$

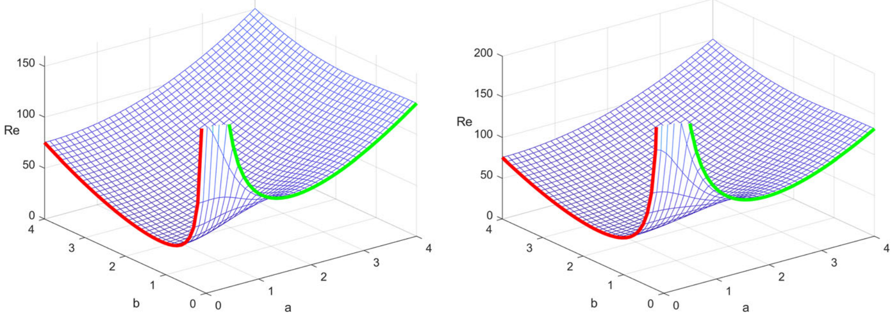

Fig. 8 Plane parabolic energy Orr-Reynolds number $\Re = \Re_c$ as function of the wave numbers a and b, for system (17) with RF and RR boundary conditions (from top to bottom). The absolute minimum of each surface is achieved on the streamwise perturbations ($a = 0$)

Due to the boundary conditions and the periodicity it follows

$$h_2(meas(\Omega)) = 0$$

and so $h_2 = 0$. Integrating $(16)_1$ over Ω we have

$$-\int_\Omega f' w d\,\Omega + 2m \int_\Omega \Delta u d\,\Omega = \int_\Omega \lambda_x d\,\Omega + \int_\Omega h_1 d\,\Omega.$$

As before we deduce

$$-\int_\Omega f' w d\,\Omega = h_1(meas(\Omega)).$$

Computing this integral and using the boundary conditions and the divergence-free equation, we have

$$-\int_\Omega f' w d\,\Omega = \int_\Omega f w_z d\,\Omega = -\int_\Omega [(fu)_x + (fv)_y] d\,\Omega = 0,$$

and so also $h_1 = 0$. Therefore the Euler-Lagrange equations are given by (14) for any boundary condition.

We define

$$\zeta = v_x - u_y$$

(it is linked to the toroidal part of the decomposition of the velocity vector $\mathbf{u}$ in the poloidal, toroidal and the mean flow, see [30, 31]) and take the third component of the *double curl* of (14) and the third component of the *curl* of (14). We obtain the system of the Euler-Lagrange equations written in terms of ζ and w:

$$\begin{cases} f'(\zeta_y + 2w_{xz}) + f'' w_x + 2m\Delta\Delta w = 0 \\ f' w_y + 2m\Delta\zeta = 0, \end{cases} \tag{17}$$

with the boundary conditions

$$w = w_z = 0, \ \ \zeta = 0 \tag{18}$$

on $z = \pm 1$ in the RR case,

$$w = w_z = 0, \ \ \zeta = 0 \text{ on } z = -1 \text{ and } w = w_{zz} = 0, \ \ \zeta_z = 0 \text{ on } z = 1 \tag{19}$$

in the RF case,

and

$$w = w_{zz} = 0, \ \ \zeta_z = 0, \tag{20}$$

on $z = \pm 1$ in the FF case.

By solving this system with the Chebyshev collocation method, and using 80 polynomials, we obtain that the critical Reynolds numbers are reached for streamwise perturbations, $\mathrm{Re}_c = \mathrm{Re}^y$.

In Table 1 we report the critical energy Orr-Reynolds numbers obtained for streamwise Re^y, and for spanwise perturbations Re^x, corresponding to the solutions of system (17).

In the next subsection, we prove that the streamwise perturbations are stable for any Reynolds number. This means that the previous results, even if numerically correct (see for example the case of the Couette flow with rigid boundary conditions studied by Joseph [18]) do not match the physics of the problem [17]. This is due to a possible wrong choice of the space of the fields competing for a maximum problem (see below).

Table 1 We report the Reynolds numbers and wave numbers for spanwise ($\Re^x$, a) and streamwise ($\Re^y$, b) perturbations, obtained from system (17) for different basic laminar flows and boundary conditions. The case FR corresponds to consider stress-free perturbations at the bottom plane

	Couette		Poiseuille		Parabolic	
RR	$\Re^x = 44.30$	$a = 1.89$	$\Re^x = 87.59$	$a = 2.09$	$\Re^x = 84.95$	$a = 1.92$
	$\Re^y = 20.66$	$b = 1.55$	$\Re^y = 49.59$	$b = 2.05$	$\Re^y = 39.86$	$b = 1.58$
RF	$\Re^x = 34.88$	$a = 1.57$	$\Re^x = 62.64$	$a = 1.70$	$\Re^x = 72.98$	$a = 1.62$
	$\Re^y = 12.93$	$b = 1.04$	$\Re^y = 22.25$	$b = 1.15$	$\Re^y = 30.63$	$b = 1.13$
FF	$\Re^x = 26.34$	$a = 1.21$	$\Re^x = 48.89$	$a = 1.39$	$\Re^x = 50.51$	$a = 1.27$
			$\Re^y = 19.22$	$b = 1.14$		
FR	$\Re^x = 34.88$	$a = 1.57$	$\Re^x = 62.64$	$a = 1.70$	$\Re^x = 61.73$	$a = 1.57$
	$\Re^y = 12.93$	$b = 1.04$	$\Re^y = 22.25$	$b = 1.15$	$\Re^y = 21.19$	$b = 1.02$

3.1 Stability of streamwise perturbations for any Reynolds number

We assume that the perturbations are *streamwise*, i.e. they *do not depend on x* ($\frac{\partial}{\partial x} \equiv 0$).

Therefore the perturbation Eqs. (3) become

$$\begin{cases} u_t = -\mathbf{u}\cdot\nabla u + \mathrm{Re}^{-1}\Delta u - f'w \\ v_t = -\mathbf{u}\cdot\nabla v + \mathrm{Re}^{-1}\Delta v - \dfrac{\partial p}{\partial y} \\ w_t = -\mathbf{u}\cdot\nabla w + \mathrm{Re}^{-1}\Delta w - \dfrac{\partial p}{\partial z} \\ v_y + w_z = 0. \end{cases} \tag{21}$$

We use the *classical energy norm* and show that the streamwise perturbations cannot destabilize the basic laminar flows: Couette, Poiseuille and laminar parabolic flow (see [17]).

We multiply $(21)_1$ by u and integrate over Ω (now $\Omega = \Omega_{yz} = [0, \frac{2\pi}{b}] \times [-1, 1]$). Besides, we multiply $(21)_2$ and $(21)_3$ by v and w and integrate over Ω. By taking into account of the solenoidality of $\mathbf{u}$, the boundary conditions and the periodicity, we have

$$\frac{d}{dt}\frac{\|u\|^2}{2} = -(f'u, w) - \mathrm{Re}^{-1}\|\nabla u\|^2,$$

$$\frac{d}{dt}\Big(\frac{\|v\|^2}{2} + \frac{\|w\|^2}{2}\Big) = -\mathrm{Re}^{-1}[\|\nabla v\|^2 + \|\nabla w\|^2].$$

By using the Wirtinger inequality [30, 32],

$$\begin{aligned} &\frac{d}{dt}\Big(\frac{\|v\|^2}{2} + \frac{\|w\|^2}{2}\Big) = -\mathrm{Re}^{-1}[\|\nabla v\|^2 + \|\nabla w\|^2] \\ &\le -C(\|v\|^2 + \|w\|^2). \end{aligned} \tag{22}$$

and integrating we have

$$\|v\|^2 + \|w\|^2 \le H_0 e^{-2Ct}, \quad H_0 = \|v_0\|^2 + \|w_0\|^2, \tag{23}$$

where C is a positive constant which depends on the domain and the boundary conditions: $C = \dfrac{1}{\mathrm{Re}}\min\{\dfrac{\pi^2}{4}, b^2\}$ for FF boundaries, $C = \dfrac{\pi^2}{16\mathrm{Re}}$ for RF boundaries, and $C = \dfrac{\pi^2}{4\mathrm{Re}}$ for RR boundaries. v_0 and w_0 are the initial values of v and w.

Now we consider the equation depending on u and define $M = \max_{[-1,1]} |f'(z)|$. We have the following inequalities:

$$\begin{aligned}\frac{d}{dt}\frac{\|u\|^2}{2} &= -(f'u, w) - \mathrm{Re}^{-1}\|\nabla u\|^2 \leq M\|u\|\|w\| + \\ &-\mathrm{Re}^{-1}\|\nabla u\|^2 \leq M(\frac{\|u\|^2}{2\epsilon} + \frac{\epsilon}{2}\|w\|^2) + \\ &-\mathrm{Re}^{-1}\|\nabla u\|^2 \leq M(\frac{\|u\|^2}{2\epsilon} + \frac{\epsilon}{2}\|w\|^2) + \\ &-C\|u\|^2 = (\frac{M}{2\epsilon} - C)\|u\|^2 + \frac{\epsilon}{2}M\|w\|^2 \\ &= -\frac{C}{2}\|u\|^2 + \frac{M^2}{C}\frac{\|w\|^2}{2}.\end{aligned} \tag{24}$$

where $\epsilon = \dfrac{M}{C}$.

We use this inequality and (23) to obtain

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &\leq -C\|u\|^2 \quad + \frac{M^2}{C}\|w\|^2 \leq -C\|u\|^2 + \\ &+ \frac{M^2}{C}(\|v\|^2 + \|w\|^2) \leq -C\|u\|^2 + \frac{M^2}{C}H_0e^{-2Ct}.\end{aligned} \tag{25}$$

Integrating last inequality, we have

$$\|u\|^2 \leq e^{-Ct}[k - \frac{M^2}{C^2}H_0e^{-Ct}] = ke^{-Ct} - \frac{M^2}{C^2}H_0e^{-2Ct}, \tag{26}$$

with $k = K_0 + \dfrac{M^2}{C^2}H_0, \quad K_0 = \|u_0\|^2$, and u_0 are the initial value of u.

We introduce the classical energy

$$L(t) = \frac{1}{2}[\|u\|^2 + \|v\|^2 + \|w\|^2], \tag{27}$$

and observe that the initial energy is given by $L_0 = \dfrac{H_0 + K_0}{2}$. Adding the (23) and the (26) we finally have:

$$\begin{aligned}L(t) &\leq H_0e^{-2Ct} + (K_0 + \frac{M^2}{C^2}H_0)e^{-Ct} - \frac{M^2}{C^2}H_0e^{-2Ct} \\ &\leq L_0e^{-2Ct} + (L_0 + \frac{M^2}{C^2}L_0)e^{-Ct} \\ &= L_0(e^{-2Ct} + e^{-Ct} + \frac{M^2}{C^2}e^{-Ct}).\end{aligned} \tag{28}$$

This inequality implies nonlinear exponential stability of the basic laminar flows (Couette, Poiseuille and parabolic) with respect to the streamwise perturbations *for any* Reynolds number.
Therefore, we have proved:

Theorem 3.1 *Assuming the perturbations to the basic shear flows* (2) *are streamwise, then we have nonlinear stability according to* (28) *for any Reynolds number.*

From this Theorem we have $\mathrm{Re}^y = +\infty$, i.e., *the streamwise perturbations cannot destabilize the basic flows.* This contradicts the numerical results we have reached (cf. Joseph [18], Joseph and Carmi [19] and Busse [8], in the RR case).

3.2 Possible solution of the contradiction

Probably the contradiction we have obtained in the previous subsection is due to the choice of the space of *kinematically admissible* perturbations where we look for the maximum [17]. Falsaperla et al. [17] note that this space is too large and likely contains perturbations which are not admissible as *physical perturbations* competing for the maximum.

How can we solve this contradiction?

As in Falsaperla et al. [17], we first observe that if the streamwise perturbations (now $w_x = 0$ and $\zeta_x = 0$) are stable for any Reynolds number, than, from $(17)_1$, $m = 0$ and this implies that $\zeta_y = 0$ *for any basic flow and any boundary condition.* Equation $(17)_2$ implies that also $w_y = 0$. If now we consider plan-form perturbations (see Chandrasekhar [33], p.24 formula (111)) we have $v = \dfrac{1}{a^2+b^2}[w_{yz} - \zeta_x] = 0.$ So, as in [17], we are led to speculate the

Conjecture:

A *possible answer* to the contradiction is this: we introduce the *subspace* $\mathcal{S}_0$ *of the physical admissible perturbations* which is the subspace of $\mathcal{S}$ consisting of maximizing functions $u, v, w \in \mathcal{S}$ such that $v = 0$ and we *conjecture* that the maximum m is assumed among the functions of this subspace, for any basic flow and any boundary condition. Furthermore, we observe that in the numerator of (12) does not appear explicitly the field v.

With this conjecture, we see immediately that the streamwise perturbations are always stable and the maximum is achieved on the spanwise perturbations.

In fact, from the Reynolds-Orr equation

$$\dot{V} = -(f'w, u) - \mathrm{Re}^{-1}[\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla w\|^2], \tag{29}$$

we have the inequality

$$\begin{aligned} \dot{V} &\le -\mathrm{Re}^{-1}\|\nabla v\|^2 - (f'w, u) - \mathrm{Re}^{-1}[\|\nabla u\|^2 + \|\nabla w\|^2] \\ &\le -\mathrm{Re}^{-1}\|\nabla v\|^2 - (m - \mathrm{Re}^{-1})[\|\nabla u\|^2 + \|\nabla w\|^2], \end{aligned} \tag{30}$$

where

$$m = \max_{\mathcal{S}_0} \frac{-(f'w, u)}{\|\nabla u\|^2 + \|\nabla w\|^2}. \tag{31}$$

The Euler-Lagrange equations of this maximum problem are given by

$$- f'w\mathbf{i} - f'u\mathbf{k} + 2m\Delta\mathbf{u} = \nabla\lambda, \tag{32}$$

where λ is a Lagrange multiplier, $\nabla\lambda = (\lambda_x, 0, \lambda_z)^T$.

We take the third component of the double curl of (32) and the third component of the curl of (32). We obtain the system of the Euler-Lagrange equations written in terms of ζ and w:

$$\begin{cases} f'(\zeta_y + 2w_{xz}) + f''w_x + 2m\Delta\Delta w = 0 \\ f'w_y + 2m\Delta\zeta = 0, \end{cases} \tag{33}$$

with the boundary conditions (20), (19), (18), where now $\zeta = -u_y$.

We take the second component of the double curl of (32) to get

$$f'\zeta_z + f''\zeta - f'w_{xy} = 0. \tag{34}$$

From this equation and $(33)_2$ we have that ζ and all its derivative with respect to z are zero on the boundaries. Let's prove this in a particular case: Couette between rigid planes and for FF boundary conditions. The proof in the case of the other laminar flows and different boundary conditions is done in a similar way.

In the case of RR Couette, from (33) and (34), we have:

$$\begin{cases} \zeta_y + 2w_{xz} + 2m\Delta\Delta w = 0 \\ w_y + 2m\Delta\zeta = 0 \\ \zeta_z - w_{xy} = 0. \end{cases} \tag{35}$$

On the boundaries $z = \pm 1$ we have $\zeta = 0$, from $(35)_3$ evaluated on $z = \pm 1$, we have $\zeta' = 0$ ($\zeta' = \dfrac{d\zeta}{dz}$). From $(35)_2$, evaluated on $z = \pm 1$, we have $\zeta'' = 0$. Now if we differentiate $(35)_2$ with respect to z and evaluate the result on the boundaries we have $\zeta''' = 0$. From this, if we differentiate twice $(35)_3$ with respect to z we have that the second derivative of w with respect to z is zero on the boundaries. And so, from $(35)_2$ differentiated twice with respect to z we have $\zeta'''' = 0$, ad so on.

In the case of FF Couette we have $\zeta' = 0$ on the boundaries. From $\zeta = -u_y$ and the solenoidality of velocity field, we have $\zeta_x = -u_{xy} = w_{zy}$. Moreover, by taking the Laplacian of $(35)_3$ we have $\Delta\zeta_z = \Delta w_{xy}$. Evaluating this on $z = \pm 1$, we have $\Delta\zeta_z = 0$ on the boundaries. From $(35)_2$ we now have $w_{yz} = \zeta_x = 0$. This implies that $\zeta = 0$ on the boundaries. $(35)_2$, evaluated on $z = \pm 1$, implies that $\zeta'' = 0$ on $z = \pm 1$. From this and $\Delta\zeta_z = 0$ on the boundaries, we also have $\zeta''' = 0$ on the boundaries. Now we take the Laplacian of $(35)_2$ and we get also $\zeta'''' = 0$ on $z = \pm 1$. And so on.

This implies that $\zeta \equiv 0$, hence $u_y = 0$ and from (33)$_2$ also $w_y = 0$. Therefore, $u = u(x, z)$, $v = 0$, $w = w(x, z)$ and the less stabilizing perturbations which satisfy the equation

$$2f' w_{xz} + f'' w_x + 2m\Delta\Delta w = 0, \tag{36}$$

with boundary conditions conditions (20), (19), (18), are the spanwise perturbations, as Orr [4] had supposed for RR Couette case. Moreover, if we consider streamwise perturbations, from the equation

$$m\|\Delta w\|^2 = 0,$$

we immediately find $m = 0$, i.e. $\mathrm{Re}^y = +\infty$. We report these results in Figs. 9, 10,11 (similar graphs can be done in the case FR), where the critical Reynolds number versus wave numbers for spanwise perturbations are shown.

4 Conclusion

We study linear instability and nonlinear stability in the L_2-energy norm for laminar Couette, Poiseuille and parabolic flows when at least one of the planes bounding the layer is stress-free. We also recall the classic case of rigid boundaries for Couette and Poiseuille flows.

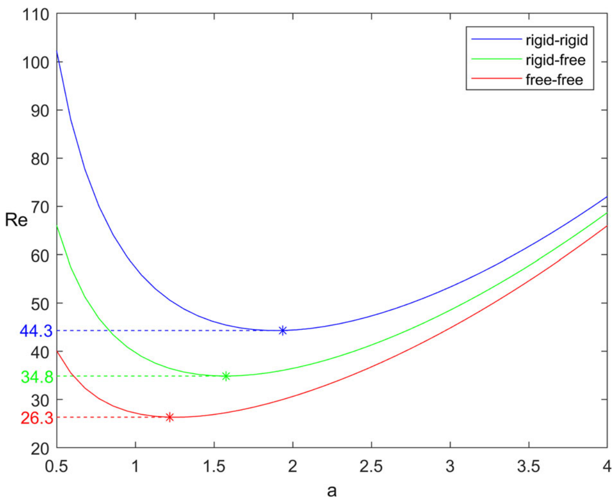

Fig. 9 The energy Orr-Reynolds number, for *Couette* shear flow, as function of the wavenumber a, for Eq. (36) with rigid-rigid, rigid-free, free-free boundary conditions (from the top to the bottom curve). For each curve, the minimum is highlighted by an asterisk and its numerical value is marked on the y-axis. Note that the minimum of the curve corresponding to the rigid-free boundary conditions is greater than the one corresponding to the free-free boundary conditions and lower than the one corresponding to the rigid-rigid boundary conditions

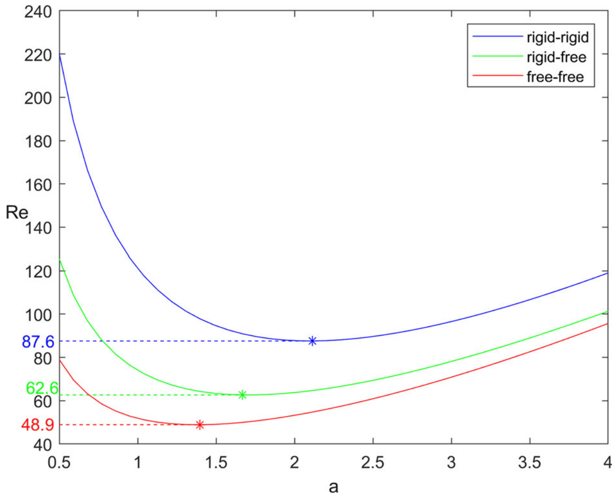

Fig. 10 The energy Orr-Reynolds number, for *Poiseuille* shear flow, as function of the wavenumber a, for Eq. (36) with rigid-rigid, rigid-free, free-free boundary conditions (from the top to the bottom curve). For each curve, the minimum is highlighted by an asterisk and its numerical value is marked on the y-axis. Note that the minimum of the curve corresponding to the rigid-free boundary conditions is greater than the one corresponding to the free-free boundary conditions and lower than the one corresponding to the rigid-rigid boundary conditions

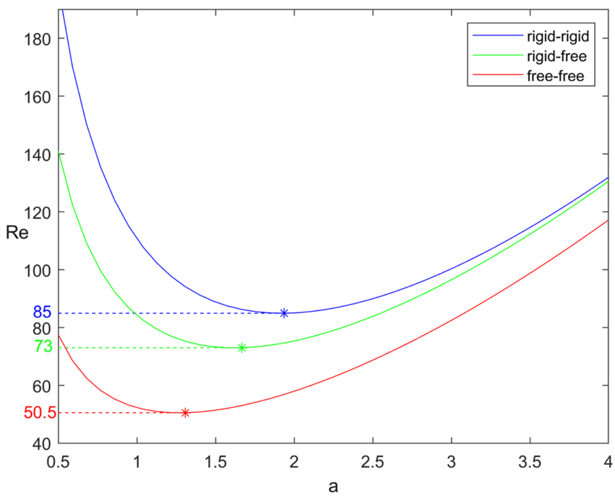

Fig. 11 The energy Orr-Reynolds number, for *parabolic shear flow*, as function of the wavenumber a, for Eq. (36) with rigid-rigid, rigid-free, free-free boundary conditions (from the top to the bottom curve). For each curve, the minimum is highlighted by an asterisk and its numerical value is marked on the y-axis. Note that the minimum of the curve corresponding to the rigid-free boundary conditions is greater than the one corresponding to the free-free boundary conditions and lower than the one corresponding to the rigid-rigid boundary conditions

We observe that the classical Couette and Poiseuille basic motions are obtained with RR boundary conditions, however it is possible to study the stability also with respect to stress-free perturbations. In fact, as can be verified, such basic motions can be obtained when one of the two planes has an assigned tangential stress (stress-free perturbations), for example in the case of the Couette motion $U(z) = z$, the value of $U(z)$ can be assigned in the lower plane - rigid plane - and its first derivative in the upper plane, or both have a fixed tangential stress (in this case, for the purpose of uniqueness of the motion, an assigned average condition must be requested).

By using the energy norm, we prove nonlinear stability conditions with respect to streamwise perturbations for any Reynolds number, and for any boundary condition RR, RF and FF i.e. we have $\text{Re}^y = +\infty$. On the other hand, the numerical calculations made with the Chebyshev collocation method give critical Reynolds numbers $\text{Re}^y < +\infty$ for non-linear energy stability on streamwise perturbations. This, as seen in the work of Falsaperla et al. [16], gives rise to a contradiction. We make a conjecture to overcome this contradiction: we introduce a space of physical perturbations competing for the maximum problem and we prove that the least stabilizing perturbations are two-dimensional (spanwise perturbations), for any boundary conditions RR, RF, FF.

We note that the critical Reynolds numbers we obtain for each basic flow Couette, Poiseuille and parabolic, show the stabilizing effect of the *no slip* boundaries as Figs. 9, 10 and 11 also indicate. This happens also in the Bénard problem, see Chandrasekhar [33] for a fluid in a layer between rigid or stress-free boundaries. Moreover, we observe that the least critical Reynolds numbers (on the spanwise perturbations) are given for Couette case (for any boundary conditions).

We observe that in Table 1, in the FF case, we left two empty lines because our calculations, for streamwise perturbations, give some problems of convergence and numerical instability for $b \to 0$.

Some open open problems are:

- To prove the conjecture;
- To consider laminar flows with Navier boundary conditions;
- The study the same problem in magnetohydrodynamics;
- To study the same problem for a mixture of immiscible fluids;
- To consider the stability of laminar flows for reological fluids (non-Newtonian fluids);
- To better understand the transition growth mechanism [27, 34].

Acknowledgements The research that led to the present paper was partially supported by the following Grants: 2017YBKNCE of national project PRIN of Italian Ministry for University and Research, PTR DMI-53722122113 and "ASDeA" of the University of Catania. The authors acknowledge also support from the project PONSCN 00451 CLARA-CLoud plAtform and smart underground imaging for natural Risk Assessment, SmartCities and Communities and Social Innovation. We also thank the group GNFM of INdAM for financial support.

Funding Open access funding provided by Università degli Studi di Catania within the CRUI-CARE Agreement.

Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Couette, M.: Études sur le frottement des liquides. Ann. Chim. Phys. **6**(21), 433–510 (1890)
2. Poiseuille, J.L.M.: Experimentelle Untersuchungen ber die Bewegung der Flssigkeiten in Rhren von sehr kleinen Durchmessern. Annalen der Physik **134**(3), 424–448 (1843)
3. Reynolds, O.: An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. Proc. R. Soc. Lond. **35**, 84–99 (1883)
4. Orr, W. M'F.: The stability or instability of the steady motions of a perfect liquid and of a viscous liquid. Proc. Roy. Irish Acad. A **27** 9–68 and 69–138 (1907)
5. Sommerfeld, A.: Ein Beitrg zur hydrodynamischen Erklaerung der turbulenten Fluessigkeitsbewegungen. In: Prooceedings 4th International Congress of Mathematicians, Rome, vol III, pp. 116–124 (1908)
6. Squire, H.B.: On the stability of three-dimensional disturbances of viscous flow between parallel walls. Proc. Roy. Soc. A **142**, 621–628 (1933)
7. Joseph, D.D.: Stability of Fluid Motions I. Springer-Verlag, Berlin Heidelberg New York (1976)
8. Busse, F.H.: A Property of the energy stability limit for plane parallel shear flow. Arch. Rat. Mech. Anal. **47**(1), 28–35 (1972)
9. Romanov, V.: Stability of plane-parallel Couette flow. Funct. Anal. Appl. **7**, 137–146 (1973)
10. Orszag, S.A.: Accurate solution of the Orr-Sommerfeld stability equation. J. Fluid Mech. **50**, 689–703 (1971)
11. Barkley, D., Tuckerman, L.S.: Stability analysis of perturbed plane Couette flow. Phys. Fluids **11**, 1187–1195 (1999)
12. Barkley, D., Tuckerman, L.S.: Computational study of turbulent laminar patterns in Couette flow. Phys. Rev. Letters **94**, 014502-1-01450–24 (2005)
13. Barkley, D., Tuckerman, L.S.: Mean flow of turbulent-laminar patterns in plane Couette flow. J. Fluid Mech. **576**, 109–137 (2007)
14. Prigent, A., Grégoire, G., Chat, H., Dauchot, O.: Long-wavelength modulation of turbulent shear flows. Physica D **174**, 100–113 (2003)
15. Falsaperla, P., Giacobbe, A., Mulone, G.: Nonlinear stability results for plane Couette and Poiseuille flows. Physi. Rev. E **100**(1), 013113 (2019). https://doi.org/10.1103/PhysRevE.100.013113
16. Falsaperla, P., Mulone, G., Perrone, C.: Stability of Hartmann shear flows in an open inclined channel. Nonlinear Anal. Real World Appl. **64**, 103446 (2022). https://doi.org/10.1016/j.nonrwa.2021.103446
17. Falsaperla, P., Mulone, G., Perrone, C.: Energy stability of plane Couette and Poiseuille flows: A conjecture. European J. Mech. / B Fluids **93**, 93–100 (2022). https://doi.org/10.1016/j.euromechflu.2022.01.006
18. Joseph, D.D.: Eigenvalue bounds for the Orr-Sommerfeld equation. J. Fluid Mech. **33**(part 3), 617–621 (1966)
19. Joseph, D.D., Carmi, S.: Stability of Poiseuille flow in pipes, annuli and channels. Quart. App. Math. **26**, 575–579 (1969)
20. Falsaperla, P., Giacobbe, A., Mulone, G.: Stability of laminar flows in an inclined open channel. Ricerche di Matematica **70**, 67–79 (2021). https://doi.org/10.1007/s11587-020-00487-8
21. Tan, H.: Applying the free-slip boundary condition with an adaptive Cartesian cut-cell method for complex geometries. Num. Heat Trans. Part B: Fundamentals **74**(4), 661–684 (2018)

22. Cerquaglia, M.L., DeliÃge, G., Boman, R., Terrapon, V., Ponthot, J.-P.: Free-slip boundary conditions for simulating free-surface incompressible flows through the particle finite element method. Int. J. Numer. Meth. Engng **110**, 921–946 (2017)
23. Rao, I.J., Rajagopal, K.R.: The effect of the slip boundary condition on the flow of fluids in a channel. Acta Mech. **135**, 113–126 (1999)
24. Khaled, A.-R.A., Vafai, K.: The effect of the slip condition on Stokes and Couette flows due to an oscillating wall: exact solutions. Int. J. Non-Linear Mech. **39**, 795–809 (2004)
25. Straughan, B.: The Energy Method, Stability and Nonlinear Convection. Applied Mathematical Sciences, vol. 91, 2nd edn. Springer-Verlag, New York (2004)
26. Rionero, S., Mulone, G.: On the non-linear stability of parallel shear flows. Continuum Mech. Thermodyn. **3**, 1–11 (1991)
27. Reddy, S.C., Schmid, P.J., Baggett, J.S., Henningson, D.S.: On stability of streamwise streaks and transition thresholds in plane channel flows. J. Fluid Mech. **365**, 269–303 (1998)
28. Moffatt, K.: Fixed points of turbulent dynamical systems and suppression of nonlinearity. In: Lumley, J. (ed.) Whither Turbulence, vol. 250. Springer, Berlin (1990)
29. Drazin, P.G., Reid, W.H.: Hydrodynamic Stability. Cambridge Monographs on Mechanics, 2nd edn. Cambridge University Press, Cambridge (2004)
30. Kaiser, R., Mulone, G.: A note on nonlinear stability of plane parallel shear flows. J. Math. Anal. Appl. **302**, 543–556 (2005)
31. Kaiser, R., Tilgner, A., vonWahl, W.: A generalized energy functional for plane Couette flow. SIAM J. Math. Anal. **37**(2), 438–454 (2005)
32. Kaiser, R., Xu, L.: Nonlinear stability of the rotating Bénard problem, the case $Pr = 1$. NoDEA Nonlinear Differ. Equ. Appl. **5**, 283–307 (1998)
33. Chandrasekhar, S.: Hydrodynamic and Hydromagnetic Stability. Oxford University Press, Oxford (1961)
34. Brandt, L.: The lift-up effect: The linear mechanism behind transition and turbulence in shear flows. Eur. J. Mech. B/Fluids **47**, 80–96 (2014)