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H^1 solutions for a Kuramoto–Sinelshchikov–Cahn–Hilliard type equation

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Abstract
The Kuramoto–Sinelshchikov–Cahn–Hilliard equation models the spinodal decomposition of phase separating systems in an external field, the spatiotemporal evolution of the morphology of steps on crystal surfaces and the growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension. In this paper, we prove the well-posedness of the Cauchy problem, associated with this equation.

Keywords Existence · Uniqueness · Stability · Kuramoto–Sinelshchikov–Cahn–Hilliard type equation · Cauchy problem

Mathematics Subject Classification 35G25 · 35K55

1 Introduction

In this study, we investigate the well-posedness od the classical solution of the following Cauchy problem:

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$$\begin{cases} \partial_t u + \nu \partial_x u + \kappa \partial_x u^2 + q \partial_x u^3 + r \partial_x u^4 + h \partial_x u^5 + m \partial_x u^6 \\ \quad + \alpha \partial_x^3 u + \beta^2 \partial_x^4 u + \gamma \partial_x^2 u + \tau \partial_x^2 (u^2) - \delta^2 \partial_x^2 (u^3) = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with $a, \kappa, q, r, h, m, \alpha, \beta, \gamma, \tau, \delta \in \mathbb{R}$, with $\beta, \tau, \delta \neq 0$. On the initial datum, we assume

$$u_0 \in H^1(\mathbb{R}). \tag{1.2}$$

(1.1) occurs in many branches of mechanics and physics. For example, taking $\nu = q = h = m = \alpha = 0$, (1.1) reads

$$\partial_t u + \kappa \partial_x u^2 + r \partial_x u^4 + \beta^2 \partial_x^4 u + \gamma \partial_x^2 u + \tau \partial_x^2 (u^2) - \delta^2 \partial_x^2 (u^3) = 0, \tag{1.3}$$

which is known as the convective Cahn–Hilliard equation. It models the spinodal decomposition of phase separating systems in an external field [35,62,84], the spatiotemporal evolution of the morphology of steps on crystal surfaces [73], and the growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension [39–41,43,66], where the constant κ, r are the driving forces.

For instance, in the case of a growing crystal surface with strongly anisotropic surface tension, the function u represents is the surface slope, while the constants κ, r are the growth driving force proportional to the difference between the bulk chemical potentials of the solid and fluid phases.

(1.3) was also obtained by Watson [81] as a small-slope approximation of the crystal-growth model obtained in [32].

From a mathematical point of view, the coarsening dynamics for (1.3) has been studied in the limit $0 < \kappa \ll 1$, $r = \tau = 0$ in [35,41] and analytically in [82]. In [1], a numerical scheme is studied for (1.3), while the existence of the periodic solution are analyzed in [33,53]. in [62,69], the existence of exact solutions for (1.3) and the its viscous form have been investigated. In [27], the authors proved the well-posedness of the classical solution of (1.1), under assumption

$$u_0 \in H^\ell(\mathbb{R}), \quad \ell \in \{2, 3, 4\}. \tag{1.4}$$

Moreover, [41] shows that, taking $r = \tau = 0$, when κ tends to ∞, (1.3) reduces to the Kuramoto–Sivashinsky equation (see (1.9)). Physically, it means that, with the growth of the driving force, there must be a transition from the coarsening dynamics to a chaotic spatiotemporal behavior.

Taking $\kappa = r = \tau = 0$ in (1.3), we obtain that

$$\partial_t u + \beta^2 \partial_x^4 u + \gamma \partial_x^2 u - \delta^2 \partial_x^2 (u^3) = 0, \tag{1.5}$$

which is known as the Cahn–Hilliard equation. It describes the spinodal decomposition in phase-separating systems [10,11]. It also describes the coarsening dynamics of the faceting of thermodynamically unstable surfaces [46,77]. Krekhov [50] shows that (1.5) can be an effective tool in technological applications to design nanostructured materials.

From a mathematical point of view, in [2], the the existence of some extremely slowly evolving solutions for (1.5) is proven, considering an boundary domain, while, in [8,37], the problem of a global attractor is studied, In [27], the well-posedness of the classical solution of under Assumption (1.4) is proven. In [42,85], numerical schemes for (1.5) are analyzed, while, in [80], an approximate analytical solution is studied.

Observe that (1.5) is has been much studied, as shown in the papers [9,34,86] and their references.

Taking $\nu = q = r = h = m = \beta = \gamma = \tau = \delta = 0$ in (1.1), we obtain the following equation:

$$\partial_t u + \kappa \partial_x u^2 + \alpha \partial_x^3 u = 0, \tag{1.6}$$

which is known as the Korteweg–de Vries equation [49]. It has a very wide range of applications, such as magnetic fluid waves, ion sound waves, and longitudinal astigmatic waves.

From a mathematical point of view, in [15,17,47], the Cauchy problem for (1.6) is studied, while in [51], the author reviewed the travelling wave solutions for (1.6). Moreover, in [18,61,74], the convergence of the solution of (1.6) to the unique entropy one of the Burgers equation is proven.

Taking $\nu = \kappa = r = h = m = \beta = \gamma = \tau = \delta = 0$, (1.1) becomes

$$\partial_t u + q \partial_x u^3 + \alpha \partial_x^3 u = 0, \tag{1.7}$$

which is known as the modified Korteweg–de Vries equation.

[3,4,19,58–60] show that (1.7) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. In [15,47], the Cauchy problem for (1.7) is studied, while, in [20,74], the convergence of the solution of (1.7) to the unique entropy solution of the following scalar conservation law

$$\partial_t u + q \partial_x u^3 = 0. \tag{1.8}$$

Assuming $\nu = q = r = h = m = \tau = \delta = 0$, (1.1) reads

$$\partial_t u + \kappa \partial_x u^2 + \alpha \partial_x^3 u + \beta^2 \partial_x^4 u + \gamma \partial_x^2 u = 0. \tag{1.9}$$

(1.9) arises in interesting physical situations, for example as a model for long waves on a viscous fluid owing down an inclined plane [79] and to derive drift waves in a plasma [31]. (1.9) was derived also independently by Kuramoto [54–56] as a model for phase turbulence in reaction–diffusion systems and by Sivashinsky [76] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

(1.9) also describes incipient instabilities in a variety of physical and chemical systems [13,44,57]. Moreover, (1.9), which is also known as the Benney–Lin equation [6,63], was derived by Kuramoto in the study of phase turbulence in the Belousov–Zhabotinsky reaction [64].

The dynamical properties and the existence of exact solutions for (1.9) have been investigated in [36,48,52,70,71,83]. In [5,12,38], the control problem for (1.9) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [14], the problem of global exponential stabilization of (1.9) with periodic boundary conditions is analyzed. In [45], it is proposed a generalization of optimal control theory for (1.9), while in [65] the problem of global boundary control of (1.9) is considered. In [72], the existence of solitonic solutions for (1.9) is proven. In [7,21,78], the well-posedness of the Cauchy problem for (1.9) is proven, using the energy space technique, a priori estimates together with an application of the Cauchy–Kovalevskaya and the fixed point method, respectively. In particular, in [21], the well-posedness of (1.1) is proven assuming

$$u_0 \in H^2(\mathbb{R}). \tag{1.10}$$

In [28,67,68], the initial-boundary value problem for (1.9) is studied, using a priori estimates together with an application of the Cauchy–Kovalevskaya and the energy space technique, respectively.

Finally, following [22,61,74], in [23], the convergence of the solution of (1.9) to the unique entropy one of the Burgers equation is proven.

The main result of this paper is the following theorem.

Theorem 1.1 *Fix $T > 0$. Assume (1.2). There exists a unique solution u of (1.1), such that*

$$u \in L^{\infty}(0, T; H^1(\mathbb{R})) \cap L^4(0, T; W^{1,4}(\mathbb{R})). \tag{1.11}$$

Moreover, if u_1 and u_2 are two solutions of (1.1), we have that

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{C(T)t} \left\|u_{1,0} - u_{2,0}\right\|_{L^2(\mathbb{R})}, \tag{1.12}$$

for some suitable $C(T) > 0$, and every $0 \le t \le T$.

Observe that Theorem 1.1 holds also when $\tau = \delta = 0$, which corresponds the Kuramoto–Sivashinsky equation. Moreover, even if the equation is of the fourth order, the proof of Theorem 1.1 is based on the Aubin–Lions Lemma due to the functional setting (see [26,29,30,75]).

The paper is organized as follows. In Sect. 2, we prove several a priori estimates on a vanishing viscosity approximation of (1.1). Those play a key role in the proof of our main result, that is given in Sect. 3.

2 Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following problem [24,25]:

$$\begin{cases} \partial_t u_\varepsilon + \nu\partial_x u_\varepsilon + \kappa\partial_x u_\varepsilon^2 + q\partial_x u_\varepsilon^3 + r\partial_x u_\varepsilon^4 + h\partial_x u_\varepsilon^5 & \\ \quad +m\partial_x u_\varepsilon^6 + \alpha\partial_x^3 u_\varepsilon + \beta^2\partial_x^4 u_\varepsilon + \gamma\partial_x^2 u_\varepsilon & \\ \quad +\tau\partial_x^2(u_\varepsilon^2) - \delta^2\partial_x^2(u_\varepsilon^3) = \varepsilon\partial_x^6 u, & t > 0, \quad x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\left\|u_{\varepsilon,0}\right\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \quad \sqrt{\varepsilon}\left\|u_{\varepsilon,0}\right\|_{L^2(\mathbb{R})} \leq C_0, \tag{2.2}$$

where C_0 is a positive constant, independent on ε.

Let us prove some a priori estimates on u_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

We begin by proving the following result.

Lemma 2.1 *Fix $T > 0$. We have that*

$$\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&+4\delta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\\
&+2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T).
\end{aligned} \tag{2.3}$$

for every $0 \leq t \leq T$. In particular, we have that

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{2.4}$$

for every $0 \leq t \leq T$.

Proof Let $0 \leq t \leq T$. We begin by observing that

$$\partial_x u_\varepsilon^2 = 2u_\varepsilon \partial_x u, \quad \partial_x u_\varepsilon^3 = 3u_\varepsilon^2 \partial_x u. \tag{2.5}$$

Multiplying (2.1) by $2u_\varepsilon$, thanks to (2.5), an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon dx\\
&= -2\nu\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon dx - 4\kappa\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon dx - 6q\int_{\mathbb{R}} u_\varepsilon^3\partial_x u_\varepsilon dx\\
&\quad - 8r\int_{\mathbb{R}} u_\varepsilon^4\partial_x u_\varepsilon dx - 10h\int_{\mathbb{R}} u_\varepsilon^5\partial_x u_\varepsilon dx - 12m\int_{\mathbb{R}} u_\varepsilon^6\partial_x u_\varepsilon dx\\
&\quad - 2\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon dx - 2\beta^2\int_{\mathbb{R}} u_\varepsilon\partial_x^4 u_\varepsilon dx - 2\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\quad - 2\tau\int_{\mathbb{R}} u_\varepsilon\partial_x^2(u_\varepsilon)^2 dx + 2\delta^2\int_{\mathbb{R}} u_\varepsilon\partial_x^2(u_\varepsilon^3) dx + 2\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^6 u_\varepsilon dx\\
&= 2\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 2\beta^2\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx - 2\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\quad + 4\tau\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx - 6\delta^2\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x^5 u_\varepsilon dx\\
&= -2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon dx + 4\tau\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx
\end{aligned}$$

$$
\begin{aligned}
&- 6\delta^2 \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx \\
&= -2\beta^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx + 4\tau \int_{\mathbb{R}} u_\varepsilon (\partial_x u_\varepsilon)^2 dx \\
&- 6\delta^2 \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\frac{d}{dt} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + 6\delta^2 \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= -2\gamma \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx + 4\tau \int_{\mathbb{R}} u_\varepsilon (\partial_x u_\varepsilon)^2 dx.
\end{aligned} \tag{2.6}
$$

Due to the Young inequality,

$$
\begin{aligned}
2|\gamma| \int_{\mathbb{R}} |u_\varepsilon||\partial_x^2 u_\varepsilon| dx &= \int_{\mathbb{R}} \left|\frac{2\gamma u_\varepsilon}{\beta}\right| \left|\beta \partial_x^2 u_\varepsilon\right| dx \\
&\le \frac{2\gamma^2}{\beta^2} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\
4|\tau| \int_{\mathbb{R}} |u_\varepsilon|(\partial_x u_\varepsilon)^2 dx &= 4 \int_{\mathbb{R}} |\delta u_\varepsilon \partial_x u_\varepsilon| \left|\frac{\tau \partial_x u_\varepsilon}{\delta}\right| dx \\
&\le 2\delta^2 \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{2\tau^2}{\delta^2} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently, by (2.6), we have that

$$
\begin{aligned}
&\frac{d}{dt} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{3\beta^2}{2} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad + 4\delta^2 \|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le \frac{2\gamma^2}{\beta^2} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{2\tau^2}{\delta^2} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.7}
$$

Observe that

$$
\frac{2\tau^2}{\delta^2} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} = \frac{2\tau^2}{\delta^2} \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x u_\varepsilon dx = -\frac{2\tau^2}{\delta^2} \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx.
$$

Therefore, by the Young inequality,

$$\begin{aligned}\frac{2\tau^2}{\delta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &\le \int_{\mathbb{R}}\left|\frac{2\tau^2 u_\varepsilon}{\beta\delta^2}\right|\left|\beta\partial_x^2 u_\varepsilon\right| dx\\ &\le \frac{2\tau^4}{\beta^2\delta^4}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}\tag{2.8}$$

It follows from (2.7) that

$$\begin{aligned}&\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+4\delta^2\|u_\varepsilon(t,\cdot)\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

By (2.2) and the the Gronwall Lemma, we get

$$\begin{aligned}&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\quad+4\delta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\|u_\varepsilon(s,\cdot)\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &\quad+2\varepsilon e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C_0 e^{C_0 t}\le C(T),\end{aligned}$$

which gives (2.3).

Finally, we prove (2.4). Thanks to (2.3) and (2.8), we have that

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C(T)+C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,T)$, thanks to (2.3), we have that

$$\int_0^t\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\le C(T)t+C_0\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T),$$

that is (2.4). □

Lemma 2.2 *Fix $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T).\tag{2.9}$$

In particular, we have that

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T),\tag{2.10}$$

for every $0 \le t \le T$. *Moreover,*

$$\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \le C(T), \tag{2.11}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. Multiplying (2.1) by $-2\partial_x^2 u_\varepsilon$, thanks to (2.5), an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= -2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx\\
&= 2\nu \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 dx + 4\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 6q \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&\quad + 8r \int_{\mathbb{R}} u_\varepsilon^3 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 10h \int_{\mathbb{R}} u_\varepsilon^4 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&\quad + 12m \int_{\mathbb{R}} u_\varepsilon^5 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2\alpha \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx\\
&\quad + 2\beta^2 \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\gamma \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + 2\tau \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^2 (u_\varepsilon)^2 dx - 2\delta^2 \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^2 (u_\varepsilon^3) dx - 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon dx\\
&= 4\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 6q \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 8r \int_{\mathbb{R}} u_\varepsilon^3 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&\quad + 10h \int_{\mathbb{R}} u_\varepsilon^4 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 12m \int_{\mathbb{R}} u_\varepsilon^5 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&\quad - 2\beta^2 \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad - 4\tau \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx - 6\delta^2 \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx\\
&\quad + 2\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon dx\\
&= 4\kappa \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 6q \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 8r \int_{\mathbb{R}} u_\varepsilon^3 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&\quad + 10h \int_{\mathbb{R}} u_\varepsilon^4 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 12m \int_{\mathbb{R}} u_\varepsilon^5 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&\quad - 2\beta^2 \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad - 4\tau \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx - 6\delta^2 \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx\\
&\quad - 2\varepsilon \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore, we have that

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = 4\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 6q\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 8r\int_{\mathbb{R}} u_\varepsilon^3\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx\\
&\qquad + 10h\int_{\mathbb{R}} u_\varepsilon^4\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 12m\int_{\mathbb{R}} u_\varepsilon^5\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 2\gamma\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad - 4\tau\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx - 6\delta^2\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^3 u_\varepsilon dx.
\end{aligned}
\tag{2.12}
$$

Due to the Young inequality,

$$
\begin{aligned}
&4|\kappa|\int_{\mathbb{R}} |u_\varepsilon\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \le 2\kappa^2\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx + 2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le \kappa^2\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + \kappa^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&6|q|\int_{\mathbb{R}} |u_\varepsilon^2\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \le 3q^2\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + 3\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&8|r|\int_{\mathbb{R}} |u_\varepsilon|^3|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \le 8\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |ru_\varepsilon^2\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx\\
&\quad \le 4r^2\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + 4\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&10|h|\int_{\mathbb{R}} u_\varepsilon^4|\partial_x u_\varepsilon|||\partial_x^2 u_\varepsilon| dx \le 10\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |hu_\varepsilon^2\partial_x||\partial_x^2 u_\varepsilon| dx\\
&\quad \le 5h^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + 5\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&12|m|\int_{\mathbb{R}} |u_\varepsilon|^5|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \le 12\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |mu_\varepsilon^2\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx\\
&\quad \le 6m^2\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + 6\|\partial_x u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&4|\tau|\int_{\mathbb{R}} |u_\varepsilon\partial_x u_\varepsilon||\partial_x^3 u_\varepsilon| dx = \int_{\mathbb{R}}\left|\frac{4\tau u_\varepsilon\partial_x u_\varepsilon}{\beta}\right|\left|\beta\partial_x^3 u_\varepsilon\right| dx\\
&\quad \le \frac{8\tau^2}{\beta^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx + \frac{\beta^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le \frac{4\tau^2}{\beta^2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + \frac{4\tau^2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&6\delta^2\int_{\mathbb{R}} |u_\varepsilon^2\partial_x u_\varepsilon||\partial_x^3 u_\varepsilon| dx = \int_{\mathbb{R}}\left|\frac{6\delta^2 u_\varepsilon^2\partial_x u_\varepsilon}{\beta}\right|\left|\beta\partial_x^3 u_\varepsilon\right| dx\\
&\quad \le \frac{18\delta^4}{\beta^2}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx + \frac{\beta^2}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.12) that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx\\
&\qquad+C_0\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx+C_0\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+C_0\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+C_0\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \qquad (2.13)
\end{aligned}$$

[15, Lemma 2.6] says that

$$\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx\le 4\left\|u_\varepsilon(t,\cdot)\right\|^4_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \qquad (2.14)$$

Consequently, by (2.3) and (2.14),

$$\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dx\le C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \qquad (2.15)$$

Therefore, by (2.13) and (2.15), we have that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+C(T)\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C_0\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

(2.2), (2.3), (2.4) and an integration on $(0,t)$ give

$$\begin{aligned}
&\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds+2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le C_0+C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\right)\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\qquad+C(T)\left(\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right)\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\qquad+C_0\int_0^t\left\|\partial_x u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}+\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right). \qquad (2.16)
\end{aligned}$$

Due to the Young inequality,

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} &\le \frac{1}{2}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{2},\\ \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})} &\le \frac{D_1}{2}\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} + \frac{1}{2D_1}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})},\end{aligned}$$

where D_1 is a positive constant, which will be specified later. It follows from (2.16) that

$$\begin{aligned}&\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\le C(T)\left(1+\left(1+\frac{1}{D_1}\right)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + D_1\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\right).\end{aligned} \tag{2.17}$$

We prove (2.9). Thanks to (2.3), (2.17) and the Hölder inequality,

$$\begin{aligned}u_\varepsilon^2(t,x) &= 2\int_{-\infty}^x u_\varepsilon\partial_x u_\varepsilon dy \le 2\int_{\mathbb{R}} |u_\varepsilon||\partial_x u_\varepsilon| dx \le 2\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &\le C(T)\sqrt{\left(1+\left(1+\frac{1}{D_1}\right)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + D_1\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\right)}.\end{aligned}$$

Hence,

$$(1 - C(T)D_1)\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\left(1+\frac{1}{D_1}\right)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0.$$

Taking

$$D_1 = \frac{1}{2C(T)}, \tag{2.18}$$

we have that

$$\frac{1}{2}\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,$$

which gives (2.9).

(2.10) follows from (2.9) and (2.18).

Finally, we prove (2.11). We begin by observing that [16, Lemma 2.3] says that

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \le 6\left(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (2.3) and (2.10),

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \le C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (2.3), we have (2.11). □

Lemma 2.3 *Fix $T>0$. There exists a constant $C(T)>0$, independent on ε, such that*

$$\begin{aligned}\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\beta^2\varepsilon\int_0^t\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &+\varepsilon^2\int_0^t\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le C(T),\end{aligned} \tag{2.19}$$

for every $0\le t\le T$.

Proof Let $0\le t\le T$. We begin by defining the following equation:

$$f'(u_\varepsilon) := 2\kappa u_\varepsilon + 3qu_\varepsilon^2 + 4ru_\varepsilon^3 + 5hu_\varepsilon^4 + 6mu_\varepsilon^5. \tag{2.20}$$

Thanks to (2.20), (2.1) reads

$$\begin{aligned}&\partial_t u_\varepsilon + \nu\partial_x u_\varepsilon + f'(u_\varepsilon)\partial_x u_\varepsilon + \alpha\partial_x^3 u_\varepsilon + \beta^2\partial_x^4 u_\varepsilon\\ &\quad+\gamma\partial_x^2 u_\varepsilon + \tau\partial_x^2(u_\varepsilon^2) - \delta^2\partial_x^2(u_\varepsilon^3) = \varepsilon\partial_x^6 u\end{aligned} \tag{2.21}$$

Multiplying (2.21) by $2\varepsilon\partial_x^4 u_\varepsilon$, thanks to (2.5), an integration on $\mathbb{R}$ gives

$$\begin{aligned}\varepsilon\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t u_\varepsilon dx\\ &= -2\varepsilon\nu\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx\\ &\quad - 2\alpha\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx - 2\beta^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad - 2\gamma\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon dx - 2\tau\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^2(u_\varepsilon^2)dx\\ &\quad + 2\delta^2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^2(u_\varepsilon^3)dx + 2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_x^6 u_\varepsilon dx\\ &= 2\varepsilon\nu\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon dx - 2\varepsilon\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon\partial_x^4 u_\varepsilon dx\\ &\quad - 2\beta^2\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\gamma\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\quad + 4\tau\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^5 u_\varepsilon dx - 6\delta^2\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^5 u_\varepsilon dx\\ &\quad - 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\end{aligned}$$

$$
\begin{aligned}
&= -2\varepsilon \int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\beta^2\varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + 2\gamma\varepsilon \left\| \partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 4\tau\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx\\
&\quad - 6\delta^2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx - 2\varepsilon \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\varepsilon \frac{d}{dt} \left\| \partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = -2\varepsilon \int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^4 u_\varepsilon dx + 2\gamma\varepsilon \left\| \partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad + 4\tau\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx + 6\delta^2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon dx.
\end{aligned} \tag{2.22}
$$

Since $0 < \varepsilon < 1$, thanks to (2.9), (2.10) and the Young inequality,

$$
\begin{aligned}
&2\varepsilon \int_{\mathbb{R}} |f'(u_\varepsilon)\partial_x u_\varepsilon||\partial_x^4 u_\varepsilon| dx \le \varepsilon \int_{\mathbb{R}} (f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 + \varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le \left\| f'\right\|^2_{L^\infty(-C(T),C(T))} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + \varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&4|\tau|\varepsilon \int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon||\partial_x^5 u_\varepsilon| dx \le 4|\tau|\varepsilon \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^5 u_\varepsilon| dx\\
&\quad \le C(T)\varepsilon \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^5 u_\varepsilon| dx \le \varepsilon C(T) \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2} \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + \frac{\varepsilon}{2} \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&6\delta^2\varepsilon \int_{\mathbb{R}} u_\varepsilon^2 |\partial_x u_\varepsilon||\partial_x^5 u_\varepsilon| dx \le 6\delta^2\varepsilon \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^5 u_\varepsilon| dx\\
&\quad \le C(T)\varepsilon \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^5 u_\varepsilon| dx \le \varepsilon C(T) \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2} \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + \frac{\varepsilon}{2} \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.22) that

$$
\begin{aligned}
&\varepsilon \frac{d}{dt} \left\| \partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\beta^2\varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon \left\| \partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T) + \varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2|\gamma|\varepsilon \left\| \partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0, t)$, by (2.2), (2.3) and (2.10), we get

$$\begin{aligned}\varepsilon \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &+ 2\beta^2\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \varepsilon\int_0^t \left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\le C_0 + \varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2|\gamma|\varepsilon\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T),\end{aligned}$$

which gives (2.19). □

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

We begin by proving the following lemma.

Lemma 3.1 *Fix $T > 0$. Then,*

$$\text{the sequence}\{u_\varepsilon\}_{\varepsilon>0} \text{ is compact in } L^2_{loc}((0,\infty)\times\mathbb{R}). \tag{3.1}$$

Consequently, there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u \in L^2_{loc}((0,\infty)\times\mathbb{R})$ such that, for each compact subset K of $(0,\infty)\times\mathbb{R})$,

$$u_{\varepsilon_k} \to u \text{ in } L^2(K) \text{ and a.e.} \tag{3.2}$$

Moreover, u is a solution of (1.1), satisfying (1.11).

Proof We begin by proving (3.1). To prove (3.1), we rely on the Aubin–Lions Lemma (see [22,29,30,75]). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second is continuous. Owing to the Aubin–Lions Lemma [75], to prove (3.1), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{loc}(\mathbb{R})), \tag{3.3}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{loc}(\mathbb{R})). \tag{3.4}$$

We prove (3.3). Thanks to Lemmas 2.1 and 2.2,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0}\text{is uniformly bounded in} L^\infty(0,T;H^1(\mathbb{R})),$$

which gives (3.3).

We prove (3.4). Observe that, by (2.1) and (2.5),

$$\begin{aligned}\partial_t u_\varepsilon = &-\partial_x \left(\nu u_\varepsilon + \alpha\partial_x^2 u_\varepsilon + \beta^2\partial_x^3 u_\varepsilon + \gamma\partial_x u_\varepsilon + 2\tau u_\varepsilon\partial_x u_\varepsilon - 3\delta^2 u_\varepsilon^2\partial_x u_\varepsilon - \varepsilon\partial_x^5 u_\varepsilon\right)\\ &- f'(u_\varepsilon)\partial_x u_\varepsilon,\end{aligned}$$

where $f'(u_\varepsilon)$ is defined in (2.20). Thanks to Lemmas 2.1, 2.2 and 2.3, we have that

$$\begin{aligned}&\nu^2\left\|u_\varepsilon^2\right\|_{L^2((0,T)\times\mathbb{R})},\ \alpha^2\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})},\ \beta^4\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^2(\mathbb{R})} \le C(T),\\ &\gamma^2\left\|\partial_x u_\varepsilon\right\|^2_{L^2(\mathbb{R})},\ \varepsilon^2\left\|\partial_x^5 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T).\end{aligned} \tag{3.5}$$

We claim that

$$\begin{aligned}&4\tau^2\int_0^T\!\!\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dtdx \le C(T),\\ &9\delta^4\int_0^T\!\!\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dtdx \le C(T).\end{aligned} \tag{3.6}$$

Thanks to (2.9) and (2.10),

$$\begin{aligned}&4\tau^2\int_0^T\!\!\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dtdx \le 4\tau^2\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} dt \le C(T),\\ &9\delta^4\int_0^T\!\!\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 dtdx \le 9\delta^4\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})}\int_0^T \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} dt \le C(T).\end{aligned}$$

Therefore, by (3.5) and (3.6), we have that

$$\begin{aligned}&\left\{\partial_x\left(\nu u_\varepsilon + \alpha\partial_x^2 u_\varepsilon + \beta^2\partial_x^3 u_\varepsilon + \gamma\partial_x u_\varepsilon + 2\tau u_\varepsilon\partial_x u_\varepsilon - 3\delta^2 u_\varepsilon^2\partial_x u_\varepsilon - \varepsilon\partial_x^5 u_\varepsilon\right)\right\}_{\varepsilon>0}\\ &\text{is bounded in } H^1((0,T)\times\mathbb{R}).\end{aligned} \tag{3.7}$$

We have that

$$\int_0^T\!\!\int_{\mathbb{R}} (f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 dtdx \le C(T). \tag{3.8}$$

Thanks to (2.9) and (2.10),

$$\int_0^T\!\!\int_{\mathbb{R}} (f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 dtdx \le \left\|f'\right\|^2_{L^\infty(-C(T),C(T))}\int_0^T \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} dt \le C(T).$$

Therefore, (3.4) follows from (3.7) and (3.8).

Thanks to the Aubin–Lions Lemma, (3.1) and (3.2) hold.

Consequently, u is solution of (1.1) and, thanks to Lemmas 2.1 and 2.2, (1.11) holds. □

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a solution of (1.1) such that (1.11) holds.

Let u_1 and u_2 two solutions of (1.1), which verify (1.11), that is

$$\begin{cases} \partial_t u_1 + \nu\partial_x u_1 + f'(u_1)\partial_x u_1 + \alpha\partial_x^3 u_1 + \beta^2\partial_x^4 u_1 \\ \quad +\gamma\partial_x^2 u_1 + \tau\partial_x^2(u_1^2) - \delta^2\partial_x^2(u_1^3) = 0, & t>0, \quad x\in\mathbb{R}, \\ u_1(0,x) = u_{1,0}(x), & x\in\mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t u_2 + \nu\partial_x u_2 + f'(u_2)\partial_x u_2 + \alpha\partial_x^3 u_2 + \beta^2\partial_x^4 u_2 \\ \quad +\gamma\partial_x^2 u_2 + \tau\partial_x^2(u_2)^2 - \delta^2\partial_x^2(u_2^3) = 0, & t>0, \quad x\in\mathbb{R}, \\ u_2(0,x) = u_{2,0}(x), & x\in\mathbb{R}, \end{cases}$$

where $f'(u)$ is defined in (2.20). Then, the function

$$\omega = u_1 - u_2 \tag{3.9}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t\omega + \nu\partial_x\omega + f'(u_1)\partial_x u_1 - f'(u_2)\partial_x u_2 + \alpha\partial_x^3\omega + \beta^2\partial_x^4\omega \\ \quad +\gamma\partial_x^2\omega + \tau\partial_x^2(u_1^2 - u_2^2) - \delta^2\partial_x^2(u_1^3 - u_2^3) = 0, & t>0, \quad x\in\mathbb{R}, \\ \omega(0,x) = u_{1,0}(x) - u_{2,0}(x), & x\in\mathbb{R}. \end{cases} \tag{3.10}$$

Observe that, thanks to (3.9),

$$\begin{aligned} f'(u_1)\partial_x u_1 - f'(u_2)\partial_x u_2 &= f'(u_1)\partial_x u_1 - f'(u_1)\partial_x u_2 + f'(u_1)\partial_x u_2 - f'(u_2)\partial_x u_2 \\ &= f'(u_1)\partial_x\omega + \left(f'(u_1) - f'(u_2)\right)\partial_x u_2. \end{aligned}$$

Therefore, (3.10) is equivalent to the following equation:

$$\begin{aligned} &\partial_t\omega + \nu\partial_x\omega + f'(u_1)\partial_x\omega + \left(f'(u_1) - f'(u_2)\right)\partial_x u_2 + \alpha\partial_x^3\omega \\ &\quad + \beta^2\partial_x^4\omega + \gamma\partial_x^2\omega + \tau\partial_x^2(u_1^2 - u_2) - \delta^2\partial_x^2(u_1^3 - u_2^3) = 0. \end{aligned} \tag{3.11}$$

Since $u_1,\ u_2 \in L^\infty((0,T);H^1)$, there exists a constant $C(T)$, such that

$$\begin{aligned} \|u_1\|_{L^\infty((0,T)\times\mathbb{R})},\ \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} &\le C(T), \\ \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_2(t,\cdot)\|_{L^2(\mathbb{R})} &\le C(T), \end{aligned} \tag{3.12}$$

for every $0\le t\le T$. Moreover, by (2.20), $f'\in C^1(\mathbb{R})$. Consequently, there exists ξ between u_1 and u_2, such that

$$f'(u_1) - f'(u_2) = f''(\xi)(u_1 - u_2) = f''(\xi)\omega, \quad u_1 < \xi < u_2 \quad \text{or} \quad u_2 < \xi < u_1, \tag{3.13}$$

while, by (3.12),

$$\begin{aligned} |f'(u_1)| &\leq \|f'\|_{L^\infty(-C(T),\,C(T))} \leq C(T), \\ |f''(\xi)| &\leq \|f''\|_{L^\infty(-C(T),\,C(T))} \leq C(T). \end{aligned} \tag{3.14}$$

Since

$$\begin{aligned} &2\nu \int_{\mathbb{R}} \omega \partial_x \omega = 0, \\ &2\alpha \int_{\mathbb{R}} \omega \partial_x^3 \omega dx = -2\alpha \int_{\mathbb{R}} \partial_x \omega \partial_x^2 \omega dx = 0, \\ &2\beta^2 \int_{\mathbb{R}} \omega \partial_x^4 \omega dx = -2\beta^2 \int_{\mathbb{R}} \partial_x \omega \partial_x^3 \omega dx = 2\beta^2 \left\| \partial_x^2 \omega(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\ &2\tau \int_{\mathbb{R}} \partial_x^2 (u_1^2 - u_2^2) \omega dx = -2\tau \int_{\mathbb{R}} \partial_x (u_1^2 - u_2^2) \partial_x \omega dx = 2\tau \int_{\mathbb{R}} \left(u_1^2 - u_2^2\right) \partial_x^2 \omega dx, \\ &-2\delta^2 \int_{\mathbb{R}} \partial_x^2 \left(u_1^3 - u_2^3\right) \omega dx = 2\delta^2 \int_{\mathbb{R}} \partial_x \left(u_1^3 - u_2^3\right) \partial_x \omega dx = -2\delta^2 \int_{\mathbb{R}} (u_1^3 - u_2^3) \partial_x^2 \omega dx, \end{aligned}$$

multiplying (3.11) by 2ω, (3.14) and an integration on $(0, t)$ give

$$\begin{aligned} &\frac{d}{dt} \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\| \partial_x^2 \omega(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ &\quad = 2 \int_{\mathbb{R}} f'(u_1) \partial_x \omega \omega dx + 2 \int_{\mathbb{R}} f''(\xi) \partial_x u_2 \omega^2 dx - 2\gamma \int_{\mathbb{R}} \omega \partial_x^2 \omega dx \\ &\qquad - 2\tau \int_{\mathbb{R}} \left(u_1^2 - u_2^2\right) \partial_x^2 \omega dx + 2\delta^2 \int_{\mathbb{R}} (u_1^3 - u_2^3) \partial_x^2 \omega dx. \end{aligned} \tag{3.15}$$

Thanks to (3.9),

$$\begin{aligned} u_1^2 - u_2^2 &= (u_1 + u_2)(u_1 - u_2) = (u_1 + u_2)\omega, \\ u_1^3 - u_2^3 &= (u_1 - u_2)(u_1^2 + u_2^2 + u_1 u_2) = (u_1^2 + u_2^2 + u_1 u_2)\omega. \end{aligned}$$

Therefore, by (3.15),

$$\begin{aligned} &\frac{d}{dt} \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \left\| \partial_x^2 \omega(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ &\quad = 2 \int_{\mathbb{R}} f'(u_1) \omega \partial_x \omega dx + 2 \int_{\mathbb{R}} f''(\xi) \omega^2 \partial_x u_2 dx - 2\gamma \int_{\mathbb{R}} \omega \partial_x^2 \omega dx \\ &\qquad - 2\tau \int_{\mathbb{R}} (u_1 + u_2) \omega \partial_x^2 \omega dx + 2\delta^2 \int_{\mathbb{R}} (u_1^2 + u_2^2 + u_1 u_2) \omega \partial_x^2 \omega dx. \end{aligned} \tag{3.16}$$

Due to (3.12), (3.14) and the Young inequality,

$$
\begin{aligned}
&2\int_{\mathbb{R}} f'(u_1)|\partial_x\omega||\omega|dx \leq 2\left\|f''\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x\omega||\omega|dx\\
&\quad\leq 2C(T)\int_{\mathbb{R}}|\partial_x\omega||\omega|dx \leq \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&2\int_{\mathbb{R}}|f''(\xi)||\partial_x u_2|\omega^2dx \leq 2\left\|f''\right\|_{L^\infty(-C(T),C(T))}\int_{\mathbb{R}}|\partial_x u_2|\omega^2dx\\
&\quad\leq 2C(T)\int_{\mathbb{R}}|\omega\partial_x u_2||\omega|dx \leq \int_{\mathbb{R}}\omega^2(\partial_x u_2)^2dx + C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\leq \|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u_2(t,\cdot)\|_{L^2(\mathbb{R})}+C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C(T)\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}+C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
&2|\gamma|\int_{\mathbb{R}}\omega\partial_x^2\omega dx = \int_{\mathbb{R}}\left|\frac{2\gamma\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx\\
&\quad\leq \frac{2\gamma^2}{\beta^2}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2|\tau|\int_{\mathbb{R}}|u_1+u_2||\omega||\partial_x^2\omega|dx \leq C(T)\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx\\
&\quad=\int_{\mathbb{R}}\left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx \leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&2\delta^2\int_{\mathbb{R}}|u_1^2+u_2^2+u_1u_2||\omega||\partial_x^2\omega|dx \leq C(T)\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx\\
&\quad=\int_{\mathbb{R}}\left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx \leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (3.16) that

$$
\begin{aligned}
&\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}+\|\partial_x\omega(t,\cdot)\|_{L^2(\mathbb{R})}.
\end{aligned}
\tag{3.17}
$$

Observe that

$$\omega^2(t,x)=2\int_{-\infty}^x\omega\partial_x\omega dy \leq 2\int_{\mathbb{R}}|\omega||\partial_x\omega|dx.$$

Therefore, by the Young inequality,

$$\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\leq\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (3.17),

$$
\begin{aligned}
&\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\leq C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\|\partial_x\omega(t,\cdot)\|_{L^2(\mathbb{R})}.
\end{aligned}
\tag{3.18}
$$

Observe that

$$C(T)\left\|\partial_x\omega(t,\cdot)\right\|_{L^2(\mathbb{R})} = C(T)\int_{\mathbb{R}}\partial_x\omega\partial_x\omega dx = -C(T)\int_{\mathbb{R}}\omega\partial_x^2\omega dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}C(T)\left\|\partial_x\omega(t,\cdot)\right\|_{L^2(\mathbb{R})} &\le 2\int_{\mathbb{R}}\left|\frac{C(T)\sqrt{3}\omega}{2\beta}\right|\left|\frac{\beta\partial_x^2\omega}{\sqrt{3}}\right|dx\\ &\le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{3}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (3.18) that

$$\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{6}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T)\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma and (3.10) give

$$\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2 e^{C(T)t}}{6}\int_0^t e^{-C(T)s}\left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds \le e^{C(T)t}\left\|\omega_0\right\|^2_{L^2(\mathbb{R})}. \tag{3.19}$$

(1.12) follows from (3.9) and (3.19). □

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