A lattice-theoretic characterization of pure subgroups of Abelian groups

M. Ferrara¹ · M. Trombetti²

Received: 22 January 2021 / Accepted: 23 March 2021
© The Author(s) 2021

Abstract

Let G be an abelian group. The aim of this short paper is to describe a way to identify pure subgroups H of G by looking only at how the subgroup lattice $\mathcal{L}(H)$ embeds in $\mathcal{L}(G)$. It is worth noticing that all results are carried out in a local nilpotent context for a general definition of purity.

Keywords Subgroup lattice · Lattice-theoretic characterization · Pure subgroup

Mathematics Subject Classification 20E15 · 06B15

1 Introduction

The aim of this short note is to define a sublattice embedding property which, into the universe of subgroup lattices of abelian groups, is satisfied precisely by all the elements corresponding to pure subgroups. Recall that a subgroup H of an abelian group G is said *pure* in G if $G^n \cap H = H^n$ for all positive integers n , or, equivalently, if every $h \in H$ having a n th root in G admits a n th root in H (for the main properties of these subgroups see for instance [3]). This work is motivated by that of Černikov in [1] (see

---

The authors are supported by GNSAGA (INdAM) and are members of the non-profit association “Advances in Group Theory and Applications” (www.advgrouptheory.com). The first author is supported by the project “Groups: overlappings between Algebra and Geometry, Logic and Mathematics Education” (VALERE: VAnviteLli pEr la RicErca).

✉ M. Trombetti
marco.trombetti@unina.it

M. Ferrara
maria.ferrara1@unicampania.it

¹ Dipartimento di Matematica e Fisica, Università degli Studi della Campania “Luigi Vanvitelli”, viale Lincoln 5, Caserta, Italy

² Dipartimento di Matematica e Applicazioni “Renato Caccioppoli”, Università degli Studi di Napoli Federico II Complesso Universitario Monte S. Angelo, Via Cintia, Naples, Italy

also the recent survey [2]), where he describes the structure of abelian groups whose pure subgroups admit a complement. An immediate consequence of our main result is that any projectivity between abelian groups preserves the property considered by Černikov.

Notice that all results are carried out in a local nilpotent context for a general definition of pure subgroup. In particular, the final result of the paper (Corollary 2.6) shows that pure subgroups are characterized by a lattice-theoretic property even in the universe of groups whose subgroup lattice is isomorphic to that of a nilpotent group.

Most of our notation is standard and can be found in [3] and in [4].

2 The main result

In order to make things clear we split the main definition into smaller pieces. Let L be a lattice with maximum 1 and minimum 0.

- (a) An element c of L is said *cyclic* if the interval $[c/0]$ is distributive and satisfies the maximal condition. If $[c/0]$ is infinite, c is also said *infinite cyclic*, otherwise it is said *finite cyclic*. Furthermore, if $[c/0]$ is a finite chain we say that c is *primary cyclic*.

A well known consequence of a theorem of Ore is that a subgroup H of a group G is cyclic if and only if it is a cyclic element of the subgroup lattice $\mathcal{L}(G)$ of G (see for instance [4, Corollary 1.2.4]). It also follows that H is a primary cyclic element of $\mathcal{L}(G)$ if and only if H is a cyclic p -group for some prime p .

- (b) Let $c \leq d$ be elements of L . Then c is said to *split* in d if $[d/0] \simeq L' \times [c/0]$ for some lattice L' . If $[c/0] \simeq [d/0]$ or $c = 0$, we say that c *improperly splits* in d , otherwise we say that it does so *properly*.
- (c) Let $c, d \in L$. We say that c and d have the *same cyclic structure* if there exists a lattice isomorphism $\varphi : [c/0] \longrightarrow [d/0]$ such that x does not properly split in $x \vee \varphi(x)$ whenever $[x/0]$ is a chain.

It is clear that x properly splits in $x \vee \varphi(x)$ if and only if $\varphi(x)$ does so. This remark can be easily employed to show that the relation defined by “having the same cyclic structure” is an equivalence relation. In terms of subgroup lattices of locally nilpotent groups, this relation “identifies” those cyclic subgroups which are isomorphic.

Lemma 2.1 *Let C and D be finite cyclic subgroups of a locally nilpotent group G . Then C and D have the same cyclic structure as elements of $\mathcal{L}(G)$ if and only if they are isomorphic.*

Proof Suppose first C is isomorphic to D and let φ be an isomorphism. Then φ induces a lattice isomorphism between $\mathcal{L}(C)$ and $\mathcal{L}(D)$. If X is a subgroup of C with totally ordered subgroup lattice, then it is a p -subgroup for some prime p . Thus $\langle X, \varphi(X) \rangle$ is a p -group and so X does not properly split in it (see [4, Theorem 1.6.5]).

Conversely, assume C and D have the same cyclic structure and let

$$\varphi : \mathcal{L}(C) \longrightarrow \mathcal{L}(D)$$

be a lattice isomorphism with the property prescribed by the definition (c). Let $p \in \pi(C)$ and let C_p be the Sylow p -subgroup of C . Now, Corollary 1.2.8 of [4] shows that $\varphi(C_p)$ is a Sylow q -subgroup of D for some prime q . If $p \neq q$, then $\langle C_p, \varphi(C_p) \rangle = C_p \times \varphi(C_p)$, against the property of φ . Thus $p = q$ and $\pi(C) \subseteq \pi(D)$ by the arbitrariness of $p \in \pi(C)$. By symmetry, $\pi(C) = \pi(D)$. Finally, Corollary 1.2.8 of [4] yields that C and D are isomorphic. $\square$

Corollary 2.2 *Let C and D be cyclic subgroups of a locally nilpotent group G . Then C and D have the same cyclic structure as elements of $\mathcal{L}(G)$ if and only if they are isomorphic.*

Proof By Lemma 2.1, it only remains to prove that if C and D are infinite cyclic subgroups of G , then they have the same cyclic structure as elements of $\mathcal{L}(G)$. However, it is clear that $\mathcal{L}(\mathbb{Z})$ has no subgroup X such that $\mathcal{L}(X)$ is a chain and so the condition in definition (c) is trivially satisfied. $\square$

The proofs of the above lemma and its corollary make it clear that in a locally nilpotent group any projectivity between two its cyclic subgroups with the property prescribed by definition (c) comes from an isomorphism.

We are now in a position to give our main definition.

- (d) An element $x \in L$ is said to be *pure* in L if for every infinite or primary cyclic element c of L such that $x \wedge c \neq 0$, there is a cyclic element y of L with $x \wedge c \leq y \leq x$ and such that $[y/x \wedge c]$ has the same cyclic structure of $[c/x \wedge c]$.

Before proving our main result we generalize the definition of pure subgroups. A subgroup H of a group G is *pure* in G if, for all positive integers n , every $h \in H$ having a n th root in G admits a n th root in H .

Lemma 2.3 *Let H be a pure subgroup of a group G . Then H is a pure element of $\mathcal{L}(G)$.*

Proof Let $C = \langle c \rangle$ be a cyclic subgroup of G which is either infinite cyclic or of prime power order, and such that $\langle c^n \rangle = H \cap C \neq 1$ for some smallest positive integer n . The purity of H in G yields the existence of $h \in H$ with $h^n = c$. The hypothesis on C shows that $\langle h \rangle / \langle c^n \rangle$ and $\langle c \rangle / \langle c^n \rangle$ are cyclic groups of order n , so they are isomorphic and hence they share the same cyclic structure by (argue similarly to the first part of the proof of Lemma 2.1). $\square$

Theorem 2.4 *Let H be a subgroup of a locally nilpotent group G . Then H is a pure element of $\mathcal{L}(G)$ if and only if H is a pure subgroup of G .*

Proof By Lemma 2.3 we may assume that H is a pure element of $\mathcal{L}(G)$. Suppose by contradiction that H is not pure in G , so there is a smallest positive integer n such that there are $g \in G$ and $h \in H$ with $g^n = h$, but there is no $h_1 \in H$ with $h_1^n = h$. Since G is locally nilpotent, it is possible to assume that $\langle g \rangle$ is either infinite cyclic or of order a power of a prime p ; in the latter case, the minimality of n shows that n itself is a power of p . Moreover,

$$\langle h \rangle = \langle g^n \rangle \leq \langle g \rangle \cap H = \langle g^m \rangle \leq \langle g \rangle,$$

where m can be chosen dividing n , so again the minimality of n shows that $m = n$.

Now, the definition of pure element yields the existence of $k \in H$ such that $\langle h \rangle \leq \langle k \rangle$, while $\langle k \rangle / \langle h \rangle$ and $\langle g \rangle / \langle h \rangle$ have the same cyclic structure. It follows from Lemma 2.1 that the latter two finite cyclic groups are actually isomorphic. This easily yields that h has a n th root in H , a contradiction proving the result. $\square$

Corollary 2.5 *Let G_1, G_2 be locally nilpotent groups and let φ be a projectivity from G_1 to G_2 . If H is any pure subgroup of G_1 , then H^φ is a pure subgroup of G_2 .*

Corollary 2.6 *Let φ be a projectivity between the locally nilpotent group G_1 and the group G_2 . Then H is a pure subgroup of G_1 if and only if H^φ is a pure subgroup of G_2 .*

Proof If H^φ is a pure subgroup of G_2 , then it is also a pure element of $\mathcal{L}(G_2)$ by Lemma 2.3. Thus H is a pure element of $\mathcal{L}(G_1)$ and so a pure subgroup of G_1 by Theorem 2.4.

Suppose H is a pure subgroup of G_1 and, by contradiction, that H^φ is not pure in G_2 . Then there is a smallest positive integer n such that there are $g_2 \in G_2$ and $h_2 \in H^\varphi$ with $g_2^n = h_2$, but there is no $x \in H^\varphi$ with $x^n = h_2$.

Put $H_2 = \langle h_2 \rangle$ and suppose first that either $\langle g_2 \rangle$ is infinite cyclic or $\langle g_2 \rangle$ has prime power order p^m . Then, by minimality of n , we have that $H_2 = \langle g_2 \rangle \cap H^\varphi = \langle g_2^n \rangle (\neq 1)$. Moreover, by hypothesis, there is a cyclic subgroup $\langle c_2 \rangle = C_1^\varphi$ such that $H_2 \leq \langle c_2 \rangle \leq H$ and $\langle c_2 \rangle / H_2$ has the same cyclic structure of $\langle g_2 \rangle / H_2$.

Assume now that $\langle g_2 \rangle$ is cyclic of order a power of p ; in particular, $\langle g_2 \rangle^{\varphi^{-1}}$ is cyclic of order a power of a prime q . Since $\langle g_2 \rangle^{\varphi^{-1}} / H_2^{\varphi^{-1}}$ and $C_1 / H_2^{\varphi^{-1}}$ have the same cyclic structure, then the order of $C_1 / H_2^{\varphi^{-1}}$ is a power of q (see Lemma 2.1) and so C_1 is cyclic of order a power of q . It follows that $\langle c_2 \rangle$ has order a power of the prime p and so we have that $\langle c_2 \rangle / H_2$ and $\langle g_2 \rangle / H_2$ are isomorphic, a clear contradiction.

Let's now deal with the case in which $\langle g_2 \rangle$ is infinite. Since

$$\langle C_1, \langle g_2 \rangle^{\varphi^{-1}} \rangle / H_2^{\varphi^{-1}}$$

is nilpotent and finite, it follows that $\langle c_2, g_2 \rangle / H_2$ is a direct product

$$K_1 / H_2 \times \dots \times K_t / H_2$$

of primary groups and P-groups with relatively prime orders (see for instance [4, Exercise 2.2.7]). By definition (c), the Sylow (primary) subgroups of $\langle c_2 \rangle / H_2$ and $\langle g_2 \rangle / H_2$ must correspond each other and must lie in the same direct factor K_i / H_2 . If K_i / H_2 is a primary group, then these Sylow subgroups are trivially isomorphic; if K_i / H_2 is a P-group, then they cannot have distinct orders since otherwise one would act non-trivially on the other contradicting the fact that $H_2 \leq Z(\langle c_2, g_2 \rangle)$. Therefore, in any case corresponding Sylow subgroups are isomorphic and hence such are $\langle c_2 \rangle / H_2$ and $\langle g_2 \rangle / H_2$, again a contradiction.

Finally, suppose H_2 has finite order and notice that the minimality of n shows that $\pi(n) \subseteq \pi(H_2)$. Let p be a prime, denote by n_p the maximum power of p dividing n ,

by $\langle h_2(p) \rangle$ the Sylow p -subgroup of H_2 and by $g_2(p)$ an element of $\langle g_2 \rangle$ such that $g_2(p)^{n_p} = h_2(p)$. What we have already proved shows that there is a p -element $c_2(p)$ of H_2 such that $c_2(p)^{n_p} = h_2(p)$. The subgroup generated by all these $c_2(p)$ has a subgroup lattice which is lattice-isomorphic to that of a finite nilpotent group, so it is a direct product of primary groups and P-groups with relatively prime orders (see for instance [4, Exercise 2.2.7]). Since the socles of the subgroups $\langle c_2(p) \rangle$'s commute with each other, it easily follows that the $c_2(p)$'s belong to distinct direct factors and so they also commute with each other. It is now clear that h_2 admits an n th root in H , this is our final contradiction. $\square$

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Chernikov, S.N.: Groups with systems of complemented subgroups. *Math. Sb. (N.S.)* **35**, 93–128 (1954)
2. Ferrara, M., Trombetti, M.: Infinite groups with many complemented subgroups, to appear
3. Robinson, D.J.S.: *A Course in the Theory of Groups*. Springer, Berlin (1995)
4. Schmidt, R.: *Subgroup Lattices of Groups*. de Gruyter, Berlin (1994)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.