Ricerche di Matematica (2022) 71:493–509
https://doi.org/10.1007/s11587-020-00543-3

A two-weight Sobolev inequality for Carnot-Carathéodory spaces

Angela Alberico[1] · **Patrizia Di Gironimo**[2]

Received: 3 August 2020 / Revised: 10 October 2020 / Accepted: 15 October 2020 /
Published online: 4 November 2020

Abstract
Let $X = \{X_1, X_2, \ldots, X_m\}$ be a system of smooth vector fields in $\mathbb{R}^n$ satisfying the Hörmander's finite rank condition. We prove the following Sobolev inequality with reciprocal weights in Carnot-Carathéodory space $\mathbb{G}$ associated to system X

$$\left(\frac{1}{\int_{B_R} K(x)\,dx}\int_{B_R}|u|^t K(x)\,dx\right)^{1/t} \le C\,R\left(\frac{1}{\int_{B_R}\frac{1}{K(x)}\,dx}\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2},$$

where Xu denotes the horizontal gradient of u with respect to X. We assume that the weight K belongs to Muckenhoupt's class A_2 and Gehring's class G_τ, where τ is a suitable exponent related to the homogeneous dimension.

Keywords Carnot-Carathéodory spaces · Weighted Sobolev inequalities · Muckenhoupt and Gehring weights

Mathematics Subject Classification 35R03 · 39B62

1 Introduction

This paper is devoted to study some basic functional and geometric properties of general families of vector fields that include the Hörmander's type as a special case.

✉ Patrizia Di Gironimo
pdigironimo@unisa.it

Angela Alberico
a.alberico@iac.cnr.it

1 Istituto per le Applicazioni del Calcolo "M. Picone" (IAC), Consiglio Nazionale delle Ricerche (CNR), Via P. Castellino 111, 80131 Naples, Italy

2 Dipartimento di Matematica, Università degli Studi di Salerno, Via Giovanni Paolo II, 132, 84084 Fisciano, SA, Italy

Similar to their Euclidean counterparts, such properties play an important role in the analysis of the relevant differential operators (both linear and nonlinear).

We are concerned with a two-weight Sobolev type inequality on $\mathbb{G}$, where $\mathbb{G}$ denotes the Carnot-Carathèodory space (Ω, d) (suitably defined - see Sect. 2.1) associated to a system of smooth vector fields $X = \{X_1, X_2, \ldots, X_m\}$ on $\mathbb{R}^n$ satisfying the Hörmander's finite rank condition. This fact introduces a kind of degeneracy different from that Euclidean one. Here, Ω is an open (Euclidean) bounded and connected set of $\mathbb{R}^n$, $n \geq 2$, and d is the metric generated by X.

Let $u \in \mathrm{Lip}\,(\mathbb{G})$. We denote by $Xu = (X_1u, \ldots, X_mu)$ the horizontal gradient of u with respect to the system X, where X_j plays the role of the first order differential operator acting on u given by

$$X_ju(x) = \langle X_j(x), \nabla u(x)\rangle \qquad \text{for } \; j = 1, \ldots, m.$$

Set

$$|Xu| = \Big(\sum_{j=1}^{m}(X_ju)^2\Big)^{1/2},$$

the length of the horizontal gradient of u. We refer to [5,12] for more details.

In our paper we prove a two-weight Sobolev type inequality where the weights K and K^{-1} form a 2-admissible pair (K^{-1}, K), namely

1) K is locally doubling in Ω and K^{-1} belongs to $A_2(\mathbb{G})$.
2) Given a compact set $V \subset \Omega$ there exist $t > 2$ and $\overline{C} \geq 1$ such that, for every ball B with center in V and $0 < r < 1$, it holds

$$r\left(\frac{\int_{rB} K(x)\,dx}{\int_B K(x)\,dx}\right)^{1/t} \leq \overline{C}\left(\frac{\int_{rB} K^{-1}(x)\,dx}{\int_B K^{-1}(x)\,dx}\right)^{1/2}. \tag{1.1}$$

Note that inequality (1.1) is the Chanillo-Wheeden condition (see [8]), with exponents t and 2, adapted to the Carnot-Carathèodory geometry (see [18]).

Our main result reads as follows.

Theorem 1.1 *Let K be in $A_2(\mathbb{G}) \cap G_\tau(\mathbb{G})$ with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$. Let $t > 2$. Then, for every $u \in \mathcal{C}_0^1(B_R)$, there exists a constant $C \geq 1$ such that*

$$\left(\frac{1}{\int_{B_R} K(x)\,dx}\int_{B_R} |u|^t K(x)\,dx\right)^{1/t} \leq C\,R\left(\frac{1}{\int_{B_R}\frac{1}{K(x)}\,dx}\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2} \tag{1.2}$$

with

$$C = c(Q, n, t, q)\,\overline{C}\,[K^{-1}]_{A_2}^{\frac{1}{2}}\,[K]_{A_2}^{\frac{1}{t'}-\frac{1}{q'}},$$

where $\overline{C}$ is the constant in (1.1), $2 < q < t$, and B_R denotes the ball centered at the origin with radius $R > 0$. Here, $[K^{-1}]_{A_2}$ and $[K]_{A_2}$ stand for A_2 constants of K^{-1} and K, respectively.

By properties of Muchenoupt's class $A_p(\mathbb{G})$, we have that since $K \in A_2(\mathbb{G})$, then $K^{-1} \in A_2(\mathbb{G})$. Moreover, by [12, Theorem 4.8], the assumption that K belongs to $A_2(\mathbb{G}) \cap G_\tau(\mathbb{G})$, with $\tau = 1 + \frac{2(Q-1)}{n+2-Q}$, guarantees that the pair $\left(K^{-1}, K\right)$ satisfies condition (1.1). Thus, one deduces that $\left(K^{-1}, K\right)$ is a 2-admissible pair in Ω. We emphasize that the 2-admissible property of $\left(K^{-1}, K\right)$ will be used in the proof of Theorem 1.1.

The tools used to obtain inequality (1.2) are the classical ones of the Euclidean case. Neverthless, here we deal with a degeneracy into the geometry due to the presence of a differential operator Xu different from the classical gradient ∇u. In particular, this fact causes a change of metric on R^n and consequently some of the results valid for Euclidean metric have been enlarged to Carnot-Caratheodory metric.

Let us emphasize that more general weighted inequalities for Euclidean case have been extensively investigated, and are the subject of a rich literature (see e.g. [1–4,6,8–11,14,15,26]).

In the Euclidean setting, Theorem 1.1 generalizes similar result contained in [2], where the authors prove a weighted Sobolev inequality of the same type as (1.2), with the weight $K(x)$ related to the function $|u|^t$ and the weight $K^{-1}(x)$ to the gradient $|\nabla u|^2$.

Problems of this kind, involving weighted Sobolev inequalities for Carnot-Caratheodory space $\mathbb{G}$, have been systematically studied in the literature (see e.g. [7,12,16,17,19]).

The result of Theorem 1.1 is a particular case of that contained in [12, Corollary 3.4] with $v(x)$ replaced by $K(x)$ and $w(x)$ replaced by $K^{-1}(x)$. In [12] the authors show the following more general weighted Sobolev inequality

$$\left(\frac{1}{\int_{B_R} v(x)\,dx}\int_{B_R} |u|^t v(x)\,dx\right)^{1/t} \leq C\,R\left(\frac{1}{\int_{B_R} w(x)\,dx}\int_{B_R} |Xu|^p w(x)\,dx\right)^{1/p}, \tag{1.3}$$

where $1 < p < t < \infty$, $C > 0$ is a constant, and (w, v) is a p-admissible pair in Ω. Herein, we prove inequality (1.2) by using different techniques which rely upon a combination of an estimate for fractional integral of first order with other some properties of $A_2(\mathbb{G})$ and $G_\tau(\mathbb{G})$ classes. Moreover, in contrast with the result in [12, Corollary 3.4], we give the explicit value of constant C in our inequality (1.2).

Our paper is organized as follows. In Sect. 2 we give some preliminary results. Actually, in Sect. 2.1 we recall definition and basic properties of Hörmander vector fields, including Carnot-Caratheodory spaces; in Sect. 2.2 we discuss the theory of Muckenhoupt's and Gehring's weights. In Sect. 3 we present the machinery we need to work with the inequality we are interested in. Finally, we prove our main theorem.

2 Preliminary results

2.1 Carnot Carathéodory spaces

Let Ω be an open (Euclidean) bounded and connected subset in $\mathbb{R}^n$, with $n \geq 2$. Let $X = \{X_1, \ldots, X_m\}$ be a system of $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^n$.

We denote by $Lie\,[X_1, \ldots, X_m]$ the *Lie algebra* generated by $X_1, \ldots, X_m$ and by their commutators of any order. We say that a field Z belongs to $Lie\,[X_1, \ldots, X_m]$ if and only if Z is a finite linear combination of terms of this type

$$[X_{i_1}[X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}]]]$$

for $k \in \mathbb{N}$, $1 \leq i_h \leq m$, $1 \leq h \leq k$.

We define, for any fixed $x \in \mathbb{R}^n$, the *Lie rank* as

$$rank\ Lie\,[X_1, \ldots, X_m] = \dim V(x),$$

where $V(x) = \{Z(x) : Z \in Lie\,[X_1, \ldots, X_m]\}$ is a subspace of $\mathbb{R}^n$. Henceforth, we assume that X satisfies the following *Hörmander's finite rank condition* in Ω

$$rank\,Lie\,[X_1, \ldots, X_m] = n, \tag{2.1}$$

namely there exist a neighborhood Ω_0 of $\overline{\Omega}$ and $m \in \mathbb{N}$ such that the family of commutators of the vector fields in X up to length m span $\mathbb{R}^n$ at every point of Ω_0.

Let $\mathcal{C}_X$ be the family of absolutely continuous curves $\gamma : [a, b] \to \mathbb{R}^n$ such that there exist measurable functions $c_j : [a, b] \to \mathbb{R}$, with $j = 1, \ldots, m$, fulfilling

$$\sum_{j=1}^{m} c_j(t)^2 \leq 1 \qquad \text{and} \qquad \gamma\,'(t) = \sum_{j=1}^{m} c_j(t) X_j(\gamma(t)) \qquad \text{for a.e. } t \in [a, b].$$

We define *Carnot-Carathéodory distance d* as

$$d(x, y) = \inf\{T > 0 : \exists\, \gamma \in \mathcal{C}_X, \gamma(0) = x, \gamma(T) = y\} \qquad \text{for } x, y \in \Omega. \tag{2.2}$$

Note that, owing to Hörmander's finite rank condition (2.1), d is a metric. This fact is not true in general. The *Carnot-Carathéodory space* $\mathbb{G}$ is the pair (Ω, d) associated to a system of $\mathcal{C}^\infty$ vector fields $X = \{X_1, \cdots, X_m\}$ on $\mathbb{R}^n$ fulfilling (2.1).

For $x \in \mathbb{R}^n$ and $R > 0$, set $B(x, R) = \{y \in \mathbb{R}^n : d(x, y) < R\}$. The basic properties of these balls have been obtained by Nagel, Stein and Wainger in [25]. In particular, in the following proposition, the authors prove that the metric d is locally Hölder continuous with respect to the Euclidean metric.

Proposition 2.1 *([25, Proposition 1.1]) Let $X_1, \cdots X_m$ be as above. Then, for any compact set $E \subset\subset \Omega$, there are positive constants c_1, c_2 and $\lambda \in (0, 1]$ such that*

$$c_1|x - y| \leq d(x, y) \leq c_2|x - y|^\lambda$$

for every $x, y \in E$.

Thanks to Proposition 2.1, the topology of Carnot-Carathéodory induced by d on Ω coincides with the Euclidean ones. In the sequel, all the distances will be understood in the sense of the Carnot-Carathéodory metric d. In particular, all the balls will be defined with respect to d.

We denote by $|\cdot|$ the Lebesgue measure in $(\mathbb{R}^n, d)$ and, by $⨍_B f(x)\,dx$, the average of a function f on the ball B, i.e.

$$⨍_B f(x)\,dx = \frac{1}{|B|}\int_B f(x)\,dx.$$

Note that the Lebesgue measure locally satisfies the following *doubling condition* (see e.g. [25]).

Proposition 2.2 *For any compact set* $E \subset\subset \Omega$, *if* $x_0 \in E$, *there exist a constant* $C_d \geq 1$, *called doubling constant, and* $R_0 > 0$ *such that*

$$|B(x_0, 2R)| \leq C_d |B(x_0, R)|$$

for $0 < R < R_0$.

Let $Y_1, \ldots, Y_l$ be the collection of the X_j's and of those commutators which are needed to generate $\mathbb{R}^n$. To each Y_i it is associated a formal "degree" $deg(Y_i) \geq 1$, namely the corresponding order of the commutator. Set $I = (i_1, \ldots, i_n)$, with $1 \leq i_j \leq l$, an n-tuple of integers. We define (see also [25]) the *degree* of I as

$$\tilde{d}(I) = \sum_{j=1}^{n} deg(Y_{i_j}).$$

For a given compact set $E \subset \mathbb{R}^n$, we define Q by

$$Q = \sup\{\tilde{d}(I) : |a_I(x)| \neq 0, x \in E\},$$

the local homogeneous dimension of E with respect to system X, where $a_I(x) = det(Y_{i_1}, \ldots, Y_{i_n})$.

We define by

$$Q(x) = \inf\{\tilde{d}(I) : |a_I(x)| \neq 0\}$$

the homogeneous dimension at $x \in \mathbb{R}^n$ with respect to X. It is obvious that $3 \leq n \leq Q(x) \leq Q$.

Just to give an idea, we consider in $\mathbb{R}^3$ the system (see [13])

$$X = \{X_1, X_2, X_3\} = \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, x_1\frac{\partial}{\partial x_3}\right\}.$$

It is easy to see that $l = 4$ and

$$\{Y_1, Y_2, Y_3, Y_4\} = \{X_1, X_2, X_3, [X_1, X_3]\}.$$

Moreover, $Q(x) = 3$ for all $x \neq 0$, whereas for any compact set E containing the origin, $Q(0) = Q = 4$.

Let Y be a metric space and μ a Borel measure in Y. Assume μ finite on bounded sets and satisfying the doubling condition on every open, bounded subset Ω in Y. We say that Q is a *homogeneous dimension* relative to Ω, if there exists a positive constant C such that

$$\frac{\mu(B)}{\mu(B_0)} \geq C \left(\frac{R}{R_0}\right)^Q$$

for any ball B_0 having center in Ω and radius $R_0 <$ diam, and any ball B centered in $x_0 \in B_0$ and having radius $R \leq R_0$.

It is well known that the doubling condition implies the existence of the homogeneous dimension Q. However, Q is not unique and it may change with Ω. Obviously, any $Q' \geq Q$ it is also a homogeneous dimension.

For a bounded open set Ω containing a family of vector fields satisfying the Hörmander's finite rank condition, the homogeneous dimension of the Carnot-Carathéodory space $\mathbb{G}$, defined with the Lebesgue measure, is given by $Q = \log_2 C_d$, where C_d is the doubling constant.

2.2 Some properties of A_p and G_q classes

In this section, we recall a few properties of Muckenhoupt's and Gehring's classes (see [22,24,27,28]).

We recall that a weight is a positive function in $L^1_{loc}(\mathbb{R}^n)$. We say that a weight w is doubling in Ω if

$$\int_{2B} w(x)\,dx \leq C \int_B w(x)\,dx,$$

where the constant C is independent by the ball $B \subset \Omega$.

We say that w is locally doubling in Ω if for each compact set $V \subset \Omega$ and $\bar{R} > 0$ there exists $C_{V,\bar{R}}$ such that

$$\int_{2B} w(x)\,dx \leq C_{V,\bar{R}} \int_B w(x)\,dx,$$

where the ball B has center in V and radius $R < \bar{R}$ and $2B$ is the ball concentric with B and having radius 2-times that of B.

We say that a weight w belongs to the class $A_p(\mathbb{G})$ (briefly, $w \in A_p(\mathbb{G})$) for some $p \in (1, +\infty)$ if

$$[w]_{A_p} = \sup_B \left(⨍_B w(x)\,dx\right)\left(⨍_B w(x)^{1-p'}\,dx\right)^{p-1} \tag{2.3}$$

is finite, where the supremum is taken over all balls $B \subset \Omega$. Here, p' denotes the Hölder conjugate of p. The quantity $[w]_{A_p}$ is called the A_p constant of w.

When $p = 1$, we say that $w \in A_1(\mathbb{G})$ if there exists a constant $c \geq 1$ such that, for every ball $B \subset \Omega$,

$$\fint_B w(x)\,dx \leq c \operatorname{ess\,inf}_B w.$$

If a weight belongs to a class A_p, it is called a Muckenhoupt weight.

A weight w is said to belong to the class $G_q(\mathbb{G})$ (briefly, $w \in G_q(\mathbb{G})$) for some $q \in (1, +\infty)$ if

$$[w]_{G_q} = \sup_B \frac{\left(\fint_B w(x)^q\,dx\right)^{\frac{1}{q}}}{\fint_B w(x)\,dx}$$

is finite. The quantity $[w]_{G_q}$ is called the G_q constant of w.

If a weight belongs to a class G_q, it is called a Gehring weight.

Here, we recall some properties of A_p classes with respect to dyadic cubes which we will be used to prove Theorem 1.1.

We use a grid $\mathcal{D}_h$ of dyadic cubes $\mathcal{Q}$, which are "almost balls", where h is a large negative integer which indexes the edgelengths $l(Q)$ of the smallest cubes $\mathcal{Q} \in \mathcal{D}_h$. In other words, the smallest edgelengths are λ^h for an appropriate geometric constant $\lambda > 1$ and each cube in the grid has edgelength λ^k for some $k \geq h$.

In particular, we will make use of a grid of dyadic cubes in the ball B_R in the same spirit of [29], where it is proved that there exists a constant $\lambda > 1$ such that, for every $h \in \mathbb{Z}$, there are points $x_j^k \in B_R$ and a family of cubes $\mathcal{D}_h = \{\mathcal{Q}_j^k\}$ for $j \in \mathbb{N}$ and $k = h, h+1, \ldots$ such that

i) $B(x_j^k, \lambda^k) \subset \mathcal{Q}_j^k \subset B(x_j^k, \lambda^{k+1})$.

ii) For each $k = h, h+1, \ldots,$ the family $\{\mathcal{Q}_j^k\}$ is pairwise disjoint in j and $B_R = \bigcup_j Q_j^k$.

iii) If $h \leq k < l$, then either $\mathcal{Q}_j^k \cap \mathcal{Q}_j^l = \emptyset$ or $\mathcal{Q}_j^k \subset \mathcal{Q}_j^l$.

We call the family $\mathcal{D} = \bigcup_{h \in \mathbb{Z}} \mathcal{D}_h$ a dyadic cube decomposition of B_R and we refer to its sets as dyadic cubes which will be denoted by $\mathcal{Q}$. We observe explicitly that being $\mathcal{D}$ a decomposition of B_R, then any dyadic cube $\mathcal{Q} \in \mathcal{D}$ is contained in the ball B_R.

By [29], making use of (2.3), one can deduce the following lemma.

Lemma 2.3 *Let $w \in A_2(\mathbb{G})$ and let $\mathcal{Q}$ and $\mathcal{Q}_0$ dyadic cubes in $\mathbb{R}^n$ such that $\mathcal{Q} \subset \mathcal{Q}_0$. If $\beta > 1$, then*

$$\sum_{\mathcal{Q} \subset \mathcal{Q}_0} \left(\int_{\mathcal{Q}} w\,dx\right)^{\beta} \leq \left(c(Q, n)[w]_{A_2}\right)^{\beta - 1} \left(\int_{\mathcal{Q}_0} w\,dx\right)^{\beta}. \tag{2.4}$$

Another important property of $A_p(\mathbb{G})$ classes is given by the following proposition (see [20], [30, Chapter 5, p. 195]).

Proposition 2.4 *If $w \in A_p(\mathbb{G})$, then, for any nonnegative f,*

$$\left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} f(x)\,dx\right)^p \le [w]_{A_p}\ \frac{1}{\int_{\mathcal{Q}} w(x)\,dx}\int_{\mathcal{Q}} |f(x)|^p w(x)\,dx \qquad \forall \mathcal{Q}\subset \mathbb{G}. \tag{2.5}$$

2.3 Some preliminary estimates

In order to prove our main theorem, let us prove some preliminary results.

The first lemma yields an estimate of the fractional integral of order 1 (see e.g. [5]). In general, the fractional integral of order $\alpha \in (0, Q)$ of a locally integrable function g in $\mathbb{R}^n$ is defined as

$$I_\alpha g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{d(x,y)^{Q-\alpha}}\,dy \qquad \text{for } x \in \mathbb{R}^n. \tag{2.6}$$

Lemma 2.5 *Let $g \in L^1_{loc}(\mathbb{G})$ and assume that $g \ge 0$. Then*

$$I_1 g(x) \le c_0 \sum_{\mathcal{Q}\in\mathcal{D}} \left(|\mathcal{Q}|^{\frac{1}{Q}-1}\int_{3\mathcal{Q}} g(y)\,dy\right)\chi_{\mathcal{Q}}(x) \qquad \forall x \in \Omega, \tag{2.7}$$

where c_0 is an absolute constant.

Proof Thanks to a dyadic cube decomposition, we discretize the operator I_1

$$\begin{aligned} I_1 g(x) &= \sum_{k\in Z}\left(\int_{2^{k-1}<d(x,y)\le 2^k} \frac{g(y)}{d(x,y)^{Q-1}}dy\right) \\ &\le c_0 \sum_{k\in Z}\sum_{\substack{\mathcal{Q}\in\mathcal{D}\\ l(\mathcal{Q})=2^k}}\left[\left(\frac{1}{l(\mathcal{Q})^{Q-1}}\int_{d(x,y)\le l(\mathcal{Q})} g(y)dy\right)\chi_{\mathcal{Q}}(x)\right] \\ &\le c_0 \sum_{\mathcal{Q}\in\mathcal{D}}\left[\left(|\mathcal{Q}|^{\frac{1-Q}{Q}}\int_{3\mathcal{Q}} g(y)dy\right)\chi_{\mathcal{Q}}(x)\right], \end{aligned}$$

where the last inequality follows by $|\mathcal{Q}| = l(\mathcal{Q})^Q$ and, moreover, by $B(x, l(\mathcal{Q})) \subset 3\mathcal{Q}$ if $x \in \mathcal{Q}$. Hence, inequality (2.7) is proved. □

Let us consider a Dirichlet problem in this form

$$\begin{cases} \Delta_{\mathbb{G}}\varphi = f(x) & \text{in } B_R \\ \varphi = 0 & \text{on } \partial B_R, \end{cases} \tag{2.8}$$

where $\Delta_{\mathbb{G}}$ denotes the canonical sub-Laplacian operator defined as $\Delta_{\mathbb{G}} = \sum_{j=1}^m X_j^2$, with $\{X_1, ..., X_m\}$ the family of smooth vector fields on $\mathbb{R}^n$ satisfying the Hörmander's finite rank condition.

Let $\mathcal{F}_\alpha(B_R)$ be the anisotropic Hölder space, with $\alpha \in (0, 1)$, defined by

$$\mathcal{F}_\alpha(B_R) = \left\{ f : B_R \to \mathbb{R} : \sup_{\substack{x,y \in B_R \\ x \neq y}} \frac{f(x) - f(y)}{d(x, y)^\alpha} < \infty \right\}, \tag{2.9}$$

where d is the Carnot-Carathéodory distance given by (2.2).

In [21, Theorem 3.2], the authors proved that, if $f \in \mathcal{F}_\alpha(B_R)$, then there exists a unique solution $\varphi \in \mathcal{C}^2(B_R) \cap \mathcal{C}^1(\overline{B_R})$ to problem (2.8), represented by the formula

$$\varphi(x) = \int_{B_R} \Delta_{\mathbb{G}}\, \varphi\, \Gamma_x(y)\, dy. \tag{2.10}$$

Here, $\Gamma_x(y)$ is the fundamental solution of the sub-Laplacian. Thanks to [21, Theorem 2.2], there exists a positive constant c such that

$$\Gamma_x(y) = c\, d(x, y)^{2-Q}. \tag{2.11}$$

Consequently, combining (2.10) and (2.11) yields

$$\varphi(x) = c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-2}} dy. \tag{2.12}$$

The next lemma gives an estimate of the gradient of the solution to problem (2.8) through the fractional integral of order 1.

Lemma 2.6 *Let $f \in \mathcal{F}_\alpha(B_R)$ and let φ be the solution to problem (2.8). Then, there exists a positive constant c such that*

$$|X\varphi(x)| \leq c\, I_1 f(x), \tag{2.13}$$

where $I_1(f)$ denotes the fractional integral of order 1 of f.

Proof Owing to (2.12), it follows that

$$X_j\varphi(x) = c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} X_j(d(x, y))dy. \tag{2.14}$$

Thus,

$$\begin{aligned} |X\varphi(x)| &= \Big(\sum_{j=1}^n |X_j\varphi(x)|^2 \Big)^{\frac{1}{2}} \\ &= \Big(\sum_{j=1}^n \Big| c \int_{B_R} \frac{f(y)}{d(x, y)^{Q-1}} X_j(d(x, y))dy \Big|^2 \Big)^{\frac{1}{2}}. \end{aligned} \tag{2.15}$$

Since $\left|X_j(d(x,y))\right| = 1$ (see [23]), by (2.15) and (2.6) one can deduce that

$$
\begin{aligned}
|X\varphi(x)| &\le \left(\sum_{j=1}^{n}\left|c\int_{B_R}\frac{f(y)}{d(x,y)^{Q-1}}dy\right|^2\right)^{\frac{1}{2}} \\
&\le n\,c\int_{B_R}\frac{f(y)}{d(x,y)^{Q-1}}dy = c\,I_1 f(x), \qquad (2.16)
\end{aligned}
$$

where the second inequality is due to the fact that $a^2+b^2 \le (a+b)^2$. □

3 Proof of main result

The following preliminary lemma will be use in the proof of Theorem 1.1.

Lemma 3.1 *If $K \in A_2(\mathbb{G})$ and $u \in \mathcal{C}_0^1(B_R)$, then*

$$
\begin{aligned}
S_1 &= \left[\sum_{\mathcal{Q}\in\mathcal{D}}\left(\int_{\mathcal{Q}}K(x)\,dx\right)^{\frac{q'}{t'}}\left(\frac{1}{\int_{\mathcal{Q}}K(x)\,dx}\int_{3\mathcal{Q}}|u|^{t-1}K(x)\,dx\right)^{q'}\right]^{\frac{1}{q'}} \\
&\le C\left(\int_{B_R}|u|^t K(x)\,dx\right)^{\frac{1}{t'}}, \qquad (3.1)
\end{aligned}
$$

where $2 < q < t$ and $C = c(Q,n,t,q)[K]_{A_2}^{\frac{1}{t'}-\frac{1}{q'}}$.

Proof For each $h \in \mathbb{Z}$, we set

$$
\mathcal{C}^h = \left\{\mathcal{Q}\text{ dyadic cube} : 2^h < \frac{1}{\int_{\mathcal{Q}}K(x)\,dx}\int_{\mathcal{Q}}|u|^{t-1}K(x)\,dx \le 2^{h+1}\right\}. \qquad (3.2)
$$

Note that, if $\mathcal{Q}$ is any dyadic cube such that $|u|^{t-1}K(x)$ is not identically zero on $\mathcal{Q}$, then $\mathcal{Q}$ belongs to only one collection $\mathcal{C}^h$.

For each $h \in \mathbb{Z}$, let us build the collection $\{\mathcal{Q}_j^h\}_j$ of pairwise disjoint maximal dyadic cubes (maximal with respect to inclusion) in $\mathcal{C}^h$. If $\mathcal{Q} \in \mathcal{C}^h$, then there exists $j \in \mathbb{N}$ such that $\mathcal{Q} \subset \mathcal{Q}_j^h$. Note also that for each fixed h, the cubes $\mathcal{Q}_j^h$ are disjoint with respect to j. Nevertheless, they may not be disjoint for different values of h.

By (3.2),

$$S_1 \le \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{\mathcal{Q}\in\mathcal{C}^h} \left(\int_{\mathcal{Q}} K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}$$

$$\le \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{j\in\mathbb{N}} \sum_{\mathcal{Q}\subset\mathcal{Q}_j^h} \left(\int_{\mathcal{Q}} K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}. \tag{3.3}$$

By Lemma 2.3, since $\dfrac{q'}{t'} > 1$, we have

$$\sum_{\mathcal{Q}\subset\mathcal{Q}_j^h} \left(\int_{\mathcal{Q}} K(x)\,dx \right)^{\frac{q'}{t'}} \le \left(c(Q,n)[K]_{A_2} \right)^{\frac{q'}{t'}-1} \left(\int_{\mathcal{Q}_j^h} K(x)\,dx \right)^{\frac{q'}{t'}}. \tag{3.4}$$

By (3.3) and (3.4), we deduce

$$S_1 \le \left(\left(c(Q,n)[K]_{A_2} \right)^{\frac{q'}{t'}-1} \sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{j\in\mathbb{N}} \left(\int_{\mathcal{Q}_j^h} K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}. \tag{3.5}$$

Since $\mathcal{Q}_j^h \in \mathcal{C}^h$, by (3.2)

$$\frac{1}{\int_{\mathcal{Q}_j^h} K(x)\,dx} \int_{\mathcal{Q}_j^h} |u|^{t-1} K(x)\,dx > 2^h.$$

Thus,

$$\frac{1}{\int_{\mathcal{Q}_j^h} K(x)\,dx} \int_{\mathcal{Q}_j^h \cap \{x\in B_R : |u| > 2^{h-10}\}} |u|^{t-1} K(x)\,dx \ge C_1\, 2^h, \tag{3.6}$$

where $C_1 = C_1(Q,n)$ is a constant. Consequently,

$$\int_{\mathcal{Q}_j^h} K(x)\,dx \le C_1\, 2^{-h} \int_{\mathcal{Q}_j^h \cap \{x\in B_R : |u| > 2^{h-10}\}} |u|^{t-1} K(x)\,dx. \tag{3.7}$$

Owing to (3.5) and (3.7),

$$S_1 \le C_2 \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)q'} \sum_{j\in\mathbb{N}} \left(2^{-h} \int_{\mathcal{Q}_j^h \cap \{x\in B_R :\ |u| > 2^{h-10}\}} |u|^{t-1} K(x)\,dx \right)^{\frac{q'}{t'}} \right)^{\frac{1}{q'}}, \tag{3.8}$$

where $C_2 = C_1(Q,n)\,\left(c(Q,n)[K]_{A_2}\right)^{\frac{1}{t'}-\frac{1}{q'}}$.

By (3.8), we have

$$S_1 \le C_2 \left(\sum_{h\in\mathbb{Z}} 2^{(h+1)-\frac{h}{t'}} \sum_{j\in\mathbb{N}} \int_{Q_j^h \cap \{x\in B_R:\,|u|>2^{h-10}\}} |u|^{t-1} K(x)\,dx \right)^{\frac{1}{t'}}$$
$$= 2^{\frac{1}{t'}} C_2 \left(\sum_{h\in\mathbb{Z}} 2^{\frac{h}{t}} \sum_{j\in\mathbb{N}} \int_{Q_j^h \cap \{x\in B_R:\,|u|>2^{h-10}\}} |u|^{t-1} K(x)\,dx \right)^{\frac{1}{t'}}$$
$$\le 2^{\frac{1}{t'}} C_2 \left(\sum_{h\in\mathbb{Z}} 2^{h} \int_{\{x\in B_R:\,|u|>2^{h-10}\}} |u|^{t-1} K(x)\,dx \right)^{\frac{1}{t'}}$$
$$= 2^{\frac{1}{t'}} C_2 \left(\int_{B_R} |u|^{t-1} K(x) \sum_{\{h\in\mathbb{Z}:\,2^h<2^{10}|u|\}} 2^h\,dx \right)^{\frac{1}{t'}}, \tag{3.9}$$

where the first inequality is a consequence of the fact that $\sum_h a_h^{\frac{q'}{t'}} \le \left[\sum_h a_h\right]^{\frac{q'}{t'}}$, the third one holds because, fixed $h \in \mathbb{Z}$, Q_j^h are disjoint in j, the fourth one is due to Fubini's type Theorem. To conclude the proof, we have to evaluate the quantity

$$\sum_{\{h\in\mathbb{Z}:\,2^h<2^{10}|u|\}} 2^h. \tag{3.10}$$

Set $H = \log_2(2^{10}|u|)$. Thus, (3.10) yields

$$\sum_{h=-\infty}^{H} 2^h = \sum_{h=-H}^{+\infty} \left(\frac{1}{2}\right)^{h} = \sum_{h=-H}^{+\infty} \left(\frac{1}{2}\right)^{h+H-H} = \left(\frac{1}{2}\right)^{-H} \sum_{h=-H}^{\infty} \left(\frac{1}{2}\right)^{h+H}$$
$$= \left(\frac{1}{2}\right)^{-H} \sum_{m=0}^{+\infty} \left(\frac{1}{2}\right)^{m} = \left(\frac{1}{2}\right)^{-H} 2 = 2^{\log_2(2^{10}|u|)}\,2 = 2^{11}|u|. \tag{3.11}$$

Then, by (3.9) and (3.11), we obtain

$$S_1 \le C_3 \left(\int_{B_R} |u|^t K(x)\,dx \right)^{\frac{1}{t'}},$$

with $C_3 = 2^{\frac{1}{t'}+11} C_1(Q,n)\,\left(c(Q,n)[K]_{A_2}\right)^{\frac{1}{t'}-\frac{1}{q'}}$ and inequality (3.1) is proved. □

Now we are in position to prove our main result.

Proof of Theorem 1.1. By Theorem 3.2 of [21], there exists a solution φ to the following Dirichlet problem for sub-Laplacian

$$\begin{cases} \Delta_{\mathbb{G}}\varphi = |u|^{t-1}K(x) & \text{in } B_R \\ \varphi = 0 & \text{on } \partial B_R, \end{cases}$$

with $u \in \mathcal{C}_0^1(B_R)$. By Lemma 2.6, we get

$$|X\varphi(x)| \le c\, I_1(|u|^{t-1}K(x)) \qquad \forall x \in B_R, \tag{3.12}$$

where c is a positive constant.

Thanks to Lemma 2.5, it follows that

$$I_1(|u|^{t-1}K)(x) \le c_0 \sum_{\mathcal{Q}\in\mathcal{D}} \left(|\mathcal{Q}|^{\frac{1}{n}-1} \int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\,dy\right)\chi_{\mathcal{Q}}(x) \qquad \forall x \in B_R, \tag{3.13}$$

where c_0 is an absolute constant.

Combining (3.12) and (3.13) yields

$$\begin{aligned}
&\int_{B_R} |u(x)|^t K(x)\,dx \\
&= \int_{B_R} |u(x)||u(x)|^{t-1}K(x)\,dx = \int_{B_R} |u(x)|\Delta_{\mathbb{G}}\varphi\,dx \\
&\le \int_{B_R} |Xu||X\varphi|\,dx \le c\int_{B_R} |Xu| I_1(|u|^{t-1}K)(x)\,dx \\
&\le C_6 \int_{B_R} |Xu(x)| \sum_{\mathcal{Q}\in\mathcal{D}} \left(|\mathcal{Q}|^{\frac{1}{n}-1}\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\,dy\right)\chi_{\mathcal{Q}}(x)\,dx \\
&= C_6 \int_{B_R} \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}-1}\,|Xu(x)|\,\chi_{\mathcal{Q}}(x)\left(\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\,dy\right)\,dx \\
&= C_6 \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}-1}\int_{B_R\cap\mathcal{Q}} |Xu(x)|\,dx\left(\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\,dy\right) \\
&= C_6 \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}}\left(\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |Xu|\,dx\right)\left(\int_{3\mathcal{Q}} |u(y)|^{t-1}K(y)\,dy\right),
\end{aligned} \tag{3.14}$$

where $C_6 = c\,c_0$. Note that the last inequality is the consequence of the fact that $B_R \cap \mathcal{Q} = \mathcal{Q}$.

By (2.5),

$$\frac{1}{|\mathcal{Q}|}\int_{\mathcal{Q}} |Xu|\,dx \le \left[K^{-1}\right]_{A_2}^{\frac{1}{2}}\left(\frac{1}{\int_{\mathcal{Q}}\frac{1}{K(x)}\,dx}\int_{\mathcal{Q}}\frac{|Xu|^2}{K(x)}\,dx\right)^{\frac{1}{2}}. \tag{3.15}$$

Coupling inequalities (3.14) and (3.15) tells us that

$$\int_{B_R} |u|^t K(x)\, dx N$$
$$\le C_6 \left[K^{-1}\right]^{\frac{1}{2}}_{A_2} \sum_{\mathcal{Q}\in\mathcal{D}} |\mathcal{Q}|^{\frac{1}{n}} \left(\frac{1}{\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx} \int_{\mathcal{Q}} \frac{|Xu|^2}{K(x)}\, dx\right)^{\frac{1}{2}} \left(\int_{3\mathcal{Q}} |u|^{t-1} K(y)\, dy\right). \tag{3.16}$$

By (3.16), the following chain of inequality holds

$$\int_{B_R} |u|^t K(x)\, dx$$
$$\le C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \sum_{\mathcal{Q}\in\mathcal{D}} \left(\int_{\mathcal{Q}} K(x)\, dx\right)^{-1/t} \left(\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx\right)^{1/2}$$
$$\times \left(\frac{1}{\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx} \int_{\mathcal{Q}} \frac{|Xu|^2}{K(x)}\, dx\right)^{1/2} \left(\int_{3\mathcal{Q}} |u|^{t-1} K(x)\, dx\right)$$
$$= C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \sum_{\mathcal{Q}\in\mathcal{D}} \left(\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx\right)^{1/2}$$
$$\left(\frac{1}{\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx} \int_{\mathcal{Q}} \frac{|Xu|^2}{K(x)}\, dx\right)^{1/2} \times \left(\int_{\mathcal{Q}} K(x)\, dx\right)^{1/t'-1} \int_{3\mathcal{Q}} |u|^{t-1} K(x)\, dx$$
$$\le C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \left[\sum_{\mathcal{Q}\in\mathcal{D}} \left(\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx\right)^{q/2}\right.$$
$$\left.\left(\frac{1}{\int_{\mathcal{Q}} \frac{1}{K(x)}\, dx} \int_{\mathcal{Q}} \frac{|Xu|^2}{K(x)}\, dx\right)^{q/2}\right]^{1/q}$$
$$\times \left[\sum_{\mathcal{Q}\in\mathcal{D}} \left(\int_{\mathcal{Q}} K(x)\, dx\right)^{q'/t'} \left(\frac{1}{\int_{\mathcal{Q}} K(x)\, dx} \int_{3\mathcal{Q}} |u|^{t-1} K(x)\, dx\right)^{q'}\right]^{1/q'}$$
$$= C_7 |B_R|^{1/n} \frac{\left(\int_{B_R} K(x)\, dx\right)^{1/t}}{\left(\int_{B_R} \frac{1}{K(x)}\, dx\right)^{1/2}} \left[\sum_{\mathcal{Q}\in\mathcal{D}} \left(\int_{\mathcal{Q}} \frac{|Xu|^2}{K(x)}\, dx\right)^{q/2}\right]^{1/q}$$
$$\left[\sum_{\mathcal{Q}\in\mathcal{D}} \left(\int_{\mathcal{Q}} K(x)\, dx\right)^{q'/t'} \left(\frac{1}{\int_{\mathcal{Q}} K(x)\, dx} \int_{3\mathcal{Q}} |u|^{t-1} K(x)\, dx\right)^{q'}\right]^{1/q'}$$

$$
\le C_7|B_R|^{1/n}\frac{\left(\int_{B_R}K(x)\,dx\right)^{1/t}}{\left(\int_{B_R}\frac{1}{K(x)}\,dx\right)^{1/2}}\left(\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2}\Bigg[\sum_{Q\in\mathcal{D}}\left(\int_Q K(x)\,dx\right)^{q'/t'}
\left(\frac{1}{\int_Q K(x)\,dx}\int_{3Q}|u|^{t-1}K(x)\,dx\right)^{q'}\Bigg]^{1/q'}
$$

$$
= C_7|B_R|^{1/n}\frac{\left(\int_{B_R}K(x)\,dx\right)^{1/t}}{\left(\int_{B_R}\frac{1}{K(x)}\,dx\right)^{1/2}}\left(\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2}S_1, \tag{3.17}
$$

where the first inequality follows by Chanillo-Wheeden condition (1.1), the second one holds since $1/t = 1 - 1/t'$, the third one is due to Hölder's inequality, for $2 < q < t$, and the fifty one comes from the fact that $\mathcal{D}$ is a decomposition of B_R. Here, constant $C_7 = C_6\,\overline{C}\,\left[K^{-1}\right]_{A_2}^{\frac{1}{2}}$. The quantity S_1 is introduced in Lemma 3.1 above.

Combining (3.17) and (3.1) shows that

$$
\left(\int_{B_R}|u|^t K(x)\,dx\right)^{1/t} \le C_8|B_R|^{1/n}\frac{\left(\displaystyle\int_{B_R}K(x)\,dx\right)^{1/t}}{\left(\displaystyle\int_{B_R}\frac{1}{K(x)}\,dx\right)^{1/2}}\left(\int_{B_R}\frac{|Xu|^2}{K(x)}\,dx\right)^{1/2}, \tag{3.18}
$$

where $C_8 = c(Q, n, t, q)\,\overline{C}\,[K^{-1}]_{A_2}^{\frac{1}{2}}[K]_{A_2}^{\frac{1}{t'}-\frac{1}{q'}}$. Then, inequality (1.2) follows. □

Acknowledgements This research was partly supported by GNAMPA of the Italian INdAM (National Institute of High Mathematics).

Funding Open access funding provided by Università degli Studi di Salerno within the CRUI-CARE Agreement.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Alberico, A.: Moser type inequalities for higher-order derivatives in Lorentz spaces. Potential Anal. **28**, 389–400 (2008)
2. Alberico, A., Alberico, T., Sbordone, C.: A Sobolev inequality with reciprocal weights. Nonlinear Anal. **75**, 5348–5356 (2012)
3. Alberico, A., Cianchi, A., Pick, L., Slavíková, L.: Sharp Sobolev type embeddings on the entire euclidean space. Commun. Pure Appl. Anal. **17**, 2011–2037 (2018)
4. Alberico, T., Cianchi, A., Sbordone, C.: Fractional integrals and A_p-weights: a sharp estimate. C. R. Acad. Sci. Paris **17**, 2011–2037 (2009)
5. Bonfiglioli, A., Lanconelli, E., Uguzzoni, F.: Stratified Lie Groups and Potential Theory for Their Sub-Laplacians. Springer, New York (2007)
6. Caso, L., Di Gironimo, P., Monsurró, S., Transirico, M.: Uniqueness results for higher order elliptic equations in weighted Sobolev spaces. Int. J. Differ. Equ. **2018**, 1–16 (2018). https://doi.org/10.1155/2018/6259307
7. Capogna, D., Danielli, D., Garofalo, N.: An embedding theorem and Harnack inequality for nonlinear subelliptic equations. Commun. Partial Differ. Equ. **18**, 1765–1794 (1993)
8. Chanillo, S., Wheeden, R.L.: Harnack's inequality and mean-value inequalities for solutions of degenerate elliptic equations. Commun. Partial Differ. Equ. **11**, 1111–1134 (1986)
9. Cianchi, A., Edmunds, D.E., Gurka, P.: On weighted Poincaré inequalities. Math. Nachr. **180**, 15–41 (1996)
10. Cianchi, A., Edmunds, D.E.: On fractional integration in weighted Lorentz spaces. Q. J. Math. Oxf. **2**, 439–451 (1997)
11. Cruz-Uribe, D., Di Gironimo, P., Sbordone, C.: On the continuity of solutions to degenerate elliptic equations. J. Differ. Equ. **250**, 2671–2686 (2011)
12. Cruz-Uribe, D., Moen, K., Naibo, V.: Regularity of solutions to degenerate $p-$Laplacian equations. J. Math. Anal. Appl. **401**, 458–478 (2013)
13. Danielli, D., Garofalo, N., Phuc, N.C.: Inequalities of Hardy-Sobolev type in Carnot-Carathéodory spaces. Sobolev spaces in mathematics I, 117–151, Int. Math. Ser. (N.Y.), **8**. Springer, New York (2009)
14. Di Gironimo, P.: ABP inequality and weak Harnack inequality for fully nonlinear elliptic operators with coefficients in weighted spaces. Far East J. Math. Sci. **64**, 1–21 (2012)
15. Di Gironimo, P.: Harnack inequality for fully nonlinear elliptic equations with coefficients in weighted spaces. J. Anal. Appl. **15**, 1–19 (2017)
16. Di Gironimo, P., Giannetti, F.: Higher integrability of minimizers of degenerate functionals in Carnot–Caratheodory spaces. Ann. Acad. Sci. Fenn. **45**, 293–303 (2020). https://doi.org/10.5186/aasfm.20.4509
17. Di Gironimo, P., Giannetti, F.: Existence an regularity of the solutions to degenerate elliptic equations in Carnot-Caratheodory spaces. Banach J. Math. Anal. (2020). https://doi.org/10.1007/s43037-020-00069-8
18. Franchi, B., Lu, G., Wheeden, R.L.: Representation formulas and weighted Poincar inequalities for Hr̈mander vector fields. Ann. Inst. Fourier (Grenoble) **45**, 577–604 (1995)
19. Franchi, B., Gallot, S., Wheeden, R.L.: Sobolev and isoperimetric inequalities for degenerate metrics. Math. Ann. **300**, 557–571 (1994)
20. Garcia Cuerva, J., Rubio de Francia, J.: Weighted Norm Inequalities and Related Topics. North Holland Math. Studies, vol. 116. North Holland, Amsterdam (1985)
21. Garofalo, N., Ruzhanskyb, M., Suragan, D.: On Green functions for Dirichlet sub-Laplacians on H-typegroups. J. Math. Anal. Appl. **452**, 896–905 (2017)
22. Gehring, F.W.: The L^p-integrability of the partial derivatives of a quasiconformal mapping. Acta Math. **130**, 265–277 (1973)
23. Monti, R., Serra, Cassano F.: Surface measures in Carnot–Carathéodory spaces. Calc. Var. Partial Differ. Equ. **13**, 339–376 (2001)
24. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972)
25. Nagel, A., Stein, M.E., Wainger, S.: Balls and metrics defined by vector fields I. Basic properties. Acta Math. **155**, 103–147 (1985)
26. Pérez, C.: Sharp L^p-weighted Sobolev inequalities. Ann. Inst. Fourier, Grenoble **45**, 809–824 (1995)

27. Stromberg, J.-O., Torchinsky, A.: Weighted Hardy Spaces. Volume 1381 of Lecture Notes in Mathematics. Springer, Berlin (1989)
28. Torchinsky, A.: Real-Variable Methods in Harmonic Analysis. Pure and Applied Mathematics, vol. 123. Academic Press, Orlando (1986)
29. Sawyer, E.T., Wheeden, R.L.: Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. Am. J. Math. **114**, 813–874 (1992)
30. Stein, E.M.: Harmonic Analysis. Princeton University Press, Princeton (1993)