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On the Cauchy problem for a class of hyperbolic operators with triple characteristics

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Abstract
The Cauchy problem for a class of hyperbolic operators with triple characteristics is analyzed. Some a priori estimates in Sobolev spaces with negative indexes are proved. Subsequently, an existence result for the Cauchy problem is obtained.

Keywords Cauchy problem · Hyperbolic equations · Pseudodifferential operators

Mathematics Subject Classification 35B45 · 35S05 · 35L30

1 Introduction

In the past, many authors studied widely hyperbolic operators with double characteristics, both in the case when there is no transition between different types on the set where the principal symbol vanishes of order 2 (see for instance [5,8] for a general survey) and when there is transition (see [1–4]). The operators are called effectively hyperbolic if the propagation cone C is transversal to the manifold of multiple points (see [8]). Moreover, if this occurs and lower order terms satisfy a generic Ivrii-Petkov vanishing condition, we have well posedness in C^∞ (see [7]).

The aim of the paper is to analyze the following class of operators with triple characteristics

$$P(x_0, D) = D_{x_0}^3 - (D_{x_1}^2 + x_1^2 D_{x_2}^2) D_{x_0} - b x_1^3 D_{x_2}^3, \quad \text{in } \Omega =]0, +\infty[\times \mathbb{R}^2,$$

where $D_{x_j} = \frac{1}{i}\partial_{x_j}$, $j = 0, 1, 2$, under hyperbolicity assumptions, namely $|b| \leq \frac{2}{3}$. Such a class of operators has been considered in [6], for example operators whose propagation cone is not transversal to the triple characteristic manifold. The authors

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prove a well posedness result in the Gevrey category for a simple hyperbolic operator with triple characteristics and whose propagation cone is not transversal to the triple manifold. Furthermore they estimate the precise Gevrey threshold, by exhibiting a special class of solutions, through which we can violate weak necessary solvability conditions. More precisely, let $x = (x_0, x')$ where $x' = (x_1, x_2)$, let $\xi = (\xi_0, \xi')$, where $\xi' = (\xi_1, \xi_2)$. In [6], the authors study the well posedness of the following Cauchy problem

$$\begin{cases} Pu = 0, \quad \text{in } \Omega =]0, +\infty[\times\mathbb{R}^2, \\ D_{x_j} u(0, x') = \phi_j(x'), \ \ j = 0, 1, 2, \end{cases}$$

with $\phi_j(x') \in \gamma^{(s)}(\mathbb{R}^2)$, $j = 0, 1, 2$, where $\gamma^{(s)}(\mathbb{R}^2)$ is the Gevrey s class. They obtained that the Cauchy problem for P is well posed in the Gevrey 2 class assuming that $b^2 < \frac{4}{27}$. Moreover, if $s > 2$, it is possible to choose $b \in \left]0, \frac{2}{3\sqrt{3}}\right[$ such that the Cauchy problem for P is not locally solvable at the origin in the Gevrey s class.

In this paper, instead, we investigate on the well posedness of the Cauchy problem

$$\begin{cases} Pu = f, \quad \text{in } \Omega, \\ D_{x_j} u(0, x') = 0, \ \ j = 0, 1, 2, \end{cases} \tag{1}$$

with $f \in H^r(\Omega)$, in the Sobolev spaces, obtaining an existence result for solutions.

Let us set

$$Q = -\partial^3_{x_0} + \left(\partial^2_{x_1} + x_1^2\partial^2_{x_2}\right)\partial_{x_0} + bx_1^3\partial^3_{x_2}, \quad \text{in } \Omega.$$

It results

$$Pu = iQu, \quad \text{in } \Omega.$$

As a consequence, problem (1) becomes

$$\begin{cases} Qu = g, \quad \text{in } \Omega, \\ \partial_{x_j} u(0, x') = 0, \ \ j = 0, 1, 2, \end{cases} \tag{2}$$

where we set $g = if$, in Ω, with g real function. The main result of the paper is the following.

Theorem 1 *Let $f \in H^r_{loc}(\overline{\Omega})$, with $r \geq 5$. For every $h, T > 0$, the Cauchy problem* (1) *admits a solution $u \in H^{r-2}(\Omega_{h,T})$, where $\Omega_{h,T} = [0, h[\times] - T, T[^2$.*

The rest of the paper is organized as follows. Section 2 deals with some preliminary notations and definitions. In Sect. 3 some a priori estimates are established. Section 4 is devoted to obtain a priori estimates in Sobolev spaces with negative indexes. Finally, the existence result for solutions to the Cauchy problem are proved in Sect. 5.

2 Notations and preliminaries

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}_0^3$. Let ∂^α be the derivative of order $|\alpha|$, let $\partial_{x_j}^h$ be the derivative of order h with respect to x_j and let ∂_{x_j,x_p}^h be the derivative of order h with respect to x_j and x_p.

We indicate the L^2-scalar product, the L^2-norm and the H^r-norm ($r \in \mathbb{N}_0$) by $(\cdot,\cdot)$, $\|\cdot\|$ and $\|\cdot\|_{H^r}$ respectively.

Let Ω be an open subset of $\mathbb{R}^3$. Let $C_0^\infty(\overline{\Omega})$ be the space of the restrictions to $\overline{\Omega}$ of functions belonging to $C_0^\infty(\mathbb{R}^3)$. For each $K \subseteq \overline{\Omega}$ compact set, let $C_0^\infty(K)$ be the set of functions $\varphi \in C_0^\infty(\overline{\Omega})$ having support contained in K. Let $S(\mathbb{R}^3)$ be the space of rapidly decreasing functions. In particular, let $S(\overline{\Omega})$ be the space of the restrictions to $\overline{\Omega}$ of functions belonging to $S(\mathbb{R}^3)$.

Let $\Omega = [0, +\infty[\times]a_1, b_1[\times] - \infty, +\infty[$ and let $s \in \mathbb{R}$, let us denote by $\|\cdot\|_{H^{0,0,s}(\overline{\Omega})}$ the norm given by

$$\|u\|^2_{H^{0,0,s}(\overline{\Omega})} = \int_0^{+\infty} dx_0 \int_{a_1}^{b_1} dx_1 \int_{-\infty}^{+\infty} \frac{1}{2\pi}(1+|\xi_2|^2)^s |\widehat{u}(x_0,x_1,\xi_2)|^2 d\xi_2, \quad \forall u \in C_0^\infty(\overline{\Omega}),$$

where the Fourier transform is performed only with respect to the variable x_2. Moreover, let us denote by A_s the pseudodifferential operator given by

$$A_s u(x) = \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{ix_2\cdot\xi_2}(1+|\xi_2|^2)^{\frac{s}{2}} \widehat{u}(x_0,x_1,\xi_2) d\xi_2, \quad \forall u \in C_0^\infty(\overline{\Omega}). \tag{3}$$

Let us recall that $A_s : C_0^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$. For every $\varphi(x_2) \in C_0^\infty(\mathbb{R})$, the operator $\varphi A_s u$ extends to a linear continuous operator from $H^{0,0,r}_{comp.}(\overline{\Omega})$ to $H^{0,0,r-s}_{loc}(\overline{\Omega})$, where $r, s \in \mathbb{R}$. In particular, in $\Omega_k = [0, k[\times]a_1, b_1[\times] - \infty, +\infty[$, for $k > 0$, we denote by $H^{0,0,s}(\Omega_k)$ the space of all $u \in H^{0,0,s}(\Omega)$ such that supp $u \subseteq \Omega_k$. Moreover, denoted by $\mathcal{U}_{x_2}$ the projection of supp u on the axis x_2, if supp $\varphi \subseteq \mathbb{R}\backslash\mathcal{U}_{x_2}$, then $\varphi A_s u$ is regularizing with respect to the variable x_2, namely it results:

$$\|\varphi A_s u\|_{H^{0,0,r}} \le c\|u\|_{H^{0,0,r'}}, \quad \forall r, r' \in \mathbb{R},\ u \in C^\infty(\overline{\Omega}).$$

The norms $\|u\|_{H^{0,0,s}(\Omega)}$ and $\|A_s u\|_{L^2(\Omega)}$ are equivalent for any $s \in \mathbb{R}$.

Let $s \in \mathbb{R}$ and $p \ge 0$. Let $H^{p,s}(\mathbb{R}^3)$ be the space of distributions U into $\mathbb{R}^3$ such that

$$\|u\|^2_{H^{p,s}(\mathbb{R}^3)} = \frac{1}{2\pi}\sum_{|h|\le p}\int_{\mathbb{R}^3}(1+|\xi_2|^2)^s |\partial^h_{x_0,x_1}\widehat{U}(x_0,x_1,\xi_2)|^2 dx_0 dx_1 d\xi_2 < +\infty.$$

At last, let $H^{p,s}(\Omega)$ be the space of the restrictions to Ω of elements of $H^{p,s}(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_{H^{p,s}(\Omega)} = \inf_{\substack{U \in H^{p,s}(\mathbb{R}^3) \\ U|_\Omega = u}} \|U\|_{H^{p,s}(\mathbb{R}^3)}.$$

3 A priori estimates

The following preliminary result holds (see [1], Lemma 3.1).

Lemma 1 *Let $u \in S(\overline{\Omega})$ and let $p, \alpha_0, \alpha_1, \alpha_2 \in \mathbb{N}_0$. Then*

$$\|x_0^{\frac{p}{2}} \partial^{\alpha_0,\alpha_1,\alpha_2} u\| \leq \frac{2}{p+1} \|x_0^{\frac{p+2}{2}} \partial^{\alpha_0+1,\alpha_1,\alpha_2} u\|. \tag{4}$$

Now, we establish a useful estimate.

Lemma 2 *Let $u \in C_0^\infty([0,+\infty[\times\mathbb{R}^2)$ such that $\operatorname{supp} u \subseteq [0,h[\times]-T,T[^2$. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}\varphi \subseteq \mathbb{R}\setminus]-nT,nT[$, with $n \geq 2$. For every $r \leq 0$, $s \in \mathbb{R}$ and $p \geq s+r$, it results*

$$\|\varphi A_s u\|_{L^2(\Omega)} \leq \frac{c_{p,r,s}}{[(n-1)T]^p} \|u\|_{H^{0,0,r}(\Omega)}.$$

Proof In order to obtain the claim, we follow analogous techniques used in the proof of Lemma 3.2 in [3]. For the reader's convenience, we present the demonstration. We have

$$\begin{aligned}
(\varphi A_s u)(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix_2\xi_2}\varphi(x_2)(1+|\xi_2|^2)^{\frac{s}{2}}\widehat{u}(x_0,x_1,\xi_2)d\xi_2 \\
&= \frac{1}{2\pi}\int\int_{\mathbb{R}^2} e^{i(x_2-y_2)\xi_2}\varphi(x_2)(1+|\xi_2|^2)^{\frac{s}{2}}u(x_0,x_1,y_2)dy_2d\xi_2 \\
&= \frac{i^m}{2\pi}\int\int_{\mathbb{R}^2} e^{i(x_2-y_2)\xi_2}\frac{\varphi(x_2)u(x_0,x_1,y_2)}{(x_2-y_2)^m}\partial_{\xi_2}^m(1+|\xi_2|^2)^{\frac{s}{2}}dy_2d\xi_2 \\
&= \frac{i^m\varphi(x_2)}{2\pi}\int_{-\infty}^{+\infty}\partial_{\xi_2}^m(1+|\xi_2|^2)^{\frac{s}{2}}d\xi_2\int_{-\infty}^{+\infty} e^{i(x_2-y_2)\xi_2} \\
&\qquad u(x_0,x_1,y_2)\frac{\psi\left(\frac{x_2-y_2}{(n-1)T}\right)}{(x_2-y_2)^m}dy_2,
\end{aligned} \tag{5}$$

where $m \in \mathbb{N}$ and $\psi \in C^\infty(\mathbb{R})$ such that $\psi(\tau) = 1$ if $|\tau| \geq 1$, $\psi(\tau) = 0$ if $|\tau| \leq \frac{1}{2}$.

By using (5), we get

$$(\varphi A_s u)(x) = \frac{i^m \varphi(x_2)}{2\pi} \int_{-\infty}^{+\infty} \partial_{\xi_2}^m (1+|\xi_2|^2)^{\frac{s}{2}} u(x_0, x_1, x_2) * \left(\psi\left(\frac{x_2}{(n-1)T} \right) \frac{e^{ix_2\xi_2}}{x_2^m} \right) d\xi_2,$$

and also

$$\mathcal{F}_{x_2}(\varphi A_s u)(x_0, x_1, \eta_2) = \frac{i^m \widehat{\varphi}(\eta_2)}{2\pi} * \int_{-\infty}^{+\infty} \partial_{\xi_2}^m (1+|\xi_2|^2)^{\frac{s}{2}} \widehat{u}(x_0, x_1, \eta_2) \cdot \mathcal{F}_{x_2}\left(\psi\left(\frac{x_2}{(n-1)T} \right) \frac{e^{ix_2\xi_2}}{x_2^m} \right) d\xi_2, \quad (6)$$

where

$$\mathcal{F}_{x_2}\left(\psi\left(\frac{x_2}{(n-1)T} \right) \frac{e^{ix_2\xi_2}}{x_2^m} \right) = \int_{-\infty}^{+\infty} e^{ix_2(\xi_2-\eta_2)} \psi\left(\frac{x_2}{(n-1)T} \right) \frac{1}{x_2^m} dx_2,$$

Easily, we deduce

$$\begin{aligned} &(1+(\xi_2-\eta_2)^r)\mathcal{F}_{x_2}\left(\psi\left(\frac{x_2}{(n-1)T} \right) \frac{e^{ix_2\xi_2}}{x_2^m} \right) \\ &= \int_{-\infty}^{+\infty} e^{ix_2(\xi_2-\eta_2)} \psi\left(\frac{x_2}{(n-1)T} \right) \frac{1}{x_2^m} dx_2 \\ &+ i^r \sum_{j=0}^{r} \binom{r}{j} \int_{-\infty}^{+\infty} e^{ix_2(\xi_2-\eta_2)} \partial_{x_2}^j \psi\left(\frac{x_2}{(n-1)T} \right) \partial_{x_2}^{r-j} \frac{1}{x_2^m} dx_2, \end{aligned}$$

and, then,

$$\left| \mathcal{F}_{x_2}\left(\psi\left(\frac{x_2}{(n-1)T} \right) \frac{e^{ix_2\xi_2}}{x_2^m} \right) \right| \leq \frac{c_{r,m}}{(1+(\xi_2-\eta_2)^2)^{\frac{r}{2}}} \left(\frac{1}{(n-1)T} \right)^{m-2}. \quad (7)$$

Making use of (6) and (7), we obtain

$$\begin{aligned} \|\varphi A_s u\| &= \|\mathcal{F}_{x_2}(\varphi A_s u)\| \\ &\leq \frac{1}{2\pi} \|\widehat{\varphi}\|_{L^1(\mathbb{R})} \left\| \int_{-\infty}^{+\infty} \partial_{\xi_2}^m (1+|\xi_2|^2)^{\frac{s}{2}} \widehat{u}(x_0, x_1, \eta_2) \right. \\ &\qquad \left. \mathcal{F}_{x_2}\left(\psi\left(\frac{x_2}{(n-1)T} \right) \frac{e^{ix_2\xi_2}}{x_2^m} \right) d\xi_2 \right\|_{L^2(\Omega)} \\ &\leq c \int_{-\infty}^{+\infty} \left\| \partial_{\xi_2}^m (1+|\xi_2|^2)^{\frac{s}{2}} \widehat{u}(x_0, x_1, \eta_2) \right. \end{aligned}$$

$$\mathcal{F}_{x_2}\left(\psi\left(\frac{x_2}{(n-1)T}\right)\frac{e^{ix_2\xi_2}}{x_2^m}\right)\Big\|_{L^2(\Omega)} d\xi_2$$

$$\leq \frac{c_{r,m}}{[(n-1)T]^{m-2}}\int_{-\infty}^{+\infty}\frac{\left\|\partial_{\xi_2}^m(1+|\xi_2|^2)^{\frac{s}{2}}\widehat{u}(x_0,x_1,\eta_2)\right\|_{L^2(\Omega)}}{(1+(\xi_2-\eta_2)^2)^{\frac{r}{2}}}d\xi_2.$$

From the previous inequality and the Peetre inequality (see [9], pag. 17), it follows

$$\|\varphi A_s u\|_{L^2(\Omega)} \leq \frac{c_{r,m,s}}{[(n-1)T]^{m-2}}$$
$$\int_{-\infty}^{+\infty}\frac{\left\|(1+|\xi_2|^2)^{\frac{s}{2}-\frac{m+1}{2}}\widehat{u}(x_0,x_1,\eta_2)\right\|_{L^2(\Omega)}}{(1+(\xi_2-\eta_2)^2)^{\frac{r}{2}}}d\xi_2. \tag{8}$$

If $m \geq s+r+2$, setting $p = m-2$ in (8), it results

$$\|\varphi A_s u\|_{L^2(\Omega)} \leq \frac{c_{p,r,s}}{[(n-1)T]^p}\|u\|_{H^{0,0,r}(\Omega)},$$

where $c_{p,r,s}$ is independent of n and T. □

Taking into account Lemma 2, we deduce

Lemma 3 *Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(\tau) = 0$, for $|\tau| \leq 1$. For every $\varepsilon > 0$, for every $r \leq 0$ and $s \in \mathbb{R}$ there exists $n > 1$ such that*

$$\left\|\varphi\left(\frac{x_2}{(n-1)T}\right)A_s u\right\|_{L^2(\Omega)} \leq \varepsilon\|u\|_{H^{0,0,r}(\Omega)}.$$

In the following, we establish a priori estimates in $L^2(\Omega_T)$, where $\Omega_{h,T} = [0,h[\times]-T,T[^2$, for functions belonging to $C_0^\infty(\Omega_{h,T})$.

Theorem 2 *For every $h, T > 0$, there exists a positive constant c such that*

$$\|\partial_{x_0}u\| + \|u\| \leq c\left(\|\partial_{x_1}Qu\| + \|\partial_{x_2}Qu\|\right), \quad \forall u \in C_0^\infty(\overline{\Omega}) : \operatorname{supp} u \subseteq \Omega_{h,T}. \tag{9}$$

Proof By means of a translation with respect to x_2 in T, we consider the function

$$v(x_0,x_1,x_2) = u(x_0,x_1,x_2-T), \quad \text{in } \Omega_T' =]0,+\infty[\times]-T,T[\times]0,2T[.$$

We extend the function v in even manner in $]-2T,2T[$. It results

$$v(x_0,x_1,-x_2) = v(x_0,x_1,x_2), \quad \text{in } \Omega_T'' =]0,+\infty[\times]-T,T[\times]-2T,2T[.$$

We consider the following Fourier development of the function v:

$$v(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} c_n(x_0, x_1)\frac{e^{in\omega_0 x_2}}{2\sqrt{T}} = \sum_{n=-\infty}^{+\infty} v_n(x_0, x_1, x_2),$$

where $\omega_0 = \frac{2\pi}{4T} = \frac{\pi}{2T}$ and

$$c_n(x_0, x_1) = \frac{1}{2\sqrt{T}}\int_{-2T}^{2T} v(x_0, x_1, x_2)e^{-in\omega_0 x_2}dx_2.$$

We remark that the Fourier coefficients c_n are real. We apply the operator Q to v_n obtaining

$$\begin{aligned} Qv_n(x_0, x_1, x_2) &= \frac{e^{in\omega_0 x_2}}{2\sqrt{T}}\Big[- \partial_{x_0}^3 c_n(x_0, x_1) + \partial_{x_1}^2 \partial_{x_0} c_n(x_0, x_1) \\ &\quad -n^2\omega_0^2 x_1^2 \partial_{x_0} c_n(x_0, x_1) - in^3\omega_0^3 b x_1^3 c_n(x_0, x_1)\Big] \\ &= L_n c_n(x_0, x_1)\frac{e^{in\omega_0 x_2}}{2\sqrt{T}}, \end{aligned}$$

where we set

$$\begin{aligned} L_n c_n(x_0, x_1) = &-\partial_{x_0}^3 c_n(x_0, x_1) + \partial_{x_1}^2 \partial_{x_0} c_n(x_0, x_1) - n^2\omega_0^2 x_1^2 \partial_{x_0} c_n(x_0, x_1) \\ &-in^3\omega_0^3 b x_1^3 c_n(x_0, x_1). \end{aligned}$$

It results

$$Qv(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} L_n c_n(x_0, x_1)\frac{e^{in\omega_0 x_2}}{2\sqrt{T}}.$$

We estimate the Fourier coefficients $c_n(x_0, x_1)$ by means of $L_n c_n(x_0, x_1)$ in L^2. To this aim, let us consider the inner products

$$\begin{aligned} &(L_n c_n, x_0\partial_{x_0}^2 c_n) + (x_0\partial_{x_0}^2 c_n, L_n c_n) \\ &\quad = -2(\partial_{x_0}^3 c_n, x_0\partial_{x_0}^2 c_n) + 2(\partial_{x_0}\partial_{x_1}^2 c_n, x_0\partial_{x_0}^2 c_n) - 2n^2\omega_0^2(x_1^2\partial_{x_0} c_n, x_0\partial_{x_0}^2 c_n) \\ &\quad = 2\|\partial_{x_0}^2 c_n\|^2 - 2(\partial_{x_1}\partial_{x_0} c_n, x_0\partial_{x_0}^2\partial_{x_1} c_n) + 2n^2\omega_0^2\|x_1\partial_{x_0} c_n\|^2 \\ &\quad = 2\|\partial_{x_0}^2 c_n\|^2 + 2\|\partial_{x_1}\partial_{x_0} c_n\|^2 + 2n^2\omega_0^2\|x_1\partial_{x_0} c_n\|^2. \end{aligned} \tag{10}$$

From which we have

$$\|\partial^2_{x_0} c_n\|^2 + \|\partial_{x_1}\partial_{x_0} c_n\|^2 + n^2\omega_0^2\|x_1\partial_{x_0} c_n\|^2 \le c\|x_0 L_n c_n\|^2. \tag{11}$$

Let us evaluate the inner products

$$(L_n\partial_{x_1} c_n, x_0\partial^2_{x_0}\partial_{x_1} c_n) + (x_0\partial^2_{x_0}\partial_{x_1} c_n, L_n\partial_{x_1} c_n).$$

Proceeding as in (10), we obtain

$$\begin{aligned}
&\|\partial^2_{x_0}\partial_{x_1} c_n\|^2 + \|\partial^2_{x_1}\partial_{x_0} c_n\|^2 + n^2\omega_0^2\|x_1\partial_{x_0}\partial_{x_1} c_n\|^2 \\
&\quad = (L_n\partial_{x_1} c_n, x_0\partial^2_{x_0}\partial_{x_1} c_n) + (x_0\partial^2_{x_0}\partial_{x_1} c_n, L_n\partial_{x_1} c_n) \\
&\quad = (\partial_{x_1} L_n c_n, x_0\partial^2_{x_0}\partial_{x_1} c_n) + (x_0\partial^2_{x_0}\partial_{x_1} c_n, \partial_{x_1} L_n c_n) \\
&\quad +4(x_1 n^2\omega_0^2\partial_{x_0} c_n, x_0\partial^2_{x_0}\partial_{x_1} c_n).
\end{aligned}$$

Hence, we deduce

$$\begin{aligned}
&\frac{1}{n^2\omega_0^2}\|\partial^2_{x_0}\partial_{x_1} c_n\|^2 + \frac{1}{n^2\omega_0^2}\|\partial^2_{x_1}\partial_{x_0} c_n\|^2 + \|x_1\partial_{x_0}\partial_{x_1} c_n\|^2 \\
&\le \frac{2}{n^2\omega_0^2}\|x_0\partial_{x_1} L_n c_n\|\|\partial^2_{x_0}\partial_{x_1} c_n\| + 4\|x_0 x_1\partial_{x_0} c_n\|\|\partial^2_{x_0}\partial_{x_1} c_n\|.
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
&\frac{1}{n^2\omega_0^2}\|\partial^2_{x_0}\partial_{x_1} c_n\|^2 + \frac{1}{n^2\omega_0^2}\|\partial^2_{x_1}\partial_{x_0} c_n\|^2 + \|x_1\partial_{x_0}\partial_{x_1} c_n\|^2 \\
&\quad \le \frac{c}{n^2\omega_0^2}\|x_0\partial_{x_1} L_n c_n\|^2 + c n^2\omega_0^2\|x_1\partial_{x_0} c_n\|^2.
\end{aligned} \tag{12}$$

Making use of (11) and (12), we get

$$\begin{aligned}
&\|\partial^2_{x_0} c_n\|^2 + \|\partial_{x_1}\partial_{x_0} c_n\|^2 + n^2\omega_0^2\|x_1\partial_{x_0} c_n\|^2 \\
&\quad + \frac{1}{n^2\omega_0^2}\|\partial^2_{x_0}\partial_{x_1} c_n\|^2 + \frac{1}{n^2\omega_0^2}\|\partial^2_{x_1}\partial_{x_0} c_n\|^2 \\
&\quad + \|x_1\partial_{x_0}\partial_{x_1} c_n\|^2 \le c\|x_0 L_n c_n\|^2 + \frac{c}{n^2\omega_0^2}\|x_0\partial_{x_1} L_n c_n\|^2 \\
&\quad = \frac{c}{n^2\omega_0^2}\|x_0\partial_{x_1} L_n c_n\|^2 + \frac{c}{n^2\omega_0^2}\|i x_0 n\omega_0 L_n c_n\|^2.
\end{aligned} \tag{13}$$

Let us consider $v \in C_0^\infty(]0, +\infty[\times] - T, T[\times]0, 2T[)$ and we still denote by v its even extension in $]0, +\infty[\times] - T, T[\times] - 2T, 2T[$. Let us develop v in Fourier's series with respect to x_2:

$$v(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} c_n(x_0, x_1) \frac{e^{in\omega_0 x_2}}{2\sqrt{T}},$$

from which it follows

$$Qv(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} L_n c_n(x_0, x_1) \frac{e^{in\omega_0 x_2}}{2\sqrt{T}},$$

Hence, it results

$$x_1 \partial_{x_0} \partial_{x_1} v(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} x_1 \partial_{x_0} \partial_{x_1} c_n(x_0, x_1) \frac{e^{in\omega_0 x_2}}{2\sqrt{T}}.$$

Applying the Parseval inequality, we have

$$\begin{aligned} \|x_1 \partial_{x_0} \partial_{x_1} v\|^2 &= \left\| \|x_1 \partial_{x_0} \partial_{x_1} v\|^2_{]-2T,2T[} \right\|^2_{]0,+\infty[\times]-T,T[} \\ &= \left\| \sum_{n=-\infty}^{+\infty} |x_1 \partial_{x_0} \partial_{x_1} c_n|^2 \right\|^2_{]0,+\infty[\times]-T,T[} \\ &\le \sum_{n=-\infty}^{+\infty} \|x_1 \partial_{x_0} \partial_{x_1} c_n\|^2. \end{aligned} \tag{14}$$

Taking into account (13) and (14), we obtain

$$\begin{aligned} \|x_1 \partial_{x_0} \partial_{x_1} v\|^2 &\le \sum_{n=-\infty}^{+\infty} \|x_1 \partial_{x_0} \partial_{x_1} c_n\|^2 \\ &\le \sum_{n=-\infty}^{+\infty} \left[\frac{c}{n^2 \omega_0^2} \|x_0 \partial_{x_1} L_n c_n\|^2 + \frac{c}{n^2 \omega_0^2} \|i x_0 n \omega_0 L_n c_n\|^2 \right] \\ &\le \frac{c}{\omega_0^2} \sum_{n=-\infty}^{+\infty} \frac{1}{n^2} \left[\|x_0 \partial_{x_1} L_n c_n\|^2 + \|i n x_0 \omega_0 L_n c_n\|^2 \right]. \end{aligned} \tag{15}$$

We remark that

$$x_0 \partial_{x_1} Qv(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} x_0 \partial_{x_1} L_n c_n(x_0, x_1) \frac{e^{in\omega_0 x_2}}{2\sqrt{T}}.$$

For the Parseval inequality, it results

$$\begin{aligned} \|x_0 \partial_{x_1} Qv\|^2 &= \left\| \|x_0 \partial_{x_1} Qv\|^2_{]-2T,2T[} \right\|^2_{]0,+\infty[\times]-T,T[} \\ &= \left\| \sum_{n=-\infty}^{+\infty} |x_0 \partial_{x_1} L_n c_n|^2 \right\|^2_{]0,+\infty[\times]-T,T[}. \end{aligned} \tag{16}$$

Moreover, we remark that

$$x_0 \partial_{x_2} Qv(x_0, x_1, x_2) = \sum_{n=-\infty}^{+\infty} in\omega_0 x_0 L_n c_n(x_0, x_1) \frac{e^{in\omega_0 x_2}}{2\sqrt{T}}.$$

Applying, again, the Parseval inequality, we have

$$\begin{aligned} \|x_0 \partial_{x_2} Qv\|^2 &= \left\| \|x_0 \partial_{x_2} Qv\|^2_{]-2T,2T[} \right\|^2_{]0,+\infty[\times]-T,T[} \\ &= \left\| \sum_{n=-\infty}^{+\infty} |in\omega_0 x_0 L_n c_n|^2 \right\|^2_{]0,+\infty[\times]-T,T[}. \end{aligned} \tag{17}$$

Making use of (15), (16) and (17), we obtain

$$\begin{aligned} \|x_1 \partial_{x_0} \partial_{x_1} v\|^2 &\le c \sum_{n=-\infty}^{+\infty} \frac{1}{n^2} \left[\|x_0 \partial_{x_1} L_n c_n\|^2 + \|in\omega_0 L_n c_n\|^2 \right] \\ &\le c \left[\|x_0 \partial_{x_1} Qv\|^2 + \|x_0 \partial_{x_2} Qv\|^2 \right] \end{aligned} \tag{18}$$

On the other hand, it results

$$\begin{aligned} 0 &= \int_{\Omega''_T} \partial_{x_1} x_1 \left(\partial_{x_0} v\right)^2 dx \\ &= \int_{\Omega''_T} \left(\partial_{x_0} v\right)^2 dx + \int_{\Omega''_T} 2x_1 \partial_{x_0} v \partial_{x_0} \partial_{x_1} v dx. \end{aligned}$$

From which it follows

$$\begin{aligned} \|\partial_{x_0} v\|^2 &= -2 \int_{\Omega''_T} x_1 \left(\partial_{x_0} v\right) \left(\partial_{x_0} \partial_{x_1} v\right) dx \\ &\le 2\|\partial_{x_0} v\| \|x_1 \partial_{x_0} \partial_{x_1} v\|. \end{aligned}$$

Hence, we have

$$\|\partial_{x_0} v\|^2 \le 4\|x_1 \partial_{x_0} \partial_{x_1} v\|^2. \tag{19}$$

From (18) and (19), we deduce

$$\|\partial_{x_0} v\|^2 \le c \left(\|x_0 \partial_{x_1} Qv\|^2 + \|x_0 \partial_{x_2} Qv\|^2 \right).$$

By using Lemma 1, it results

$$\|\partial_{x_0} v\|^2 + \|v\|^2 \le c \left(\|x_0 \partial_{x_1} Qv\|^2 + \|x_0 \partial_{x_2} Qv\|^2 \right). \tag{20}$$

Let us remark

$$x_0\partial_{x_1}Qv = \sum_{n=-\infty}^{+\infty} x_0\partial_{x_1}L_nc_n(x_0,x_1)\frac{e^{in\omega_0x_2}}{2\sqrt{T}},$$
$$x_0\partial_{x_2}Qv = \sum_{n=-\infty}^{+\infty} in\omega_0x_0L_nc_n(x_0,x_1)\frac{e^{in\omega_0x_2}}{2\sqrt{T}}.$$

As a consequence, we have

$$\|x_0\partial_{x_1}Qv\|^2_{]-2T,2T[} = \sum_{n=-\infty}^{+\infty} |x_0\partial_{x_1}L_nc_n|^2, \tag{21}$$
$$\|x_0\partial_{x_2}Qv\|^2_{]-2T,2T[} = \sum_{n=-\infty}^{+\infty} |in\omega_0x_0L_nc_n|^2. \tag{22}$$

Furthermore, we obtain

$$(x_0\partial_{x_1}Qv)(x_2) + (x_0\partial_{x_1}Qv)(-x_2) = \sum_{n=-\infty}^{+\infty} 2x_0\partial_{x_1}L_nc_n\frac{\cos(n\omega_0x_2)}{2\sqrt{T}}, \tag{23}$$
$$(x_0\partial_{x_2}Qv)(x_2) + (x_0\partial_{x_2}Qv)(-x_2) = \sum_{n=-\infty}^{+\infty} 2in\omega_0x_0L_nc_n\frac{\cos(n\omega_0x_2)}{2\sqrt{T}}. \tag{24}$$

By using (21), (22), (23) and (24), we deduce

$$\begin{aligned}
&\|x_0\partial_{x_1}Qv\|^2_{]-2T,2T[} + \|x_0\partial_{x_2}Qv\|^2_{]-2T,2T[}\\
&= \sum_{n=-\infty}^{+\infty} |x_0\partial_{x_1}L_nc_n|^2 + \sum_{n=-\infty}^{+\infty} |in\omega_0x_0L_nc_n|^2\\
&= \|x_0(\partial_{x_1}Qv)(x_2) + x_0(\partial_{x_1}Qv)(-x_2)\|^2_{]-2T,2T[}\\
&\quad+\|x_0(\partial_{x_2}Qv)(x_2) + x_0(\partial_{x_2}Qv)(-x_2)\|^2_{]-2T,2T[}\\
&= 2\|x_0(\partial_{x_1}Qv)(x_2) + x_0(\partial_{x_1}Qv)(-x_2)\|^2_{]0,2T[}\\
&\quad+2\|x_0(\partial_{x_2}Qv)(x_2) + x_0(\partial_{x_2}Qv)(-x_2)\|^2_{]0,2T[}\\
&\le 2\big(\|x_0(\partial_{x_1}Qv)(x_2)\|^2_{]0,2T[} + \|x_0(\partial_{x_1}Qv)(-x_2)\|^2_{]0,2T[}\\
&\quad+\|x_0(\partial_{x_2}Qv)(x_2)\|^2_{]0,2T[} + \|x_0(\partial_{x_2}Qv)(-x_2)\|^2_{]0,2T[}\big).
\end{aligned} \tag{25}$$

Now, we want to estimate directly the norms. Let us start from

$$x_0\left(\partial_{x_1}Qv\right)(x_2) = \sum_{n=-\infty}^{+\infty} x_0\partial_{x_1}L_nc_n(x_0,x_1)\frac{e^{in\omega_0x_2}}{2\sqrt{T}}.$$

It results

$$\left\|x_0\left(\partial_{x_1}Qv\right)(x_2)\right\|^2_{]0,2T[}$$
$$=\frac{1}{4T}\sum_{n=-\infty}^{+\infty}\sum_{h=-\infty}^{+\infty}x_0^2\partial_{x_1}L_nc_n(x_0,x_1)\left(\partial_{x_1}L_hc_h(x_0,x_1)\right)^*\int_0^{2T}e^{in\omega_0x_2}e^{-ih\omega_0x_2}dx_2.$$

Let us compute the other norm remembering that

$$x_0\left(\partial_{x_1}Qv\right)(-x_2)=\sum_{n=-\infty}^{+\infty}x_0\partial_{x_1}L_nc_n(x_0,x_1)\frac{e^{-in\omega_0x_2}}{2\sqrt{T}}.$$

We have

$$\left\|x_0\left(\partial_{x_1}Qv\right)(-x_2)\right\|^2_{]0,2T[}$$
$$=\frac{1}{4T}\sum_{n=-\infty}^{+\infty}\sum_{h=-\infty}^{+\infty}x_0^2\partial_{x_1}L_nc_n(x_0,x_1)\left(\partial_{x_1}L_hc_h(x_0,x_1)\right)^*\int_0^{2T}e^{-in\omega_0x_2}e^{ih\omega_0x_2}dx_2.$$

From which, it follows

$$\left\|x_0\left(\partial_{x_1}Qv\right)(x_2)\right\|^2_{]0,2T[}=\left\|x_0\left(\partial_{x_1}Qv\right)(-x_2)\right\|^2_{]0,2T[}.$$

Moreover, making use of (25) and (20), we obtain

$$\begin{aligned}\|\partial_{x_0}v\|^2_{\Omega''_T}+\|v\|^2_{\Omega''_T}&\le c\left(\|x_0\partial_{x_1}Qv\|^2_{\Omega''_T}+\|x_0\partial_{x_2}Qv\|^2_{\Omega''_T}\right)\\&\le 2c\left(\|x_0\partial_{x_1}Qv\|^2_{\Omega'_T}+\|x_0\partial_{x_2}Qv\|^2_{\Omega'_T}\right).\end{aligned}$$

Since $v(x_0,x_1,x_2)=u(x_0,x_1,x_2-T)$, for every $(x_0,x_1,x_2)\in\Omega'_T$, we have

$$\|\partial_{x_0}u\|_{\Omega_T}+\|u\|_{\Omega_T}\le c\left(\|x_0\partial_{x_1}Qu\|_{\Omega_T}+\|x_0\partial_{x_2}Qu\|_{\Omega_T}\right),\tag{26}$$

from which the claim follows. □

Let us remark that the positive constant c in (26) does not depend on T but only on x_1. As a consequence, the following result holds:

Corollary 1 *For every* $h, T>0$, *there exists a positive constant* c *such that*

$$\|\partial_{x_0}u\|+\|u\|\le c\left(\|\partial_{x_1}Qu\|+\|\partial_{x_2}Qu\|\right),\tag{27}$$

for every $u\in C_0^\infty(\overline{\Omega})$ *such that* $\operatorname{supp}u\subseteq[0,T[\times]-T,T[\times]-nT,nT[$, *for every* $n\in\mathbb{N}$.

4 A priori estimate in Sobolev spaces

In the following, we establish a priori estimate in the Sobolev spaces.

Theorem 3 *For every $s > 0$, it results*

$$\|\partial_{x_0} u\|_{H^{0,0,-s}} + \|u\|_{H^{0,0,-s}} \leq c\left(\|Qu\|_{H^{0,1,-s}} + \|Qu\|_{H^{0,0,-s+1}}\right), \quad \forall u \in C_0^\infty(\Omega_{h,T}), \tag{28}$$

where $\Omega_{h,T} = [0, h[\times] - T, T[^2$.

Proof Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi(x) = 1$ in $[-(n-1)T, (n-1)T]$ and $\operatorname{supp}\varphi \subseteq]-nT, nT[$, with $n > 1$. For every $u \in C_0^\infty(\Omega_{h,T})$, we set $v_s = \varphi A_s u$, where A_s is the pseudodifferential operator defined as:

$$A_s u = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_2\xi_2}(1 + |\xi_2|^2)^{-\frac{s}{2}}\widehat{u}(x_0, x_1, \xi_2)d\xi_2,$$

with $s > 0$. Applying (27) to v_s, we have

$$\begin{aligned}
\|\partial_{x_0} v_s\| + \|v_s\| &\leq c\left(\|\partial_{x_1} Q v_s\| + \|\partial_{x_2} Q v_s\|\right)\\
&= c\left(\|\partial_{x_1} Q\varphi A_s u\| + \|\partial_{x_2} Q\varphi A_s u\|\right)\\
&\leq c\left(\|\varphi\partial_{x_1} Q A_s u\| + \|\partial_{x_1}[Q,\varphi]A_s u\|\right)\\
&\quad + c\left(\|\partial_{x_2}\varphi Q A_s u\| + \|\partial_{x_2}[Q,\varphi]A_s u\|\right)\\
&\leq c\left(\|\varphi\partial_{x_1} A_s Qu\| + \|\partial_{x_1}[Q,\varphi]A_s u\|\right)\\
&\quad + c\left(\|\varphi\partial_{x_2} A_s Qu\| + \|[\partial_{x_2},\varphi]QA_s u\| + \|\partial_{x_2}[Q,\varphi]A_s u\|\right)\\
&= c\Big(\|\varphi A_s\partial_{x_1} Qu\| + \|\varphi A_s\partial_{x_2} Qu\| + \|R_1 Qu\|\\
&\quad + \|R_2 u\| + \|R_3\partial_{x_0} u\|\Big) + c\|[Q,\varphi]A_s\partial_{x_1} u\|,
\end{aligned} \tag{29}$$

where R_1, R_2 and R_3 are regularizing operators with respect to the variable x_2 of type

$$R_i = \psi\left(\frac{x_2}{(n-1)T}\right)A_s, \quad i = 1, 2, 3, \tag{30}$$

with $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 0$ in $[-1, 1]$, as in Lemma 3, and having used $\partial_{x_1} QA_s u = A_s\partial_{x_1} Qu$ and $\partial_{x_2} QA_s u = A_s\partial_{x_2} Qu$.

Making use of Lemmas 1, 3 and (29), we deduce

$$\begin{aligned}
\|\partial_{x_0} u\|_{H^{0,0,-s}} + \|u\|_{H^{0,0,-s}} &\leq c\left(\|Qu\|_{H^{0,1,-s}} + \|Qu\|_{H^{0,0,-s+1}} + \|Qu\|_{H^{0,0,-s}}\right)\\
&\quad + c\left(\|R_4\partial_{x_1} u\| + \|R_5\partial_{x_1}\partial_{x_0} u\|\right),
\end{aligned} \tag{31}$$

where R_4 and R_4 are regularizing operators with respect to the variable x_2 of type (30).

Now, written the operator Q as:

$$Qu = L(\partial_{x_0}u) + x_1^2\partial_{x_2}^2\partial_{x_0}u + bx_1^3\partial_{x_2}^3 u,$$

where L is the wave operator, namely $L = \partial_{x_0}^2 + \partial_{x_1}^2$, it results

$$(L(\partial_{x_0}u), \partial_{x_0}^2 u) = (Qu, \partial_{x_0}^2 u) - (x_1^2\partial_{x_2}^2\partial_{x_0}u, \partial_{x_0}^2 u) - (bx_1^3\partial_{x_2}^3 u, \partial_{x_0}^2 u).$$

Integrating by parts, we have easily:

$$\|\partial_{x_0}^2 u\| + \|\partial_{x_1}\partial_{x_0}u\| \le c\left(\|\partial_{x_2}^2\partial_{x_0}u\| + \|\partial_{x_2}^3 u\| + \|Qu\|\right).$$

Making use of Lemma 1, it follows

$$\|\partial_{x_0}u\| + \|\partial_{x_1}u\| + \|\partial_{x_0}^2 u\| + \|\partial_{x_1}\partial_{x_0}u\| \le c\left(\|\partial_{x_2}^2\partial_{x_0}u\| + \|\partial_{x_2}^3 u\| + \|Qu\|\right). \tag{32}$$

Taking into account (31), (32) and Lemma 3, we deduce

$$\begin{aligned}\|\partial_{x_0}u\|_{H^{0,0,-s}} + \|u\|_{H^{0,0,-s}} &\le c\left(\|Qu\|_{H^{0,1,-s}} + \|Qu\|_{H^{0,0,-s+1}} + \|Qu\|_{H^{0,0,-s}}\right)\\ &\quad + c\left(\|\partial_{x_1}u\|_{H^{0,0,-s-3}} + \|\partial_{x_1}\partial_{x_0}u\|_{H^{0,0,-s-3}}\right)\\ &\le c\left(\|Qu\|_{H^{0,1,-s}} + \|Qu\|_{H^{0,0,-s+1}}\right)\\ &\quad + c\left(\|u\|_{H^{0,0,-s}} + \|\partial_{x_0}u\|_{H^{0,0,-s}} + \|Qu\|_{H^{0,0,-s}}\right).\end{aligned}$$

From which we have

$$\|\partial_{x_0}u\|_{H^{0,0,-s}} + \|u\|_{H^{0,0,-s}} \le c\left(\|Qu\|_{H^{0,1,-s}} + \|Qu\|_{H^{0,0,-s+1}}\right),$$

namely (28). □

5 Proof of Theorem 1

For every $u \in C_0^\infty(\Omega_{h,T})$, where $\Omega_{h,T} = [0, h[\times] - T, T[^2$, let $\psi = {}^tQu = Qu$ and let $F(\psi) = (f, u)$. It results

$$|F(\psi)| \le \|f\|_{H^{0,0,s}(\Omega_{h,T})}\|u\|_{H^{0,0,s}(\Omega_{h,T})}.$$

Making use of (28), it follows

$$\begin{aligned}|F(\psi)| &\le c\|f\|_{H^{0,0,s}(\Omega_{h,T})}\left(\|{}^tQu\|_{H^{0,1,-s}(\Omega_{h,T})} + \|{}^tQu\|_{H^{0,0,-s+1}(\Omega_{h,T})}\right)\\ &\le c'\|\psi\|_{H^{0,1,-s+1}(\Omega_{h,T})}.\end{aligned}$$

Hence, the functional F can be extended in $H^{0,1,-s+1}(\Omega_{h,T})$ and, therefore, there exists $w \in H^{0,-1,s-1}(\Omega_{h,T})$ such that

$$F(\psi) = (w, \psi) = (w, {}^tQu) = (g, u), \quad \forall u \in C_0^\infty(\Omega_{h,T}).$$

Then, we have

$$Qw = g, \quad \text{in } \mathcal{D}'(\Omega_{h,T}).$$

Written $Qw = L(\partial_{x_0} w) + x_1^2 \partial_{x_2}^2 \partial_{x_0} w + b x_1^3 \partial_{x_2}^3 w$, we obtain

$$L(\partial_{x_0} w) = g - x_1^2 \partial_{x_2}^2 \partial_{x_0} w - b x_1^3 \partial_{x_2}^3 w.$$

For $s > 4$, we deduce $\partial_{x_0} w \in H^{0,0,s-1}$ and, hence, $u \in H^{1,s-1}$. Repeating the same procedure more times, we have that if $g \in H^r$ then $w \in H^{r-2}$. Therefore, if $r \geq 5$, we have

$$(w, {}^tQu) = (g, u), \quad \forall u \in C_0^\infty(\Omega_{h,T}). \tag{33}$$

Choosen a suitable u, for instance, such that $u(0, x') = 0$, $\partial_{x_0} u(0, x') = 0$ and $\partial_{x_0}^2 u(0, x') = \varphi(x')$, with $\varphi \in C_0^\infty(]-T, T[^2)$, integrating by parts in the left-hand side of (33), we obtain

$$(Qw, u) + \int_{[-T,T]^2} \varphi(x') w dx' = (g, u).$$

As a consequence, we get

$$\int_{[-T,T]^2} \varphi(x') w dx' = 0.$$

For the arbitrariness of φ, it follows

$$w(0, x') = 0.$$

Instead, choosing $u \in C_0^\infty(\Omega_{h,T})$ such that $u(0, x') = 0$, $\partial_{x_0} u(0, x') = \varphi(x')$ and $\partial_{x_0}^2 u(0, x') = 0$, with $\varphi \in C_0^\infty(]-T, T[^2)$, and proceeding as above, it results

$$\partial_{x_0} w(0, x') = 0.$$

Finally, if we chose $u \in C_0^\infty(\Omega_{h,T})$ such that $u(0, x') = \varphi(x')$, $\partial_{x_0} u(0, x') = 0$ and $\partial_{x_0}^2 u(0, x') = 0$, with $\varphi \in C_0^\infty(]-T, T[^2)$, we obtain

$$\partial_{x_0}^2 w(0, x') = 0.$$

Then we have proved that there exists $w \in H^{r-2}(\Omega_{h,T})$, with $r \geq 5$, such that

$$(w, {}^t Qu) = (g, u), \quad \forall u \in C_0^\infty(\Omega_{h,T}),$$

$w(0, x') = 0$, $\partial_{x_0} w(0, x') = 0$ and $\partial_{x_0}^2 w(0, x') = 0$. Hence, if $g \in H^r$, with $r \geq 5$, there exists a solution $w \in H^{r-2}$ to the problem

$$\begin{cases} Qw = g, & \text{in } \Omega_{h,T} \\ w(0, x') = 0, \ \partial_{x_0} w(0, x') = 0, \ \partial_{x_0}^2 w(0, x') = 0 \end{cases}$$

where $g = if$, with $f \in H^r_{loc}(\Omega)$. Therefore there exists a solution to problem (1) also in $\Omega_{h,T}$.

6 Conclusions

The paper deals with a class of hyperbolic operators with triple characteristics. A priori estimate in Sobolev spaces with negative indexes are obtained. Thanks to this estimate, the existence of solutions to the associated Cauchy problem can be established.

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