Generalizing algebraically defined norms

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Abstract

We extend the algebraic construction of finite dimensional varying exponent $L^{p(\cdot)}$ space norms, defined in terms of Cauchy polynomials to a more general setting, including varying exponent $L^{p(\cdot)}$ spaces. This boils down to reformulating the Musielak–Orlicz or Nakano space norm in an algebraic fashion where the infimum appearing in the definition of the norm should become a (uniquely attained) minimum. The latter may easily fail, as turns out, and in this connection we examine the Fatou type semicontinuity conditions on the modulars. Norms defined by ODEs are applied in studying such semicontinuity properties of $L^{p(\cdot)}$ space norms with $p(\cdot)$ unbounded.

Keywords Modular spaces · Musielak–Orlicz spaces · Variable exponent Lebesgue spaces · Fixed point · Non-linear integral equation

Mathematics Subject Classification 46E30 · 45G10

1 Introduction

The authors of Anatriello et al. [1] considered the variable (exponent) Lebesgue space $L^{p(\cdot)}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a set of positive Lebesgue measure and the symbol $p(\cdot)$ is a simple function of the type

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$$p(x) = \sum_{j=1}^k j \chi_{\Omega_j}(x), \quad \Omega = \bigcup_{j=1}^k \Omega_j, \quad |\Omega_j| > 0 \quad \forall j = 1, \dots, k, \\ \Omega_j \text{ pairwise disjoint}$$

and observed that the usual Luxemburg norm of a function $f \in L^{p(\cdot)}(\Omega)$ is the unique positive root of a polynomial of degree k , k being the number of values of $p(\cdot)$. Uniqueness of the positive root is due to the special form of the polynomial, which is a so-called Cauchy polynomial, i.e. a polynomial of the type

$$P(x) = x^k - \sum_{j=1}^k a_j x^{k-j} \quad (1.1)$$

where the coefficients a_j 's are non-negative. By the classical rule of signs, if (at least) one of the coefficients a_j 's is nonzero, $P(x)$ admits exactly a unique simple positive root. Historically, the importance of such root is linked to the problem, to which Cauchy gave an important contribution, to bound the modulus of the roots of polynomials. In recent years, in [1], it has been shown that—roughly speaking—when coefficients are norms, the unique positive root is again a norm; in view of old literature (see e.g. [11, Theorem 27.2, p. 123] and [3]), this root satisfies a number of norm inequalities.

In this paper we start from the same observation, but we investigate the problem in order to understand when the norms of more general variable Lebesgue spaces (and even more general class of spaces) are solutions of equations and, in particular, the unique positive solutions of equations which are not necessarily algebraic but rather fixed point equations.

Intuitively, the norm of variable Lebesgue spaces whose exponent has k values should be the unique positive solution of an equation containing the sum of k monomials, and if the exponent has a countable number of values, the norm should be the unique positive solution of an equation containing some series; finally, it is natural to expect that for general exponents $p(\cdot)$ the series should be an integral. That is, analogously as in [1], this leads to a more general construction of norms, by using a variant of (1.1) as follows:

$$R:]0, \infty[\rightarrow]0, \infty[, \quad \lambda \mapsto \int_{\Omega} \lambda^{1-p(t)} |f(t)|^{p(t)} d\mu(t) \quad (1.2)$$

or, more generally,

$$\lambda \mapsto \lambda \rho \left(\frac{|f(t)|}{\lambda} \right)$$

where the fixed point condition $R(\lambda) = \lambda$, which in case of (1.2) leads to a non-linear integral equation, defines the norm as $\|f\| = \lambda$. That is, up to suitable Fatou type semicontinuity properties of the modular which also depend on the structure of the space in question. In analyzing these Fatou type properties we apply some ideas

from Talponen [14], where norms of variable Lebesgue spaces are considered, built as solutions of suitable ODEs.

1.1 Preliminaries

In [12] the following result is proved (see Theorem 1.5, p. 2 and Lemma 2.4, p. 8).

Theorem 1.1 *If $\rho : X \rightarrow [0, \infty]$ is a convex pseudomodular on a real vector space X , namely, if ρ satisfies*

- $\rho(0) = 0$,
- $\rho(-f) = \rho(f)$,
- $\rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g)$ for $\alpha, \beta \geq 0$, $\alpha + \beta = 1$,

which implies

$$\begin{aligned} \rho(\alpha f) &\leq \rho(f) \text{ for } |\alpha| \leq 1, \\ \rho\left(\sum_{i=1}^k \alpha_i f_i\right) &\leq \sum_{i=1}^k \alpha_i \rho(f_i) \text{ for } \alpha_i \geq 0, \sum_{i=1}^k \alpha_i = 1, \end{aligned}$$

then the functional

$$\|f\|_\rho := \inf \left\{ \lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leq 1 \right\}$$

is such that for any $f, g \in X_\rho$, i.e. such that

$$\lim_{\lambda \rightarrow 0} \rho(\lambda f) = 0 \quad \lim_{\lambda \rightarrow 0} \rho(\lambda g) = 0$$

the following hold:

- $\|0\|_\rho = 0$,
- $\| -f \|_\rho = \|f\|_\rho$,
- $\|f + g\|_\rho \leq \|f\|_\rho + \|g\|_\rho$,
- $\|\alpha f\|_\rho = |\alpha| \|f\|_\rho \quad \forall \alpha \in \mathbb{R}$,
- $\rho(\alpha f) \leq \rho(\alpha g) \quad \forall \alpha > 0 \Rightarrow \|f\|_\rho \leq \|g\|_\rho$,
- $\lambda \in [0, \infty[\Rightarrow \|\lambda f\|_\rho$ is nondecreasing,
- $\|f\|_\rho < 1 \Rightarrow \rho(f) \leq \|f\|_\rho$,
- $\|f\|_\rho > 1 \Rightarrow \rho(f) \geq \|f\|_\rho$.

A typical example of norm generated by a modular is that one of Musielak–Orlicz spaces over a set $\Omega \subset \mathbb{R}^n$ of positive Lebesgue measure:

$$\|f\|_{\text{MO}, L_v^{\Phi(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dv(x) \leq 1 \right\},$$

which gives, as a particular case, the variable Lebesgue space norm with exponent $p(\cdot)$ when $\Phi(x, t) = t^{p(x)}$.

An easy consequence of Theorem 1.1 is the following proposition, where a generalization of the classical Fatou lemma plays an essential role to establish that the infimum defining $\|f\|_\rho$ is actually a minimum.

Proposition 1.2 *If $\rho : X \rightarrow [0, \infty]$ is a convex pseudomodular on a real vector space X which is left lower semicontinuous, i.e.*

$$\rho(f) \leq \liminf_{\mu \rightarrow 1^-} \rho(\mu f) \quad \forall f \in X_\rho$$

then

$$\rho\left(\frac{f}{\|f\|_\rho}\right) \leq 1 \quad \forall f \in X_\rho, \quad f \neq 0.$$

Proof If $f \in X_\rho$, then there exists $\mu > 0$ such that $\rho(\mu f) < 1$. Therefore $\lambda := 1/\mu$ belongs to the set of the λ 's defining $\|f\|_\rho$, which is therefore nonempty. Let (λ_h) be a decreasing minimizing sequence, i.e. such that $\lambda_h \downarrow \|f\|_\rho$. We have $\rho(f/\lambda_h) \leq 1$ and therefore, since also $f/\|f\|_\rho \in X_\rho$,

$$\rho\left(\frac{f}{\|f\|_\rho}\right) \leq \liminf_{\mu \rightarrow 1^-} \rho\left(\mu \frac{f}{\|f\|_\rho}\right) \leq \lim_h \rho\left(\frac{\|f\|_\rho}{\lambda_h} \frac{f}{\|f\|_\rho}\right) = \lim_h \rho\left(\frac{f}{\lambda_h}\right) \leq 1.$$

□

In spite of its simplicity, we obtained a popular property of the norm under a not usual assumption. We underline the weakness of the condition of left lower semicontinuity. The Fatou property known from the theory of Banach function spaces (see e.g. [2,10]) is equivalent to the left continuity of the modular (see the details in Theorem 2.2.2, p. 19 in [8], stated in the framework of modular function spaces), and, most of all, the elements in X must be real valued or complex valued functions, in any case must be in some lattice. The notion of left-continuity in [12] is given in general vector spaces, but, as we will see, it is stronger with respect to our needs. In the case of the Musielak–Orlicz spaces, hence, in particular, of the variable Lebesgue spaces, the left lower semicontinuity is guaranteed by the classical Fatou property. For variable Lebesgue spaces, Proposition 1.2 reduces to the first part of Prop. 2.21 in [4], for Orlicz spaces reduces to Prop. 3, p. 60 in [13], for Musielak–Orlicz spaces reduces to Lemma 3.2.4 in [6].

2 Functions whose norms are fixed points

The question of the continuity of $\lambda \in [0, \infty[\rightarrow \rho(f/\lambda)$ has been recently considered by the second named author for certain particular Musielak–Orlicz spaces, namely, certain weighted variable Lebesgue spaces. The following statement is contained in Prop. 5.3 in [15].

Proposition 2.1 *If $w : [0, 1] \rightarrow]0, \infty[$, $p : [0, 1] \rightarrow [1, \infty[$ are such that*

$$\int_0^1 |f(t)|^{p(t)} dt \leq 1 \quad \Rightarrow \quad \lambda \in]0, \infty[\rightarrow \int_0^1 w(t) \frac{|f(t)|^{p(t)}}{\lambda^{p(t)}} dt \text{ is}$$

continuous at $\lambda = 1$ for every $f \in L^{p(\cdot)}$

then there exists a constant $D > 0$ such that $w(t) \leq D/p(t)$ a.e. in $[0, 1]$.

Let us record the following consequence. First of all, by Proposition 1.2 and Theorem 1.1 (see also, e.g., Theorem 3, p. 54 in [13])

$$\int_0^1 |f(t)|^{p(t)} dt \leq 1 \quad \Leftrightarrow \quad \|f\|_{L^{p(\cdot)}(0,1)} \leq 1.$$

Now let $p(\cdot)$ be (finite and) unbounded in $[0, 1]$. Since the necessary condition on w established in Proposition 2.1 is not satisfied by the constant weight $w \equiv 1$, then there exists f , $\|f\|_{\text{MO}, L^{p(\cdot)}} \leq 1$, such that

$$R:]0, \infty[\rightarrow]0, \infty[, \quad \lambda \mapsto \int_0^1 \frac{|f(t)|^{p(t)}}{\lambda^{p(t)}} dt \text{ is not continuous at } \lambda = 1. \quad (2.1)$$

We can go further and make a more general assertion: by Prop. 2.21 in [4], there exists a function f_0 having norm $\|f_0\|_{\text{MO}, L^{p(\cdot)}} = 1$ such that $\rho(f_0) = \rho(f_0/\|f_0\|_{\text{MO}, L^{p(\cdot)}}) < 1$, and this suggests (2.1) by virtue of the following proposition, which is a complement to Proposition 1.2:

Proposition 2.2 *If $\rho : X \rightarrow [0, \infty]$ is a left lower semicontinuous, convex pseudo-modular on a real vector space X , then*

$$\rho\left(\frac{f}{\|f\|_\rho}\right) < 1 \quad \Rightarrow \quad F : \lambda \in]0, \infty[\rightarrow F(\lambda) = \rho\left(\frac{f}{\lambda}\right) \text{ is discontinuous at } \lambda = \|f\|_\rho.$$

Proof By assumption $F(\|f\|_\rho) < 1$ and by definition of $\|f\|_\rho$ for any $\varepsilon > 0$ it must be $F(\|f\|_\rho - \varepsilon) > 1$. $\square$

The discontinuity of F considered in Proposition 2.2 occurs also in Orlicz spaces theory. In fact, in [13] (see Proposition 6, p. 77 therein for details) it is proved that continuity of F is equivalent to the so-called Δ_2 condition for the Young function generating the Orlicz space, hence discontinuity may occur for not Δ_2 Young functions.

Immediate consequence of Propositions 1.2 and 2.2 is the following

Theorem 2.3 *If $\rho : X \rightarrow [0, \infty]$ is a left lower semicontinuous, convex pseudomodular on a real vector space X , if for $f \in X$, $f \neq 0$,*

$$F : \lambda \in]0, \infty[\rightarrow F(\lambda) = \rho\left(\frac{f}{\lambda}\right) \text{ is continuous at } \lambda = \|f\|_\rho,$$

then

$$\rho\left(\frac{f}{\|f\|_\rho}\right) = 1,$$

and therefore the norm $\|f\|_\rho$ is a fixed point for

$$R:]0, \infty[\rightarrow]0, \infty[, \quad \lambda \mapsto \lambda \rho \left(\frac{f}{\lambda} \right). \quad (2.2)$$

The conclusion of Theorem 2.3 is known to hold for Musielak–Orlicz spaces satisfying the Δ_2 condition (see [12], (3) p. 54), namely, in particular, in the case

$$\rho(f) = \int_{\Omega} \varphi(x, |f(x)|) dx$$

where $\varphi: \Omega \times]0, \infty[\rightarrow]0, \infty[$ is such that (see [12] p. 52)

$$\varphi(t, 2u) \leq k\varphi(t, u) + h(t) \quad \forall u \geq 0 \quad \forall t \text{ a.e. in } \Omega, \quad (2.3)$$

where $h \geq 0$, $h \in L^1(\Omega)$, $k > 0$.

3 The case of variable Lebesgue spaces

3.1 Norms in terms of algebraic equations

Let us consider the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a set of positive Lebesgue measure and $p(\cdot)$ is a bounded function (so that (2.3) holds) having at most a countable number of values:

$$p(x) = \sum_{j \in J} p_j \chi_{A_j}(x), \quad p_j \geq 1 \quad \forall j \in J, \quad A_i \cap A_j = \emptyset \text{ if } i \neq j, \quad |A_j| > 0 \quad \forall j \in J, \\ \bigcup_{j \in J} A_j = \Omega, \quad J \subset \mathbb{N}.$$

In this case the norm $\|f\|_{\text{MO}, L^{p(\cdot)}(\Omega)}$ is a solution of the equation

$$R(\lambda) = \lambda$$

where the function R is in (2.2). In this case the equation reads

$$\lambda \rho \left(\frac{f}{\lambda} \right) = \lambda$$

i.e. (next equality should be compared with that one in [15], p. 752)

$$\begin{aligned} \lambda \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx &= \lambda \sum_{j \in J} \int_{A_j} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx = \lambda \sum_{j \in J} \int_{A_j} \left| \frac{f(x)}{\lambda} \right|^{p_j} dx \\ &= \sum_{j \in J} \lambda^{1-p_j} \|f \chi_{A_j}\|_{\text{MO}, L^{p(\cdot)}(\Omega)}^{p_j} = \lambda, \end{aligned}$$

last equality being exactly the algebraic equation of degree $k \in \mathbb{N}$ in the case that $p(\cdot)$ has the values $1, \dots, k$.

3.2 Norms in terms of integral equations

In the case of a general, bounded exponent $p(\cdot)$, the above chain of equalities cannot be performed, because of possibly infinity values of the exponent. However, the equality between the extreme left and right sides is still true. In fact, we have

Proposition 3.1 *If $\Omega \subset \mathbb{R}^n$ is a set of positive Lebesgue measure and $p(\cdot)$ is a bounded function, then the value of the norm $\|f\|_{\text{MO}, L^{p(\cdot)}(\Omega)}$ is the unique solution of the equation*

$$\int_{\Omega} |f(x)|^{p(x)} \lambda^{1-p(x)} dx = \lambda, \quad \lambda > 0.$$

Proof of Proposition 3.1 We already observed that since $p(\cdot)$ is a bounded function, then (2.3) holds and we may apply Theorem 2.3. Hence by (2.2) the norm $\|f\|_{\text{MO}, L^{p(\cdot)}(\Omega)}$ is the solution of the equation $R(\lambda) = \lambda$, i.e.

$$\lambda \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx = \lambda. \quad (3.1)$$

Uniqueness of the solution is easy to see and the proof is complete. $\square$

We remark that (3.1), with the substitution $\lambda = \|f\|_{\text{MO}, L^{p(\cdot)}(\Omega)}$, is known from the general theory of variable Lebesgue spaces (see the second part of Prop. 2.21 in [4]), and that it holds also for exponents which may assume value $+\infty$, but which must be bounded in the set where they are finite (if such set exists). In particular, in the case of the space L^∞ , we remark that its norm could be seen as generated by the pseudomodular $\rho(f) = \|f\|_\infty$ which satisfies all the assumptions of Theorem 2.3. This shows that the conclusion of Theorem 2.3 holds also for spaces which do not have an absolutely continuous norm (see e.g. [2,9] for details about spaces having absolutely continuous norm).

3.3 Norms in terms of minima

Fan [5] observed that the Amemiya norm and the Luxemburg norm coincide for a given Orlicz functional. This result includes the variable exponent Lebesgue spaces as a special case.

Recall that the Amemiya norm is defined as follows in the latter case:

$$\|f\|_{A, p(\cdot)} := \inf_{K > 0} \frac{1}{K} \left(1 + \int_{\Omega} (K|f(x)|)^{p(x)} dx \right).$$

It is easy to see that the term inside the infimum tends to infinity as $K \searrow 0$ or $K \nearrow \infty$ for an essentially non-zero function f . On the other hand, by convexity

considerations the infimum is actually a minimum occurring for a unique value of K . Thus, a natural question occurs: What is the value of K , depending on f and the space $L^{p(\cdot)}(\mu)$? It turns out that there is a neat expression for the minimum. We may rewrite the above definition in a form more compatible with our approach here:

$$\|f\|_{A,p(\cdot)} = \inf_{\lambda>0} \lambda \left(1 + \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \right).$$

Here the function $\lambda \mapsto \lambda$ is convex and $\lambda \mapsto \int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx$ is strictly convex, provided that $f \neq 0$ and $p(\cdot) > 1$. The latter is due to the fact that $]0, \infty[\rightarrow]0, \infty[$, $x \mapsto ax^{-p}$ is strictly convex for every $a > 0$ and $p > 0$.

Proposition 3.2 *Consider an exponent $p(\cdot) > 1$ (uniformly) bounded from above, $f \in L^{p(\cdot)}(\Omega)$, $f \neq 0$, and let $K > 0$ be the value where the minimum in the Amemiya norm is attained. Then*

$$K = \left(\|f\|_{\text{MO}, L_v^{p(\cdot)}(\Omega)} \right)^{-1}$$

where v is the special weight generating the measure given by

$$v(A) = \int_A p(x) - 1 \, dx, \quad A \in \Sigma.$$

The assumption regarding the boundedness of $p(\cdot)$ is not completely innocent. These issues are discussed in the next section.

Proof of Proposition 3.2 We first prove the statement in the special case where both $p(\cdot)$ and $f(\cdot)$ are simple functions. In this case the integrals involved are actually finite sums.

According to the considerations above, we will find the unique value of λ by means of differentiation.

$$\frac{d}{d\lambda} \left(\lambda + \int_{\Omega} \frac{|f(x)|^{p(x)}}{\lambda^{p(x)-1}} dx \right) = 0.$$

This is equivalent to an equality having a familiar form:

$$\int_{\Omega} (p(x) - 1) \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx = 1,$$

that is

$$\lambda = \|f\|_{\text{MO}, L^{p(\cdot)}(v)}$$

corresponds to the unique value K where the minimum in the definition of the Amemiya norm is attained.

Recall that the Musielak–Orlicz norms and the Amemiya norms coincide (see e.g. [7]), so that we may consider which ever suits us. Our observation is then extended for all measurable exponents by taking limits of Musielak–Orlicz norms with simple exponents. Finally, we approximate with simple functions f to obtain the full statement. $\square$

3.4 Varying-exponent L_p functions whose norm admits an algebraic representation

The second named author introduced in [14] a method of defining var-expo $L^{p(\cdot)}[0, 1]$ norms by means of weak solutions to suitable first-order ODEs. The measure space (in this case the Lebesgue measure on $[0, 1]$) is required to be linearly ordered in order for the ODE to be defined. There the following natural subspaces of Musielak–Orlicz $L^{p(\cdot)}$ spaces were studied as well:

$$L_0^{p(\cdot)} := \overline{\bigcup_{r < \infty} \{1_{p(\cdot) < r} f : f \in L^{p(\cdot)}\}} \subset L^{p(\cdot)}.$$

Clearly, if $p(\cdot)$ is essentially bounded, then $L_0^{p(\cdot)} = L^{p(\cdot)}$. As observed in [14], the ODE-determined class $L^{p(\cdot)}$ need not be a linear space for an essentially unbounded exponent $p(\cdot)$, but nevertheless the following facts hold:

- If $p(\cdot)$ is essentially bounded or non-decreasing, then the corresponding ODE-determined class $L^{p(\cdot)}$ is a Banach space in the natural way and its norm is equivalent to the Luxemburg norm of the Musielak–Orlicz space by an isomorphism constant 2.
- The class $L_0^{p(\cdot)}$ is in any case a Banach space and its norm is equivalent to the Luxemburg norm of the Musielak–Orlicz space by an isomorphism constant 2.

Theorem 3.3 *Given a Σ -measurable function, measurable $p: \Omega \rightarrow [1, \infty)$, and $f \in L^{p(\cdot)}(\Omega, \Sigma, \mu)$, a Musielak–Orlicz space, and $\|f\|_{\text{MO}, p(\cdot)} = 1$. Consider the following conditions:*

(1) *The mapping*

$$\lambda \mapsto \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} d\mu(x)$$

is continuous at $\lambda = 1$;

(2) *The function $f \in L_0^{p(\cdot)}(\Omega, \Sigma, \mu)$;*

(3) $\int_{\Omega} |f(x)|^{p(x)} d\mu(x) = 1$.

Then (1) $\iff$ (2) $\implies$ (3). The particular value of the norm is not restrictive.

We note that the last condition (3) may hold even if the integral in (1) is infinite for every $0 < \lambda < 1$.

Proof of Theorem 3.3 Let us make the assumption of the theorem. Then it is clear that the first condition implies the last one.

The fact that

$$\lambda \mapsto \left| \frac{f(x)}{\lambda} \right|^{p(x)}$$

is decreasing for $f(x) \neq 0$ and the Monotone Convergence Theorem, we obtain that in (1) the mapping is continuous from the right.

It easily follows from the continuity of the mapping in condition (1) from the left that

$$\int_{\Omega} |1_{p(\cdot) > n}(x) f(x)|^{p(x)} d\mu(x) \longrightarrow 0 \text{ as } n \rightarrow \infty.$$

Then condition (2) follows easily.

Finally, let us check the last implication, namely, that condition (2) implies the first one. If $p(\cdot)$ is essentially bounded, then all the conditions (1)–(3) are readily true, so assume that $p(\cdot)$ is essentially unbounded. Let $f \in L_0^{p(\cdot)}(\Omega, \Sigma, \mu)$. According to the Dominated Convergence Theorem it suffices to verify that there is $0 < \lambda < 1$ such that

$$\int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} d\mu(x) < \infty.$$

To achieve this, we apply the observations in [15] to find a linear isometry

$$T: L_0^{p(\cdot)}(\Omega, \Sigma, \mu) \longrightarrow L_0^{p'(\cdot)}[0, 1]$$

where the domain is the Musielak–Orlicz space over $\Omega \subset \mathbb{R}^n$ with the Luxemburg norm and the target space is a similar space, except over the unit interval. This is obtained by a weighted composition operator. The composition is given by the lifting of the Boolean isomorphism between the measure algebras, that is, a measure-preserving bijection $\Omega \rightarrow [0, 1]$, where μ is normalized to be a probability measure. Here p' is obtained from p by applying the lifting $L: \Omega \rightarrow [0, 1]$ as follows: $p'(t) := p \circ L^{-1}(t)$. Then, by using the inequalities established in the same paper, we find an isomorphism

$$S: L_0^{p'(\cdot)}[0, 1] \rightarrow L_0^{p''(\cdot)}[0, 1]$$

by a measure-preserving rearrangement, where p'' becomes a non-decreasing rearrangement of p' .

Here $L_0^{p''(\cdot)}[0, 1] = L^{p''(\cdot)}[0, 1]$ can be regarded as an ODE-determined Banach function space introduced in [14]. The equality of the spaces is due to the fact that $p''(\cdot)$ is non-decreasing. The norm of the ODE-determined function space norm is equivalent to the Luxemburg norm in question.

Then

$$R = S \circ T: L_0^{p(\cdot)}(\Omega, \Sigma, \mu) \longrightarrow L_0^{p''(\cdot)}[0, 1]$$

defines a linear isomorphism which satisfies the following condition:

$$R(1_{p(\cdot) > \alpha} f) = 1_{p''(\cdot) > \alpha} R(f)$$

for every $\alpha \geq 1$. If μ is a probability measure we have $R(1_{p(\cdot) > \alpha}) = 1_{p''(\cdot) > \alpha}$ for every α .

Let us investigate $g = R(f)$. Since $g \in L_0^{p''(\cdot)}[0, 1]$ where p'' is non-decreasing, we observe that $\|1_{[t, 1]} g\|_{\text{ODE}, p''(\cdot)} \rightarrow 0$ as $t \rightarrow 1^-$.

We shall exploit the precise form of the differential equation,

$$\varphi_g(0) = 0, \quad \varphi'_g(t) = \frac{|g(t)|^{p''(t)}}{p''(t)} (\varphi_g(t))^{1-p''(t)}$$

whose solution, in the sense of Carathéodory, produces the ODE-determined norm:

$$\|g\|_{\text{ODE}, p''(\cdot)} := \varphi_g(1).$$

Thus, let $t_0 \in (0, 1)$ be such that $h := 1_{[t_0, 1]} g$ satisfies

$$0 < \|h\|_{\text{ODE}, p''(\cdot)} \leq \frac{1}{2}.$$

Let $\lambda = \frac{3}{4}$. Let us study the ODE which produces the norm of h :

$$\varphi_h(t_0) = 0, \quad \varphi'_h(t) = \frac{|g(t)|^{p''(t)}}{p''(t)} (\varphi_g(t))^{1-p''(t)}, \quad t \in [t_0, 1].$$

Recall that $p''(t) \rightarrow \infty$ as $t \nearrow 1$. This means that there is $t_1 \in [t_0, 1)$ such that

$$p''(t) \ll \left(\frac{\lambda}{\varphi_g(t)} \right)^{p''(t)}, \quad t \in [t_1, 1].$$

since the exponent function grows at a super-linear rate. Therefore, without loss of generality, recalling that $\varphi_g(t) \rightarrow \|h\|_{\text{ODE}, p''(\cdot)}$ as $t \nearrow 1$, we may assume that

$$\left(\frac{|g(t)|}{\lambda} \right)^{p''(t)} \leq \frac{|g(t)|^{p''(t)}}{p''(t)} (\varphi_g(t))^{1-p''(t)}, \quad t \in [t_1, 1]$$

Thus,

$$\int_{t_1}^1 \left(\frac{|g(t)|}{\lambda} \right)^{p''(t)} dt \leq \int_{t_1}^1 \frac{|g(t)|^{p''(t)}}{p''(t)} (\varphi_g(t))^{1-p''(t)} dt \leq \|h\|_{\text{ODE}, p''(\cdot)} \leq \frac{1}{2}.$$

Consequently, since $p''(\cdot)$ is bounded on $[0, t_1]$, we have

$$\int_0^1 \left(\frac{|g(t)|}{\lambda} \right)^{p''(t)} dt < \infty.$$

Finally, by using the fact that a simultaneous rearrangement of $g(\cdot)$ and $p''(\cdot)$ does not affect the value of the above integral, we obtain that

$$\int_0^1 (|T(f/\lambda)[t]|)^{p'(t)} dt = \int_0^1 \left(\frac{|T(f)[t]|}{\lambda} \right)^{p'(t)} dt < \infty.$$

By recalling the construction of T we obtain that

$$\int_{\Omega} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} d\mu(x) = \int_0^1 (|T(f/\lambda)[t]|)^{p'(t)} dt < \infty.$$

This concludes the proof. $\square$

Although the Luxemburg-norm associated to the Musielak–Orlicz space is equivalent to the ODE-determined norm, these norms have some dissimilarities. In the ODE-determined class one can approximate the value $\|f\|_{\text{ODE}, L^{p(\cdot)}}$ for any f in the class as follows: one may restrict the support of f to obtain a sequence of functions $f_n := 1_{C_n} f$ with $|f_n(t)|^{p(t)}$ essentially bounded such that $\|f_n\|_{\text{ODE}, L^{p(\cdot)}} \rightarrow \|f\|_{\text{ODE}, L^{p(\cdot)}}$ and $f_n \rightarrow f$ a.e. as $n \rightarrow \infty$. (see [14]). The similar statement does not hold true for $\|\cdot\|_{\text{MO}, L^{p(\cdot)}}$ norms. Indeed, to see this, one can construct an example such that

$$\int_0^1 |f(x)|^{p(x)} dx < 1$$

but

$$\int_0^1 \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx = \infty$$

for all $\lambda < 1$.

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