

An isoperimetric inequality in the plane with a log-convex density

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Abstract Given a positive lower semi-continuous density f on \mathbb{R}^2 the weighted volume $V_f := \int f \mathcal{L}^2$ is defined on the \mathcal{L}^2 -measurable sets in \mathbb{R}^2 . The f -weighted perimeter of a set of finite perimeter E in \mathbb{R}^2 is written $P_f(E)$. We study minimisers for the weighted isoperimetric problem

$$I_f(v) := \inf \left\{ P_f(E) : E \text{ is a set of finite perimeter in } \mathbb{R}^2 \text{ and } V_f(E) = v \right\}$$

for $v > 0$. Suppose f takes the form $f : \mathbb{R}^2 \rightarrow (0, +\infty); x \mapsto e^{h(|x|)}$ where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Let $v > 0$ and B a centred ball in \mathbb{R}^2 with $V_f(B) = v$. We show that B is a minimiser for the above variational problem and obtain a uniqueness result.

Keywords Isoperimetric problem · Log-convex density · Generalised mean curvature

Mathematics Subject Classification 49Q20

1 Introduction

Let f be a positive lower semi-continuous density on \mathbb{R}^2 . The weighted volume $V_f := \int f \mathcal{L}^2$ is defined on the \mathcal{L}^2 -measurable sets in \mathbb{R}^2 . Let E be a set of finite perimeter in \mathbb{R}^2 . The weighted perimeter of E is defined by

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$$P_f(E) := \int_{\mathbb{R}^2} f \, d|D\chi_E| \in [0, +\infty]. \tag{1.1}$$

We study minimisers for the weighted isoperimetric problem

$$I_f(v) := \inf \left\{ P_f(E) : E \text{ is a set of finite perimeter in } \mathbb{R}^2 \text{ and } V_f(E) = v \right\} \tag{1.2}$$

for $v > 0$. To be more specific we suppose that f takes the form

$$f : \mathbb{R}^2 \rightarrow (0, +\infty); x \mapsto e^{h(|x|)} \tag{1.3}$$

where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Our first main result is the following. It contains the classical isoperimetric inequality (cf. [9, 12]) as a special case; namely, when h is constant on $[0, +\infty)$.

Theorem 1.1 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Let $v > 0$ and B a centred ball in \mathbb{R}^2 with $V_f(B) = v$. Then B is a minimiser for (1.2).*

For $x \geq 0$ and $v \geq 0$ define the directional derivative of h in direction v by

$$h'_+(x, v) := \lim_{t \downarrow 0} \frac{h(x + tv) - h(x)}{t} \in \mathbb{R}$$

and define $h'_-(x, v)$ similarly for $x > 0$ and $v \leq 0$. We introduce the notation

$$\rho_+ := h'_+(\cdot, +1), \rho_- := -h'_+(\cdot, -1) \text{ and } \rho := (1/2)(\rho_+ + \rho_-)$$

on $(0, +\infty)$. The function h is locally of bounded variation and is differentiable a.e. with $h' = \rho$ a.e. on $(0, +\infty)$. Our second main result is a uniqueness theorem.

Theorem 1.2 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Suppose that $R := \inf\{\rho > 0\} \in [0, +\infty)$ and set $v_0 := V(B(0, R))$. Let $v > 0$ and E a minimiser for (1.2). The following hold:*

- (i) if $v \leq v_0$ then E is a.e. equivalent to a ball B in $\overline{B}(0, R)$ with $V(B) = V(E)$;
- (ii) if $v > v_0$ then E is a.e. equivalent to a centred ball B with $V(B) = V(E)$.

Theorem 1.1 is a generalisation of Conjecture 3.12 in [24] (due to K. Brakke) in the sense that less regularity is required of the density f : in the latter, h is supposed to be smooth on $(0, +\infty)$ as well as convex and non-decreasing. This conjecture springs in part from the observation that the weighted perimeter of a local volume-preserving perturbation of a centred ball is non-decreasing ([24] Theorem 3.10). In addition, the conjecture holds for log-convex Gaussian densities of the form $h : [0, +\infty) \rightarrow \mathbb{R}; t \mapsto e^{ct^2}$ with $c > 0$ ([3, 24] Theorem 5.2). In subsequent work partial forms of the conjecture were proved in the literature. In [19] it is shown to hold for large v provided that h is uniformly convex in the sense that $h'' \geq 1$ on $(0, +\infty)$ (see [19] Corollary

6.8). A complementary result is contained in [11] Theorem 1.1 which establishes the conjecture for small v on condition that h'' is locally uniformly bounded away from zero on $[0, +\infty)$. The above-mentioned conjecture is proved in large part in [7] (see Theorem 1.1) in dimension $n \geq 2$ (see also [4]). There it is assumed that the function h is of class C^3 on $(0, +\infty)$ and is convex and even (meaning that h is the restriction of an even function on \mathbb{R} to $[0, +\infty)$). A uniqueness result is also obtained ([7] Theorem 1.2). We obtain these results under weaker hypotheses in the 2-dimensional case and our proofs proceed along different lines.

We give a brief outline of the article. In Sect. 2 we discuss some preliminary material. In Sect. 3 we show that (1.2) admits an open minimiser E with C^1 boundary M (Theorem 3.8). The argument draws upon the regularity theory for almost minimal sets (cf. [27]) and includes an adaptation of [21] Proposition 3.1. In Sect. 4 it is shown that the boundary M is of class $C^{1,1}$ (and has weakly bounded curvature). This result is contained in [21] Corollary 3.7 (see also [8]) but we include a proof for completeness. This Section also includes the result that E may be supposed to possess spherical cap symmetry (Theorem 4.5). Section 5 contains further results on spherical cap symmetric sets useful in the sequel. The main result of Sect. 6 is Theorem 6.5 which shows that the generalised (mean) curvature is conserved along M in a weak sense. In Sect. 7 it is shown that there exist convex minimisers of (1.2). Sections 8 and 9 comprise an analytic interlude and are devoted to the study of solutions of the first-order differential equation that appears in Theorem 6.6 subject to Dirichlet boundary conditions. Section 9 for example contains a comparison theorem for solutions to a Riccati equation (Theorem 9.15 and Corollary 9.16). These are new as far as the author is aware. Section 10 concludes the proof of our main theorems.

2 Some preliminaries

Geometric measure theory. We use $|\cdot|$ to signify the Lebesgue measure on \mathbb{R}^2 (or occasionally \mathcal{L}^2). Let E be a \mathcal{L}^2 -measurable set in \mathbb{R}^2 . The set of points in E with density $t \in [0, 1]$ is given by

$$E^t := \left\{ x \in \mathbb{R}^2 : \lim_{\rho \downarrow 0} \frac{|E \cap B(x, \rho)|}{|B(x, \rho)|} = t \right\}.$$

As usual $B(x, \rho)$ denotes the open ball in \mathbb{R}^2 with centre $x \in \mathbb{R}^2$ and radius $\rho > 0$. The set E^1 is the measure-theoretic interior of E while E^0 is the measure-theoretic exterior of E . The essential boundary of E is the set $\partial^* E := \mathbb{R}^2 \setminus (E^0 \cup E^1)$.

Recall that an integrable function u on \mathbb{R}^2 is said to have bounded variation if the distributional derivative of u is representable by a finite Radon measure Du (cf. [1] Definition 3.1 for example) with total variation $|Du|$; in this case, we write $u \in \text{BV}(\mathbb{R}^2)$. The set E has finite perimeter if χ_E belongs to $\text{BV}_{\text{loc}}(\mathbb{R}^2)$. The reduced boundary $\mathcal{F}E$ of E is defined by

$$\mathcal{F}E := \left\{ x \in \text{supp}|D\chi_E| : \nu^E(x) := \lim_{\rho \downarrow 0} \frac{D\chi_E(B(x, \rho))}{|D\chi_E|(B(x, \rho))} \text{ exists in } \mathbb{R}^2 \text{ and } |\nu^E(x)| = 1 \right\}$$

(cf. [1] Definition 3.54) and is a Borel set (cf. [1] Theorem 2.22 for example). We use \mathcal{H}^k ($k \in [0, +\infty)$) to stand for k -dimensional Hausdorff measure. If E is a set of finite perimeter in \mathbb{R}^2 then

$$\mathcal{F}E \subset E^{1/2} \subset \partial^*E \text{ and } \mathcal{H}^1(\partial^*E \setminus \mathcal{F}E) = 0 \tag{2.1}$$

by [1] Theorem 3.61.

Let f be a positive locally Lipschitz density on \mathbb{R}^2 . Let E be a set of finite perimeter and U a bounded open set in \mathbb{R}^2 . The weighted perimeter of E relative to U is defined by

$$P_f(E, U) := \sup \left\{ \int_U \operatorname{div}(fX) \, dx : X \in C_c^\infty(U, \mathbb{R}^2), \|X\|_\infty \leq 1 \right\}.$$

By the Gauss–Green formula ([1] Theorem 3.36 for example) and a convolution argument,

$$\begin{aligned} P_f(E, U) &= \sup \left\{ \int_{\mathbb{R}^2} f \langle \nu^E, X \rangle \, d|D\chi_E| : X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2), \right. \\ &\quad \left. \operatorname{supp}[X] \subset U, \|X\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}^2} f \langle \nu^E, X \rangle \, d|D\chi_E| : X \in C_c(\mathbb{R}^2, \mathbb{R}^2), \right. \\ &\quad \left. \operatorname{supp}[X] \subset U, \|X\|_\infty \leq 1 \right\} \\ &= \int_U f \, d|D\chi_E| \end{aligned} \tag{2.2}$$

where we have also used [1] Propositions 1.47 and 1.23.

Lemma 2.1 *Let φ be a C^1 diffeomorphism of \mathbb{R}^2 which coincides with the identity map on the complement of a compact set and $E \subset \mathbb{R}^2$ with $\chi_E \in \operatorname{BV}(\mathbb{R}^2)$. Then*

- (i) $\chi_{\varphi(E)} \in \operatorname{BV}(\mathbb{R}^2)$;
- (ii) $\partial^*\varphi(E) = \varphi(\partial^*E)$;
- (iii) $\mathcal{H}^1(\mathcal{F}\varphi(E) \Delta \varphi(\mathcal{F}E)) = 0$.

Proof Part (i) follows from [1] Theorem 3.16 as φ is a proper Lipschitz function. Given $x \in E^0$ we claim that $y := \varphi(x) \in \varphi(E)^0$. Let M stand for the Lipschitz constant of φ and L stand for the Lipschitz constant of φ^{-1} . Note that $B(y, r) \subset \varphi(B(x, Lr))$ for each $r > 0$. As φ is a bijection and using [1] Proposition 2.49,

$$\begin{aligned} |\varphi(E) \cap B(y, r)| &\leq |\varphi(E) \cap \varphi(B(x, Lr))| \\ &= |\varphi(E \cap B(x, Lr))| \leq M^2|E \cap B(x, Lr)|. \end{aligned}$$

This means that

$$\frac{|\varphi(E) \cap B(y, r)|}{|B(y, r)|} \leq (LM)^2 \frac{|E \cap B(x, Lr)|}{|B(x, Lr)|}$$

for $r > 0$ and this proves the claim. This entails that $\varphi(E^0) \subset [\varphi(E)]^0$. The reverse inclusion can be seen using the fact that φ is a bijection. In summary $\varphi(E^0) = [\varphi(E)]^0$. The corresponding identity for E^1 can be seen in a similar way. These identities entail (ii). From (2.1) and (ii) we may write $\mathcal{F}\varphi(E) \cup N_1 = \varphi(\mathcal{F}E) \cup \varphi(N_2)$ for \mathcal{H}^1 -null sets N_1, N_2 in \mathbb{R}^2 . Item (iii) follows. \square

Curves with weakly bounded curvature. Suppose the open set E in \mathbb{R}^2 has C^1 boundary M . Denote by $n : M \rightarrow \mathbb{S}^1$ the inner unit normal vector field. Given $p \in M$ we choose a tangent vector $t(p) \in \mathbb{S}^1$ in such a way that the pair $\{t(p), n(p)\}$ forms a positively oriented basis for \mathbb{R}^2 . There exists a local parametrisation $\gamma_1 : I \rightarrow M$ where $I = (-\delta, \delta)$ for some $\delta > 0$ of class C^1 with $\gamma_1(0) = p$. We always assume that γ_1 is parametrised by arc-length and that $\dot{\gamma}_1(0) = t(p)$ where the dot signifies differentiation with respect to arc-length. Let X be a vector field defined in some neighbourhood of p in M . Then

$$(D_t X)(p) := \left. \frac{d}{ds} \right|_{s=0} (X \circ \gamma_1)(s) \tag{2.3}$$

if this limit exists and the divergence $\text{div}^M X$ of X along M at p is defined by

$$\text{div}^M X := \langle D_t X, t \rangle \tag{2.4}$$

evaluated at p . Suppose that X is a vector field in $C^1(U, \mathbb{R}^2)$ where U is an open neighbourhood of p in \mathbb{R}^2 . Then

$$\text{div} X = \text{div}^M X + \langle D_n X, n \rangle \tag{2.5}$$

at p . If $p \in M \setminus \{0\}$ let $\sigma(p)$ stand for the angle measured anti-clockwise from the position vector p to the tangent vector $t(p)$; $\sigma(p)$ is uniquely determined up to integer multiples of 2π .

Let E be an open set in \mathbb{R}^2 with $C^{1,1}$ boundary M . Let $x \in M$ and $\gamma_1 : I \rightarrow M$ a local parametrisation of M in a neighbourhood of x . There exists a constant $c > 0$ such that

$$|\dot{\gamma}_1(s_2) - \dot{\gamma}_1(s_1)| \leq c|s_2 - s_1|$$

for $s_1, s_2 \in I$; a constraint on average curvature (cf. [10, 18]). That is, $\dot{\gamma}_1$ is Lipschitz on I . So $\dot{\gamma}_1$ is absolutely continuous and differentiable a.e. on I with

$$\dot{\gamma}_1(s_2) - \dot{\gamma}_1(s_1) = \int_{s_1}^{s_2} \ddot{\gamma}_1 ds \tag{2.6}$$

for any $s_1, s_2 \in I$ with $s_1 < s_2$. Moreover, $|\ddot{\gamma}_1| \leq c$ a.e. on I (cf. [1] Corollary 2.23). As $\langle \dot{\gamma}_1, \dot{\gamma}_1 \rangle = 1$ on I we see that $\langle \dot{\gamma}_1, \ddot{\gamma}_1 \rangle = 0$ a.e. on I . The (geodesic) curvature k_1 is then defined a.e. on I via the relation

$$\ddot{\gamma}_1 = k_1 n_1 \tag{2.7}$$

as in [18]. The curvature k of M is defined \mathcal{H}^1 -a.e. on M by

$$k(x) := k_1(s) \tag{2.8}$$

whenever $x = \gamma_1(s)$ for some $s \in I$ and $k_1(s)$ exists. We sometimes write $H(\cdot, E) = k$.

Let E be an open set in \mathbb{R}^2 with C^1 boundary M . Let $x \in M$ and $\gamma_1 : I \rightarrow M$ a local parametrisation of M in a neighbourhood of x . In case $\gamma_1 \neq 0$ let θ_1 stand for the angle measured anti-clockwise from e_1 to the position vector γ_1 and σ_1 stand for the angle measured anti-clockwise from the position vector γ_1 to the tangent vector $t_1 = \dot{\gamma}_1$. Put $r_1 := |\gamma_1|$ on I . Then $r_1, \theta_1 \in C^1(I)$ and

$$\dot{r}_1 = \cos \sigma_1; \tag{2.9}$$

$$r_1 \dot{\theta}_1 = \sin \sigma_1; \tag{2.10}$$

on I provided that $\gamma_1 \neq 0$. Now suppose that M is of class $C^{1,1}$. Let α_1 stand for the angle measured anti-clockwise from the fixed vector e_1 to the tangent vector t_1 (uniquely determined up to integer multiples of 2π). Then $t_1 = (\cos \alpha_1, \sin \alpha_1)$ on I so α_1 is absolutely continuous on I . In particular, α_1 is differentiable a.e. on I with $\dot{\alpha}_1 = k_1$ a.e. on I . This means that $\alpha_1 \in C^{0,1}(I)$. In virtue of the identities $r_1 \cos \sigma_1 = \langle \gamma_1, t_1 \rangle$ and $r_1 \sin \sigma_1 = -\langle \gamma_1, n_1 \rangle$ we see that σ_1 is absolutely continuous on I and $\sigma_1 \in C^{0,1}(I)$. By choosing an appropriate branch we may assume that

$$\alpha_1 = \theta_1 + \sigma_1 \tag{2.11}$$

on I . We may choose σ in such a way that $\sigma \circ \gamma_1 = \sigma_1$ on I .

Flows. Recall that a diffeomorphism $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is said to be proper if $\varphi^{-1}(K)$ is compact whenever $K \subset \mathbb{R}^2$ is compact. Given $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ there exists a 1-parameter group of proper C^∞ diffeomorphisms $\varphi : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ as in [20] Lemma 2.99 that satisfy

$$\begin{aligned} \partial_t \varphi(t, x) &= X(\varphi(t, x)) \text{ for each } (t, x) \in \mathbb{R} \times \mathbb{R}^2; \\ \varphi(0, x) &= x \text{ for each } x \in \mathbb{R}^2. \end{aligned} \tag{2.12}$$

We often use φ_t to refer to the diffeomorphism $\varphi(t, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$.

Lemma 2.2 *Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and φ be the corresponding flow as above. Then*

(i) *there exists $R \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2)$ and $K > 0$ such that*

$$\varphi(t, x) = \begin{cases} x + tX(x) + R(t, x) & \text{for } x \in \text{supp}[X]; \\ x & \text{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R(t, x)| \leq Kt^2$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^2$;

(ii) there exists $R^{(1)} \in C^\infty(\mathbb{R} \times \mathbb{R}^2, M_2(\mathbb{R}))$ and $K_1 > 0$ such that

$$d\varphi(t, x) = \begin{cases} I + t dX(x) + R^{(1)}(t, x) & \text{for } x \in \text{supp}[X]; \\ I & \text{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R^{(1)}(t, x)| \leq K_1 t^2$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^2$;

(iii) there exists $R^{(2)} \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ and $K_2 > 0$ such that

$$J_2 d\varphi(t, x) = \begin{cases} 1 + t \operatorname{div} X(x) + R^{(2)}(t, x) & \text{for } x \in \text{supp}[X]; \\ 1 & \text{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R^{(2)}(t, x)| \leq K_2 t^2$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Let $x \in \mathbb{R}^2$, v a unit vector in \mathbb{R}^2 and M the line through x perpendicular to v . Then

(iv) there exists $R^{(3)} \in C^\infty(\mathbb{R} \times \mathbb{R}^2, \mathbb{R})$ and $K_3 > 0$ such that

$$J_1 d^M \varphi(t, x) = \begin{cases} 1 + t (\operatorname{div}^M X)(x) + R^{(3)}(t, x) & \text{for } x \in \text{supp}[X]; \\ 1 & \text{for } x \notin \text{supp}[X]; \end{cases}$$

where $|R^{(3)}(t, x)| \leq K_3 t^2$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Proof (i) First notice that $\varphi \in C^\infty(\mathbb{R} \times \mathbb{R}^2)$ by [16] Theorem 3.3 and Exercise 3.4. The statement for $x \notin \text{supp}[X]$ follows by uniqueness (cf. [16] Theorem 3.1); the assertion for $x \in \text{supp}[X]$ follows from Taylor’s theorem. (ii) follows likewise: note, for example, that

$$[\partial_{t_i} d\varphi]_{\alpha\beta}|_{t=0} = X_{,\beta\delta}^\alpha X^\delta + X_{,\gamma}^\alpha X_{,\beta}^\gamma$$

where the subscript \cdot signifies partial differentiation. (iii) follows from (ii) and the definition of the 2-dimensional Jacobian (cf. [1] Definition 2.68). (iv) Using [1] Definition 2.68 together with the Cauchy–Binet formula [1] Proposition 2.69, $J_1 d^M \varphi(t, x) = |d\varphi(t, x)v|$ for $t \in \mathbb{R}$ and the result follows from (ii). \square

Let I be an open interval in \mathbb{R} containing 0. Let $Z : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; (t, x) \mapsto Z(t, x)$ be a continuous time-dependent vector field on \mathbb{R}^2 with the properties

- (Z.1) $Z(t, \cdot) \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ for each $t \in I$;
- (Z.2) $\text{supp}[Z(t, \cdot)] \subset K$ for each $t \in I$ for some compact set $K \subset \mathbb{R}^2$.

By [16] Theorems I.1.1, I.2.1, I.3.1, I.3.3 there exists a unique flow $\varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

- (F.1) $\varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is of class C^1 ;
- (F.2) $\varphi(0, x) = x$ for each $x \in \mathbb{R}^2$;
- (F.3) $\partial_t \varphi(t, x) = Z(t, \varphi(x, t))$ for each $(t, x) \in I \times \mathbb{R}^2$;
- (F.4) $\varphi_t := \varphi(t, \cdot) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a proper diffeomorphism for each $t \in I$.

Lemma 2.3 *Let Z be a time-dependent vector field with the properties (Z.1)–(Z.2) and φ be the corresponding flow. Then*

(i) *for $(t, x) \in I \times \mathbb{R}^2$,*

$$d\varphi(t, x) = \begin{cases} I + t dZ_0(x) + tR(t, x) & \text{for } x \in K; \\ I & \text{for } x \notin K; \end{cases}$$

where $\sup_K |R(t, \cdot)| \rightarrow 0$ as $t \rightarrow 0$.

Let $x \in \mathbb{R}^2$, v a unit vector in \mathbb{R}^2 and M the line through x perpendicular to v . Then

(ii) *for $(t, x) \in I \times \mathbb{R}^2$,*

$$J_1 d^M \varphi(t, x) = \begin{cases} 1 + t(\operatorname{div}^M Z_0)(x) + tR^{(1)}(t, x) & \text{for } x \in K; \\ 1 & \text{for } x \notin K. \end{cases}$$

where $\sup_K |R^{(1)}(t, \cdot)| \rightarrow 0$ as $t \rightarrow 0$.

Proof (i) We first remark that the flow $\varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to Z is continuously differentiable in t, x in virtue of (Z.1) by [16] Theorem I.3.3. Put $y(t, x) := d\varphi(t, x)$ for $(t, x) \in I \times \mathbb{R}^2$. By [16] Theorem I.3.3,

$$\dot{y}(t, x) = dZ(t, \varphi(t, x))y(t, x)$$

for each $(t, x) \in I \times \mathbb{R}^2$ and $y(0, x) = I$ for each $x \in \mathbb{R}^2$ where I stands for the 2×2 -identity matrix. For $x \in K$ and $t \in I$,

$$\begin{aligned} d\varphi(t, x) &= I + d\varphi(t, x) - d\varphi(0, x) \\ &= I + t\dot{y}(0, x) + t \left\{ \frac{d\varphi(t, x) - d\varphi(0, x)}{t} - \dot{y}(0, x) \right\} \\ &= I + t dZ(0, x) + t \left\{ \frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \right\} \\ &= I + t dZ_0(x) + t \left\{ \frac{y(t, x) - y(0, x)}{t} - \dot{y}(0, x) \right\}. \end{aligned}$$

Applying the mean-value theorem component-wise and using uniform continuity of the matrix \dot{y} in its arguments we see that

$$\frac{y(t, \cdot) - y(0, \cdot)}{t} - \dot{y}(0, \cdot) \rightarrow 0$$

uniformly on K as $t \rightarrow 0$. This leads to (i). Part (ii) follows as in Lemma 2.2. □

Let E be a set of finite perimeter in \mathbb{R}^2 with $V_f(E) < +\infty$. The first variation of weighted volume resp. perimeter along $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is defined by

$$\delta V_f(X) := \left. \frac{d}{dt} \right|_{t=0} V_f(\varphi_t(E)), \tag{2.13}$$

$$\delta P_f^+(X) := \lim_{t \downarrow 0} \frac{P_f(\varphi_t(E)) - P_f(E)}{t}, \tag{2.14}$$

whenever the limit exists. By Lemma 2.1 the f -perimeter in (2.14) is well-defined. *Convex functions.* Suppose that $h : [0, +\infty) \rightarrow \mathbb{R}$ is a convex function. For $x \geq 0$ and $v \geq 0$ define

$$h'_+(x, v) := \lim_{t \downarrow 0} \frac{h(x + tv) - h(x)}{t} \in \mathbb{R}$$

and define $h'_-(x, v)$ similarly for $x > 0$ and $v \leq 0$. For future use we introduce the notation

$$\rho_+ := h'(\cdot, +1), \rho_- := -h'(\cdot, -1) \text{ and } \rho := (1/2)(\rho_+ + \rho_-)$$

on $(0, +\infty)$. It holds that h is differentiable a.e. and $h' = \rho$ a.e. on $(0, +\infty)$. Define $[\rho] := \rho_+ - \rho_-$. Then $[\rho] \geq 0$ and vanishes a.e. on $(0, +\infty)$.

Lemma 2.4 *Suppose that the function f takes the form (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a convex function. Then*

- (i) *the directional derivative $f'_+(x, v)$ exists in \mathbb{R} for each $x \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$;*
- (ii) *for $v \in \mathbb{R}^2$,*

$$f'_+(x, v) = \begin{cases} f(x)h'_+(|x|, \operatorname{sgn}(x, v)) \frac{|(x,v)|}{|x|} & \text{for } x \in \mathbb{R}^2 \setminus \{0\}; \\ f(0)h'_+(0, +1)|v| & \text{for } x = 0; \end{cases}$$

- (iii) *if M is a C^1 hypersurface in \mathbb{R}^2 such that $\cos \sigma \neq 0$ on M then f is differentiable \mathcal{H}^1 -a.e. on M and*

$$(\nabla f)(x) = f(x)\rho(|x|) \frac{\langle x, \cdot \rangle}{|x|}$$

for \mathcal{H}^1 -a.e. $x \in M$.

Proof The assertion in (i) follows from the monotonicity of chords property while (ii) is straightforward. (iii) Let $x \in M$ and $\gamma_1 : I \rightarrow M$ be a C^1 -parametrisation of M near x as above. Now $r_1 \in C^1(I)$ and $r'_1(0) = \cos \sigma(x) \neq 0$ so we may assume that $r_1 : I \rightarrow r_1(I) \subset (0, +\infty)$ is a C^1 diffeomorphism. The differentiability set $D(h)$ of h has full Lebesgue measure in $[0, +\infty)$. It follows by [1] Proposition 2.49 that $r_1^{-1}(D(h))$ has full measure in I . This entails that f is differentiable \mathcal{H}^1 -a.e. on $\gamma_1(I) \subset M$. □

3 Existence and C^1 regularity

We start with an existence theorem.

Theorem 3.1 Assume that f is a positive radial lower-semicontinuous non-decreasing density on \mathbb{R}^2 which diverges to infinity. Then for each $v > 0$,

- (i) (1.2) admits a minimiser;
- (ii) any minimiser of (1.2) is essentially bounded.

Proof See [22] Theorems 3.3 and 5.9. □

But the bulk of this section will be devoted to a discussion of C^1 regularity.

Proposition 3.2 Let f be a positive locally Lipschitz density on \mathbb{R}^2 . Let $E \subset \mathbb{R}^2$ be a bounded set with finite perimeter. Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then

$$\delta V_f(X) = \int_E \operatorname{div}(fX) \, dx = - \int_{\mathcal{F}E} f \langle \nu^E, X \rangle \, d\mathcal{H}^1.$$

Proof Let $t \in \mathbb{R}$. By the area formula ([1] Theorem 2.71 and (2.74)),

$$V_f(\varphi_t(E)) = \int_{\varphi_t(E)} f \, dx = \int_E (f \circ \varphi_t) J_2 d(\varphi_t)_x \, dx \quad (3.1)$$

and

$$\begin{aligned} V_f(\varphi_t(E)) - V_f(E) &= \int_E (f \circ \varphi_t) J_2 d\varphi_t - f \, dx \\ &= \int_E (f \circ \varphi_t)(J_2 d\varphi_t - 1) \, dx + \int_E f \circ \varphi_t - f \, dx. \end{aligned}$$

The density f is locally Lipschitz and in particular differentiable a.e. on \mathbb{R}^2 (see [1] 2.3 for example). By the dominated convergence theorem and Lemma 2.2,

$$\begin{aligned} \delta V_f(X) &= \int_E \left\{ f \operatorname{div}(X) + \langle \nabla f, X \rangle \right\} \, dx = \int_E \operatorname{div}(fX) \, dx \\ &= - \int_{\mathcal{F}E} f \langle \nu^E, X \rangle \, d\mathcal{H}^1 \end{aligned}$$

by the generalised Gauss–Green formula [1] Theorem 3.36. □

Proposition 3.3 Let f be a positive locally Lipschitz density on \mathbb{R}^2 . Let $E \subset \mathbb{R}^2$ be a bounded set with finite perimeter. Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Then there exist constants $C > 0$ and $\delta > 0$ such that

$$|P_f(\varphi_t(E)) - P_f(E)| \leq C|t|$$

for $|t| < \delta$.

Proof Let $t \in \mathbb{R}$. By Lemma 2.1 and [1] Theorem 3.59,

$$P_f(\varphi_t(E)) = \int_{\mathbb{R}^2} f \, d|D\chi_{\varphi_t(E)}| = \int_{\mathcal{F}\varphi_t(E)} f \, d\mathcal{H}^1 = \int_{\varphi_t(\mathcal{F}E)} f \, d\mathcal{H}^1.$$

As $\mathcal{F}E$ is countably 1-rectifiable ([1] Theorem 3.59) we may use the generalised area formula [1] Theorem 2.91 to write

$$P_f(\varphi_t(E)) = \int_{\mathcal{F}E} (f \circ \varphi_t) J_1 d^{\mathcal{F}E}(\varphi_t)_x \, d\mathcal{H}^1.$$

For each $x \in \mathcal{F}E$ and any $t \in \mathbb{R}$,

$$|(f \circ \varphi_t)(x) - f(x)| \leq K|\varphi(t, x) - x| \leq K\|X\|_\infty|t|$$

where K is the Lipschitz constant of f on $\text{supp}[X]$. The result follows upon writing

$$\begin{aligned} P_f(\varphi_t(E)) - P_f(E) &= \int_{\mathcal{F}E} (f \circ \varphi_t)(J_1 d^{\mathcal{F}E}(\varphi_t)_x - 1) \\ &\quad + [f \circ \varphi_t - f] \, d\mathcal{H}^1 \end{aligned} \tag{3.2}$$

and using Lemma 2.2. □

Lemma 3.4 *Let f be a positive locally Lipschitz density on \mathbb{R}^2 . Let $E \subset \mathbb{R}^2$ be a bounded set with finite perimeter and $p \in \mathcal{F}E$. For any $r > 0$ there exists $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(p, r)$ such that $\delta V_f(X) = 1$.*

Proof By (2.2) and [1] Theorem 3.59 and (3.57) in particular,

$$P_f(E, B(p, r)) = \int_{B(p,r) \cap \mathcal{F}E} f \, d\mathcal{H}^1 > 0$$

for any $r > 0$. By the variational characterisation of the f -perimeter relative to $B(p, r)$ we can find $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[Y] \subset B(p, r)$ such that

$$0 < \int_{E \cap B(p,r)} \text{div}(fY) \, dx = - \int_{\mathcal{F}E \cap B(p,r)} f \langle \nu^E, Y \rangle \, d\mathcal{H}^1 =: c$$

where we make use of the generalised Gauss–Green formula (cf. [1] Theorem 3.36). Put $X := (1/c)Y$. Then $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(p, r)$ and $\delta V_f(X) = 1$ according to Proposition 3.2. □

Proposition 3.5 *Let f be a positive lower semi-continuous density on \mathbb{R}^2 . Let U be a bounded open set in \mathbb{R}^2 with Lipschitz boundary. Let E, F_1, F_2 be bounded sets in \mathbb{R}^2 with finite perimeter. Assume that $E \Delta F_1 \subset\subset U$ and $E \Delta F_2 \subset\subset \mathbb{R}^2 \setminus \bar{U}$. Define*

$$F := [F_1 \cap U] \cup [F_2 \setminus U].$$

Then F is a set of finite perimeter in \mathbb{R}^2 and

$$P_f(E) + P_f(F) = P_f(F_1) + P_f(F_2).$$

Proof The function $\chi_E|_U \in \text{BV}(U)$ and $D(\chi_E|_U) = (D\chi_E)|_U$. We write χ_E^U for the boundary trace of $\chi_E|_U$ (see [1] Theorem 3.87); then $\chi_E^U \in L^1(\partial U, \mathcal{H}^1 \llcorner \partial U)$ (cf. [1] Theorem 3.88). We use similar notation elsewhere. By [1] Corollary 3.89,

$$\begin{aligned} D\chi_E &= D\chi_E \llcorner U + (\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2 \setminus \bar{U}); \\ D\chi_F &= D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_{F_2}^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2 \setminus \bar{U}); \\ D\chi_{F_1} &= D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_E^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2 \setminus \bar{U}); \\ D\chi_{F_2} &= D\chi_E \llcorner U + (\chi_E^U - \chi_{F_2}^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2 \setminus \bar{U}). \end{aligned}$$

From the definition of the total variation measure ([1] Definition 1.4),

$$\begin{aligned} |D\chi_E| &= |D\chi_E \llcorner U + (\chi_E^U - \chi_E^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2 \setminus \bar{U})|; \\ |D\chi_F| &= |D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_{F_2}^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2 \setminus \bar{U})|; \\ |D\chi_{F_1}| &= |D\chi_{F_1} \llcorner U + (\chi_{F_1}^U - \chi_E^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_E \llcorner (\mathbb{R}^2 \setminus \bar{U})|; \\ |D\chi_{F_2}| &= |D\chi_E \llcorner U + (\chi_E^U - \chi_{F_2}^{\mathbb{R}^2 \setminus \bar{U}})_v^U \mathcal{H}^1 \llcorner \partial U + D\chi_{F_2} \llcorner (\mathbb{R}^2 \setminus \bar{U})|; \end{aligned}$$

where we also use the fact that $\chi_{F_1}^U = \chi_E^U$ as $E \Delta F_1 \subset\subset U$ and similarly for F_2 . The result now follows. □

Proposition 3.6 *Assume that f is a positive locally Lipschitz density on \mathbb{R}^2 . Let $v > 0$ and suppose that the set E is a bounded minimiser of (1.2). Let U be a bounded open set in \mathbb{R}^2 . There exist constants $C > 0$ and $\delta > 0$ with the following property. For any $x \in U$ and $0 < r < \delta$,*

$$P_f(E) - P_f(F) \leq C |V_f(E) - V_f(F)| \tag{3.3}$$

where F is any set with finite perimeter in \mathbb{R}^2 such that $E \Delta F \subset\subset B(x, r)$.

Proof The proof follows that of [21] Proposition 3.1. We assume to the contrary that

$$\begin{aligned} &(\forall C > 0)(\forall \delta > 0)(\exists x \in U)(\exists r \in (0, \delta))(\exists F \subset \mathbb{R}^2) \\ &\quad [F \Delta E \subset\subset B(x, r) \wedge \Delta P_f > C |\Delta V_f|] \end{aligned} \tag{3.4}$$

in the language of quantifiers where we have taken some liberties with notation.

Choose $p_1, p_2 \in \mathcal{F}E$ with $p_1 \neq p_2$. Choose $r_0 > 0$ such that the open balls $B(p_1, r_0)$ and $B(p_2, r_0)$ are disjoint. Choose vector fields $X_j \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X_j] \subset B(p_j, r_0)$ such that

$$\delta V_f(X_j) = 1 \text{ and } |P_f(\varphi_t^{(j)}(E)) - P_f(E)| \leq a_j|t| \text{ for } |t| < \delta_j \text{ and } j = 1, 2 \tag{3.5}$$

as in Lemma 3.4 and Proposition 3.3. Put $a := \max\{a_1, a_2\}$. By (3.5),

$$V_f(\varphi_t^{(j)}(E)) - V_f(E) = t + o(t) \text{ as } t \rightarrow 0 \text{ for } j = 1, 2.$$

So there exist $\varepsilon > 0$ and $1 > \eta > 0$ such that

$$\begin{aligned} t - \eta|t| < V_f(\varphi_t^{(j)}(E)) - V_f(E) < t + \eta|t|; \\ |P_f(\varphi_t^{(j)}(E)) - P_f(E)| < (a + 1)|t|; \end{aligned} \tag{3.6}$$

for $|t| < \varepsilon$ and $j = 1, 2$. In particular,

$$\begin{aligned} |V_f(\varphi_t^{(j)}(E)) - V_f(E)| > (1 - \eta)|t|; \\ |P_f(\varphi_t^{(j)}(E)) - P_f(E)| < \frac{1+a}{1-\eta}|V_f(\varphi_t^{(j)}(E)) - V_f(E)| \text{ for } |t| < \varepsilon; \end{aligned} \tag{3.7}$$

for $|t| < \varepsilon$ and $j = 1, 2$.

In (3.4) choose $C = (1 + a)/(1 - \eta)$ and $\delta > 0$ such that

- (a) $0 < 2\delta < \text{dist}(B(p_1, r_0), B(p_2, r_0))$,
- (b) $\sup\{V_f(B(x, \delta)) : x \in U\} < (1 - \eta)\varepsilon$.

Choose x, r and F_1 as in (3.4). In light of (a) we may assume that $B(x, r) \cap B(p_1, r_0) = \emptyset$. By (b),

$$|V_f(F_1) - V_f(E)| \leq V_f(B(x, r)) \leq V_f(B(x, \delta)) < (1 - \eta)\varepsilon. \tag{3.8}$$

From (3.6) and (3.8) we can find $t \in (-\varepsilon, \varepsilon)$ such that with $F_2 := \varphi_t^{(1)}(E)$,

$$V_f(F_2) - V_f(E) = -\left\{V_f(F_1) - V_f(E)\right\} \tag{3.9}$$

by the intermediate value theorem. From (3.4),

$$P_f(F_1) < P_f(E) - C|V_f(F_1) - V_f(E)| \tag{3.10}$$

while from (3.7),

$$P_f(F_2) < P_f(E) + C|V_f(F_2) - V_f(E)|. \tag{3.11}$$

Let F be the set

$$F := \left[F_1 \setminus B(p_1, r_0) \right] \cup \left[B(p_1, r_0) \cap F_2 \right].$$

Note that $E \Delta F_2 \subset\subset B(p_1, r_0)$. By Proposition 3.5, F is a bounded set of finite perimeter in \mathbb{R}^2 and

$$P_f(E) + P_f(F) = P_f(F_1) + P_f(F_2).$$

We then infer from (3.10), (3.11) and (3.9) that

$$\begin{aligned} P_f(F) &= P_f(F_1) + P_f(F_2) - P_f(E) \\ &< P_f(E) - C|V_f(F_1) - V_f(E)| + P_f(E) \\ &\quad + C|V_f(F_2) - V_f(E)| - P_f(E) = P_f(E). \end{aligned}$$

On the other hand, $V_f(F) = V_f(F_1) + V_f(F_2) - V_f(E) = V_f(E)$ by (3.9). We therefore obtain a contradiction to the f -isoperimetric property of E . \square

Let E be a set of finite perimeter in \mathbb{R}^2 and U a bounded open set in \mathbb{R}^2 . The minimality excess is the function ψ defined by

$$\psi(E, U) := P(E, U) - v(E, U) \tag{3.12}$$

where

$$v(E, U) := \inf\{P(F, U) : F \text{ is a set of finite perimeter with } F \Delta E \subset\subset U\}$$

as in [27] (1.9). We recall that the boundary of E is said to be almost minimal in \mathbb{R}^2 if for each bounded open set U in \mathbb{R}^2 there exists $T > 0$ and a positive constant K such that for every $x \in U$ and $r \in (0, T)$,

$$\psi(E, B(x, r)) \leq Kr^2. \tag{3.13}$$

This definition corresponds to [27] Definition 1.5.

Theorem 3.7 *Assume that f is a positive locally Lipschitz density on \mathbb{R}^2 . Let $v > 0$ and assume that E is a bounded minimiser of (1.2). Then the boundary of E is almost minimal in \mathbb{R}^2 .*

Proof Let U be a bounded open set in \mathbb{R}^2 and $C > 0$ and $\delta > 0$ as in Proposition 3.6. The open δ -neighbourhood of U is denoted $I_\delta(U)$. Let $x \in U$ and $r \in (0, \delta)$. Put $V := I_{2\delta}(U)$. For the sake of brevity write $m := \inf_{B(x,r)} f$ and $M := \sup_{B(x,r)} f$. Let F be a set of finite perimeter in \mathbb{R}^2 such that $F \Delta E \subset\subset B(x, r)$. By Proposition 3.6,

$$\begin{aligned}
 & P(E, B(x, r)) - P(F, B(x, r)) \\
 & \leq \frac{1}{m} P_f(E, B(x, r)) - \frac{1}{M} P_f(F, B(x, r)) \\
 & = \frac{1}{m} (P_f(E, B(x, r)) - P_f(F, B(x, r))) + \left(\frac{1}{m} - \frac{1}{M}\right) P_f(F, B(x, r)) \\
 & \leq \frac{1}{m} (P_f(E, B(x, r)) - P_f(F, B(x, r))) + \frac{M - m}{m^2} P_f(F, B(x, r)) \\
 & \leq \frac{C}{\inf_V f} |V_f(E) - V_f(F)| + (2Lr) \frac{\sup_V f}{(\inf_V f)^2} P(F, B(x, r)) \\
 & \leq C\pi r^2 \frac{\sup_V f}{\inf_V f} + (2Lr) \frac{\sup_V f}{(\inf_V f)^2} P(F, B(x, r))
 \end{aligned}$$

where L stands for the Lipschitz constant of the restriction of f to V . We then derive that

$$\psi(E, B(x, r)) \leq C\pi r^2 \frac{\sup_V f}{\inf_V f} + (2Lr) \frac{\sup_V f}{(\inf_V f)^2} \nu(E, B(x, r)).$$

By [13] (5.14), $\nu(E, B(x, r)) \leq \pi r$. The inequality in (3.13) now follows. □

Theorem 3.8 *Assume that f is a positive locally Lipschitz density on \mathbb{R}^2 . Let $v > 0$ and suppose that E is a bounded minimiser of (1.2). Then there exists a set $\tilde{E} \subset \mathbb{R}^2$ such that*

- (i) \tilde{E} is a bounded minimiser of (1.2);
- (ii) \tilde{E} is equivalent to E ;
- (iii) \tilde{E} is open and $\partial\tilde{E}$ is a C^1 hypersurface in \mathbb{R}^2 .

Proof By [13] Proposition 3.1 there exists a Borel set F equivalent to E with the property that

$$\partial F = \{x \in \mathbb{R}^2 : 0 < |F \cap B(x, \rho)| < \pi\rho^2 \text{ for each } \rho > 0\}.$$

By Theorem 3.7 and [27] Theorem 1.9, ∂F is a C^1 hypersurface in \mathbb{R}^2 (taking note of differences in notation). The set

$$\tilde{E} := \{x \in \mathbb{R}^2 : |F \cap B(x, \rho)| = \pi\rho^2 \text{ for some } \rho > 0\}$$

satisfies (i)–(iii). □

4 Weakly bounded curvature and spherical cap symmetry

Theorem 4.1 *Assume that f is a positive locally Lipschitz density on \mathbb{R}^2 . Let $v > 0$ and suppose that E is a bounded minimiser of (1.2). Then there exists a set $\tilde{E} \subset \mathbb{R}^2$ such that*

- (i) \tilde{E} is a bounded minimiser of (1.2);
- (ii) \tilde{E} is equivalent to E ;

(iii) \tilde{E} is open and $\partial\tilde{E}$ is a $C^{1,1}$ hypersurface in \mathbb{R}^2 .

Proof We may assume that E has the properties listed in Theorem 3.8. Put $M := \partial E$. Let $x \in M$ and U a bounded open set containing x . Choose $C > 0$ and $\delta > 0$ as in Proposition 3.6. Let $0 < r < \delta$ and $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(x, r)$. Then

$$P_f(E) - P_f(\varphi_t(E)) \leq C|V_f(E) - V_f(\varphi_t(E))|$$

for each $t \in \mathbb{R}$. From the identity (3.2),

$$\begin{aligned} - \int_M (f \circ \varphi_t)(J_1 d^M(\varphi_t)_x - 1) d\mathcal{H}^1 &\leq C|V_f(E) - V_f(\varphi_t(E))| \\ &+ \int_M [f \circ \varphi_t - f] d\mathcal{H}^1 \\ &\leq C|V_f(E) - V_f(\varphi_t(E))| + \sqrt{2}K \|X\|_\infty \mathcal{H}^1(M \cap \text{supp}[X])t \end{aligned}$$

where K stands for the Lipschitz constant of f restricted to U . On dividing by t and taking the limit $t \rightarrow 0$ we obtain

$$\begin{aligned} - \int_M f \text{div}^M X d\mathcal{H}^1 &\leq C \left| \int_M f \langle n, X \rangle d\mathcal{H}^1 \right| \\ &+ \sqrt{2}K \|X\|_\infty \mathcal{H}^1(M \cap \text{supp}[X]) \end{aligned}$$

upon using Lemma 2.2 and Proposition 3.2. Replacing X by $-X$ we derive that

$$\left| \int_M f \text{div}^M X d\mathcal{H}^1 \right| \leq C_1 \|X\|_\infty \mathcal{H}^1(M \cap \text{supp}[X])$$

where $C_1 = C\|f\|_{L^\infty(U)} + \sqrt{2}K$. Let $\gamma_1 : I \rightarrow M$ be a local C^1 parametrisation of M near x . Suppose that $Y \in C_c^1(I, \mathbb{R}^2)$ with $\text{supp}[Y] \subset I$ and that $\gamma_1(I) \subset M \cap B(x, r)$. Note that there exists $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(x, r)$ such that $X \circ \gamma_1 = Y$ on I . The above estimate entails that

$$\left| \int_I (f \circ \gamma_1) \langle \dot{Y}, t \rangle ds \right| \leq C_1 |\text{supp}[Y]| \|Y\|_\infty.$$

This means that the function $(f \circ \gamma_1)t$ belongs to $\text{BV}(I)$ and this implies in turn that $t \in \text{BV}(I)$. For $s_1, s_2 \in I$ with $s_1 < s_2$,

$$\begin{aligned} |t(s_2) - t(s_1)| &= |Dt((s_1, s_2))| \leq |Dt((s_1, s_2))| \\ &= \sup \left\{ \int_{(s_1, s_2)} \langle t, \dot{Y} \rangle ds : Y \in C_c^1((s_1, s_2)) \text{ and } \|Y\|_\infty \leq 1 \right\} \\ &\leq c \sup \left\{ \int_{(s_1, s_2)} (f \circ \gamma_1) \langle t, \dot{Y} \rangle ds : Y \in C_c^1((s_1, s_2)) \text{ and } \|Y\|_\infty \leq 1 \right\} \\ &\leq cC_1 |s_2 - s_1| \end{aligned}$$

where $1/c = \inf_{\bar{J}} f > 0$. It follows that M is of class $C^{1,1}$. □

We turn to the topic of spherical cap symmetrisation. Denote by S^1_τ the centred circle in \mathbb{R}^2 with radius $\tau > 0$. We sometimes write S^1 for S^1_1 . Given $x \in \mathbb{R}^2$, $v \in S^1$ and $\alpha \in (0, \pi]$ the open cone with vertex x , axis v and opening angle 2α is the set

$$C(x, v, \alpha) := \left\{ y \in \mathbb{R}^2 : \langle y - x, v \rangle > |y - x| \cos \alpha \right\}.$$

Let E be an \mathcal{L}^2 -measurable set in \mathbb{R}^2 and $\tau > 0$. The τ -section E_τ of E is the set $E_\tau := E \cap S^1_\tau$. Put

$$L(\tau) = L_E(\tau) := \mathcal{H}^1(E_\tau) \text{ for } \tau > 0 \tag{4.1}$$

and $p(E) := \{\tau > 0 : L(\tau) > 0\}$. The function L is \mathcal{L}^1 -measurable by [1] Theorem 2.93. Given $\tau > 0$ and $0 < \alpha \leq \pi$ the spherical cap $C(\tau, \alpha)$ is the set

$$C(\tau, \alpha) := \begin{cases} S^1_\tau \cap C(0, e_1, \alpha) & \text{if } 0 < \alpha < \pi; \\ S^1_\tau & \text{if } \alpha = \pi; \end{cases}$$

and has \mathcal{H}^1 -measure $s(\tau, \alpha) := 2\alpha\tau$. The spherical cap symmetral E^{sc} of the set E is defined by

$$E^{sc} := \bigcup_{\tau \in p(E)} C(\tau, \alpha) \tag{4.2}$$

where $\alpha \in (0, \pi]$ is determined by $s(\tau, \alpha) = L(\tau)$. Observe that E^{sc} is a \mathcal{L}^2 -measurable set in \mathbb{R}^2 and $V_f(E^{sc}) = V_f(E)$. Note also that if B is a centred open ball then $B^{sc} = B \setminus \{0\}$. We say that E is spherical cap symmetric if $\mathcal{H}^1((E \Delta E^{sc})_\tau) = 0$ for each $\tau > 0$. This definition is broad but suits our purposes.

The result below is stated in [22] Theorem 6.2 and a sketch proof given. A proof along the lines of [2] Theorem 1.1 can be found in [23]. First, let B be a Borel set in $(0, +\infty)$; then the annulus $A(B)$ over B is the set $A(B) := \{x \in \mathbb{R}^2 : |x| \in B\}$.

Theorem 4.2 *Let E be a set of finite perimeter in \mathbb{R}^2 . Then E^{sc} is a set of finite perimeter and*

$$P(E^{sc}, A(B)) \leq P(E, A(B)) \tag{4.3}$$

for any Borel set $B \subset (0, \infty)$ and the same inequality holds with E^{sc} replaced by any set F that is \mathcal{L}^2 -equivalent to E^{sc} .

Corollary 4.3 *Let f be a positive lower semi-continuous radial function on \mathbb{R}^2 . Let E be a set of finite perimeter in \mathbb{R}^2 . Then $P_f(E^{sc}) \leq P_f(E)$.*

Proof Assume that $P_f(E) < +\infty$. We remark that f is Borel measurable as f is lower semi-continuous. Let (f_h) be a sequence of simple Borel measurable radial

functions on \mathbb{R}^2 such that $0 \leq f_h \leq f$ and $f_h \uparrow f$ on \mathbb{R}^2 as $h \rightarrow \infty$. By Theorem 4.2,

$$P_{f_h}(E^{sc}) = \int_{\mathbb{R}^2} f_h d|D\chi_{E^{sc}}| \leq \int_{\mathbb{R}^2} f_h d|D\chi_E| = P_{f_h}(E)$$

for each h . Taking the limit $h \rightarrow \infty$ the monotone convergence theorem gives $P_f(E^{sc}) \leq P_f(E)$. \square

Lemma 4.4 *Let E be an \mathcal{L}^2 -measurable set in \mathbb{R}^2 such that $E \setminus \{0\} = E^{sc}$. Then there exists an \mathcal{L}^2 -measurable set F equivalent to E such that*

- (i) $\partial F = \{x \in \mathbb{R}^2 : 0 < |F \cap B(x, \rho)| < |B(x, \rho)| \text{ for any } \rho > 0\}$;
- (ii) F is spherical cap symmetric.

Proof Put

$$E_1 := \{x \in \mathbb{R}^2 : |E \cap B(x, \rho)| = |B(x, \rho)| \text{ for some } \rho > 0\};$$

$$E_0 := \{x \in \mathbb{R}^2 : |E \cap B(x, \rho)| = 0 \text{ for some } \rho > 0\}.$$

We claim that E_1 is spherical cap symmetric. For take $x \in E_1$ with $\tau = |x| > 0$ and $|\theta(x)| \in (0, \pi]$. Now $|E \cap B(x, \rho)| = |B(x, \rho)|$ for some $\rho > 0$. Let $y \in \mathbb{R}^2$ with $|y| = \tau$ and $|\theta(y)| < |\theta(x)|$. Choose a rotation $O \in \text{SO}(2)$ such that $OB(x, \rho) = B(y, \rho)$. As $E \setminus \{0\} = E^{sc}$, $|E \cap B(y, \rho)| = |O(E \cap B(x, \rho))| = |E \cap B(x, \rho)| = |B(x, \rho)| = |B(y, \rho)|$. The claim follows. It follows in a similar way that $\mathbb{R}^2 \setminus E_0$ is spherical cap symmetric. It can then be seen that the set $F := (E_1 \cup E) \setminus E_0$ inherits this property. As in [13] Proposition 3.1 the set F is equivalent to E and enjoys the property in (i). \square

Theorem 4.5 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Then there exists an \mathcal{L}^2 -measurable set \tilde{E} with the properties*

- (i) \tilde{E} is a minimiser of (1.2);
- (ii) $L_{\tilde{E}} = L$ a.e. on $(0, +\infty)$;
- (iii) \tilde{E} is open, bounded and has $C^{1,1}$ boundary;
- (iv) $\tilde{E} \setminus \{0\} = \tilde{E}^{sc}$.

Proof Let E be a bounded minimiser for (1.2). Then $E_1 := E^{sc}$ is a bounded minimiser of (1.2) by Corollary 4.3 and $L_E = L_{E_1}$ on $(0, +\infty)$. Now put $E_2 := F$ with F as in Lemma 4.4. Then $L_{E_2} = L$ a.e. on $(0, +\infty)$ as E_2 is equivalent to E_1 , E_2 is a bounded minimiser of (1.2) and E_2 is spherical cap symmetric. Moreover, $\partial E_2 = \{x \in \mathbb{R}^2 : 0 < |E_2 \cap B(x, \rho)| < |B(x, \rho)| \text{ for any } \rho > 0\}$. As in the proof of Theorem 3.8, ∂E_2 is a C^1 hypersurface in \mathbb{R}^2 . Put

$$\tilde{E} := \{x \in \mathbb{R}^2 : |E_2 \cap B(x, \rho)| = |B(x, \rho)| \text{ for some } \rho > 0\}.$$

Then \tilde{E} is equivalent to E_2 so that (ii) holds, and is a bounded minimiser of (1.2); \tilde{E} is open and $\partial \tilde{E} = \partial E_2$ is C^1 . In fact, $\partial \tilde{E}$ is of class $C^{1,1}$ by Theorem 4.1. As E_2

is spherical cap symmetric the same is true of \tilde{E} . But \tilde{E} is open which entails that $\tilde{E} \setminus \{0\} = \tilde{E}^{sc}$. □

5 More on spherical cap symmetry

Let

$$H := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$$

stand for the open upper half-plane in \mathbb{R}^2 and

$$S : \mathbb{R}^2 \rightarrow \mathbb{R}^2; x = (x_1, x_2) \mapsto (x_1, -x_2)$$

for reflection in the x_1 -axis. Let $O \in \text{SO}(2)$ represent rotation anti-clockwise through $\pi/2$.

Lemma 5.1 *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Let $x \in M \setminus \{0\}$. Then*

- (i) $Sx \in M \setminus \{0\}$;
- (ii) $n(Sx) = Sn(x)$;
- (iii) $\cos \sigma(Sx) = -\cos \sigma(x)$.

Proof (i) The closure \bar{E} of E is spherical cap symmetric. The spherical cap symmetrals \bar{E} is invariant under S from the representation (4.2). (ii) is a consequence of this last observation. (iii) Note that $t(Sx) = O^*n(Sx) = O^*Sn(x)$. Then

$$\begin{aligned} \cos \sigma(Sx) &= \langle Sx, t(Sx) \rangle = \langle Sx, O^*Sn(x) \rangle = \langle x, SO^*Sn(x) \rangle \\ &= \langle x, On(x) \rangle = -\langle x, O^*n(x) \rangle = \cos \sigma(x) \end{aligned}$$

as $SO^*S = O$ and $O = -O^*$. □

We introduce the projection $\pi : \mathbb{R}^2 \rightarrow [0, +\infty)$; $x \mapsto |x|$.

Lemma 5.2 *Let E be an open set in \mathbb{R}^2 with boundary M and assume that $E \setminus \{0\} = E^{sc}$.*

- (i) *Suppose $0 \neq x \in \mathbb{R}^2 \setminus \bar{E}$ and $\theta(x) \in (0, \pi]$. Then there exists an open interval I in $(0, +\infty)$ containing τ and $\alpha \in (0, \theta(x))$ such that $A(I) \setminus \bar{S}(\alpha) \subset \mathbb{R}^2 \setminus \bar{E}$.*
- (ii) *Suppose $0 \neq x \in E$ and $\theta(x) \in [0, \pi)$. Then there exists an open interval I in $(0, +\infty)$ containing τ and $\alpha \in (\theta(x), \pi)$ such that $A(I) \cap S(\alpha) \subset E$.*
- (iii) *For each $0 < \tau \in \pi(M)$, M_τ is the union of two closed spherical arcs in \mathbb{S}_τ^1 symmetric about the x_1 -axis.*

Proof (i) We can find $\alpha \in (0, \theta(x))$ such that $\mathbb{S}_\tau^1 \setminus S(\alpha) \subset \mathbb{R}^2 \setminus \bar{E}$ as can be seen from definition (4.2). This latter set is compact so $\text{dist}(\mathbb{S}_\tau^1 \setminus S(\alpha), \bar{E}) > 0$. This means that the ε -neighbourhood of $\mathbb{S}_\tau^1 \setminus S(\alpha)$ is contained in $\mathbb{R}^2 \setminus \bar{E}$ for $\varepsilon > 0$ small. The claim

follows. (ii) Again from (4.2) we can find $\alpha \in (\theta(x), \pi)$ such that $\overline{\mathbb{S}_\tau^1 \cap S(\alpha)} \subset E$ and the assertion follows as before.

(iii) Suppose x_1, x_2 are distinct points in M_τ with $0 \leq \theta(x_1) < \theta(x_2) \leq \pi$. Suppose y lies in the interior of the spherical arc joining x_1 and x_2 . If $y \in \mathbb{R}^2 \setminus \overline{E}$ then $x_2 \in \mathbb{R}^2 \setminus \overline{E}$ by (i) and hence $x_2 \notin M$. If $y \in E$ we obtain the contradiction that $x_1 \in E$ by (ii). Therefore $y \in M$. We infer that the closed spherical arc joining x_1 and x_2 lies in M_τ . The claim follows noting that M_τ is closed. \square

Lemma 5.3 *Let E be an open set in \mathbb{R}^2 with C^1 boundary M . Let $x \in M$. Then*

$$\liminf_{E \ni y \rightarrow x} \left\langle \frac{y - x}{|y - x|}, n(x) \right\rangle \geq 0.$$

Proof Assume for a contradiction that

$$\liminf_{E \ni y \rightarrow x} \left\langle \frac{y - x}{|y - x|}, n(x) \right\rangle \in [-1, 0).$$

There exists $\eta \in (0, 1)$ and a sequence (y_h) in E such that $y_h \rightarrow x$ as $h \rightarrow \infty$ and

$$\left\langle \frac{y_h - x}{|y_h - x|}, n(x) \right\rangle < -\eta \tag{5.1}$$

for each $h \in \mathbb{N}$. Choose $\alpha \in (0, \pi/2)$ such that $\cos \alpha = \eta$. As M is C^1 there exists $r > 0$ such that

$$B(x, r) \cap C(x, -n(x), \alpha) \cap E = \emptyset.$$

By choosing h sufficiently large we can find $y_h \in B(x, r)$ with the additional property that $y_h \in C(x, -n(x), \alpha)$ by (5.1). We are thus led to a contradiction. \square

Lemma 5.4 *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. For each $0 < \tau \in \pi(M)$,*

- (i) $|\cos \sigma|$ is constant on M_τ ;
- (ii) $\cos \sigma = 0$ on $M_\tau \cap \{x_2 = 0\}$;
- (iii) $\langle Ox, n(x) \rangle \leq 0$ for $x \in M_\tau \cap H$
- (iv) $\cos \sigma \leq 0$ on $M_\tau \cap H$;

and if $\cos \sigma \neq 0$ on M_τ then

- (v) $\tau \in p(E)$;
- (vi) M_τ consists of two disjoint singletons in \mathbb{S}_τ^1 symmetric about the x_1 -axis;
- (vii) $L(\tau) \in (0, 2\pi\tau)$;
- (viii) $M_\tau = \{(\tau \cos(L(\tau)/2\tau), \pm\tau \sin(L(\tau)/2\tau)\}$.

Proof (i) By Lemma 5.2, M_τ is the union of two closed spherical arcs in \mathbb{S}_τ^1 symmetric about the x_1 -axis. In case $M_\tau \cap \overline{H}$ consists of a singleton the assertion follows from Lemma 5.1. Now suppose that $M_\tau \cap \overline{H}$ consists of a spherical arc in \mathbb{S}_τ^1 with non-empty

interior. It can be seen that $\cos \sigma$ vanishes on the interior of this arc as $0 = r'_1 = \cos \sigma_1$ in a local parametrisation by (2.9). By continuity $\cos \sigma = 0$ on M_τ . (ii) follows from Lemma 5.1. (iii) Let $x \in M_\tau \cap H$ so $\theta(x) \in (0, \pi)$. Then $S(\theta(x)) \cap \mathbb{S}^1_\tau \subset \bar{E}$ as \bar{E} is spherical cap symmetric. Then

$$0 \leq \lim_{S(\theta(x)) \cap \mathbb{S}^1_\tau \ni y \rightarrow x} \left\langle \frac{y-x}{|y-x|}, n(x) \right\rangle = -\langle Ox, n(x) \rangle$$

by Lemma 5.3. (iv) The adjoint transformation O^* represents rotation clockwise through $\pi/2$. Let $x \in M_\tau \cap H$. By (iii),

$$0 \geq \langle Ox, n(x) \rangle = \langle x, O^*n(x) \rangle = \langle x, t(x) \rangle = \tau \cos \sigma(x)$$

and this leads to the result. (v) As $\cos \sigma \not\equiv 0$ on M_τ we can find $x \in M_\tau \cap H$. We claim that $\mathbb{S}^1_\tau \cap S(\theta(x)) \subset E$. For suppose that $y \in \mathbb{S}^1_\tau \cap S(\theta(x))$ but $y \notin E$. We may suppose that $0 \leq \theta(y) < \theta(x) < \pi$. If $y \in \mathbb{R}^2 \setminus \bar{E}$ then $x \in \mathbb{R}^2 \setminus \bar{E}$ by Lemma 5.2. On the other hand, if $y \in M$ then the spherical arc in H joining y to x is contained in M again by Lemma 5.2. This arc also has non-empty interior in \mathbb{S}^1_τ . Now $\cos \sigma = 0$ on its interior so $\cos(\sigma(x)) = 0$ by (i) contradicting the hypothesis. A similar argument deals with (vi) and this together with (v) in turn entails (vii) and (viii). \square

Lemma 5.5 *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Suppose that $0 \in M$. Then*

- (i) $(\sin \sigma)(0+) = 0$;
- (ii) $(\cos \sigma)(0+) = -1$.

Proof (i) Let γ_1 be a C^1 parametrisation of M in a neighbourhood of 0 with $\gamma_1(0) = 0$ as above. Then $n(0) = n_1(0) = e_1$ and hence $t(0) = t_1(0) = -e_2$. By Taylor's Theorem $\gamma_1(s) = \gamma_1(0) + t_1(0)s + o(s) = -e_2s + o(s)$ for $s \in I$. This means that $r_1(s) = |\gamma_1(s)| = s + o(s)$ and

$$\cos \theta_1 = \frac{\langle e_1, \gamma_1 \rangle}{r_1} = \frac{\langle e_1, \gamma_1 \rangle}{s} \frac{s}{r_1} \rightarrow 0$$

as $s \rightarrow 0$ which entails that $(\cos \theta_1)(0-) = 0$. Now t_1 is continuous on I so $t_1 = -e_2 + o(1)$ and $\cos \alpha_1 = \langle e_1, t_1 \rangle = o(1)$. We infer that $(\cos \alpha_1)(0-) = 0$. By (2.11), $\cos \alpha_1 = \cos \sigma_1 \cos \theta_1 - \sin \sigma_1 \sin \theta_1$ on I and hence $(\sin \sigma_1)(0-) = 0$. We deduce that $(\sin \sigma)(0+) = 0$. Item (ii) follows from (i) and Lemma 5.4. \square

The set

$$\Omega := \pi \left[(M \setminus \{0\}) \cap \{\cos \sigma \neq 0\} \right] \tag{5.2}$$

plays an important rôle in the proof of Theorem 1.1.

Lemma 5.6 *Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$. Then Ω is an open set in $(0, +\infty)$.*

Proof Suppose $0 < \tau \in \Omega$. Choose $x \in M_\tau \cap \{\cos \sigma \neq 0\}$. Let $\gamma_1 : I \rightarrow M$ be a local C^1 parametrisation of M in a neighbourhood of x such that $\gamma_1(0) = x$ as before. By shrinking I if necessary we may assume that $r_1 \neq 0$ and $\cos \sigma_1 \neq 0$ on I . Then the set $\{r_1(s) : s \in I\} \subset \Omega$ is connected and so an interval in \mathbb{R} (see for example [25] Theorems 6.A and 6.B). By (2.9), $r'_1(0) = \cos \sigma_1(0) = \cos \sigma(p) \neq 0$. This means that the set $\{r_1(s) : s \in I\}$ contains an open interval about τ . \square

6 Generalised (mean) curvature

Given a set E of finite perimeter in \mathbb{R}^2 the first variation $\delta V_f(Z)$ resp. $\delta P_f^+(Z)$ of weighted volume and perimeter along a time-dependent vector field Z are defined as in (2.13) and (2.14).

Proposition 6.1 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Let E be a bounded open set in \mathbb{R}^2 with C^1 boundary M . Let Z be a time-dependent vector field. Then*

$$\delta P_f^+(Z) = \int_M f'_+(\cdot, Z_0) + f \operatorname{div}^M Z_0 \, d\mathcal{H}^1$$

where $Z_0 := Z(0, \cdot) \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$.

Proof The identity (3.2) holds for each $t \in I$ with M in place of $\mathcal{F}E$. The assertion follows on appealing to Lemma 2.3 and Lemma 2.4 with the help of the dominated convergence theorem. \square

Given $X, Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ let ψ resp. χ stand for the 1-parameter group of C^∞ diffeomorphisms of \mathbb{R}^2 associated to the vector fields X resp. Y as in (2.12). Let I be an open interval in \mathbb{R} containing the point 0. Suppose that the function $\sigma : I \rightarrow \mathbb{R}$ is C^1 . Define a flow via

$$\varphi : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; (t, x) \mapsto \chi(\sigma(t), \psi(t, x)).$$

Lemma 6.2 *The time-dependent vector field Z associated with the flow φ is given by*

$$Z(t, x) = \sigma'(t)Y(\chi(\sigma(t), \psi(t, x))) + d\chi(\sigma(t), \psi(t, x))X(\psi(t, x)) \tag{6.1}$$

for $(t, x) \in I \times \mathbb{R}^2$ and satisfies (Z.1) and (Z.2).

Proof For $t \in I$ and $x \in \mathbb{R}^2$ we compute using (2.12),

$$\partial_t \varphi(t, x) = (\partial_t \chi)(\sigma(t), \psi(t, x))\sigma'(t) + d\chi(\sigma(t), \psi(t, x))\partial_t \psi(t, x)$$

and this gives (6.1). Put $K_1 := \operatorname{supp}[X]$, $K_2 := \operatorname{supp}[Y]$ and $K := K_1 \cup K_2$. Then (Z.2) holds with this choice of K . \square

Let E be a bounded open set in \mathbb{R}^2 with C^1 boundary M . Define $\Lambda := (M \setminus \{0\}) \cap \{\cos \sigma = 0\}$ and

$$\Lambda_1 := \{x \in M : \mathcal{H}^1(\Lambda \cap B(x, \rho)) = \mathcal{H}^1(M \cap B(x, \rho)) \text{ for some } \rho > 0\}. \quad (6.2)$$

For future reference put $\Lambda_1^\pm := \Lambda_1 \cap \{x \in M : \pm \langle x, n \rangle > 0\}$.

Lemma 6.3 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Let E be a bounded open set in \mathbb{R}^2 with $C^{1,1}$ boundary M and suppose that $E \setminus \{0\} = E^{sc}$. Then*

- (i) Λ_1 is a countable disjoint union of well-separated open circular arcs centred at 0;
- (ii) $\mathcal{H}^1(\overline{\Lambda_1} \setminus \Lambda_1) = 0$;
- (iii) f is differentiable \mathcal{H}^1 -a.e. on $M \setminus \overline{\Lambda_1}$.

The term well-separated in (i) means the following: if Γ is an open circular arc in Λ_1 with $\Gamma \cap (\Lambda_1 \setminus \Gamma) = \emptyset$ then $d(\Gamma, \Lambda_1 \setminus \Gamma) > 0$.

Proof (i) Let $x \in \Lambda_1$ and $\gamma_1 : I \rightarrow M$ a $C^{1,1}$ parametrisation of M near x . By shrinking I if necessary we may assume that $\gamma_1(I) \subset M \cap B(x, \rho)$ with ρ as in (6.2). So $\cos \sigma = 0$ \mathcal{H}^1 -a.e. on $\gamma_1(I)$ and hence $\cos \sigma_1 = 0$ a.e. on I . This means that $\cos \sigma_1 = 0$ on I as $\sigma_1 \in C^{0,1}(I)$ and that r_1 is constant on I by (2.9). Using (2.10) it can be seen that $\gamma_1(I)$ is an open circular arc centred at 0. By compactness of M it follows that Λ_1 is a countable disjoint union of open circular arcs centred on 0. The well-separated property flows from the fact that M is C^1 . (ii) follows as a consequence of this property. (iii) Let $x \in M \setminus \overline{\Lambda_1}$ and $\gamma_1 : I \rightarrow M$ a $C^{1,1}$ parametrisation of M near x with properties as before. We assume that x lies in the upper half-plane H . By shrinking I if necessary we may assume that $\gamma_1(I) \subset (M \setminus \overline{\Lambda_1}) \cap H$. Let $s_1, s_2, s_3 \in I$ with $s_1 < s_2 < s_3$. Then $y := \gamma_1(s_2) \in M \setminus \overline{\Lambda_1}$. So $\mathcal{H}^1(M \cap \{\cos \sigma \neq 0\} \cap B(y, \rho)) > 0$ for each $\rho > 0$. This means that for small $\eta > 0$ the set $\gamma_1((s_2 - \eta, s_2 + \eta)) \cap \{\cos \sigma \neq 0\}$ has positive \mathcal{H}^1 -measure. Consequently, $r_1(s_3) - r_1(s_1) = \int_{s_1}^{s_3} \cos \sigma_1 ds < 0$ bearing in mind Lemma 5.4. This shows that r_1 is strictly decreasing on I . So h is differentiable a.e. on $r_1(I) \subset (0, +\infty)$ in virtue of the fact that h is convex and hence locally Lipschitz. This entails (iii). □

Proposition 6.4 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a minimiser of (1.2). Assume that E is a bounded open set in \mathbb{R}^2 with C^1 boundary M and suppose that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Lambda_1} \neq \emptyset$. Then there exists $\lambda \in \mathbb{R}$ such that for any $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$,*

$$0 \leq \int_M \left\{ f'_+(\cdot, X) + f \operatorname{div}^M X - \lambda f \langle n, X \rangle \right\} d\mathcal{H}^1.$$

Proof Let $X \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Let $x \in M$ and $r > 0$ such that $M \cap B(x, r) \subset M \setminus \overline{\Lambda_1}$. Choose $Y \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $\operatorname{supp}[Y] \subset B(x, r)$ as in Lemma 3.4. Let ψ resp. χ stand for the 1-parameter group of C^∞ diffeomorphisms of \mathbb{R}^2 associated to the vector

fields X resp. Y as in (2.12). For each $(s, t) \in \mathbb{R}^2$ the set $\chi_s(\psi_t(E))$ is an open set in \mathbb{R}^2 with C^1 boundary and $\partial(\chi_s \circ \psi_t)(E) = (\chi_s \circ \psi_t)(M)$ by Lemma 2.1. Define

$$\begin{aligned} V(s, t) &:= V_f(\chi_t(\psi_s(E))) - V_f(E), \\ P(s, t) &:= P_f(\chi_t(\psi_s(E))), \end{aligned}$$

for $(s, t) \in \mathbb{R}^2$. We write $F = (\chi_t \circ \psi_s)(E)$. Arguing as in Proposition 3.2,

$$\begin{aligned} \partial_t V(s, t) &= \lim_{h \rightarrow 0} (1/h) \{V_f(\chi_h(F)) - V_f(F)\} = \int_F \operatorname{div}(fY) \, dx \\ &= \int_E (\operatorname{div}(fY) \circ \chi_t \circ \psi_s) J_2 d(\chi_t \circ \psi_s)_x \, dx \end{aligned}$$

with an application of the area formula (cf. [1] Theorem 2.71). This last varies continuously in (s, t) . The same holds for partial differentiation with respect to s . Indeed, put $\eta := \chi_t \circ \psi_s$. Then noting that $J_2 d(\eta \circ \psi_h) = (J_2 d\eta) \circ \psi_h J_2 d\psi_h$ and using the dominated convergence theorem,

$$\begin{aligned} \partial_s V(s, t) &= \lim_{h \rightarrow 0} (1/h) \left\{ V_f(\eta(\psi_h(E))) - V_f(\eta(E)) \right\} \\ &= \lim_{h \rightarrow 0} (1/h) \left\{ \int_E (f \circ \eta \circ \psi_h) J_2 d(\eta \circ \psi_h)_x \, dx - \int_E (f \circ \eta) J_2 d\eta_x \, dx \right\} \\ &= \lim_{h \rightarrow 0} (1/h) \left\{ \int_E [(f \circ \eta \circ \psi_h) - (f \circ \eta)] J_2 d(\eta \circ \psi_h)_x \, dx \right. \\ &\quad \left. + \int_E (f \circ \eta) [(J_2 d\eta \circ \psi_h - J_2 d\eta)] J_2 d\psi_h \, dx \right. \\ &\quad \left. + \int_E (f \circ \eta) J_2 d\eta [J_2 d\psi_h - 1] \, dx \right\} \\ &= \int_E \langle \nabla(f \circ \eta), X \rangle J_2 d\eta_x \, dx + \int_E (f \circ \eta) \langle \nabla J_2 d\eta, X \rangle \, dx \\ &\quad + \int_E (f \circ \eta) J_2 d\eta \operatorname{div} X \, dx \end{aligned}$$

where the explanation for the last term can be found in the proof of Proposition 3.2. In this regard we note that $d(d\chi_t)$ (for example) is continuous on $I \times \mathbb{R}^2$ (cf. [1] Theorem 3.3 and Exercise 3.2) and in particular $\nabla J_2 d\chi_t$ is continuous on $I \times \mathbb{R}^2$. The expression above also varies continuously in (s, t) as can be seen with the help of the dominated convergence theorem. This means that $V(\cdot, \cdot)$ is continuously differentiable on \mathbb{R}^2 . Note that

$$\partial_t V(0, 0) = \int_E \operatorname{div}(fY) \, dx = 1$$

by choice of Y . By the implicit function theorem there exists $\eta > 0$ and a C^1 function $\sigma : (-\eta, \eta) \rightarrow \mathbb{R}$ such that $\sigma(0) = 0$ and $V(s, \sigma(s)) = 0$ for $s \in (-\eta, \eta)$; moreover,

$$\begin{aligned} \sigma'(0) &= -\partial_s V(0, 0) = -\int_E \left\{ \langle \nabla f, X \rangle + f \operatorname{div} X \right\} dx \\ &= -\int_E \operatorname{div}(fX) dx = \int_M f \langle n, X \rangle d\mathcal{H}^1 \end{aligned}$$

by the Gauss–Green formula (cf. [1] Theorem 3.36).

The mapping

$$\varphi : (-\eta, \eta) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2; t \mapsto \chi(\sigma(t), \psi(t, x))$$

satisfies conditions (F.1)–(F.4) above with $I = (-\eta, \eta)$ where the associated time-dependent vector field Z is given as in (6.1) and satisfies (Z.1) and (Z.2); moreover, $Z_0 = Z(0, \cdot) = \sigma'(0)Y + X$. Note that $Z_0 = X$ on $M \setminus B(x, r)$.

The mapping $I \rightarrow \mathbb{R}; t \mapsto P_f(\varphi_t(E))$ is right-differentiable at $t = 0$ as can be seen from Proposition 6.1 and has non-negative right-derivative there. By Proposition 6.1 and Lemma 6.3,

$$\begin{aligned} 0 \leq \delta P_f^+(Z) &= \int_M f'_+(\cdot, Z_0) + f \operatorname{div}^M Z_0 d\mathcal{H}^1 \\ &= \int_{M \setminus \overline{\Lambda_1}} f'_+(\cdot, Z_0) + f \operatorname{div}^M Z_0 d\mathcal{H}^1 \\ &\quad + \int_{\Lambda_1} f'_+(\cdot, X) + f \operatorname{div}^M X d\mathcal{H}^1 \\ &= \int_{M \setminus \overline{\Lambda_1}} \sigma'(0) \langle \nabla f, Y \rangle + \langle \nabla f, X \rangle \\ &\quad + \sigma'(0) f \operatorname{div}^M Y + f \operatorname{div}^M X d\mathcal{H}^1 \\ &\quad + \int_{\Lambda_1} f'_+(\cdot, X) + f \operatorname{div}^M X d\mathcal{H}^1 \\ &= \int_M f'_+(\cdot, X) + f \operatorname{div}^M X d\mathcal{H}^1 \\ &\quad + \sigma'(0) \int_M f'_+(\cdot, Y) + f \operatorname{div}^M Y d\mathcal{H}^1. \end{aligned} \tag{6.3}$$

The identity then follows upon inserting the expression for $\sigma'(0)$ above with $\lambda = -\int_M f'_+(\cdot, Y) + f \operatorname{div}^M Y d\mathcal{H}^1$. The claim follows for $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ by a density argument. \square

Theorem 6.5 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a minimiser of (1.2). Assume that E is a bounded open set in \mathbb{R}^2 with $C^{1,1}$ boundary M and suppose that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Lambda_1} \neq \emptyset$. Then there exists $\lambda \in \mathbb{R}$ such that*

- (i) $k + \rho \sin \sigma + \lambda = 0$ \mathcal{H}^1 -a.e. on $M \setminus \overline{\Lambda_1}$;
- (ii) $\rho_- - \lambda \leq k \leq \rho_+ - \lambda$ on Λ_1^+ ;
- (iii) $-\rho_+ - \lambda \leq k \leq -\rho_- - \lambda$ on Λ_1^- .

The expression $k + \rho \sin \sigma$ is called the generalised (mean) curvature of M .

Proof (i) Let $x \in M$ and $r > 0$ such that $M \cap B(x, r) \subset M \setminus \overline{\Lambda_1}$. Choose $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ with $\text{supp}[X] \subset B(x, r)$. We know from Lemma 6.3 that f is differentiable \mathcal{H}^1 -a.e. on $\text{supp}[X]$. Let λ be as in Proposition 6.4. Replacing X by $-X$ we deduce from Proposition 6.4 that

$$0 = \int_M \left\{ \langle \nabla f, X \rangle + f \operatorname{div}^M X - \lambda f \langle n, X \rangle \right\} d\mathcal{H}^1.$$

The divergence theorem on manifolds (cf. [1] Theorem 7.34) holds also for $C^{1,1}$ manifolds. So

$$\begin{aligned} \int_M \langle \nabla f, X \rangle + f \operatorname{div}^M X \, d\mathcal{H}^1 &= \int_M \partial_n f \langle n, X \rangle + \langle \nabla^M f, X \rangle + f \operatorname{div}^M X \, d\mathcal{H}^1 \\ &= \int_M \partial_n f \langle n, X \rangle + \operatorname{div}^M(fX) \, d\mathcal{H}^1 \\ &= \int_M \partial_n f \langle n, X \rangle - Hf \langle n, X \rangle \, d\mathcal{H}^1 \\ &= \int_M fu \{ \partial_n \log f - H \} \, d\mathcal{H}^1 \end{aligned}$$

where $u = \langle n, X \rangle$. Combining this with the equality above we see that

$$\int_M uf \{ \partial_n \log f - H - \lambda \} \, d\mathcal{H}^1 = 0$$

for all $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$. This leads to the result.

(ii) Let $x \in M$ and $r > 0$ such that $M \cap B(x, r) \subset \Lambda_1^+$. Let $\phi \in C^1(\mathbb{S}_r^1)$ with support in $\mathbb{S}_r^1 \cap B(x, r)$. We can construct $X \in C_c^1(\mathbb{R}^2, \mathbb{R}^2)$ with the property that $X = \phi n$ on $M \cap B(x, r)$. By Lemma 2.4,

$$f'_+(\cdot, X) = fh'_+(|x|, \operatorname{sgn}\langle x, X \rangle) |\langle n, X \rangle| = fh'_+(|x|, \operatorname{sgn} \phi(x, n)) |\phi|$$

on Λ_1 . Let us assume that $\phi \geq 0$. As $\langle \cdot, n \rangle > 0$ on Λ_1^+ we have that $f'_+(\cdot, X) = f\phi h'_+(|x|, +1) = f\phi\rho_+$ so by Proposition 6.4,

$$\begin{aligned} 0 &\leq \int_M \left\{ f'_+(\cdot, X) + f \operatorname{div}^M X - \lambda f \langle n, X \rangle \right\} d\mathcal{H}^1 \\ &= \int_M f\phi \{ \rho_+ - k - \lambda \} \, d\mathcal{H}^1. \end{aligned}$$

We conclude that $\rho_+ - k - \lambda \geq 0$ on $M \cap B(x, r)$. Now assume that $\phi \leq 0$. Then $f'_+(\cdot, X) = -f\phi h'_+(|x|, -1) = f\phi\rho_-$ so

$$0 \leq \int_M f\phi \{ \rho_- - k - \lambda \} d\mathcal{H}^1$$

and hence $\rho_- - k - \lambda \leq 0$ on $M \cap B(x, r)$. This shows (ii).

(iii) The argument is similar. Assume in the first instance that $\phi \geq 0$. Then $f'_+(\cdot, X) = f\phi h'_+(|x|, -1) = -f\phi\rho_-$ so

$$0 \leq \int_M f\phi \{ -\rho_- - k - \lambda \} d\mathcal{H}^1.$$

We conclude that $-\rho_- - k - \lambda \geq 0$ on $M \cap B(x, r)$. Next suppose that $\phi \leq 0$. Then $f'_+(\cdot, X) = -f\phi h'_+(|x|, +1) = -f\phi\rho_+$ so

$$0 \leq \int_M f\phi \{ -\rho_+ - k - \lambda \} d\mathcal{H}^1$$

and $-\rho_+ - k - \lambda \leq 0$ on $M \cap B(x, r)$. □

Let E be an open set in \mathbb{R}^2 with C^1 boundary M and assume that $E \setminus \{0\} = E^{sc}$ and that Ω is as in (5.2). Bearing in mind Lemma 5.4 we may define

$$\theta_2 : \Omega \rightarrow (0, \pi); \tau \mapsto L(\tau)/2\tau; \tag{6.4}$$

$$\gamma : \Omega \rightarrow M; \tau \mapsto (\tau \cos \theta_2(\tau), \tau \sin \theta_2(\tau)). \tag{6.5}$$

The function

$$u : \Omega \rightarrow [-1, 1]; \tau \mapsto \sin(\sigma(\gamma(\tau))). \tag{6.6}$$

plays a key role.

Theorem 6.6 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open with $C^{1,1}$ boundary M and that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Delta_1} \neq \emptyset$ and let λ be as in Theorem 6.5. Then $u \in C^{0,1}(\Omega)$ and*

$$u' + (1/\tau + \rho)u + \lambda = 0$$

a.e. on Ω .

Proof Let $\tau \in \Omega$ and x a point in the open upper half-plane such that $x \in M_\tau$. There exists a $C^{1,1}$ parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. Put $u_1 := \sin \sigma_1$ on I . By shrinking the open interval I if necessary we may assume that $r_1 : I \rightarrow r_1(I)$ is a diffeomorphism and that $r_1(I) \subset\subset \Omega$. Note that $\gamma = \gamma_1 \circ r_1^{-1}$ and $u = u_1 \circ r_1^{-1}$ on $r_1(I)$. It follows that $u \in C^{0,1}(\Omega)$. By (2.9),

$$u' = \frac{\dot{u}_1}{\dot{r}_1} \circ r_1^{-1} = \dot{\sigma}_1 \circ r_1^{-1}$$

a.e. on $r_1(I)$. As $\dot{\alpha}_1 = k_1$ a.e. on I and using the identity (2.10) we see that $\dot{\sigma}_1 = \dot{\alpha}_1 - \dot{\theta}_1 = k_1 - (1/r_1) \sin \sigma_1$ a.e. on I . Thus,

$$u' = k - (1/\tau) \sin(\sigma \circ \gamma) = k - (1/\tau)u$$

a.e. on $r_1(I)$. By Theorem 6.5 there exists $\lambda \in \mathbb{R}$ such that $k + \rho \sin \sigma + \lambda = 0$ \mathcal{H}^1 -a.e. on M . So

$$u' = -\rho(\tau)u - \lambda - (1/\tau)u = -(1/\tau + \rho(\tau))u - \lambda$$

a.e. on $r_1(I)$. The result follows. □

Lemma 6.7 *Suppose that E is a bounded open set in \mathbb{R}^2 with C^1 boundary M and that $E \setminus \{0\} = E^{sc}$. Then*

- (i) $\theta_2 \in C^1(\Omega)$;
- (ii) $\theta'_2 = -\frac{1}{\tau} \frac{u}{\sqrt{1-u^2}}$ on Ω .

Proof Let $\tau \in \Omega$ and x a point in the open upper half-plane such that $x \in M_\tau$. There exists a C^1 parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. By shrinking the open interval I if necessary we may assume that $r_1 : I \rightarrow r_1(I)$ is a diffeomorphism and that $r_1(I) \subset\subset \Omega$. It then holds that

$$\theta_2 = \theta_1 \circ r_1^{-1} \text{ and } \sigma \circ \gamma = \sigma_1 \circ r_1^{-1}$$

on $r_1(I)$ by choosing an appropriate branch of θ_1 . It follows that $\theta_2 \in C^1(\Omega)$. By the chain-rule, (2.10) and (2.9),

$$\begin{aligned} \theta'_2 &= \frac{\dot{\theta}_1}{\dot{r}_1} \circ r_1^{-1} = \left(\frac{1}{r_1} \tan \sigma_1 \right) \circ r_1^{-1} \\ &= (1/\tau) \tan(\sigma \circ \gamma) \end{aligned}$$

on $r_1(I)$. By Lemma 5.4, $\cos(\sigma \circ \gamma) = -\sqrt{1-u^2}$ on Ω . This entails (ii). □

7 Convexity

Lemma 7.1 *Let E be a bounded open set in \mathbb{R}^2 with $C^{1,1}$ boundary M and assume that $E \setminus \{0\} = E^{sc}$. Put $d := \sup\{|x| : x \in M\} > 0$ and $b := (d, 0)$. Let $\gamma_1 : I \rightarrow M$ be a $C^{1,1}$ parametrisation of M near b with $\gamma_1(0) = b$. Then*

$$\lim_{\delta \downarrow 0} \left\{ \text{ess sup}_{[-\delta, \delta]} k_1 \right\} \geq 1/d.$$

Proof For $s \in I$,

$$\gamma_1(s) = de_1 + se_2 + \int_0^s \left\{ \dot{\gamma}_1(u) - \dot{\gamma}_1(0) \right\} du$$

and

$$\dot{\gamma}_1(u) - \dot{\gamma}_1(0) = \int_0^u k_1 n_1 dv$$

by (2.6). By the Fubini–Tonelli Theorem,

$$\gamma_1(s) = de_1 + se_2 + \int_0^s (s - u)k_1(u)n_1(u) du = de_1 + se_2 + R(s)$$

for $s \in I$. Assume for a contradiction that

$$\lim_{\delta \downarrow 0} \left\{ \text{ess sup}_{[-\delta, \delta]} k_1 \right\} < l < 1/d$$

for some $l \in \mathbb{R}$. Then we can find $\delta > 0$ such that $k_1 < l$ a.e. on $[-\delta, \delta]$. So

$$\langle R(s), e_1 \rangle = \int_0^s (s - u)k_1(u)\langle n_1(u), e_1 \rangle du > -(1/2)s^2l(1 + o(1))$$

as $s \downarrow 0$ and

$$\begin{aligned} r_1(s)^2 - d^2 &= 2d\langle R(s), e_1 \rangle + s^2 + o(s^2) \\ &> -dls^2(1 + o(1)) + s^2 + o(s^2) \end{aligned}$$

as $s \downarrow 0$. Alternatively,

$$\frac{r_1(s)^2 - d^2}{s^2} > 1 - dl + o(1).$$

As $1 - dl > 0$ we can find $s \in I$ with $r_1(s) > d$, contradicting the definition of d . \square

Lemma 7.2 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open with $C^{1,1}$ boundary M and that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Lambda_1} \neq \emptyset$. Then $\lambda \leq -1/d - \rho_-(d) < 0$ with λ as in Theorem 6.5.*

Proof We write M as the disjoint union $M = (M \setminus \overline{\Lambda_1}) \cup \overline{\Lambda_1}$. Let b be as above. Suppose that $b \in \overline{\Lambda_1}$. Then $b \in \Lambda_1$; in fact, $b \in \Lambda_1^-$. By Theorem 6.5, $\lambda \leq -\rho_- - k$ at b . By Lemma 7.1, $\lambda \leq -1/d - \rho_-(d)$ upon considering an appropriate sequence in M converging to b . Now suppose that b lies in the open set $M \setminus \overline{\Lambda_1}$ in M . Let $\gamma_1 : I \rightarrow M$ be a $C^{1,1}$ parametrisation of M near b with $\gamma_1(I) \subset M \setminus \overline{\Lambda_1}$. By Theorem 6.5, $k_1 + \rho(r_1) \sin \sigma_1 + \lambda = 0$ a.e. on I . Now $\sin \sigma_1(s) \rightarrow 1$ as $s \rightarrow 0$. In light of Lemma 7.1, $1/d + \rho(d^-) + \lambda \leq 0$ and $\lambda \leq -1/d - \rho_-(d)$. \square

Theorem 7.3 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open with $C^{1,1}$ boundary M and that $E \setminus \{0\} = E^{sc}$. Suppose that $M \setminus \overline{\Lambda_1} \neq \emptyset$. Then E is convex.*

Proof The proof runs along similar lines as [22] Theorem 6.5. By Theorem 6.5, $k + \rho \sin \sigma + \lambda = 0$ \mathcal{H}^1 -a.e. on $M \setminus \overline{\Lambda_1}$. By Lemma 7.2,

$$0 \leq k + \rho_-(d) + \lambda \leq k - 1/d$$

and $k \geq 1/d$ \mathcal{H}^1 -a.e. on $M \setminus \overline{\Lambda_1}$. On Λ_1^+ , $k \geq \rho_- - \lambda \geq \rho_- + \rho_-(d) + 1/d > 0$; on the other hand, $k < 0$ on Λ_1^+ . So in fact $\Lambda_1^+ = \emptyset$. If $b \in \Lambda_1^-$ then $k = 1/d$. On $\Lambda_1^- \cap B(0, d)$, $k \geq -\rho_+ - \lambda \geq -\rho_+ + \rho_-(d) + 1/d \geq 1/d$. Therefore $k \geq 1/d > 0$ \mathcal{H}^1 -a.e. on M . The set E is then convex by a modification of [26] Theorem 1.8 and Proposition 1.4. It is sufficient that the function f (here α_1) in the proof of the former theorem is non-decreasing. □

8 A reverse Hermite–Hadamard inequality

Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let h be a primitive of ρ on $[a, b]$ so that $h \in C^{0,1}([a, b])$ and introduce the functions

$$f : [a, b] \rightarrow \mathbb{R}; x \mapsto e^{h(x)}; \tag{8.1}$$

$$g : [a, b] \rightarrow \mathbb{R}; x \mapsto xf(x). \tag{8.2}$$

Then

$$g' = (1/x + \rho)g = f + g\rho \tag{8.3}$$

a.e. on (a, b) . Define

$$m = m(\rho, a, b) := \frac{g(b) - g(a)}{\int_a^b g dt}. \tag{8.4}$$

If ρ takes the constant value $\mathbb{R} \ni \lambda \geq 0$ on $[a, b]$ we use the notation $m(\lambda, a, b)$ and we write $m_0 = m(0, a, b)$. A computation gives

$$m_0 = m(0, a, b) = A(a, b)^{-1} \tag{8.5}$$

where $A(a, b) := (a + b)/2$ stands for the arithmetic mean of a and b .

Lemma 8.1 *Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Then $m_0 \leq m$.*

Proof Note that g is convex on $[a, b]$ as can be seen from (8.3). By the Hermite-Hadamard inequality (cf. [15, 17]),

$$\frac{1}{b-a} \int_a^b g \, dt \leq \frac{g(a) + g(b)}{2}. \tag{8.6}$$

The inequality $(b-a)(g(a) + g(b)) \leq (a+b)(g(b) - g(a))$ entails

$$\int_a^b g \, dt \leq \frac{a+b}{2}(g(b) - g(a))$$

and the result follows on rearrangement. □

Lemma 8.2 *Let $0 \leq a < b < +\infty$ and $\lambda > 0$. Then $m(\lambda, a, b) < \lambda + A(a, b)^{-1}$.*

Proof First suppose that $\lambda = 1$ and take $h : [a, b] \rightarrow \mathbb{R}; t \mapsto t$. In this case,

$$\int_a^b g \, dt = \int_a^b t e^t \, dt = (b-1)e^b - (a-1)e^a$$

and

$$m(1, a, b) = \frac{be^b - ae^a}{(b-1)e^b - (a-1)e^a}.$$

The inequality in the statement is equivalent to

$$(a+b)(be^b - ae^a) < ((b-1)e^b - (a-1)e^a)(2+a+b)$$

which in turn is equivalent to the statement $\tanh[(b-a)/2] < (b-a)/2$ which holds for any $b > a$.

For $\lambda > 0$ take $h : [a, b] \rightarrow \mathbb{R}; t \mapsto \lambda t$. Substitution gives

$$\int_a^b g \, dt = (1/\lambda)^2 [(\lambda b - 1)e^{\lambda b} - (\lambda a - 1)e^{\lambda a}] \text{ and}$$

$$g(b) - g(a) = (1/\lambda)[\lambda b e^{\lambda b} - \lambda a e^{\lambda a}]$$

so from above

$$m(\lambda, a, b) = \lambda m(1, \lambda a, \lambda b) < \lambda \left\{ 1 + A(\lambda a, \lambda b)^{-1} \right\} = \lambda + A(a, b)^{-1}.$$

□

Theorem 8.3 *Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Then*

(i) $m(\rho, a, b) \leq \rho(b-a) + A(a, b)^{-1}$;

(ii) equality holds if and only if $\rho \equiv 0$ on $[a, b]$.

Proof (i) Define $h := \int_a^\cdot \rho \, d\tau$ on $[a, b]$ so that $h' = \rho$ a.e. on (a, b) . Define $h_1 : [a, b] \rightarrow \mathbb{R}; t \mapsto h(b) - \rho(b-)(b - t)$. Then $h_1(b) = h(b), h_1' = \rho(b-) \geq \rho = h'$ a.e. on (a, b) and hence $h \geq h_1$ on $[a, b]$. We derive

$$\int_a^b g \, dt = \int_a^b t e^{h(t)} \, dt \geq \int_a^b t e^{h_1(t)} \, dt = \int_a^b g_1 \, dt$$

and

$$\begin{aligned} g(b) - g(a) &= b e^{h(b)} - a e^{h(a)} = b e^{h_1(b)} - a e^{h(a)} \\ &\leq b e^{h_1(b)} - a e^{h_1(a)} = g_1(b) - g_1(a) \end{aligned}$$

with obvious notation. This entails that $m(\rho, a, b) \leq m(\rho(b-), a, b)$ and the result follows with the help of Lemma 8.2.

(ii) Suppose that $\rho \not\equiv 0$ on $[a, b]$. If ρ is constant on $[a, b]$ the assertion follows from Lemma 8.2. Assume then that ρ is not constant on $[a, b]$. Then $h \not\equiv h_1$ on $[a, b]$ in the above notation and $\int_a^b t e^{h(t)} \, dt > \int_a^b t e^{h_1(t)} \, dt$ which entails strict inequality in (i). \square

With the above notation define

$$\hat{m} = \hat{m}(\rho, a, b) := \frac{g(a) + g(b)}{\int_a^b g \, dt}. \tag{8.7}$$

A computation gives

$$\hat{m}_0 := \hat{m}(0, a, b) = \frac{2}{b - a}. \tag{8.8}$$

Lemma 8.4 *Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Then $\hat{m} \geq \hat{m}_0$.*

Proof This follows by the Hermite-Hadamard inequality (8.6). \square

We prove a reverse Hermite-Hadamard inequality.

Theorem 8.5 *Let $0 \leq a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Then*

- (i) $(b - a)\hat{m}(\rho, a, b) \leq 2 + a\rho(a+) + b\rho(b-);$
- (ii) equality holds if and only if $\rho \equiv 0$ on $[a, b]$.

This last inequality can be written in the form

$$\frac{g(a) + g(b)}{2 + a\rho(a+) + b\rho(b-)} \leq \frac{1}{b - a} \int_a^b g \, dt;$$

comparing with (8.6) justifies naming this a reverse Hermite-Hadamard inequality.

Proof (i) We assume in the first instance that $\rho \in C^1((a, b))$. We prove the above result in the form

$$\int_a^b g \, dt \geq (b - a) \frac{g(a) + g(b)}{2 + a\rho(a) + b\rho(b)}. \tag{8.9}$$

Put

$$w := \frac{(t - a)(g(a) + g)}{2 + a\rho(a) + t\rho}$$

for $t \in [a, b]$ so that

$$\int_a^b w' \, dt = (b - a) \frac{g(a) + g(b)}{2 + a\rho(a) + b\rho(b)}.$$

Then using (8.3),

$$\begin{aligned} w' &= \frac{(g(a) + g + (t - a)g')(2 + a\rho(a) + t\rho) - (t - a)(g(a) + g)(\rho + t\rho')}{(2 + a\rho(a) + t\rho)^2} \\ &= \frac{(g(a) - ag' + (2 + t\rho)g)(2 + a\rho(a) + t\rho) - (t - a)(g(a) + g)(\rho + t\rho')}{(2 + a\rho(a) + t\rho)^2} \\ &= \frac{(2 + t\rho)(2 + a\rho(a) + t\rho)}{(2 + a\rho(a) + t\rho)^2} g \\ &\quad + \frac{(g(a) - ag')(2 + a\rho(a) + t\rho) - (t - a)(g(a) + g)(\rho + t\rho')}{(2 + a\rho(a) + t\rho)^2} \\ &\leq g - \frac{2g(a)}{(2 + a\rho(a) + b\rho(b))^2} (t - a)\rho \\ &\leq g \end{aligned} \tag{8.10}$$

on (a, b) as

$$g(a) - ag' = a(f(a) - (1/t + \rho)g) = a(f(a) - f - \rho g) \leq 0.$$

An integration over $[a, b]$ gives the result.

Let us now assume that $\rho \geq 0$ is a non-decreasing bounded function on $[a, b]$. Extend ρ to \mathbb{R} via

$$\tilde{\rho}(t) := \begin{cases} \rho(a+) & \text{for } t \in (-\infty, a]; \\ \rho(t) & \text{for } t \in (a, b]; \\ \rho(b-) & \text{for } t \in (b, +\infty); \end{cases}$$

for $t \in \mathbb{R}$. Let $(\psi_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers (see e.g. [1] 2.1) and set $\tilde{\rho}_\varepsilon := \tilde{\rho} \star \psi_\varepsilon$ on \mathbb{R} for each $\varepsilon > 0$. Then $\tilde{\rho}_\varepsilon \in C^\infty(\mathbb{R})$ and is non-decreasing on \mathbb{R} for each $\varepsilon > 0$. Put $\rho_\varepsilon := \tilde{\rho}_\varepsilon|_{[a,b]}$ for each $\varepsilon > 0$. Then $(\rho_\varepsilon)_{\varepsilon>0}$ converges to ρ in $L^1((a, b))$ by [1]

2.1 for example. Note that $h_\varepsilon := \int_a^\cdot \rho_\varepsilon dt \rightarrow h$ pointwise on $[a, b]$ as $\varepsilon \downarrow 0$ and that (h_ε) is uniformly bounded on $[a, b]$. Moreover, $\rho_\varepsilon(a) \rightarrow \rho(a+)$ and $\rho_\varepsilon(b) \rightarrow \rho(b-)$ as $\varepsilon \downarrow 0$. By the above result,

$$(b - a)\hat{m}(\rho_\varepsilon, a, b) \leq 2 + a\rho_\varepsilon(a) + b\rho_\varepsilon(b)$$

for each $\varepsilon > 0$. The inequality follows on taking the limit $\varepsilon \downarrow 0$ with the help of the dominated convergence theorem.

(ii) We now consider the equality case. We claim that

$$(b - a) \frac{g(a) + g(b)}{2 + a\rho(a+) + b\rho(b-)} \leq \int_a^b g dt - \frac{2g(a)}{(2 + a\rho(a+) + b\rho(b-))^2} \int_a^b (t - a)\rho dt; \tag{8.11}$$

this entails the equality condition in (ii). First suppose that $\rho \in C^1((a, b))$. In this case the inequality in (8.10) implies (8.11) upon integration. Now suppose that $\rho \geq 0$ is a non-decreasing bounded function on $[a, b]$. Then (8.11) holds with ρ_ε in place of ρ for each $\varepsilon > 0$. The inequality for ρ follows by the dominated convergence theorem. □

9 Comparison theorems for first-order differential equations

Let \mathcal{L} stand for the collection of Lebesgue measurable sets in $[0, +\infty)$. Define a measure μ on $([0, +\infty), \mathcal{L})$ by $\mu(dx) := (1/x) dx$. Let $0 \leq a < b < +\infty$. Suppose that $u : [a, b] \rightarrow \mathbb{R}$ is an \mathcal{L}^1 -measurable function with the property that

$$\mu(\{u > t\}) < +\infty \text{ for each } t > 0. \tag{9.1}$$

The distribution function $\mu_u : (0, +\infty) \rightarrow [0, +\infty)$ of u with respect to μ is given by

$$\mu_u(t) := \mu(\{u > t\}) \text{ for } t > 0.$$

Note that μ_u is right-continuous and non-increasing on $(0, \infty)$ and $\mu_u(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let u be a Lipschitz function on $[a, b]$. Define

$$Z_1 := \{u \text{ differentiable and } u' = 0\}, Z_2 := \{u \text{ not differentiable}\} \text{ and } Z := Z_1 \cup Z_2.$$

By [1] Lemma 2.96, $Z \cap \{u = t\} = \emptyset$ for \mathcal{L}^1 -a.e. $t \in \mathbb{R}$ and hence $N := u(Z) \subset \mathbb{R}$ is \mathcal{L}^1 -negligible. We make use of the coarea formula ([1] Theorem 2.93 and (2.74)),

$$\int_{[a,b]} \phi |u'| dx = \int_{-\infty}^\infty \int_{\{u=t\}} \phi d\mathcal{H}^0 dt \tag{9.2}$$

for any \mathcal{L}^1 -measurable function $\phi : [a, b] \rightarrow [0, \infty]$.

Lemma 9.1 *Let $0 \leq a < b < +\infty$ and u a Lipschitz function on $[a, b]$. Then*

- (i) $\mu_u \in \text{BV}_{\text{loc}}((0, +\infty))$;
- (ii) $D\mu_u = -u_{\#}\mu$;
- (iii) $D\mu_u^a = D\mu_u \llcorner ((0, +\infty) \setminus N)$;
- (iv) $D\mu_u^s = D\mu_u \llcorner N$;
- (v) $A := \left\{ t \in (0, +\infty) : \mathcal{L}^1(Z \cap \{u = t\}) > 0 \right\}$ is the set of atoms of $D\mu_u$ and $D\mu_u^j = D\mu_u \llcorner A$;
- (vi) μ_u is differentiable \mathcal{L}^1 -a.e. on $(0, +\infty)$ with derivative given by

$$\mu'_u(t) = - \int_{\{u=t\} \setminus Z} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}$$

for \mathcal{L}^1 -a.e. $t \in (0, +\infty)$;

- (vii) $\text{Ran}(u) \cap [0, +\infty) = \text{supp}(D\mu_u)$.

The notation above $D\mu_u^a, D\mu_u^s, D\mu_u^j$ stands for the absolutely continuous resp. singular resp. jump part of the measure $D\mu_u$ (see [1] 3.2 for example).

Proof For any $\varphi \in C_c^\infty((0, +\infty))$ with $\text{supp}[\varphi] \subset (\tau, +\infty)$ for some $\tau > 0$,

$$\begin{aligned} \int_0^\infty \mu_u \varphi' dt &= \int_{[a,b]} \varphi \circ u d\mu \\ &= \int_{[a,b]} \chi_{\{u>\tau\}} \varphi \circ u d\mu \end{aligned} \tag{9.3}$$

by Fubini’s theorem; so $\mu_u \in \text{BV}_{\text{loc}}((0, +\infty))$ and $D\mu_u$ is the push-forward of μ under u , $D\mu_u = -u_{\#}\mu$ (cf. [1] 1.70). By (9.2),

$$\begin{aligned} D\mu_u \llcorner ((0, +\infty) \setminus N)(A) &= -\mu(\{u \in A\} \setminus Z) \\ &= - \int_A \int_{\{u=t\} \setminus Z} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau} dt \end{aligned}$$

for any \mathcal{L}^1 -measurable set A in $(0, +\infty)$. In light of the above, we may identify $D\mu_u^a = D\mu_u \llcorner ((0, +\infty) \setminus N)$ and $D\mu_u^s = D\mu_u \llcorner N$. The set of atoms of $D\mu_u$ is defined by $A := \{t \in (0, +\infty) : D\mu_u(\{t\}) \neq 0\}$. For $t > 0$,

$$\begin{aligned} D\mu_u(\{t\}) &= D\mu_u^s(\{t\}) = (D\mu_u \llcorner N)(\{t\}) \\ &= -u_{\#}\mu(N \cap \{t\}) = -\mu(Z \cap \{u = t\}) \end{aligned}$$

and this entails (v). The monotone function μ_u is a good representative within its equivalence class and is differentiable \mathcal{L}^1 -a.e. on $(0, +\infty)$ with derivative given by the density of $D\mu_u$ with respect to \mathcal{L}^1 by [1] Theorem 3.28. Item (vi) follows from (9.2) and (iii). Item (vii) follows from (ii). □

Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let $\eta \in \{\pm 1\}^2$. We study solutions to the first-order linear ordinary differential equation

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (a, b) \text{ with } u(a) = \eta_1 \text{ and } u(b) = \eta_2 \tag{9.4}$$

where $u \in C^{0,1}([a, b])$ and $\lambda \in \mathbb{R}$. In case $\rho \equiv 0$ on $[a, b]$ we use the notation u_0 .

Lemma 9.2 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let $\eta \in \{\pm 1\}^2$. Then*

- (i) *there exists a solution (u, λ) of (9.4) with $u \in C^{0,1}([a, b])$ and $\lambda = \lambda_\eta \in \mathbb{R}$;*
- (ii) *the pair (u, λ) in (i) is unique;*
- (iii) *λ_η is given by*

$$-\lambda_{(1,1)} = \lambda_{(-1,-1)} = m; \quad \lambda_{(1,-1)} = -\lambda_{(-1,1)} = \hat{m};$$

- (iv) *if $\eta = (1, 1)$ or $\eta = (-1, -1)$ then u is uniformly bounded away from zero on $[a, b]$.*

Proof (i) For $\eta = (1, 1)$ define $u : [a, b] \rightarrow \mathbb{R}$ by

$$u(t) := \frac{m \int_a^t g \, ds + g(a)}{g(t)} \text{ for } t \in [a, b] \tag{9.5}$$

with m as in (8.4). Then $u \in C^{0,1}([a, b])$ and satisfies (9.4) with $\lambda = -m$. For $\eta = (1, -1)$ set $u = (-\hat{m} \int_a^t g \, ds + g(a))/g$ with $\lambda = \hat{m}$. The cases $\eta = (-1, -1)$ and $\eta = (-1, 1)$ can be dealt with using linearity. (ii) We consider the case $\eta = (1, 1)$. Suppose that (u_1, λ_1) resp. (u_2, λ_2) solve (9.4). By linearity $u := u_1 - u_2$ solves

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (a, b) \text{ with } u(a) = u(b) = 0$$

where $\lambda = \lambda_1 - \lambda_2$. An integration gives that $u = (-\lambda \int_a^t g \, ds + c)/g$ for some constant $c \in \mathbb{R}$ and the boundary conditions entail that $\lambda = c = 0$. The other cases are similar. (iii) follows as in (i). (iv) If $\eta = (1, 1)$ then $u > 0$ on $[a, b]$ from (9.5) as $m > 0$. □

The boundary condition $\eta_1 \eta_2 = -1$.

Lemma 9.3 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let (u, λ) solve (9.4) with $\eta = (1, -1)$. Then*

- (i) *there exists a unique $c \in (a, b)$ with $u(c) = 0$;*
- (ii) *$u' < 0$ a.e. on $[a, c]$ and u is strictly decreasing on $[a, c]$;*
- (iii) *$D\mu_u^s = 0$.*

Proof (i) We first observe that $u' \leq -\hat{m} < 0$ a.e. on $\{u \geq 0\}$ in view of (9.4). Suppose $u(c_1) = u(c_2) = 0$ for some $c_1, c_2 \in (a, b)$ with $c_1 < c_2$. We may assume that $u \geq 0$ on $[c_1, c_2]$. This contradicts the above observation. Item (ii) is plain. For any \mathcal{L}^1 -measurable set B in $(0, +\infty)$, $D\mu_u^s(B) = \mu(\{u \in B\} \cap Z) = 0$ using Lemma 9.1 and (ii). \square

Lemma 9.4 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let (u, λ) solve (9.4) with $\eta = (1, -1)$. Assume that*

- (a) u is differentiable at both a and b and that (9.4) holds there;
- (b) $u'(a) < 0$ and $u'(b) < 0$;
- (c) ρ is differentiable at a and b .

Put $v := -u$. Then

- (i) $\int_{\{v=1\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} \geq \int_{\{u=1\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}$;
- (ii) equality holds if and only if $\rho \equiv 0$ on $[a, b]$.

Proof First, $\{u = 1\} = \{a\}$ by Lemma 9.3. Further $0 < -au'(a) = 1 + a[\hat{m} + \rho(a)]$ from (9.4). On the other hand $\{v = 1\} \supset \{b\}$ and $0 < bv'(b) = -1 + b[\hat{m} - \rho(b)]$. Thus

$$\begin{aligned} & \int_{\{v=1\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} - \int_{\{u=1\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau} \\ & \geq \frac{1}{-1 + b[\hat{m} - \rho(b)]} - \frac{1}{1 + a[\hat{m} + \rho(a)]}. \end{aligned}$$

By Theorem 8.5, $0 \leq 2 + (a - b)\hat{m} + a\rho(a) + b\rho(b)$, noting that $\rho(a) = \rho(a+)$ in virtue of (c) and similarly at b . A rearrangement leads to the inequality. The equality assertion follows from Theorem 8.5. \square

Theorem 9.5 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Suppose that (u, λ) solves (9.4) with $\eta = (1, -1)$ and set $v := -u$. Assume that $u > -1$ on $[a, b]$. Then*

- (i) $-\mu'_v \geq -\mu'_u$ for \mathcal{L}^1 -a.e. $t \in (0, 1)$;
- (ii) if $\rho \not\equiv 0$ on $[a, b]$ then there exists $t_0 \in (0, 1)$ such that $-\mu'_v > -\mu'_u$ for \mathcal{L}^1 -a.e. $t \in (t_0, 1)$;
- (iii) for $t \in [-1, 1]$,

$$\mu_{u_0}(t) = \log \left\{ \frac{-(b-a)t + \sqrt{(b-a)^2 t^2 + 4ab}}{2a} \right\}$$

and $\mu_{v_0} = \mu_{u_0}$ on $[-1, 1]$;

in obvious notation.

Proof (i) The set

$$Y_u := Z_{2,u} \cup \left(\{u' + (1/x + \rho)u + \lambda \neq 0\} \setminus Z_{2,u} \right) \cup \{\rho \text{ not differentiable}\} \subset [a, b]$$

(in obvious notation) is a null set in $[a, b]$ and likewise for Y_v . By [1] Lemma 2.95 and Lemma 2.96, $\{u = t\} \cap (Y_u \cup Z_{1,u}) = \emptyset$ for a.e. $t \in (0, 1)$ and likewise for the function v . Let $t \in (0, 1)$ and assume that $\{u = t\} \cap (Y_u \cup Z_{1,u}) = \emptyset$ and $\{v = t\} \cap (Y_v \cup Z_{1,v}) = \emptyset$. Put $c := \max\{u \geq t\}$. Then $c \in (a, b)$, $\{u > t\} = [a, c)$ by Lemma 9.3 and u is differentiable at c with $u'(c) < 0$. Put $d := \max\{v \leq t\} = \max\{u \geq -t\}$. As u is continuous on $[a, b]$ it holds that $a < c < d < b$. Moreover, $u'(d) < 0$ as $v(d) = t$ and $d \notin Z_v$. Put $\tilde{u} := u/t$ and $\tilde{v} := v/t$ on $[c, d]$. Then

$$\begin{aligned} \tilde{u}' + (1/\tau + \rho)\tilde{u} + \hat{m}/t &= 0 \text{ a.e. on } (c, d) \text{ and } \tilde{u}(c) = -\tilde{u}(d) = 1; \\ \tilde{v}' + (1/\tau + \rho)\tilde{v} - \hat{m}/t &= 0 \text{ a.e. on } (c, d) \text{ and } -\tilde{v}(c) = \tilde{v}(d) = 1. \end{aligned}$$

By Lemma 9.4,

$$\begin{aligned} \int_{\{v=t\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} &\geq \int_{[c,d] \cap \{v=t\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} \\ &= (1/t) \int_{[c,d] \cap \{\tilde{v}=1\} \setminus Z_v} \frac{1}{|\tilde{v}'|} \frac{d\mathcal{H}^0}{\tau} \\ &\geq (1/t) \int_{[c,d] \cap \{\tilde{u}=1\} \setminus Z_u} \frac{1}{|\tilde{u}'|} \frac{d\mathcal{H}^0}{\tau} \\ &= \int_{\{u=t\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}. \end{aligned}$$

By Lemma 9.1,

$$-\mu'_u(t) = \int_{\{u=t\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}$$

for \mathcal{L}^1 -a.e. $t \in (0, 1)$ and a similar formula holds for v . The assertion in (i) follows.

(ii) Assume that $\rho \not\equiv 0$ on $[a, b)$. Put $\alpha := \inf\{\rho > 0\} \in [a, b)$. Note that $\max\{v \leq t\} \rightarrow b$ as $t \uparrow 1$ as $v < 1$ on $[a, b)$ by assumption. Choose $t_0 \in (0, 1)$ such that $\max\{v \leq t_0\} > \alpha$. Then for $t > t_0$,

$$a < \max\{u \geq t\} < \max\{u \geq -t_0\} = \max\{v \leq t_0\} < \max\{v \leq t\} < d;$$

that is, the interval $[c, d]$ with c, d as described above intersects (α, b) . So for \mathcal{L}^1 -a.e. $t \in (t_0, 1)$,

$$\int_{\{v=t\} \setminus Z_v} \frac{1}{|v'|} \frac{d\mathcal{H}^0}{\tau} > \int_{\{u=t\} \setminus Z_u} \frac{1}{|u'|} \frac{d\mathcal{H}^0}{\tau}.$$

by the equality condition in Lemma 9.4. The conclusion follows from the representation of μ_u resp. μ_v in Lemma 9.1.

(iii) A direct computation gives

$$u_0(\tau) = \frac{1}{b-a} \left\{ -\tau + \frac{ab}{\tau} \right\}$$

for $\tau \in [a, b]$; u_0 is strictly decreasing on its domain. This leads to the formula in (iii). A similar computation gives

$$\mu_{v_0}(t) = \log \left\{ \frac{2b}{(b-a)t + \sqrt{(b-a)^2 t^2 + 4ab}} \right\}$$

for $t \in [-1, 1]$. Rationalising the denominator results in the stated equality. □

Corollary 9.6 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Suppose that (u, λ) solves (9.4) with $\eta = (1, -1)$ and set $v := -u$. Assume that $u > -1$ on $[a, b]$. Then*

- (i) $\mu_u(t) \leq \mu_v(t)$ for each $t \in (0, 1)$;
- (ii) if $\rho \not\equiv 0$ on $[a, b]$ then $\mu_u(t) < \mu_v(t)$ for each $t \in (0, 1)$.

Proof (i) By [1] Theorem 3.28 and Lemma 9.3,

$$\begin{aligned} \mu_u(t) &= \mu_u(t) - \mu_u(1) = -D\mu_u((t, 1)) \\ &= -D\mu_u^a((t, 1)) - D\mu_u^s((t, 1)) \\ &= -\int_{(t,1)} \mu'_u ds \end{aligned}$$

for each $t \in (0, 1)$ as $\mu_u(1) = 0$. On the other hand,

$$\begin{aligned} \mu_v(t) &= \mu_v(1) + (\mu_v(t) - \mu_v(1)) = \mu_v(1) - D\mu_v((t, 1)) \\ &= \mu_v(1) - \int_{(t,1)} \mu'_v ds - D\mu_v^s((t, 1)) \end{aligned}$$

for each $t \in (0, 1)$. The claim follows from Theorem 9.5 noting that $D\mu_v^s((t, 1)) \leq 0$ as can be seen from Lemma 9.1. Item (ii) follows from Theorem 9.5 (ii). □

Corollary 9.7 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Suppose that (u, λ) solves (9.4) with $\eta = (1, -1)$. Assume that $u > -1$ on $[a, b]$. Let $\varphi \in C^1((-1, 1))$ be an odd strictly increasing function with $\varphi \in L^1((-1, 1))$. Then*

- (i) $\int_{\{u>0\}} \varphi(u) d\mu < +\infty$;
- (ii) $\int_a^b \varphi(u) d\mu \leq 0$;
- (iii) equality holds in (ii) if and only if $\rho \equiv 0$ on $[a, b]$.

In particular,

- (iv) $\int_a^b \frac{u}{\sqrt{1-u^2}} d\mu \leq 0$ with equality if and only if $\rho \equiv 0$ on $[a, b]$.

Proof (i) Put $I := \{1 > u > 0\}$. The function $u : I \rightarrow (0, 1)$ is $C^{0,1}$ and $u' \leq -\hat{m}$ a.e. on I by Lemma 9.3. It has $C^{0,1}$ inverse $v : (0, 1) \rightarrow I$, $v' = 1/(u' \circ v)$ and $|v'| \leq 1/\hat{m}$ a.e. on $(0, 1)$. By a change of variables,

$$\int_{\{u>0\}} \varphi(u) d\mu = \int_0^1 \varphi(v'/v) dt$$

from which the claim is apparent. (ii) The integral is well-defined because $\varphi(u)^+ = \varphi(u)\chi_{\{u>0\}} \in L^1((a, b), \mu)$ by (i). By Lemma 9.3 the set $\{u = 0\}$ consists of a singleton and has μ -measure zero. So

$$\begin{aligned} \int_a^b \varphi(u) d\mu &= \int_{\{u>0\}} \varphi(u) d\mu + \int_{\{u<0\}} \varphi(u) d\mu \\ &= \int_{\{u>0\}} \varphi(u) d\mu - \int_{\{v>0\}} \varphi(v) d\mu \end{aligned}$$

where $v := -u$ as φ is an odd function. We remark that in a similar way to (9.3),

$$\begin{aligned} \int_0^1 \varphi' \mu_u dt &= \int_{\{u>0\}} \{\varphi(u) - \varphi(0)\} \\ d\mu &= \int_{\{u>0\}} \varphi(u) d\mu \end{aligned}$$

using oddness of φ and an analogous formula holds with v in place of u . Thus we may write

$$\begin{aligned} \int_a^b \varphi(u) d\mu &= \int_0^1 \varphi' \mu_u dt - \int_0^1 \varphi' \mu_v dt \\ &= \int_0^1 \varphi' \{\mu_u - \mu_v\} dt \leq 0 \end{aligned}$$

by Corollary 9.6 as $\varphi' > 0$ on $(0, 1)$. (iii) Suppose that $\rho \not\equiv 0$ on $[a, b)$. Then strict inequality holds in the above by Corollary 9.6. If $\rho \equiv 0$ on $[a, b)$ the equality follows from Theorem 9.5. (iv) follows from (ii) and (iii) with the particular choice $\varphi : (-1, 1) \rightarrow \mathbb{R}; t \mapsto t/\sqrt{1-t^2}$. □

The boundary condition $\eta_1\eta_2 = 1$. Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. We study solutions of the auxilliary Riccati equation

$$w' + \lambda w^2 = (1/x + \rho)w \text{ a.e. on } (a, b) \text{ with } w(a) = w(b) = 1; \tag{9.6}$$

with $w \in C^{0,1}([a, b])$ and $\lambda \in \mathbb{R}$. If $\rho \equiv 0$ on $[a, b]$ then we write w_0 instead of w . Suppose (u, λ) solves (9.4) with $\eta = (1, 1)$. Then $u > 0$ on $[a, b]$ by Lemma 9.2 and we may set $w := 1/u$. Then $(w, -\lambda)$ satisfies (9.6).

Lemma 9.8 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Then*

- (i) *there exists a solution (w, λ) of (9.6) with $w \in C^{0,1}([a, b])$ and $\lambda \in \mathbb{R}$;*
- (ii) *the pair (w, λ) in (i) is unique;*
- (iii) *$\lambda = m$.*

Proof (i) Define $w : [a, b] \rightarrow \mathbb{R}$ by

$$w(t) := \frac{g(t)}{m \int_a^t g \, ds + g(a)} \text{ for } t \in [a, b].$$

Then $w \in C^{0,1}([a, b])$ and (w, m) satisfies (9.6). (ii) We claim that $w > 0$ on $[a, b]$ for any solution (w, λ) of (9.6). For otherwise, $c := \min\{w = 0\} \in (a, b)$. Then $u := 1/w$ on $[a, c)$ satisfies

$$u' + \left(\frac{1}{\tau} + \rho\right)u - \lambda = 0 \text{ a.e. on } (a, c) \text{ and } u(a) = 1, u(c-) = +\infty.$$

Integrating, we obtain

$$gu - g(a) - \lambda \int_a^{\cdot} g \, dt = 0 \text{ on } [a, c)$$

and this entails the contradiction that $u(c-) < +\infty$. We may now use the uniqueness statement in Lemma 9.2. (iii) follows from (ii) and the particular solution given in (i). □

We introduce the mapping

$$\omega : (0, \infty) \times (0, \infty) \rightarrow \mathbb{R}; (t, x) \mapsto -(2/t) \coth(x/2).$$

For $\xi > 0$,

$$|\omega(t, x) - \omega(t, y)| \leq \operatorname{cosech}^2[\xi/2](1/t)|x - y| \tag{9.7}$$

for $(t, x), (t, y) \in (0, \infty) \times (\xi, \infty)$ and ω is locally Lipschitzian in x on $(0, \infty) \times (0, \infty)$ in the sense of [16] I.3. Let $0 < a < b < +\infty$ and set $\lambda := A/G > 1$. Here, $A = A(a, b)$ stands for the arithmetic mean of a, b as introduced in the previous Section while $G = G(a, b) := \sqrt{|ab|}$ stands for their geometric mean. We refer to the initial value problem

$$z' = \omega(t, z) \text{ on } (0, \lambda) \text{ and } z(1) = \mu((a, b)). \tag{9.8}$$

Define

$$z_0 : (0, \lambda) \rightarrow \mathbb{R}; t \mapsto 2 \log \left\{ \frac{\lambda + \sqrt{\lambda^2 - t^2}}{t} \right\}.$$

Lemma 9.9 *Let $0 < a < b < +\infty$. Then*

- (i) $w_0(\tau) = \frac{2A\tau}{G^2 + \tau^2}$ for $\tau \in [a, b]$;
- (ii) $\|w_0\|_\infty = \lambda$;
- (iii) $\mu_{w_0} = z_0$ on $[1, \lambda]$;
- (iv) z_0 satisfies (9.8) and this solution is unique;
- (v) $\int_{\{w_0=1\}} \frac{1}{|w'_0|} \frac{d\mathcal{H}^0}{\tau} = 2 \coth(\mu((a, b))/2)$;
- (vi) $\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{dx}{x} = \pi$.

Proof (i) follows as in the proof of Lemma 9.8 with $g(t) = t$ while (ii) follows by calculus. (iii) follows by solving the quadratic equation $t\tau^2 - 2A\tau + G^2t = 0$ for τ with $t \in (0, \lambda)$. Uniqueness in (iv) follows from [16] Theorem 3.1 as ω is locally Lipschitzian with respect to x in $(0, \infty) \times (0, \infty)$. For (v) note that $|aw'_0(a)| = 1 - a/A$ and $|bw'_0(b)| = b/A - 1$ and

$$2 \coth(\mu((a, b))/2) = 2(a + b)/(b - a).$$

(vi) We may write

$$\begin{aligned} \int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{d\tau}{\tau} &= \int_a^b \frac{ab + \tau^2}{\sqrt{(a + b)^2\tau^2 - (ab + \tau^2)^2}} \frac{d\tau}{\tau} \\ &= \int_a^b \frac{ab + \tau^2}{\sqrt{(\tau^2 - a^2)(b^2 - \tau^2)}} \frac{d\tau}{\tau}. \end{aligned}$$

The substitution $s = \tau^2$ followed by the Euler substitution (cf. [14] 2.251)

$$\sqrt{(s - a^2)(b^2 - s)} = t(s - a^2)$$

gives

$$\int_a^b \frac{1}{\sqrt{w_0^2 - 1}} \frac{d\tau}{\tau} = \int_0^\infty \frac{1}{1 + t^2} + \frac{ab}{b^2 + a^2t^2} dt = \pi.$$

□

Lemma 9.10 *Let $0 < a < b < +\infty$. Then*

- (i) for $y > a$ the function $x \mapsto \frac{by - ax}{(y - a)(b - x)}$ is strictly increasing on $(-\infty, b]$;
- (ii) the function $y \mapsto \frac{(b - a)y}{(y - a)(b - y)}$ is strictly increasing on $[G, b]$;
- (iii) for $x < b$ the function $y \mapsto \frac{by - ax}{(y - a)(b - x)}$ is strictly decreasing on $[a, +\infty)$

Proof The proof is an exercise in calculus. □

Lemma 9.11 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let (w, λ) solve (9.6). Assume*

- (i) w is differentiable at both a and b and that (9.6) holds there;
- (ii) $w'(a) > 0$ and $w'(b) < 0$;
- (iii) $w > 1$ on (a, b) ;
- (iv) ρ is differentiable at a and b .

Then

$$\int_{\{w=1\} \setminus Z_w} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{\tau} \geq 2 \coth(\mu((a, b))/2)$$

with equality if and only if $\rho \equiv 0$ on $[a, b]$.

Proof At the end-points $x = a, b$ the condition (i) entails that $w' + m - \rho = 1/x = w'_0 + m_0$ so that

$$w' - w'_0 = m_0 - m + \rho \text{ at } x = a, b. \tag{9.9}$$

We consider the four cases

- (a) $w'(a) \geq w'_0(a)$ and $w'(b) \geq w'_0(b)$;
- (b) $w'(a) \geq w'_0(a)$ and $w'(b) \leq w'_0(b)$;
- (c) $w'(a) \leq w'_0(a)$ and $w'(b) \geq w'_0(b)$;
- (d) $w'(a) \leq w'_0(a)$ and $w'(b) \leq w'_0(b)$;

in turn.

- (a) Condition (a) together with (9.9) means that $m_0 - m + \rho(a) \geq 0$; that is, $m - \rho(a) \leq m_0$. By (i) and (ii), $bm - b\rho(b) - 1 = -bw'(b) > 0$; or $m - \rho(b) > 1/b$. Therefore,

$$0 < 1/b < m - \rho(b) \leq m - \rho(a) \leq 1/A$$

by (8.5). Put $x := 1/(m - \rho(b))$ and $y := 1/(m - \rho(a))$. Then

$$a < A \leq y \leq x < b.$$

We write

$$\begin{aligned} aw'(a) &= -(m - \rho(a))a + 1 = -(1/y)a + 1 > 0; \\ bw'(b) &= -(m - \rho(b))b + 1 = -(1/x)b + 1 < 0. \end{aligned}$$

Making use of assumption (iii),

$$\begin{aligned} \int_{\{w=1\} \setminus Z_w} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{x} &= \frac{1}{-(1/y)a + 1} - \frac{1}{-(1/x)b + 1} \\ &= \frac{by - ax}{(y - a)(b - x)}. \end{aligned}$$

By Lemma 9.10 (i) then (ii),

$$\int_{\{w=1\}} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{x} \geq \frac{(b-a)y}{(y-a)(b-y)} \geq \frac{(b-a)A}{(A-a)(b-A)}$$

$$= 2 \frac{a+b}{b-a} = 2 \coth(\mu((a, b))/2).$$

If equality holds then $\rho(a) = \rho(b)$ and ρ is constant on $[a, b]$. By Theorem 8.3 we conclude that $\rho \equiv 0$ on $[a, b]$.

- (b) Condition (b) together with (9.9) entails that $0 \leq m_0 - m + \rho(a)$ and $0 \leq -m_0 + m - \rho(b)$ whence $0 \leq \rho(a) - \rho(b)$ upon adding; so ρ is constant on the interval $[a, b]$ by monotonicity. Define x and y as above. Then $x = y$ and $y \geq A$. The result now follows in a similar way to case (a).
- (c) In this case,

$$\frac{1}{aw'(a)} - \frac{1}{bw'(b)} \geq \frac{1}{aw'_0(a)} - \frac{1}{bw'_0(b)}$$

$$= 2 \coth(\mu((a, b))/2)$$

by Lemma 9.9. If equality holds then $w'(b) = w'_0(b)$ so that $m_0 - m + \rho(b) = 0$ and ρ vanishes on $[a, b]$ by Theorem 8.3.

- (d) Condition (d) together with (9.9) means that $m_0 - m + \rho(b) \leq 0$; that is, $m \geq \rho(b) + m_0$. On the other hand, by Theorem 8.3, $m \leq \rho(b) + m_0$. In consequence, $m = \rho(b) + m_0$. It then follows that $\rho \equiv 0$ on $[a, b]$ by Theorem 8.3. Now use Lemma 9.9.

□

Lemma 9.12 *Let $\phi : (0, +\infty) \rightarrow (0, +\infty)$ be a convex non-increasing function with $\inf_{(0,+\infty)} \phi > 0$. Let Λ be an at most countably infinite index set and $(x_h)_{h \in \Lambda}$ a sequence of points in $(0, +\infty)$ with $\sum_{h \in \Lambda} x_h < +\infty$. Then*

$$\sum_{h \in \Lambda} \phi(x_h) \geq \phi\left(\sum_{h \in \Lambda} x_h\right)$$

and the left-hand side takes the value $+\infty$ in case Λ is countably infinite and is otherwise finite.

Proof Suppose $0 < x_1 < x_2 < +\infty$. By convexity $\phi(x_1) + \phi(x_2) \geq 2\phi(\frac{x_1+x_2}{2}) \geq \phi(x_1 + x_2)$ as ϕ is non-increasing. The result for finite Λ follows by induction. □

Theorem 9.13 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let (w, λ) solve (9.6). Assume that $w > 1$ on (a, b) . Then*

- (i) for \mathcal{L}^1 -a.e. $t \in (1, \|w\|_\infty)$,

$$- \mu'_w \geq (2/t) \coth((1/2)\mu_w); \tag{9.10}$$

(ii) if $\rho \neq 0$ on $[a, b]$ then there exists $t_0 \in (1, \|w\|_\infty)$ such that strict inequality holds in (9.10) for \mathcal{L}^1 -a.e. $t \in (1, t_0)$.

Proof (i) The set

$$Y_w := Z_{2,w} \cup \left(\{w' + mw^2 \neq (1/x + \rho)w\} \setminus Z_{2,w} \right) \cup \{\rho \text{ not differentiable}\} \subset [a, b]$$

is a null set in $[a, b]$. By [1] Lemma 2.95 and Lemma 2.96, $\{w = t\} \cap (Y_w \cap Z_{1,w}) = \emptyset$ for a.e. $t > 1$. Let $t \in (1, \|w\|_\infty)$ and assume that $\{w = t\} \cap (Y_w \cap Z_{1,w}) = \emptyset$. We write $\{w > t\} = \bigcup_{h \in \Lambda} I_h$ where Λ is an at most countably infinite index set and $(I_h)_{h \in \Lambda}$ are disjoint non-empty well-separated open intervals in (a, b) . The term well-separated means that for each $h \in \Lambda$, $\inf_{k \in \Lambda \setminus \{h\}} d(I_h, I_k) > 0$. This follows from the fact that $w' \neq 0$ on ∂I_h for each $h \in \Lambda$. Put $\tilde{w} := w/t$ on $\{w > t\}$ so

$$\tilde{w}' + (mt)\tilde{w}^2 = (1/x + \rho)\tilde{w} \text{ a.e. on } \{w > t\} \text{ and } \tilde{w} = 1 \text{ on } \{w = t\}.$$

We use the fact that the mapping $\phi : (0, +\infty) \rightarrow (0, +\infty); t \mapsto \coth t$ satisfies the hypotheses of Lemma 9.12. By Lemmas 9.11 and 9.12,

$$\begin{aligned} (0, +\infty] &\ni \int_{\{w=t\} \setminus Z_w} \frac{1}{|w'|} \frac{d\mathcal{H}^0}{x} = (1/t) \int_{\{\tilde{w}=1\}} \frac{1}{|\tilde{w}'|} \frac{d\mathcal{H}^0}{\tau} \\ &= (1/t) \sum_{h \in \Lambda} \int_{\partial I_h} \frac{1}{|\tilde{w}'|} \frac{d\mathcal{H}^0}{\tau} \\ &\geq (2/t) \sum_{h \in \Lambda} \coth((1/2)\mu(I_h)) \\ &\geq (2/t) \coth\left((1/2) \sum_{h \in \Lambda} \mu(I_h) \right) \\ &= (2/t) \coth((1/2)\mu(\{w > t\})) \\ &= (2/t) \coth((1/2)\mu_w(t)). \end{aligned}$$

The statement now follows from Lemma 9.1.

(ii) Suppose that $\rho \neq 0$ on $[a, b]$. Put $\alpha := \min\{\rho > 0\} \in [a, b]$. Now that $\{w > t\} \uparrow (a, b)$ as $t \downarrow 1$ as $w > 1$ on (a, b) . Choose $t_0 \in (1, \|w\|_\infty)$ such that $\{w > t_0\} \cap (\alpha, b) \neq \emptyset$. Then for each $t \in (1, t_0)$ there exists $h \in \Lambda$ such that $\rho \neq 0$ on I_h . The statement then follows by Lemma 9.11. □

Lemma 9.14 *Let $\emptyset \neq S \subset \mathbb{R}$ be bounded and suppose S has the property that for each $s \in S$ there exists $\delta > 0$ such that $[s, s + \delta) \subset S$. Then S is \mathcal{L}^1 -measurable and $|S| > 0$.*

Proof For each $s \in S$ put $t_s := \inf\{t > s : t \notin S\}$. Then $s < t_s < +\infty$, $[s, t_s) \subset S$ and $t_s \notin S$. Define

$$\mathcal{C} := \left\{ [s, t] : s \in S \text{ and } t \in (s, t_s) \right\}.$$

Then \mathcal{C} is a Vitali cover of S (see [6] Chapter 16 for example). By Vitali’s Covering Theorem (cf. [6] Theorem 16.27) there exists an at most countably infinite subset $\Lambda \subset \mathcal{C}$ consisting of pairwise disjoint intervals such that

$$\left| S \setminus \bigcup_{I \in \Lambda} I \right| = 0.$$

Note that $I \subset S$ for each $I \in \Lambda$. Consequently, $S = \bigcup_{I \in \Lambda} I \cup N$ where N is an \mathcal{L}^1 -null set and hence S is \mathcal{L}^1 -measurable. The positivity assertion is clear. \square

Theorem 9.15 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Let (w, λ) solve (9.6). Assume that $w > 1$ on (a, b) . Put $T := \min\{\|w_0\|_\infty, \|w\|_\infty\} > 1$. Then*

- (i) $\mu_w(t) \leq \mu_{w_0}(t)$ for each $t \in [1, T)$;
- (ii) $\|w\|_\infty \leq \|w_0\|_\infty$;
- (iii) if $\rho \not\equiv 0$ on $[a, b]$ then there exists $t_0 \in (1, \|w\|_\infty)$ such that $\mu_w(t) < \mu_{w_0}(t)$ for each $t \in (1, t_0)$.

Proof (i) We adapt the proof of [16] Theorem I.6.1. The assumption entails that $\mu_w(1) = \mu_{w_0}(1) = \mu((a, b))$. Suppose for a contradiction that $\mu_w(t) > \mu_{w_0}(t)$ for some $t \in (1, T)$.

For $\varepsilon > 0$ consider the initial value problem

$$z' = \omega(t, z) + \varepsilon \text{ and } z(1) = \mu((a, b)) + \varepsilon \tag{9.11}$$

on $(0, T)$. Choose $\nu \in (0, 1)$ and $\tau \in (t, T)$. By [16] Lemma I.3.1 there exists $\varepsilon_0 > 0$ such that for each $0 \leq \varepsilon < \varepsilon_0$ (9.11) has a continuously differentiable solution z_ε defined on $[\nu, \tau]$ and this solution is unique by [16] Theorem I.3.1. Moreover, the sequence $(z_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ converges uniformly to z_0 on $[\nu, \tau]$.

Given $0 < \varepsilon < \eta < \varepsilon_0$ it holds that $z_0 \leq z_\varepsilon \leq z_\eta$ on $[1, \tau]$ by [16] Theorem I.6.1. Note for example that $z'_0 \leq \omega(\cdot, z_0) + \varepsilon$ on $(1, \tau)$. In fact, $(z_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ decreases strictly to z_0 on $(1, \tau)$. For if, say, $z_0(s) = z_\varepsilon(s)$ for some $s \in (1, \tau)$ then $z'_\varepsilon(s) = \omega(s, z_\varepsilon(s)) + \varepsilon > \omega(s, z_0(s)) = z'_0(s)$ by (9.11); while on the other hand $z'_\varepsilon(s) \leq z'_0(s)$ by considering the left-derivative at s and using the fact that $z_\varepsilon \geq z_0$ on $[1, \tau]$. This contradicts the strict inequality.

Choose $\varepsilon_1 \in (0, \varepsilon_0)$ such that $z_\varepsilon(t) < \mu_w(t)$ for each $0 < \varepsilon < \varepsilon_1$. Now μ_w is right-continuous and strictly decreasing as $\mu_w(t) - \mu_w(s) = -\mu(\{s < w \leq t\}) < 0$ for $1 \leq s < t < \|w\|_\infty$ by continuity of w . So the set $\{z_\varepsilon < \mu_w\} \cap (1, t)$ is open and non-empty in $(0, +\infty)$ for each $\varepsilon \in (0, \varepsilon_1)$. Thus there exists a unique $s_\varepsilon \in [1, t)$ such that

$$\mu_w > z_\varepsilon \text{ on } (s_\varepsilon, t] \text{ and } \mu_w(s_\varepsilon) = z_\varepsilon(s_\varepsilon)$$

for each $\varepsilon \in (0, \varepsilon_1)$. As $z_\varepsilon(1) > \mu((a, b))$ it holds that each $s_\varepsilon > 1$. Note that $1 < s_\varepsilon < s_\eta$ whenever $0 < \varepsilon < \eta$ as $(z_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$ decreases strictly to z_0 as $\varepsilon \downarrow 0$.

Define

$$S := \left\{ s_\varepsilon : 0 < \varepsilon < \varepsilon_1 \right\} \subset (1, t).$$

We claim that for each $s \in S$ there exists $\delta > 0$ such that $[s, s + \delta) \subset S$. This entails that S is \mathcal{L}^1 -measurable with positive \mathcal{L}^1 -measure by Lemma 9.14.

Suppose $s = s_\varepsilon \in S$ for some $\varepsilon \in (0, \varepsilon_1)$ and put $z := z_\varepsilon(s) = \mu_w(s)$. Put $k := \operatorname{cosech}^2(z_0(t)/2)$. For $0 \leq \zeta < \eta < \varepsilon_1$ define

$$\Omega_{\zeta, \eta} := \left\{ (u, y) \in \mathbb{R}^2 : u \in (0, t) \text{ and } z_\zeta(u) < y < z_\eta(u) \right\}$$

and note that this is an open set in \mathbb{R}^2 . We remark that for each $(u, y) \in \Omega_{\zeta, \eta}$ there exists a unique $v \in (\zeta, \eta)$ such that $y = z_v(u)$. Given $r > 0$ with $s + r < t$ set

$$Q = Q_r := \left\{ (u, y) \in \mathbb{R}^2 : s \leq u < s + r \text{ and } |y - z| < \|z_\varepsilon - z\|_{C([s, s+r])} \right\}.$$

Choose $r \in (0, t - s)$ and $\varepsilon_2 \in (\varepsilon, \varepsilon_1)$ such that

- (a) $Q_r \subset \Omega_{0, \varepsilon_1}$;
- (b) $\|z_\varepsilon - z\|_{C([s, s+r])} < s\varepsilon/(2k)$;
- (c) $\sup_{\eta \in (\varepsilon, \varepsilon_2)} \|z_\eta - z\|_{C([s, s+r])} \leq \|z_\varepsilon - z\|_{C([s, s+r])}$;
- (d) $z_\eta < \mu_w$ on $[s + r, t]$ for each $\eta \in (\varepsilon, \varepsilon_2)$.

We can find $\delta \in (0, r)$ such that $z_\varepsilon < \mu_w < z_{\varepsilon_2}$ on $(s, s + \delta)$ as $z_{\varepsilon_2}(s) > z$; in other words, the graph of μ_w restricted to $(s, s + \delta)$ is contained in $\Omega_{\varepsilon, \varepsilon_2}$.

Let $u \in (s, s + \delta)$. Then $\mu_w(u) = z_\eta(u)$ for some $\eta \in (\varepsilon, \varepsilon_2)$ as above. We claim that $u = s_\eta$ so that $u \in S$. This implies in turn that $[s, s + \delta) \subset S$. Suppose for a contradiction that $z_\eta \not\leq \mu_w$ on $(u, t]$. Then there exists $v \in (u, t]$ such that $\mu_w(v) = z_\eta(v)$. In view of condition (d), $v \in (u, s + r)$. By [1] Theorem 3.28 and Theorem 9.13,

$$\begin{aligned} \mu_w(v) - \mu_w(u) &= D\mu_w((u, v)) = D\mu_w^a((u, v)) + D\mu_w^s((u, v)) \\ &\leq D\mu_w^a((u, v)) = \int_u^v \mu'_w \, d\tau \leq \int_u^v \omega(\cdot, \mu_w) \, d\tau. \end{aligned}$$

On the other hand,

$$z_\eta(v) - z_\eta(u) = \int_u^v z'_\eta \, d\tau = \int_u^v \omega(\cdot, z_\eta) \, d\tau + \eta(v - u).$$

We derive that

$$\begin{aligned} \varepsilon(v - u) &\leq \eta(v - u) \leq \int_u^v \left\{ \omega(\cdot, \mu_w) - \omega(\cdot, z_\eta) \right\} d\tau \\ &\leq k \int_u^v |\mu_w - z_\eta| d\mu \end{aligned}$$

using the estimate (9.7). Thus

$$\begin{aligned} \varepsilon &\leq k \frac{1}{v - u} \int_u^v |\mu_w - z_\eta| d\mu \\ &\leq (k/s) \|\mu_w - z_\eta\|_{C([u, v])} \\ &\leq (k/s) \left\{ \|\mu_w - z\|_{C([s, s+r])} + \|z_\eta - z\|_{C([s, s+r])} \right\} \\ &\leq (2k/s) \|z_\varepsilon - z\|_{C([s, s+r])} < \varepsilon \end{aligned}$$

by (b) and (c) giving rise to the desired contradiction.

By Theorem 9.13, $\mu'_w \leq \omega(\cdot, \mu_w)$ for \mathcal{L}^1 -a.e. $t \in S$. Choose $s \in S$ such that μ_w is differentiable at s and the latter inequality holds at s . Let $\varepsilon \in (0, \varepsilon_1)$ such that $s = s_\varepsilon$. For any $u \in (s, t)$,

$$\mu_w(u) - \mu_w(s) > z_\varepsilon(u) - z_\varepsilon(s).$$

We deduce that $\mu'_w(s) \geq z'_\varepsilon(s)$. But then

$$\mu'_w(s) \geq z'_\varepsilon(s) = \omega(s, z_\varepsilon(s)) + \varepsilon > \omega(s, \mu_w(s)).$$

This strict inequality holds on a set of full measure in S . This contradicts Theorem 9.13.

- (ii) Use the fact that $\|w\|_\infty = \sup\{t > 0 : \mu_w(t) > 0\}$.
- (iii) Assume that $\rho \not\equiv 0$ on $[a, b)$. Let $t_0 \in (1, \|w\|_\infty)$ be as in Lemma 9.13. Then for $t \in (1, t_0)$,

$$\begin{aligned} \mu_w(t) - \mu_w(1) &= D\mu_w((1, t]) = D\mu_w^a((1, t]) \\ &\quad + D\mu_w^s((1, t]) \leq D\mu_w^a((1, t]) \\ &= \int_{(1, t]} \mu'_w ds < \int_{(1, t]} \omega(s, \mu_w) ds \\ &\leq \int_{(1, t]} \omega(s, \mu_{w_0}) ds = \mu_{w_0}(t) - \mu_{w_0}(1) \end{aligned}$$

by Theorem 9.13, Lemma 9.9 and the inequality in (i).

□

Corollary 9.16 *Let $0 < a < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[a, b]$. Suppose that (w, λ) solves (9.6). Assume that $w > 1$ on (a, b) . Let $0 \leq \varphi \in C^1((1, +\infty))$ be strictly decreasing with $\int_a^b \varphi(w_0) d\mu < +\infty$. Then*

- (i) $\int_a^b \varphi(w) d\mu \geq \int_a^b \varphi(w_0) d\mu$;
- (ii) equality holds in (i) if and only if $\rho \equiv 0$ on $[a, b]$.

In particular,

- (iii) $\int_a^b \frac{1}{\sqrt{w^2-1}} d\mu \geq \pi$ with equality if and only if $\rho \equiv 0$ on $[a, b]$.

Proof (i) Let $\varphi \geq 0$ be a decreasing function on $(1, +\infty)$ which is piecewise C^1 . Suppose that $\varphi(1+) < +\infty$. By Tonelli’s Theorem,

$$\begin{aligned} \int_{[1,+\infty)} \varphi' \mu_w ds &= \int_{[1,+\infty)} \varphi' \left\{ \int_{(a,b)} \chi_{\{w>s\}} d\mu \right\} ds \\ &= \int_{(a,b)} \left\{ \int_{[1,+\infty)} \varphi' \chi_{\{w>s\}} ds \right\} d\mu \\ &= \int_{(a,b)} \left\{ \varphi(w) - \varphi(1) \right\} d\mu \\ &= \int_{(a,b)} \varphi(w) d\mu - \varphi(1)\mu((a, b)) \end{aligned}$$

and a similar identity holds for μ_{w_0} . By Theorem 9.15, $\int_a^b \varphi(w) d\mu \geq \int_a^b \varphi(w_0) d\mu$. Now suppose that $0 \leq \varphi \in C^1((1, +\infty))$ is strictly decreasing with $\int_a^b \varphi(w_0) d\mu < +\infty$. The inequality holds for the truncated function $\varphi \wedge n$ for each $n \in \mathbb{N}$. An application of the monotone convergence theorem establishes the result for φ .

(ii) Suppose that equality holds in (i). For $c \in (1, +\infty)$ put $\varphi_1 := \varphi \vee \varphi(c) - \varphi(c)$ and $\varphi_2 := \varphi \wedge \varphi(c)$. By (i) we deduce

$$\int_a^b \varphi_2(w) d\mu = \int_a^b \varphi_2(w_0) d\mu;$$

and hence by the above that

$$\int_{[c,+\infty)} \varphi' \left\{ \mu_w - \mu_{w_0} \right\} ds = 0.$$

This means that $\mu_w = \mu_{w_0}$ on $(c, +\infty)$ and hence on $(1, +\infty)$. By Theorem 9.15 we conclude that $\rho \equiv 0$ on $[a, b]$. (iii) flows from (i) and (ii) noting that the function $\varphi : (1, +\infty) \rightarrow \mathbb{R}; t \mapsto 1/\sqrt{t^2 - 1}$ satisfies the integral condition by Lemma 9.9. \square

The case $a = 0$. Let $0 < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[0, b]$. We study solutions to the first-order linear ordinary differential equation

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (0, b) \text{ with } u(0) = 0 \text{ and } u(b) = 1 \tag{9.12}$$

where $u \in C^{0,1}([0, b])$ and $\lambda \in \mathbb{R}$. If $\rho \equiv 0$ on $[0, b]$ then we write u_0 instead of u .

Lemma 9.17 *Let $0 < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[0, b]$. Then*

- (i) *there exists a solution (u, λ) of (9.12) with $u \in C^{0,1}([0, b])$ and $\lambda \in \mathbb{R}$;*
- (ii) *λ is given by $\lambda = -g(b)/G(b)$ where $G := \int_0^b g \, ds$;*
- (iii) *the pair (u, λ) in (i) is unique;*
- (iv) *$u > 0$ on $(0, b]$.*

Proof (i) The function $u : [a, b] \rightarrow \mathbb{R}$ given by

$$u = \frac{g(b)}{G(b)} \frac{G}{g} \tag{9.13}$$

on $[0, b]$ solves (9.12) with λ as in (ii). (iii) Suppose that (u_1, λ_1) resp. (u_2, λ_2) solve (9.12). By linearity $u := u_1 - u_2$ solves

$$u' + (1/x + \rho)u + \lambda = 0 \text{ a.e. on } (0, b) \text{ with } u(0) = u(b) = 0$$

where $\lambda = \lambda_1 - \lambda_2$. An integration gives that $u = (-\lambda G + c)/g$ for some constant $c \in \mathbb{R}$ and the boundary conditions entail that $\lambda = c = 0$. (iv) follows from the formula (9.13) and unicity. □

Lemma 9.18 *Suppose $-\infty < a < b < +\infty$ and that $\phi : [a, b] \rightarrow \mathbb{R}$ is convex. Suppose that there exists $\xi \in (a, b)$ such that*

$$\phi(\xi) = \frac{b - \xi}{b - a} \phi(a) + \frac{\xi - a}{b - a} \phi(b).$$

Then

$$\phi(c) = \frac{b - c}{b - a} \phi(a) + \frac{c - a}{b - a} \phi(b)$$

for each $c \in [a, b]$.

Proof Let $c \in (\xi, b)$. By monotonicity of chords,

$$\frac{\phi(\xi) - \phi(a)}{\xi - a} \leq \frac{\phi(c) - \phi(\xi)}{c - \xi}$$

so

$$\begin{aligned} \phi(c) &\geq \frac{c - a}{\xi - a} \phi(\xi) - \frac{c - \xi}{\xi - a} \phi(a) \\ &= \frac{c - a}{\xi - a} \left\{ \frac{b - \xi}{b - a} \phi(a) + \frac{\xi - a}{b - a} \phi(b) \right\} - \frac{c - \xi}{\xi - a} \phi(a) \end{aligned}$$

$$= \frac{b - c}{b - a} \phi(a) + \frac{c - a}{b - a} \phi(b)$$

and equality follows. The case $c \in (a, \xi)$ is similar. □

Lemma 9.19 *Let $0 < b < +\infty$ and $\rho \geq 0$ be a non-decreasing bounded function on $[0, b]$. Let (u, λ) satisfy (9.12). Then*

- (i) $u \geq u_0$ on $[0, b]$;
- (ii) if $\rho \not\equiv 0$ on $[0, b)$ then $u > u_0$ on $(0, b)$.

Proof (i) The mapping $G : [0, b] \rightarrow [0, G(b)]$ is a bijection with inverse G^{-1} . Define $\eta : [0, G(b)] \rightarrow \mathbb{R}$ via $\eta := (tg) \circ G^{-1}$. Then

$$\eta' = \frac{(tg)'}{g} \circ G^{-1} = (2 + t\rho) \circ G^{-1}$$

a.e. on $(0, G(b))$ so η' is non-decreasing there. This means that η is convex on $[0, G(b)]$. In particular, $\eta(s) \leq [\eta(G(b))/G(b)]s$ for each $s \in [0, G(b)]$. For $t \in [0, b]$ put $s := G(t)$ to obtain $tg(t) \leq (bg(b)/G(b))G(t)$. A rearrangement gives $u \geq u_0$ on $[0, b]$ noting that $u_0 : [0, b] \rightarrow \mathbb{R}; t \mapsto t/b$. (ii) Assume $\rho \not\equiv 0$ on $[0, b)$. Suppose that $u(c) = u_0(c)$ for some $c \in (0, b)$. Then $\eta(G(c)) = [\eta(G(b))/G(b)]G(c)$. By Lemma 9.18, $\eta' = 0$ on $(0, G(b))$. This implies that $\rho \equiv 0$ on $[0, b)$. □

Lemma 9.20 *Let $0 < b < +\infty$. Then $\int_0^b \frac{u_0}{\sqrt{1-u_0^2}} d\mu = \pi/2$.*

Proof The integral is elementary as $u_0(t) = t/b$ for $t \in [0, b]$. □

10 Proof of main results

Lemma 10.1 *Let $x \in H$ and v be a unit vector in \mathbb{R}^2 such that the pair $\{x, v\}$ forms a positively oriented orthogonal basis for \mathbb{R}^2 . Put $b := (\tau, 0)$ where $|x| = \tau$ and $\gamma := \theta(x) \in (0, \pi)$. Let $\alpha \in (0, \pi/2)$ such that*

$$\frac{\langle v, x - b \rangle}{|x - b|} = \cos \alpha.$$

Then

- (i) $C(x, v, \alpha) \cap H \cap \overline{C}(0, e_1, \gamma) = \emptyset$;
- (ii) for any $y \in C(x, v, \alpha) \cap H \setminus \overline{B}(0, \tau)$ the line segment $[b, y]$ intersects \mathbb{S}_τ^1 outside the closed cone $\overline{C}(0, e_1, \gamma)$.

We point out that $C(0, e_1, \gamma)$ is the open cone with vertex 0 and axis e_1 which contains the point x on its boundary. We note that $\cos \alpha \in (0, 1)$ because

$$\begin{aligned} \langle v, x - b \rangle &= -\langle v, b \rangle = -\langle (1/\tau)Ox, b \rangle \\ &= -\langle Op, e_1 \rangle = \langle x, O^*e_1 \rangle = \langle x, e_2 \rangle > 0 \end{aligned} \tag{10.1}$$

and if $|x - b| = \langle v, x - b \rangle$ then $b = x - \lambda v$ for some $\lambda \in \mathbb{R}$ and hence $x_1 = \langle e_1, x \rangle = \tau$ and $x_2 = 0$.

Proof (i) For $\omega \in \mathbb{S}^1$ define the open half-space

$$H_\omega := \{y \in \mathbb{R}^2 : \langle y, \omega \rangle > 0\}.$$

We claim that $C(x, v, \alpha) \subset H_v$. For given $y \in C(x, v, \alpha)$,

$$\langle y, v \rangle = \langle y - x, v \rangle > |y - x| \cos \alpha > 0.$$

On the other hand, it holds that $\overline{C}(0, e_1, \gamma) \cap H \subset \overline{H}_{-v}$. This establishes (i).

(ii) By some trigonometry $\gamma = 2\alpha$. Suppose that ω is a unit vector in $C(b, -e_1, \pi/2 - \alpha)$. Then $\lambda := \langle \omega, e_1 \rangle < \cos \alpha$ since upon rewriting the membership condition for $C(b, -e_1, \pi/2 - \alpha)$ we obtain the quadratic inequality

$$\lambda^2 - 2 \cos^2 \alpha \lambda + \cos \gamma > 0.$$

For ω a unit vector in $\overline{C}(0, e_1, \gamma)$ the opposite inequality $\langle \omega, e_1 \rangle \geq \cos \alpha$ holds. This shows that

$$C(b, -e_1, \pi/2 - \alpha) \cap \overline{C}(0, e_1, \gamma) \cap \mathbb{S}_\tau^1 = \emptyset.$$

The set $C(x, v, \alpha) \cap H$ is contained in the open convex cone $C(b, -e_1, \pi/2 - \alpha)$. Suppose $y \in C(x, v, \alpha) \cap H \setminus \overline{B}(0, \tau)$. Then the line segment $[b, y]$ is contained in $C(b, -e_1, \pi/2 - \alpha) \cup \{b\}$. Now the set $C(b, -e_1, \pi/2 - \alpha) \cap \mathbb{S}_\tau^1$ disconnects $C(b, -e_1, \pi/2 - \alpha) \cup \{b\}$. This entails that $(b, y] \cap C(b, -e_1, \pi/2 - \alpha) \cap \mathbb{S}_\tau^1 \neq \emptyset$. The foregoing paragraph entails that $(b, y] \cap \overline{C}(0, e_1, \gamma) \cap \mathbb{S}_\tau^1 = \emptyset$. This establishes the result. □

Lemma 10.2 *Let E be an open set in \mathbb{R}^2 such that $M := \partial E$ is a $C^{1,1}$ hypersurface in \mathbb{R}^2 . Assume that $E \setminus \{0\} = E^{sc}$. Suppose*

- (i) $x \in (M \setminus \{0\}) \cap H$;
- (ii) $\sin(\sigma(x)) = -1$.

Then E is not convex.

Proof Let $\gamma_1 : I \rightarrow M$ be a $C^{1,1}$ parametrisation of M in a neighbourhood of x with $\gamma_1(0) = x$ as above. As $\sin(\sigma(x)) = -1, n(x)$ and hence $n_1(0)$ point in the direction of x . Put $v := -t_1(0) = -t(x)$. We may write

$$\gamma_1(s) = \gamma_1(0) + st_1(0) + R_1(s) = x - sv + R_1(s)$$

for $s \in I$ where $R_1(s) = s \int_0^1 \dot{\gamma}_1(ts) - \dot{\gamma}_1(0) dt$ and we can find a finite positive constant K such that $|R_1(s)| \leq Ks^2$ on a symmetric open interval I_0 about 0 with $I_0 \subset\subset I$. Then

$$\begin{aligned} \frac{\langle \gamma_1(s) - x, v \rangle}{|\gamma_1(s) - x|} &= \frac{\langle -sv + R_1, v \rangle}{|-sv + R_1|} \\ &= \frac{1 - \langle (R_1/s), v \rangle}{|v - R_1/s|} \rightarrow 1 \end{aligned}$$

as $s \uparrow 0$. Let α be as in Lemma 10.1 with x and v as just mentioned. The above estimate entails that $\gamma_1(s) \in C(x, v, \alpha)$ for small $s < 0$. By (2.9) and Lemma 5.4 the function r_1 is non-increasing on I . In particular, $r_1(s) \geq r_1(0) = |x| =: \tau$ for $I \ni s < 0$ and $\gamma_1(s) \notin B(0, \tau)$.

Choose $\delta_1 > 0$ such that $\gamma_1(s) \in C(x, v, \alpha) \cap H$ for each $s \in [-\delta_1, 0)$. Put $\beta := \inf\{s \in [-\delta_1, 0] : r_1(s) = \tau\}$. Suppose first that $\beta \in [-\delta_1, 0)$. Then E is not convex (see Lemma 5.2). Now suppose that $\beta = 0$. Let γ be as in Lemma 10.1. Then the open circular arc $\mathbb{S}_\tau^1 \setminus \overline{C}(0, e_1, \gamma)$ does not intersect \overline{E} : for otherwise, M intersects $\mathbb{S}_\tau^1 \setminus \overline{C}(0, e_1, \gamma)$ and $\beta < 0$ bearing in mind Lemma 5.2. Choose $s \in [-\delta_1, 0)$. Then the points b and $\gamma_1(s)$ lie in \overline{E} . But by Lemma 10.1 the line segment $[b, \gamma_1(s)]$ intersects \mathbb{S}_τ^1 in $\mathbb{S}_\tau^1 \setminus \overline{C}(0, e_1, \gamma)$. Let $c \in [b, \gamma_1(s)] \cap \mathbb{S}_\tau^1$. Then $c \notin \overline{E}$. This shows that \overline{E} is not convex. But if E is convex then \overline{E} is convex. Therefore E is not convex. \square

Theorem 10.3 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Given $v > 0$ let E be a bounded minimiser of (1.2). Assume that E is open, $M := \partial E$ is a $C^{1,1}$ hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$. Put*

$$R := \inf\{\rho > 0\} \in [0, +\infty). \tag{10.2}$$

Then $\Omega \cap (R, +\infty) = \emptyset$ with Ω as in (5.2).

Proof Suppose that $\Omega \cap (R, +\infty) \neq \emptyset$. As Ω is open in $(0, +\infty)$ by Lemma 5.6 we may write Ω as a countable union of disjoint open intervals in $(0, +\infty)$. By a suitable choice of one of these intervals we may assume that $\Omega = (a, b)$ for some $0 \leq a < b < +\infty$ and that $\Omega \cap (R, +\infty) \neq \emptyset$. Let us assume for the time being that $a > 0$. Note that $[a, b] \subset \pi(M)$ and $\cos \sigma$ vanishes on $M_a \cup M_b$.

Let $u : \Omega \rightarrow [-1, 1]$ be as in (6.6). Then u has a continuous extension to $[a, b]$ and $u = \pm 1$ at $\tau = a, b$. This may be seen as follows. For $\tau \in (a, b)$ the set $M_\tau \cap \overline{H}$ consists of a singleton by Lemma 5.4. The limit $x := \lim_{\tau \downarrow a} M_\tau \cap \overline{H} \in \mathbb{S}_a^1 \cap \overline{H}$ exists as M is C^1 . There exists a $C^{1,1}$ parametrisation $\gamma_1 : I \rightarrow M$ with $\gamma_1(0) = x$ as above. By (2.9) and Lemma 5.4, r_1 is decreasing on I . So $r_1 > a$ on $I \cap \{s < 0\}$ for otherwise the C^1 property fails at x . It follows that $\gamma_1 = \gamma \circ r_1$ and $\sigma_1 = \sigma \circ \gamma \circ r_1$ on $I \cap \{s < 0\}$. Thus $\sin(\sigma \circ \gamma) \circ r_1 = \sin \sigma_1$ on $I \cap \{s < 0\}$. Now the function $\sin \sigma_1$ is continuous on I . So $u \rightarrow \sin \sigma_1(0) \in \{\pm 1\}$ as $\tau \downarrow a$. Put $\eta_1 := u(a)$ and $\eta_2 := u(b)$.

Let us consider the case $\eta = (\eta_1, \eta_2) = (1, 1)$. According to Theorem 6.5 the generalised (mean) curvature is constant \mathcal{H}^1 -a.e. on M with value $-\lambda$, say. Note that $u < 1$ on (a, b) for otherwise $\cos(\sigma \circ \gamma)$ vanishes at some point in (a, b) bearing in

mind Lemma 5.4. By Theorem 6.6 the pair (u, λ) satisfies (9.4) with $\eta = (1, 1)$. By Lemma 9.2, $u > 0$ on $[a, b]$. Put $w := 1/u$. Then $(w, -\lambda)$ satisfies (9.6) and $w > 1$ on (a, b) . By Lemma 6.7,

$$\begin{aligned} \theta_2(b) - \theta_2(a) &= \int_a^b \theta_2' d\tau = - \int_a^b \frac{u}{\sqrt{1-u^2}} \frac{d\tau}{\tau} \\ &= - \int_a^b \frac{1}{\sqrt{w^2-1}} \frac{d\tau}{\tau}. \end{aligned}$$

By Corollary 9.16, $|\theta_2(b) - \theta_2(a)| > \pi$. But this contradicts the definition of θ_2 in (6.4) as θ_2 takes values in $(0, \pi)$ on (a, b) . If $\eta = (-1, -1)$ then $\lambda > 0$ by Lemma 9.2; this contradicts Lemma 7.2.

Now let us consider the case $\eta = (-1, 1)$. Using the same formula as above, $\theta_2(b) - \theta_2(a) < 0$ by Corollary 9.7. This means that $\theta_2(a) \in (0, \pi]$. As before the limit $x := \lim_{\tau \downarrow a} M_\tau \cap \overline{H} \in \mathbb{S}_a^1 \cap \overline{H}$ exists as M is C^1 . Using a local parametrisation it can be seen that $\theta_2(a) = \theta(x)$ and $\sin(\sigma(x)) = -1$. If $\theta_2(a) \in (0, \pi)$ then E is not convex by Lemma 10.2. This contradicts Theorem 7.3. Note that we may assume that $\theta_2(a) \in (0, \pi)$. For otherwise, $\langle \gamma, e_2 \rangle < 0$ for $\tau > a$ near a , contradicting the definition of γ (6.5). If $\eta = (1, -1)$ then $\lambda > 0$ by Lemma 9.2 and this contradicts Lemma 7.2 as before.

Suppose finally that $a = 0$. By Lemma 5.5, $u(0) = 0$ and $u(b) = \pm 1$. Suppose $u(b) = 1$. Again employing the formula above, $\theta_2(b) - \theta_2(0) < -\pi/2$ by Lemma 9.19, the fact that the function $\phi : (0, 1) \rightarrow \mathbb{R}; t \mapsto t/\sqrt{1-t^2}$ is strictly increasing and Lemma 9.20. This means that $\theta_2(0) > \pi/2$. This contradicts the C^1 property at $0 \in M$. If $u(b) = -1$ then then $\lambda > 0$ by Lemma 9.2 giving a contradiction. \square

Lemma 10.4 *Let f be as in (1.3) where $h : [0, +\infty) \rightarrow \mathbb{R}$ is a non-decreasing convex function. Let $v > 0$.*

- (i) *Let E be a bounded minimiser of (1.2). Assume that E is open, $M := \partial E$ is a $C^{1,1}$ hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$. Then for any $r > 0$ with $r \geq R$, $M \setminus \overline{B}(0, r)$ consists of a finite union of disjoint centred circles.*
- (ii) *There exists a minimiser E of (1.2) such that ∂E consists of a finite union of disjoint centred circles.*

Proof (i) First observe that

$$\emptyset \neq \pi(M) = \left[\pi(M) \cap [0, r] \right] \cup \left[\pi(M) \cap (r, +\infty) \right] \setminus \Omega$$

by Lemma 10.3. We assume that the latter member is non-empty. By definition of Ω , $\cos \sigma = 0$ on $M \cap A((r, +\infty))$. Let $\tau \in \pi(M) \cap (r, +\infty)$. We claim that $M_\tau = \mathbb{S}_\tau^1$. Suppose for a contradiction that $M_\tau \neq \mathbb{S}_\tau^1$. By Lemma 5.2, M_τ is the union of two closed spherical arcs in \mathbb{S}_τ^1 . Let x be a point on the boundary of one of these spherical arcs relative to \mathbb{S}_τ^1 . There exists a $C^{1,1}$ parametrisation $\gamma_1 : I \rightarrow M$ of M in a neighbourhood of x with $\gamma_1(0) = x$ as before. By shrinking I if necessary we may assume that $\gamma_1(I) \subset A((r, +\infty))$ as $\tau > r$. By (2.9), $\dot{r}_1 = 0$ on I as $\cos \sigma_1 = 0$ on I

because $\cos \sigma = 0$ on $M \cap A((r, +\infty))$; that is, r_1 is constant on I . This means that $\gamma_1(I) \subset \mathbb{S}_r^1$. As the function $\sin \sigma_1$ is continuous on I it takes the value ± 1 there. By (2.10), $r_1 \theta_1 = \sin \sigma_1 = \pm 1$ on I . This means that θ_1 is either strictly decreasing or strictly increasing on I . This entails that the point x is not a boundary point of M_τ in \mathbb{S}_τ^1 and this proves the claim.

It follows from these considerations that $M \setminus \overline{B}(0, r)$ consists of a finite union of disjoint centred circles. Note that $f \geq e^{h(0)} =: c > 0$ on \mathbb{R}^2 . As a result, $+\infty > P_f(E) \geq cP(E)$ and in particular the relative perimeter $P(E, \mathbb{R}^2 \setminus \overline{B}(0, r)) < +\infty$. This explains why $M \setminus \overline{B}(0, r)$ comprises only finitely many circles.

(ii) Let E be a bounded minimiser of (1.2) such that E is open, $M := \partial E$ is a $C^{1,1}$ hypersurface in \mathbb{R}^2 and $E \setminus \{0\} = E^{sc}$ as in Theorem 4.5. Assume that $R > 0$. By (i), $M \setminus \overline{B}(0, R)$ consists of a finite union of disjoint centred circles. We claim that only one of the possibilities

$$M_R = \emptyset, M_R = \mathbb{S}_R^1, M_R = \{Re_1\} \text{ or } M_R = \{-Re_1\} \tag{10.3}$$

holds. To prove this suppose that $M_R \neq \emptyset$ and $M_R \neq \mathbb{S}_R^1$. Bearing in mind Lemma 5.2 we may choose $x \in M_R$ such that x lies on the boundary of M_R relative to \mathbb{S}_R^1 . Assume that $x \in H$. Let $\gamma_1 : I \rightarrow M$ be a local parametrisation of M with $\gamma_1(0) = x$ with the usual conventions. We first notice that $\cos(\sigma(x)) = 0$ for otherwise we obtain a contradiction to Theorem 10.3. As r_1 is decreasing on I and x is a relative boundary point it holds that $r_1 < R$ on $I^+ := I \cap \{s > 0\}$. As $M \setminus \overline{A}_1$ is open in M we may suppose that $\gamma_1(I^+) \subset M \setminus \overline{A}_1$. According to Theorem 6.5 the curvature k of $\gamma_1(I^+) \cap B(0, R)$ is a.e. constant as ρ vanishes on $(0, R)$. Hence $\gamma_1(I^+) \cap B(0, R)$ consists of a line or circular arc. The fact that $\cos(\sigma(x)) = 0$ means that $\gamma_1(I^+) \cap B(0, R)$ cannot be a line. So $\gamma_1(I^+) \cap B(0, R)$ is an open arc of a circle C containing x in its closure with centre on the line-segment $[0, x]$ and radius $r \in (0, R)$. By considering a local parametrisation, it can be seen that $C \cap B(0, R) \subset M$. But this contradicts the fact that $E \setminus \{0\} = E^{sc}$. In summary, $M_R \subset \{\pm Re_1\}$. Finally note that if $M_R = \{\pm Re_1\}$ then $M_R = \mathbb{S}_R^1$ by Lemma 5.2. This establishes (10.3).

Suppose that $M_R = \emptyset$. As both sets M and \mathbb{S}_R^1 are compact, $d(M, \mathbb{S}_R^1) > 0$. Assume first that $\mathbb{S}_R^1 \subset E$. Put $F := B(0, R) \setminus E$ and suppose $F \neq \emptyset$. Then F is a set of finite perimeter, $F \subset \subset B(0, R)$ and $P(F) = P(E, B(0, R))$. Let B be a centred ball with $|B| = |F|$. By the classical isoperimetric inequality, $P(B) \leq P(F)$. Define $E_1 := (\mathbb{R}^2 \setminus B) \cap (B(0, R) \cup E)$. Then $V_f(E_1) = V_f(E)$ and $P_f(E_1) \leq P_f(E)$. That is, E_1 is a minimiser of (1.2) such that ∂E_1 consists of a finite union of disjoint centred circles. Now suppose that $\mathbb{S}_R^1 \subset \mathbb{R}^2 \setminus E$. In like fashion we may redefine E via $E_1 := B \cup (E \setminus \overline{B}(0, R))$ with B a centred ball in $B(0, R)$. The remaining cases in (10.3) can be dealt with in a similar way. The upshot of this argument is that there exists a minimiser of (1.2) whose boundary M consists of a finite union of disjoint centred circles in case $R > 0$.

Now suppose that $R = 0$. By (i), $M \setminus \overline{B}(0, r)$ consists of a finite union of disjoint centred circles for any $r \in (0, 1)$. If these accumulate at 0 then M fails to be C^1 at the origin. The assertion follows. \square

Lemma 10.5 *Suppose that the function $J : [0, +\infty) \rightarrow [0, +\infty)$ is continuous non-decreasing and $J(0) = 0$. Let $N \in \mathbb{N} \cup \{+\infty\}$ and $\{t_h : h = 0, \dots, 2N + 1\}$ a sequence of points in $[0, +\infty)$ with*

$$t_0 > t_1 > \dots > t_{2h} > t_{2h+1} > \dots \geq 0.$$

Then

$$+\infty \geq \sum_{h=0}^{2N+1} J(t_h) \geq J\left(\sum_{h=0}^{2N+1} (-1)^h t_h\right).$$

Proof We suppose that $N = +\infty$. The series $\sum_{h=0}^{\infty} (-1)^h t_h$ converges by the alternating series test. For each $n \in \mathbb{N}$,

$$\sum_{h=0}^{2n+1} (-1)^h t_h \leq t_0$$

and the same inequality holds for the infinite sum. As in Step 2 in [5] Theorem 2.1,

$$+\infty \geq \sum_{h=0}^{\infty} J(t_h) \geq J(t_0) \geq J\left(\sum_{h=0}^{\infty} (-1)^h t_h\right)$$

as J is non-decreasing. □

Proof of Theorem 1.1 There exists a minimiser E of (1.2) with the property that ∂E consists of a finite union of disjoint centred circles according to Lemma 10.4. As such we may write

$$E = \bigcup_{h=0}^N A((a_{2h+1}, a_{2h}))$$

where $N \in \mathbb{N}$ and $+\infty > a_0 > a_1 > \dots > a_{2N} > a_{2N+1} > 0$. Define

$$\begin{aligned} f &: [0, +\infty) \rightarrow \mathbb{R}; t \mapsto e^{h(t)}; \\ g &: [0, +\infty) \rightarrow \mathbb{R}; t \mapsto t f(t); \\ G &: [0, +\infty) \rightarrow \mathbb{R}; t \mapsto \int_0^t g \, d\tau. \end{aligned}$$

Then $G : [0, +\infty) \rightarrow [0, +\infty)$ is a bijection with inverse G^{-1} . Define the strictly increasing function

$$J : [0, +\infty) \rightarrow \mathbb{R}; t \mapsto g \circ G^{-1}.$$

Put $t_h := G(a_h)$ for $h = 0, \dots, 2N + 1$. Then $+\infty > t_0 > t_1 > \dots > t_{2N} > t_{2N+1} >> 0$. Put $B := B(0, r)$ where $r := G^{-1}(v/2\pi)$ so that $V_f(B) = v$. Note that

$$\begin{aligned}
 v &= V_f(E) = 2\pi \sum_{h=0}^N \left\{ G(a_{2h}) - G(a_{2h+1}) \right\} \\
 &= 2\pi \sum_{h=0}^{2N+1} (-1)^h t_h.
 \end{aligned}$$

By Lemma 10.5,

$$\begin{aligned}
 P_f(E) &= 2\pi \sum_{h=0}^{2N+1} g(a_h) = 2\pi \sum_{h=0}^{2N+1} J(t_h) \\
 &\geq 2\pi J \left(\sum_{h=0}^{2N+1} (-1)^h t_h \right) \\
 &= 2\pi J(v/2\pi) = P_f(B).
 \end{aligned}$$

□

Proof of Theorem 1.2 Let $v > 0$ and E be a minimiser for (1.2). Then E is essentially bounded by Theorem 3.1. By Theorem 4.5 there exists an \mathcal{L}^2 -measurable set \tilde{E} with the properties

- (a) \tilde{E} is a minimiser of (1.2);
- (b) $L_{\tilde{E}} = L_E$ a.e. on $(0, +\infty)$;
- (c) \tilde{E} is open, bounded and has $C^{1,1}$ boundary;
- (d) $\tilde{E} \setminus \{0\} = \tilde{E}^{sc}$.

(i) Suppose that $0 < v \leq v_0$ so that $R > 0$. Choose $r \in (0, R]$ such that $V(B(0, r)) = V(E) = v$. Suppose that $\tilde{E} \setminus \overline{B}(0, R) \neq \emptyset$. By Lemma 10.4 there exists $t > R$ such that $\mathbb{S}_t^1 \subset M$. As g is strictly increasing, $g(t) > g(r)$. So $P_f(E) = P_f(\tilde{E}) \geq \pi g(t) > \pi g(r) = P_f(B(0, r))$. This contradicts the fact that E is a minimiser for (1.2). So $\tilde{E} \subset \overline{B}(0, R)$ and $L_{\tilde{E}} = 0$ on $(R, +\infty)$. By property (b), $|E \setminus \overline{B}(0, R)| = 0$. By the uniqueness property in the classical isoperimetric theorem (see for example [12] Theorem 4.11) the set E is equivalent to a ball B in $\overline{B}(0, R)$.

(ii) With $r > 0$ as before, $V(B(0, r)) = V(E) = v > v_0 = V(B(0, R))$ so $r > R$. If $\tilde{E} \setminus \overline{B}(0, r) \neq \emptyset$ we derive a contradiction in the same way as above. Consequently, $\tilde{E} = B := B(0, r)$. Thus, $L_E = L_B$ a.e. on $(0, +\infty)$; in particular, $|E \setminus B| = 0$. This entails that E is equivalent to B . □

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