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K-stability of Fano 3-folds of Picard rank 3 and degree 20

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Abstract
We prove K-stability of smooth Fano 3-folds of Picard rank 3 and degree 20 that satisfy very explicit generality condition.

Keywords K-stability · Fano threefold · Del Pezzo surface · Du Val singularities · polarized deltainvariant

Mathematics Subject Classification 14J45 · 14J30 · 32Q20

Contents

1 Introduction

Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, let C be a smooth curve in S of degree $(5, 1)$, and let $\epsilon\colon C \to \mathbb{P}^1$ be the morphism induced by the projection $S \to \mathbb{P}^1$ to the first factor. Then ϵ is a finite morphism of degree five, and we may assume that the points $([1 : 0], [0 : 1])$ and $([0 : 1], [1 : 0])$ are among its ramifications points. This assumption implies that the curve C is given by

$$u\big(x^5 + a_1x^4y + a_2x^3y^2 + a_3x^2y^3\big) = v\big(y^5 + b_1xy^4 + b_2x^2y^3 + b_3x^3y^2\big)$$

Throughout this paper, all varieties are assumed to be projective and defined over $\mathbb{C}$.

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for some $a_1, a_2, a_3, b_1, b_2, b_3$, where $([u : v], [x : y])$ are coordinates on S. Note that the ramification index of the point $([1 : 0], [0 : 1])$ can be computed as follows:

$$\begin{cases} 2 \text{ if } a_3 \neq 0, \\ 3 \text{ if } a_3 = 0 \text{ and } a_2 \neq 0, \\ 4 \text{ if } a_3 = a_2 = 0 \text{ and } a_1 \neq 0, \\ 5 \text{ if } a_3 = a_2 = a_1 = 0. \end{cases}$$

Likewise, we can compute the ramification index of the point $([0 : 1], [1 : 0])$. We may assume that

- $([1 : 0], [0 : 1])$ has the largest ramification index among all ramifications points of ϵ
- the ramification index of the point $([0 : 1], [1 : 0])$ is the second largest index.

If both these indices are 5, then $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$, the morphism ϵ does not have other ramification points, and the equation of the curve C simplifies as

$$ux^5 = vy^5.$$

In this case, we have $\mathrm{Aut}(S, C) \cong \mathbb{C}^* \rtimes \mathbb{Z}/2\mathbb{Z}$. In all other cases, this group is finite [5, Corollary 2.7].
Now, we consider embedding $S \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2$ given by

$$\big([u : v], [x : y]\big) \mapsto \big([u : v], [x^2 : xy : y^2]\big),$$

and identify S and C with their images in $\mathbb{P}^1 \times \mathbb{P}^2$. Let $\pi : X \to \mathbb{P}^1 \times \mathbb{P}^2$ be the blow up of the curve C. Then X is a smooth Fano threefold in the deformation family № 3.5 in the Mori–Mukai list and every smooth member of this family can be obtained in this way. We know from [2, Section 5.14], that

- X is K-stable if the numbers $a_1, a_2, a_3, b_1, b_2, b_3$ are general enough,
- X is K-polystable if $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0$.

However, for some $a_1, a_2, a_3, b_1, b_2, b_3$, the threefold X is not K-polystable.

Example 1 If $(a_1, a_2, a_3) = (0, 0, 0) \neq (b_1, b_2, b_3)$, then X is not K-polystable [2, Lemma 7.6].

Note also that it follows from the proof of [5, Lemma 8.7] that $\mathrm{Aut}(X) \cong \mathrm{Aut}(S, C)$. In particular, we conclude the group $\mathrm{Aut}(X)$ is finite if and only if $(a_1, a_2, a_3, b_1, b_2, b_3) \neq (0, 0, 0, 0, 0, 0)$. In this case, the threefold X is K-polystable if and only if it is K-stable. Moreover, we have

Conjecture 1 ([2]) The Fano threefold X is K-stable if and only if $(a_1, a_2, a_3) \neq (0, 0, 0)$.

Geometrically, this conjecture says that the following two conditions are equivalent:

(1) the threefold X is K-stable,
(2) the morphism $\epsilon: C \to \mathbb{P}^1$ does not have ramification points of ramification index five.

The goal of this paper is to prove the following (slightly weaker) result:

Main Theorem *If all ramification points of ϵ have ramification index two, then X is K-stable.*

Let $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the projection to the first factor and $\phi_1 = \mathrm{pr}_1 \circ \pi$. Then ϕ_1 is a fibration into del Pezzo surfaces of degree four, and every singular fiber of this fibration has Du Val singular points of types $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$, and we have the following possibilities for the singularities of a given singular fiber

(1) one singular point of type $\mathbb{A}_1$,
(2) two singular points of type $\mathbb{A}_1$,
(3) one singular point of type $\mathbb{A}_2$,
(4) one singular point of type $\mathbb{A}_1$ and one singular point of type $\mathbb{A}_2$,
(5) one singular point of type $\mathbb{A}_3$,
(6) one singular point of type $\mathbb{A}_4$.

Note that ϕ_1 has at most two singular fibers that have singular points of type $\mathbb{A}_4$. Moreover, if ϕ_1 has two singular fibers with singular points of type $\mathbb{A}_4$ then all numbers a_i and b_j vanish, so that X is K-polystable. Vice versa, if ϕ_1 has exactly one singular fiber with a point type $\mathbb{A}_1$, then the authors of [2] proved that X is not K-polystable. Moreover, they conjectured that X is K-stable in all remaining cases. Now **Main Theorem** and Conjecture 1 can be restated as follows:

Main Theorem *If every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, then X is K-stable.*

Conjecture 2 The Fano threefold X is K-stable if and only if every singular fiber of ϕ_1 has only singular points of type $\mathbb{A}_1$, $\mathbb{A}_2$ or $\mathbb{A}_3$.

2 The Proof

To prove **Main Theorem**, we suppose that each singular fiber of the fibration ϕ_1 has one or two singular points of type $\mathbb{A}_1$. Note that this fiber is a del Pezzo surface of degree 4 with Du Val singularities. We know ([7, 9]) that the Fano threefold X is K-stable if and only if for every prime divisor $\mathbf{F}$ over X we have

$$\beta(\mathbf{F}) = A_X(\mathbf{F}) - S_X(\mathbf{F}) > 0$$

where $A_X(\mathbf{F})$ is the log discrepancy of the divisor $\mathbf{F}$, and

$$S_X\big(\mathbf{F}\big) = \frac{1}{(-K_X)^3}\int\limits_0^\infty \mathrm{vol}\big(-K_X - u\mathbf{F}\big)du.$$

To show this, we fix a prime divisor $\mathbf{F}$ over X. Then we set $Z = C_X(\mathbf{F})$. If Z is an irreducible surface, then it follows from [8] that $\beta(\mathbf{F}) > 0$, see also [2, Theorem 3.17]. Therefore, we may assume that

- either Z is an irreducible curve in X,
- or Z is a point in X.

In both cases, we fix a point $O \in Z$. Let $\overline{T}$ be the fiber of ϕ_1 which contains O. Then $\overline{T}$ is a del Pezzo surface with at most Du Val singularities. Set

$$\tau(\overline{T}) = \sup\Big\{u \in \mathbb{R}_{>0} \Big| \text{ the divisor } -K_X - u\overline{T} \text{ is pseudo-effective}\Big\}$$

For $u \in [0, \tau(\overline{T})]$ let $P(u)$ be the positive part of the Zariski decomposition of the divisor $-K_X - u\overline{T}$, and let $N(u)$ be its negative part. We denote $\widetilde{S}$ to be the proper transform on X of the surface S. Then we have

$$P(u) = \begin{cases} -K_X - u\overline{T} & \text{if } u \in [0,1], \\ -K_X - u\overline{T} - (u-1)\widetilde{S} & \text{if } u \in [1,2], \end{cases} \quad \text{and}$$

$$N(u) = \begin{cases} 0 & \text{if } u \in [0,1], \\ (u-1)\widetilde{S} & \text{if } u \in [1,2], \end{cases}$$

which gives

$$S_X(\overline{T}) = \frac{1}{20}\int_0^2 P(u)^3 du = \frac{69}{80} < 1$$

Now, for every prime divisor F over the surface $\overline{T}$, we set

$$S\big(W^{\overline{T}}_{\bullet,\bullet}; F\big) = \frac{3}{(-K_X)^3}\int_0^\tau \operatorname{ord}_F\big(N(u)|_{\overline{T}}\big)\big(P(u)|_{\overline{T}}\big)^2 du + \frac{3}{(-K_X)^3}\int_0^\tau\int_0^\infty \operatorname{vol}\big(P(u)\big|_{\overline{T}} - vF\big)dvdu.$$

Then, following [1, 2], we let

$$\delta_O\big(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}\big) = \inf_{\substack{F/\overline{T} \\ O \in C_{\overline{T}}(F)}} \frac{A_T(F)}{S\big(W^{\overline{T}}_{\bullet,\bullet}; F\big)},$$

where the infimum is taken by all prime divisors over the surface $\overline{T}$ whose center on $\overline{T}$ contains O. Then it follows from [1, 2] that

$$\frac{A_X(\mathbf{F})}{S_X(\mathbf{F})} \geqslant \min\left\{\frac{1}{S_X(\overline{T})}, \delta_O\big(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}\big)\right\}.$$

Therefore, if $\beta(\mathbf{F}) \leqslant 0$, then $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) \leqslant 1$.
Let's prove that $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) > 1$. To estimate $\delta_O(T, W^{\overline{T}}_{\bullet,\bullet})$, we set $\overline{D} = P(u)|_{\overline{T}}$. We have

$$\overline{D} = \begin{cases} -K_{\overline{T}} & \text{if } u \in [0,1], \\ -K_{\overline{T}} - (u-1)\overline{C}_2 & \text{if } u \in [1,2], \end{cases}$$

where $\overline{C}_2 := \widetilde{S}|_{\overline{T}}$. Then $\overline{D}$ is ample for $u \in [0, 2)$, and

$$\overline{D}^2 = \begin{cases} 4 & \text{if } u \in [0,1], \\ 5 - u^2 & \text{if } u \in [1,2]. \end{cases}$$

By [2, Lemma 5.68] and [2, Lemma 5.69] we have

Lemma 1 *If $O \in \widetilde{S}$ then $\delta_O(X) > 1$.*

Lemma 2 *If $\overline{T}$ is smooth then $\delta_O(X) > 1$.*

Thus, to prove **Main Theorem**, we may assume that $O \notin \widetilde{S}$ and $\overline{T}$ is singular. Recall that

$$\delta_O\big(\overline{T}, \overline{D}\big) = \inf_{\substack{F/\overline{T} \\ O \in C_{\overline{T}}(F)}} \frac{A_{\overline{T}}(F)}{S_{\overline{D}}(F)}$$

where the infimum is taken by all prime divisors over $\overline{T}$ whose center on $\overline{T}$ contain O, and $S_{\overline{D}}(F) = \frac{1}{\overline{D}^2}\int_0^\infty \mathrm{vol}\big(\overline{D} - vF\big)dv$. Usually $\delta_O(\overline{T}, -K_{\overline{T}})$ is denoted by $\delta_O(\overline{T})$. Note that since $O \notin \widetilde{S}$ then for any divisor F over $\overline{T}$ then we get

$$\begin{aligned} S\big(W^{\overline{T}}_{\bullet,\bullet}; F\big) &= \frac{3}{(-K_X)^3}\bigg(\int_0^\tau \big(P(u)^2 \cdot \overline{T}\big) \cdot \mathrm{ord}_O\Big(N(u)\big|_{\overline{T}}\Big)du \\ &\quad + \int_0^\tau \int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big)dvdu\bigg) \\ &= \frac{3}{20}\int_0^\tau \int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vF\big)dvdu \\ &= \frac{3}{20}\bigg(\int_0^1 \int_0^\infty \mathrm{vol}\big(-K_{\overline{T}} - vF\big)dvdu \end{aligned}$$

$$
\begin{aligned}
&+\int_1^2\int_0^\infty \operatorname{vol}\big(-K_{\overline{T}}-(u-1)\overline{C}_2-vF\big)dvdu\Big)\\
&=\frac{3}{20}\Big(\int_0^\infty \operatorname{vol}\big(-K_{\overline{T}}-vF\big)dv\\
&+\int_0^\infty \operatorname{vol}\big(-K_T-(u-1)\overline{C}_2-vF\big)dv\Big)\\
&=\frac{3}{20}\Big(\int_0^\infty \operatorname{vol}\big(-K_{\overline{T}}-vF\big)dv+\int_0^\infty \operatorname{vol}\big(-K_{\overline{T}}-vF\big)dv\Big)\\
&=\frac{3}{10}\Big(\int_0^\infty \operatorname{vol}\big(-K_{\overline{T}}-vF\big)dv\Big)\\
&=\frac{6}{5}\Big(\frac{1}{4}\int_0^\infty \operatorname{vol}\big(-K_{\overline{T}}-vF\big)dv\Big)\\
&=\frac{6}{5}S_{\overline{T}}(F)\le\frac{6}{5}\cdot\frac{A_{\overline{T}}(F)}{\delta_O(\overline{T})}
\end{aligned}
$$

Thus, if $\delta_O(\overline{T})>6/5$, then $\delta_O(\overline{T},W^{\overline{T}}_{\bullet,\bullet})>1$. To estimate $\delta_O(\overline{T},W^{\overline{T}}_{\bullet,\bullet})$ in the case when $\delta_O(\overline{T})\le 6/5$, we define the following positive continuous function on $[1,2]$:

$$
f(u):=\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3} & \text{if } u\in[1,a],\\[2mm] \dfrac{15-3u^2}{11-u^3} & \text{if } u\in[a,2]\end{cases}
$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$. More precisely, $a\in[1.355,1.356]$. In the appendix we prove that for each O such that $\delta_O(\overline{T})\le\frac{6}{5}$ we have $\delta_O(\overline{T},\overline{D})\ge f(u)$ for every $u\in[1,2]$. So we obtain

$$
\begin{aligned}
S\big(W^{\overline{T}}_{\bullet,\bullet};F\big)&=\frac{3}{(-K_X)^3}\int_1^2\int_0^\infty \operatorname{vol}\big(P(u)|_{\overline{T}}-vF\big)dvdu\\
&+\frac{3}{(-K_X)^3}\int_0^1\int_0^\infty \operatorname{vol}\big(P(u)|_{\overline{T}}-vF\big)dvdu\\
&\le\frac{3}{20}\Big(\int_1^2\frac{(5-u^2)}{\delta_O(\overline{T},\overline{D})}du\Big)A_{\overline{T}}(F)+\frac{3}{20}\cdot\frac{4A_{\overline{T}}(F)}{\delta_O(\overline{T})}\\
&\le\frac{3}{20}\Big(\int_1^2\frac{(5-u^2)}{f(u)}du\Big)A_{\overline{T}}(F)+\frac{3}{5}A_{\overline{T}}(F)
\end{aligned}
$$

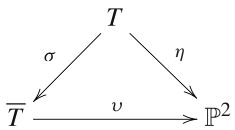
$$\leq \frac{3}{20}\Bigg(\int\limits_{1}^{1.356} (5-u^2)\frac{16+3u-9u^2+2u^3}{15-3u^2}du + \int\limits_{1.355}^{2} (5-u^2)\frac{11-u^3}{15-3u^2}du \Bigg) A_{\overline{T}}(F) + \frac{3}{5}A_{\overline{T}}(F) \leq \frac{99}{100}A_{\overline{T}}(F)$$

Thus $\frac{A_{\overline{T}}(F)}{S\left(W^{\overline{T}}_{\bullet,\bullet};F\right)} \geq \frac{100}{99}$ for every prime divisor F over $\overline{T}$ whose support on F contains O, so that $\delta_O(W^{\overline{T}}, F) \geq \frac{100}{99}$, which implies $\beta(\mathbf{F}) > 0$ and X is K-stable.

Remark 1 If O were a singular point of type $\mathbb{A}_2$ in $\overline{T}$, this approach would not work, because as is shown in Appendix A.3 we have $\delta_O(\overline{T}, \overline{D}) = \frac{15-3u^2}{u^3-6u^2+19}$ and there is prime divisor F over $\overline{T}$ such that $A_{\overline{T}}(F) = 1$ and $S(W^{\overline{T}}_{\bullet,\bullet}; F) = \frac{83}{80}$, which implies that $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) \leq \frac{80}{83}$.

Appendix A. Polarized δ-invariant via Kento Fujita's formulas

Let us use notations from Section 2. Recall that $\overline{T}$ is a Du Val del Pezzo surface, and the blow up π induces a birational morphism $\upsilon : \overline{T} \to \mathbb{P}^2$. We assume that $\overline{T}$ is singular. We have the following commutative diagram



Suppose that $u \in [1,2]$. Recall that $\overline{D} = -K_{\overline{T}} - (1-u)\overline{C}_2$. Observe that $\overline{C}_2$ is contained in the smooth locus of the surface $\overline{T}$. Let C_2 be the strict transform of the curve $\overline{C}_2$ on the surface T, set $D = -K_T - (1-u)C_2$. Note that $D = \sigma^*(\overline{D})$ so the divisor D is big and nef for $u \in [1,2]$. Recall

$$\delta_O(\overline{T}, \overline{D}) = \inf_{\substack{F/\overline{T} \\ O \in C_{\overline{T}}(F)}} \frac{A_{\overline{T}}(F)}{S_D(F)}$$

where the infimum is run over all prime divisor F over $\overline{T}$ such that $O \in C_{\overline{T}}(F)$. For every point $P \in T$, we also define

$$\delta_P(T, D) = \inf_{\substack{E/T \\ P \in C_T(E)}} \frac{A_T(E)}{S_D\big(E\big)}$$

where the infimum is run over all prime divisor E over T such that $P \in C_T(E)$. Since $D = \sigma^*(\overline{D})$ and $K_T = \sigma^*(K_{\overline{T}})$, we have

$$\delta_O(\overline{T}, \overline{D}) = \inf_{\substack{P \in T \\ O = \sigma(P)}} \delta_P(T, D)$$

So, to estimate $\delta_O(\overline{T}, \overline{D})$ it is enough to estimate $\delta_P(T, D)$ for P all points P such that $\sigma(P) = O$.
Let $\mathcal{C}$ be a smooth curve on T containing P. Set

$$\tau(\mathcal{C}) = \sup\Big\{v \in \mathbb{R}_{\geqslant 0} \,\Big|\, \text{the divisor } -K_T - v\mathcal{C} \text{ is pseudo-effective}\Big\}.$$

For $v \in [0, \tau]$, let $P(v)$ be the positive part of the Zariski decomposition of the divisor $-K_T - \mathcal{C}$, and let $N(v)$ be its negative part. Then we set

$$S\big(W^{\mathcal{C}}_{\bullet,\bullet}; P\big) = \frac{2}{D^2} \int_0^{\tau(\mathcal{C})} h_D(v) dv,$$

where

$$h_D(v) = \big(P(v) \cdot \mathcal{C}\big) \times \big(N(v) \cdot \mathcal{C}\big)_P + \frac{\big(P(v) \cdot \mathcal{C}\big)^2}{2}.$$

It follows from [1, 2] that:

$$\delta_P(T, D) \geqslant \min\left\{\frac{1}{S_D(\mathcal{C})}, \frac{1}{S(W^{\mathcal{C}}_{\bullet,\bullet}, P)}\right\}.$$

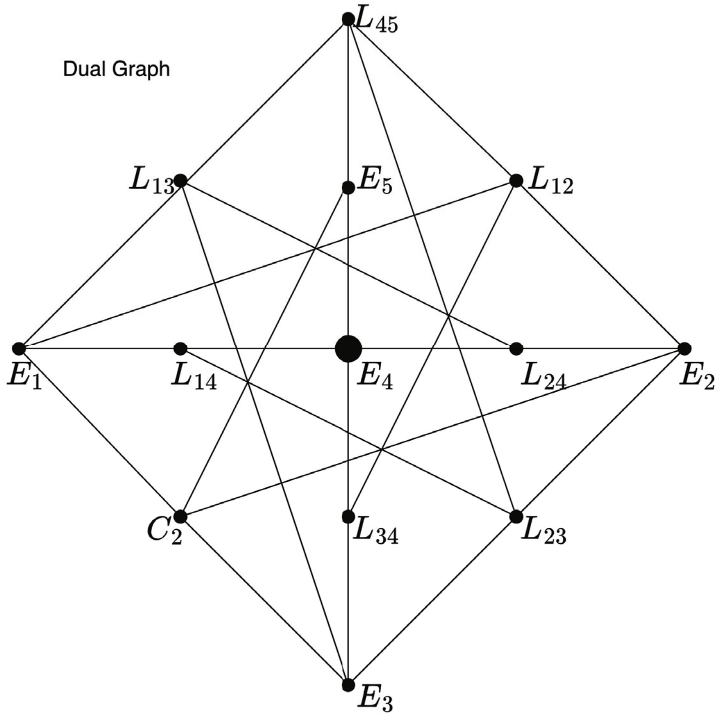
We will estimate $\delta_P(T, D)$ in the following using the notations above for a suitable choice of the curve $\mathcal{C}$, $\tau(\mathcal{C})$, $P(v)$ and $N(v)$ later in special cases.
A similar approach was taken in [3] and [4].

A.1 Polarized δ-invariant on Del Pezzo surface of degree 4 with $\mathbb{A}_1$ singularity.

Suppose that $\overline{T}$ has one singular point and this point is a singular point of type $\mathbb{A}_1$. Then η is a blow up of $\mathbb{P}^2$ at points P_1, P_2, P_3 and P_4 in general position and a point P_5 which belongs to the exceptional divisor corresponding to P_4 and no other negative curve. Suppose $\mathbf{E} := L_{14} \cup L_{24} \cup L_{24} \cup E_5$. By [6, Section 6.2] we have:

$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in E_4, \\ 6/5 \text{ if } P \in \mathbf{E} \backslash E_4, \\ 4/3 \text{ if } P \text{ belongs to two curves in } \{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}, L_{45}, C_2\}, \\ 18/13 \text{ if } P \text{ belongs to exactly one curve in } \{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}, \\ L_{45}, C_2\} \backslash \mathbf{E}, \\ 3/2, \text{ otherwise} \end{cases}$$

where E_1, E_2, E_3, E_4, E_5 are exceptional divisors corresponding to P_1, P_2, P_3, P_4, P_5 respectively, C_2 is a strict transform of a (-1)-curve coming from the conic on $\mathbb{P}^2$, L_{ij} are strict transforms of the lines passing through P_i and P_j for $(i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$ and L_{45} a strict transform of a (-1)-curve coming from a line on $\mathbb{P}^2$. The dual graph of (-1) and (-2)-curves is given in the following picture:



Lemma A.1 *Suppose P is a point on T and $D = -K_T - (u-1)C_2$ with $D^2 = 5 - u^2$ then*

$$\delta_P(T, D) = \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P \in E_4 \backslash E_5 \text{ and } u \in [1,2] \\ \dfrac{15-3u^2}{11-u^3} \text{ for } P \in E_5 \backslash E_4 \text{ and } u \in [1,2] \\ \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P \in L_{14} \backslash (E_4 \cup E_5 \cup E_1) \text{ and } u \in [1,2] \end{cases}$$

and

$$\delta_P(T, D) \geq \begin{cases} \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P = E_4 \cap E_5 \text{ and } u \in [1,a] \\ \dfrac{15-3u^2}{11-u^3} \text{ for } P = E_4 \cap E_5 \text{ and } u \in [a,2] \end{cases}$$

and

$$\delta_P(T, D) \geq \begin{cases} \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P = L_{14} \cap E_1 \text{ and } u \in [1,b] \\ \dfrac{2(15-3u^2)}{19-2u^3} \text{ for } P = L_{14} \cap E_1 \text{ and } u \in [b,3/2] \\ \dfrac{15-3u^2}{3u^3-18u^2+27u-4} \text{ for } P = L_{14} \cap E_1 \text{ and } u \in [3/2,2] \end{cases}$$

where a is a root of $3u^3 - 9u^2 + 3u + 5$ on $[1, 2]$, b is a root of $8u^3 - 24u^2 + 12u + 7$ on $[1, 3/2]$. Note that $a \in [1.355, 1.356]$, $b \in [1.261, 1.262]$.

Proof **Step 1.** Suppose $P \in E_4$. In this case we set $\mathcal{C} = E_4$. Then $\tau(\mathcal{C}) = 3 - u$. The Zariski Decomposition of the divisor $D - vE_4$ is given by:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_4 \text{ for } v \in [0, 2-u] \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 - (v-1)(L_{14} + L_{24} + L_{34}) \\ \text{for } v \in [1, 3-u] \end{cases}$$

and

$$N(v) = \begin{cases} 0 \text{ for } v \in [0, 2-u] \\ (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ (u+v-2)E_5 + (v-1)(L_{14} + L_{24} + L_{34}) \text{ for } v \in [1, 3-u] \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5-u^2-2v^2 \text{ for } v \in [0, 2-u] \\ 9+2uv-4u-4v-v^2 \text{ for } v \in [2-u, 1] \\ 2(2-v)(3-u-v) \text{ for } v \in [1, 3-u] \end{cases}$$

and

$$P(v) \cdot \mathcal{C} = \begin{cases} 2v \text{ for } v \in [0, 2-u] \\ 2-u+v \text{ for } v \in [2-u, 1] \\ 5-u-2v \text{ for } v \in [1, 3-u] \end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5-u^2-2v^2 dv + \int_{2-u}^{1} 9+2uv-4u-4v-v^2 dv + \int_1^{3-u} 2(2-v)(3-u-v)dv\Big) = \frac{16+3u-9u^2+2u^3}{15-3u^2}$$

Thus, $\delta_P(T, D) \leq \frac{15-3u^2}{16+3u-9u^2+2u^3}$ for $P \in E_4$. Note that we have:

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(5-u-2v)^2}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P = E_4 \cap E_5$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)(u+3v-2)}{2} \text{ for } v \in [2-u, 1] \\ \frac{(u+1)(5-u-2v)}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P \in E_4 \cap (L_{14} \cup L_{24} \cup L_{34})$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(3-u)(5-u-2v)}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

So we have

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv$$

$$+\int_{2-u}^{1}\frac{(2-u+v)^2}{2}dv+\int_{1}^{3-u}\frac{(5-u-2v)^2}{2}dv\Big)$$
$$=\frac{9+6u-9u^2+2u^3}{15-3u^2}\leq\frac{16+3u-9u^2+2u^3}{15-3u^2}$$

- if $P=E_4\cap E_5$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}2v^2dv+\int_{2-u}^{1}\frac{(2-u+v)(u+3v-2)}{2}dv$$
$$+\int_1^{3-u}\frac{(u+1)(5-u-2v)}{2}dv\Big)=\frac{11-u^3}{15-3u^2}$$

- if $P\in E_4\cap(L_{14}\cup L_{24}\cup L_{34})$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}2v^2dv+\int_{2-u}^{1}\frac{(2-u+v)^2}{2}dv$$
$$+\int_1^{3-u}\frac{(3-u)(5-u-2v)}{2}dv\Big)$$
$$=\frac{13+3u^3-12u^2+6u}{15-3u^2}$$
$$\leq\frac{16+3u-9u^2+2u^3}{15-3u^2}$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }P\in E_4\backslash E_5\text{ and }u\in[1,2]$$

and

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }P=E_4\cap E_5\text{ and }u\in[1,a]\\ \dfrac{15-3u^2}{11-u^3}\text{ for }P=E_4\cap E_5\text{ and }u\in[a,2]\end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$. Note that $a\in[1.355,1.356]$.
Step 2. Suppose $P\in E_5$. In this case we set $\mathcal{C}=E_5$. Then $\tau(\mathcal{C})=2$. The Zariski Decomposition of the divisor $D-vE_5$ is given by:

$$P(v)=\begin{cases}-K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4\text{ for }v\in[0,1]\\-K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4-(v-1)L_{45}\text{ for }v\in[1,u]\\-K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4-(v-1)L_{45}-(v-u)C_2\text{ for }v\in[u,2]\end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_4 \text{ for } v \in [0,1] \\ \frac{v}{2}E_4 + (v-1)L_{45} \text{ for } v \in [1,u] \\ \frac{v}{2}E_4 + (v-1)L_{45} + (v-u)C_2 \text{ for } v \in [u,2] \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5 - 4v + 2uv - u^2 - \frac{v^2}{2} \text{ for } v \in [0,1] \\ 6 - 6v + \frac{v^2}{2} + 2uv - u^2 \text{ for } v \in [1,u] \\ \frac{3(2-v)^2}{2} \text{ for } v \in [u,2] \end{cases}$$

and

$$P(v) \cdot \mathcal{C} = \begin{cases} 2 - u + v/2 \text{ for } v \in [0,1] \\ 3 - u - v/2 \text{ for } v \in [1,u] \\ 3 - 3v/2 \text{ for } v \in [u,2] \end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^1 5 - 4v + 2uv - u^2 - \frac{v^2}{2} dv + \int_1^u 6 - 6v + \frac{v^2}{2} + 2uv - u^2 dv + \int_u^2 \frac{3(2-v)^2}{3} dv\Big) = \frac{11-u^3}{15-3u^2}$$

Thus, $\delta_P(T, D) \leq \frac{15-3u^2}{11-u^3}$ for $P \in E_5$. Note that we have:

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1,u] \\ \frac{(3-3v/2)^2}{2} \text{ for } v \in [u,2] \end{cases}$$

- if $P = E_5 \cap C_2$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0,1] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1,u] \\ \frac{3(2-v)(6-4u+v)}{8} \text{ for } v \in [u,2] \end{cases}$$

- if $P = E_5 \cap L_{45}$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0, 1] \\ \frac{(6-2u-v)(2-2u+3v)}{8} \text{ for } v \in [1, u] \\ \frac{3(2-v)(v+2)}{8} \text{ for } v \in [u, 2] \end{cases}$$

So we have

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(3-u-v/2)^2}{2}dv + \int_u^2 \frac{(3-3v/2)^2}{2}dv\Big) = \frac{21+6u-18u^2+5u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}$$

- if $P = E_5 \cap C_2$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(3-u-v/2)^2}{2}dv + \int_u^2 \frac{3(2-v)(6-4u+v)}{8}dv\Big) = \frac{45-30u+2u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}$$

- if $P = E_5 \cap L_{45}$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(6-2u-v)(2-2u+3v)}{8}dv + \frac{3(2-v)(v+2)}{8}dv\Big) = \frac{26-12u^2+3u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}$$

We obtain that

$$\delta_P(T, D) = \frac{15-3u^2}{11-u^3} \text{ for } P \in E_5 \backslash E_4 \text{ and } u \in [1, 2].$$

Step 3.1. Suppose $P \in L_{14} \cup L_{24} \cup L_{34}$ and $u \in [1, 3/2]$. Without loss of generality, we can assume that $P \in L_{14}$. In this case we set $\mathcal{C} = L_{14}$. Then $\tau(\mathcal{C}) = 3 - u$. The Zariski Decomposition of the divisor $D - vL_{14}$ is given by:

$$P(v) = \begin{cases} D - vL_{14} - \frac{v}{2}E_4 \text{ for } v \in [0, 2-u] \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 \text{ for } v \in [2-u, 1] \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 - (v-1)L_{23} \text{ for } v \in [1, 4-2u] \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (v-1)L_{23} - (2u+v-4)E_5 \\ \text{ for } v \in [4-2u, 3-u] \end{cases}$$

and

$$N(v)=\begin{cases}\frac{v}{2}E_4 \text{ for } v\in[0,2-u]\\ \frac{v}{2}E_4+(u+v-2)E_1 \text{ for } v\in[2-u,1]\\ \frac{v}{2}E_4+(u+v-2)E_1+(v-1)L_{23} \text{ for } v\in[1,4-2u]\\ (u+v-2)(E_1+E_4)+(v-1)L_{23}+(2u+v-4)E_5\\ \text{ for } v\in[4-2u,3-u]\end{cases}$$

Moreover

$$P(v)^2=\begin{cases}5-2v-\frac{v^2}{2}-u^2 \text{ for } v\in[0,2-u]\\ 9-4u-6v+\frac{v^2}{2}+2uv \text{ for } v\in[2-u,1]\\ \frac{(v-2)(3v+4u-10)}{2} \text{ for } v\in[1,4-2u]\\ 2(u+v-3)^2 \text{ for } v\in[4-2u,3-u]\end{cases}$$

and

$$P(v)\cdot\mathcal{C}=\begin{cases}v/2+1 \text{ for } v\in[0,2-u]\\ 3-u-v/2 \text{ for } v\in[2-u,1]\\ 4-u-3v/2 \text{ for } v\in[1,4-2u]\\ 2(3-u-v) \text{ for } v\in[4-2u,3-u]\end{cases}$$

Thus,

$$\begin{aligned}S_D(\mathcal{C})=\frac{1}{5-u^2}\Big(&\int_0^{2-u}5-2v-\frac{v^2}{2}-u^2dv+\int_{2-u}^1 9-4u-6v+\frac{v^2}{2}+2uvdv\\ &+\int_1^{4-2u}\frac{(v-2)(3v+4u-10)}{2}dv\\ &+\int_{4-2u}^{3-u}2(u+v-3)^2dv\Big)=\frac{3u^3-12u^2+6u+13}{15-3u^2}\end{aligned}$$

Thus, $\delta_P(T,D)\leq\frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P\in L_{14}$. Note that we have:

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$h_D(v)=\begin{cases}\frac{(v/2+1)^2}{2} \text{ for } v\in[0,2-u]\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,1]\\ \frac{(4-u-3v/2)^2}{2} \text{ for } v\in[1,4-2u]\\ 2(3-u-v)^2 \text{ for } v\in[4-2u,3-u]\end{cases}$$

- if $P = L_{14} \cap E_1$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v \in [2-u, 1] \\ \frac{(8-2u-3v)(2u+v)}{8} \text{ for } v \in [1, 4-2u] \\ (3-u-v) \text{ for } v \in [4-2u, 3-u] \end{cases}$$

- if $P = L_{14} \cap L_{23}$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(8-2u-3v)(4-2u+v)}{8} \text{ for } v \in [1, 4-2u] \\ 2(2-u)(3-u-v) \text{ for } v \in [4-2u, 3-u] \end{cases}$$

So we have

- if $P \in L_{14} \backslash (E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$\begin{aligned} S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) &= \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2}dv + \int_{2-u}^{1} \frac{(3-u-v/2)^2}{2}dv \\ &\quad + \int_1^{4-2u} \frac{(4-u-3v/2)^2}{2}dv + \int_{4-2u}^{3-u} 2(3-u-v)^2 dv\Big) \\ &= \frac{21-u^3-6u}{2(15-3u^2)} \le \frac{3u^3-12u^2+6u+13}{15-3u^2} \end{aligned}$$

- if $P = L_{14} \cap E_1$ then

$$\begin{aligned} S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) &= \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2}dv \\ &\quad + \int_{2-u}^{1} \frac{(6-2u-v)(2u+3v-2)}{8}dv \\ &\quad + \int_1^{4-2u} \frac{(8-2u-3v)(2u+v)}{8}dv \\ &\quad + \int_{4-2u}^{3-u} (3-u-v)dv\Big) = \frac{19-2u^3}{2(15-3u^2)} \end{aligned}$$

- if $P = L_{14} \cap L_{23}$ then

$$\begin{aligned} S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) &= \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2}dv + \int_{2-u}^{1} \frac{(3-u-v/2)^2}{2}dv \\ &\quad + \int_1^{4-2u} \frac{(8-2u-3v)(4-2u+v)}{8}dv \end{aligned}$$

$$+\int_{4-2u}^{3-u} 2(2-u)(3-u-v)dv\Big)$$
$$=\frac{26-12u^2+3u^3}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P\in L_{14}\backslash(E_4\cup E_5\cup E_1)\text{ and }u\in[1,3/2]$$

and

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[1,b]\\ \dfrac{2(15-3u^2)}{19-2u^3}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[b,3/2]\end{cases}$$

where b is a root of $8u^3-24u^2+12u+7$ on $[1,3/2]$. Note that $b\in[1.261,1.262]$.
Step 3.2. Suppose $P\in L_{14}\cup L_{24}\cup L_{34}$ and $u\in[3/2,2]$. Without loss of generality, we can assume that $P\in L_{14}$. In this case we set $\mathcal{C}=L_{14}$. Then $\tau(\mathcal{C})=3-u$. The Zariski Decomposition of the divisor $D-vL_{14}$ is given by:

$$P(v)=\begin{cases}D-vL_{14}-\frac{v}{2}E_4\text{ for }v\in[0,2-u]\\ D-vL_{14}-\frac{v}{2}E_4-(u+v-2)E_1\text{ for }v\in[2-u,4-2u]\\ D-vL_{14}-(u+v-2)(E_1+E_4)-(2u+v-4)E_5\text{ for }v\in[4-2u,1]\\ D-vL_{14}-(u+v-2)(E_1+E_4)-(v-1)L_{23}-(2u+v-4)E_5\\ \text{ for }v\in[1,3-u]\end{cases}$$

and

$$N(v)=\begin{cases}\frac{v}{2}E_4\text{ for }v\in[0,2-u]\\ \frac{v}{2}E_4+(u+v-2)E_1\text{ for }v\in[2-u,4-2u]\\ (u+v-2)(E_1+E_4)+(2u+v-4)E_5\text{ for }v\in[4-2u,1]\\ (u+v-2)(E_1+E_4)+(v-1)L_{23}+(2u+v-4)E_5\text{ for }v\in[1,3-u]\end{cases}$$

Moreover

$$P(v)^2=\begin{cases}5-2v-\frac{v^2}{2}-u^2\text{ for }v\in[0,2-u]\\ 9-4u-6v+\frac{v^2}{2}+2uv\text{ for }v\in[2-u,4-2u]\\ 2u^2+4uv+v^2-12u-10v+17\text{ for }v\in[4-2u,1]\\ 2(u+v-3)^2\text{ for }v\in[1,3-u]\end{cases}$$

and

$$P(v)\cdot\mathcal{C}=\begin{cases}1+v/2 \text{ for } v\in[0,2-u]\\ 3-u-v/2 \text{ for } v\in[2-u,4-2u]\\ 5-2u-v \text{ for } v\in[4-2u,1]\\ 2(3-u-v) \text{ for } v\in[1,3-u]\end{cases}$$

Thus,

$$\begin{aligned}S_D(\mathcal{C})&=\frac{1}{5-u^2}\Big(\int_0^{2-u}5-2v-\frac{v^2}{2}-u^2dv+\int_{2-u}^{4-2u}9-4u-6v+\frac{v^2}{2}+2uvdv\\&\quad+\int_{4-2u}^{1}2u^2+4uv+v^2-12u-10v+17dv+\int_1^{3-u}2(u+v-3)^2dv\Big)\\&=\frac{3u^3-12u^2+6u+13}{15-3u^2}\end{aligned}$$

Thus, $\delta_P(T,D)\le\frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P\in L_{14}$. Note that we have:

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u]\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,4-2u]\\ \frac{(5-2u-v)^2}{2} \text{ for } v\in[4-2u,1]\\ 2(3-u-v)^2 \text{ for } v\in[1,3-u]\end{cases}$$

- if $P=L_{14}\cap E_1$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u]\\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v\in[2-u,4-2u]\\ \frac{(v+1)(5-2u-v)}{2} \text{ for } v\in[4-2u,1]\\ 2(3-u-v) \text{ for } v\in[1,3-u]\end{cases}$$

- if $P=L_{14}\cap L_{23}$ then

$$h_D(v)=\begin{cases}\frac{(1+v/2)^2}{2} \text{ for } v\in[0,2-u]\\ \frac{(3-u-v/2)^2}{2} \text{ for } v\in[2-u,4-2u]\\ \frac{(5-2u-v)^2}{2} \text{ for } v\in[4-2u,1]\\ 2(2-u)(3-u-v) \text{ for } v\in[1,3-u]\end{cases}$$

- if $P\in L_{14}\backslash(E_4\cup E_1\cup L_{23}\cup E_5)$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{4-2u}\frac{(3-u-v/2)^2}{2}dv$$

$$+\int_{4-2u}^{1}\frac{(5-2u-v)^2}{2}dv+\int_{1}^{3-u}2(3-u-v)^2dv\Big)$$
$$=\frac{7u^3-36u^2+48u-6}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}$$

- if $P=L_{14}\cap E_1$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv$$
$$+\int_{2-u}^{4-2u}\frac{(6-2u-v)(2u+3v-2)}{8}dv$$
$$+\int_{4-2u}^{1}\frac{(v+1)(5-2u-v)}{2}dv+\int_{1}^{3-u}2(3-u-v)dv\Big)$$
$$=\frac{3u^3-18u^2+27u-4}{15-3u^2}$$

- if $P=L_{14}\cap L_{23}$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{4-2u}\frac{(3-u-v/2)^2}{2}dv$$
$$+\int_{4-2u}^{1}\frac{(5-2u-v)^2}{2}dv+\int_{1}^{3-u}2(2-u)(3-u-v)dv\Big)$$
$$=\frac{3u^3-12u^2+26}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P\in L_{14}\backslash(E_1\cup E_4\cup E_5)\text{ and }u\in[3/2,2]$$

and

$$\delta_P(T,D)\geq\frac{15-3u^2}{3u^3-18u^2+27u-4}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[3/2,2]$$

□

Corollary A.1 *Let P be a point in T that is contained in $L_{12}\cup L_{24}\cup L_{34}\cup E_4\cup E_5$ then*

$$\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3} & \textit{for } u\in[1,a],\\ \dfrac{15-3u^2}{11-u^3} & \textit{for } u\in[a,2]\end{cases}$$

Corollary A.2 *Suppose O is a point on a del Pezzo surface $\overline{T}$ with $\mathbb{A}_1$ singularity and $\delta_O(T) \leq \frac{6}{5}$ then*

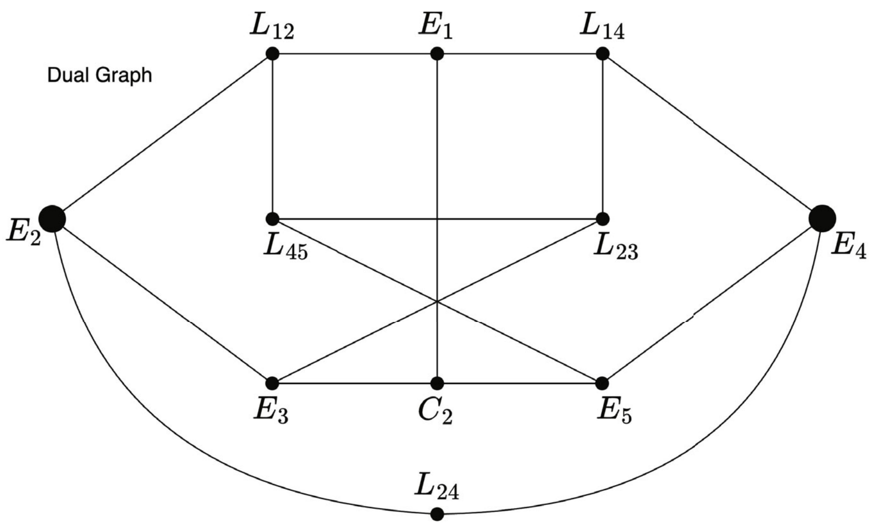
$$\delta_O(\overline{T}, \overline{D}) \geq \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } u \in [1, a], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } u \in [a, 2] \end{cases}$$

A.2 Polarized δ-invariant on Del Pezzo surface of degree 4 with two $\mathbb{A}_1$ singularities.

Suppose that $\overline{T}$ has two singular points and these points are singular point of type $\mathbb{A}_1$. Then η is a blow up of $\mathbb{P}^2$ at points P_1, P_2, and P_4 in general position and after that blowing up a point P_3 which belongs to the exceptional divisor corresponding to P_2 and a point P_5 which belongs to the exceptional divisor corresponding to P_4 and no other negative curve. By [6, Section 6.2] we have:

$$\delta_P(T) = \begin{cases} 1 \text{ if } P \in (E_2 \cup E_4 \cup L_{24}), \\ 6/5 \text{ if } P \in (E_3 \cup E_5 \cup L_{12} \cup L_{14}) \setminus (E_2 \cup E_4), \\ 4/3 \text{ if } P \in (C_2 \cap E_1) \cup (L_{23} \cap L_{45}), \\ 18/13 \text{ if } P \in (C_2 \cup E_1 \cup L_{23} \cup L_{45}) \setminus \big((C_2 \cap E_1) \cup (L_{23} \cap L_{45}) \\ \cup (E_3 \cup E_5 \cup L_{12} \cup L_{14})\big), \\ 3/2, \text{ otherwise} \end{cases}$$

where E_1, E_2, E_3, E_4, E_5 are exceptional divisors corresponding to P_1, P_2, P_3, P_4, P_5 respectively, C_2 is a strict transform of a (-1)-curve coming from the conic on $\mathbb{P}^2$, L_{ij} are strict transforms of the lines passing through P_i and P_j for $(i, j) \in \{(1, 2), (1, 4)\}$ and L_{45} and L_{23} are strict transforms of a (-1)-curve coming from lines passing through P_2 and P_4 respectively on $\mathbb{P}^2$. The dual graph of (-1) and (-2)-curves is given in the following picture:



Lemma A.2 *Suppose P is a point on T and $D = -K_T - (u-1)C_2$ with $D^2 = 5 - u^2$ then*

$$\delta_P(T, D) = \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P \in (E_2 \cup E_4)\backslash(E_3 \cup E_5) \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } P \in (E_3 \cup E_5)\backslash(E_2 \cup E_4) \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{u^3-6u^2+6u+5} \text{ for } P \in L_{24}\backslash(E_2 \cup E_4) \text{ and } u \in [1,2], \\ \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P \in L_{14}\backslash(E_4 \cup E_5 \cup E_1) \text{ and } u \in [1,2], \end{cases}$$

and

$$\delta_P(T, D) \geq \begin{cases} \dfrac{15-3u^2}{16+3u-9u^2+2u^3} \text{ for } P \in \{E_2 \cap E_3, E_4 \cap E_5\} \text{ and } u \in [1,a], \\ \dfrac{15-3u^2}{11-u^3} \text{ for } P \in \{E_2 \cap E_3, E_4 \cap E_5\} \text{ and } u \in [a,2] \end{cases}$$

and

$$\delta_P(T, D) \geq \begin{cases} \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \textit{ for } P = L_{14} \cap E_1 \textit{ and } u \in [1, b], \\ \dfrac{2(15-3u^2)}{19-2u^3} \textit{ for } P = L_{14} \cap E_1 \textit{ and } u \in [b, 3/2], \\ \dfrac{15-3u^2}{3u^3-18u^2+27u-4} \textit{ for } P = L_{14} \cap E_1 \textit{ and } u \in [3/2, 2], \end{cases}$$

where a is a root of $3u^3 - 9u^2 + 3u + 5$ *on* $[1, 2]$*, b is a root of* $8u^3 - 24u^2 + 12u + 7$ *on* $[1, 3/2]$*. Note that* $a \in [1.355, 1.356]$*,* $b \in [1.261, 1.262]$.

Proof **Step 1.** Suppose $P \in E_2 \cup E_4$. Without loss of generality we can assume that $P \in E_4$. In this case we set $\mathcal{C} = E_4$. Then $\tau(\mathcal{C}) = 3 - u$. The Zariski Decomposition of the divisor $D - vE_4$ is given by:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_4 \text{ for } v \in [0, 2-u] \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ -K_T - (u-1)C_2 - vE_4 - (u+v-2)E_5 - (v-1)(L_{14} + 2L_{24} + E_2) \\ \text{ for } v \in [1, 3-u] \end{cases}$$

and

$$N(v) = \begin{cases} 0 \text{ for } v \in [0, 2-u] \\ (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ (u+v-2)E_5 + (v-1)(L_{14} + 2L_{24} + E_2) \text{ for } v \in [1, 3-u] \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5 - u^2 - 2v^2 \text{ for } v \in [0, 2-u] \\ 9 + 2uv - 4u - 4v - v^2 \text{ for } v \in [2-u, 1] \\ 2(2-v)(3-u-v) \text{ for } v \in [1, 3-u] \end{cases}$$

and

$$P(v) \cdot \mathcal{C} = \begin{cases} 2v \text{ for } v \in [0, 2-u] \\ 2 - u + v \text{ for } v \in [2-u, 1] \\ 5 - u - 2v \text{ for } v \in [1, 3-u] \end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5 - u^2 - 2v^2 dv + \int_{2-u}^{1} 9 + 2uv - 4u - 4v - v^2 dv$$

$$+\int_1^{3-u} 2(2-v)(3-u-v)dv\Big) = \frac{16+3u-9u^2+2u^3}{15-3u^2}$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{16+3u-9u^2+2u^3}$ for $P \in E_4$. Note that we have:

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(5-u-2v)^2}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P = E_4 \cap E_5$

$$h_D(v) = \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)(u+3v-2)}{2} \text{ for } v \in [2-u, 1] \\ \frac{(u+1)(5-u-2v)}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P \in E_4 \cap (L_{14} \cup L_{24})$

$$h_D(v) \leq \begin{cases} 2v^2 \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(5-u-2v)(1-u+2v)}{2} \text{ for } v \in [1, 3-u] \end{cases}$$

So we have

- if $P \in E_4 \backslash (E_5 \cup L_{14} \cup L_{24} \cup L_{34})$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^1 \frac{(2-u+v)^2}{2} dv + \int_1^{3-u} \frac{(5-u-2v)^2}{2} dv\Big) = \frac{9+6u-9u^2+2u^3}{15-3u^2} \leq \frac{16+3u-9u^2+2u^3}{15-3u^2}$$

- if $P = E_4 \cap E_5$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^1 \frac{(2-u+v)(u+3v-2)}{2} dv + \int_1^{3-u} \frac{(u+1)(5-u-2v)}{2} dv\Big) = \frac{11-u^3}{15-3u^2}$$

- if $P \in E_4 \cap (L_{14} \cup L_{24})$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} 2v^2 dv + \int_{2-u}^1 \frac{(2-u+v)^2}{2} dv$$

$$+\int_1^{3-u}\frac{(5-u-2v)(1-u+2v)}{2}dv\Big)$$
$$=\frac{2u^3-6u^2+8}{15-3u^2}\le\frac{16+3u-9u^2+2u^3}{15-3u^2}$$

We obtain that

$$\delta_P(T,D)=\frac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }P\in E_4\backslash E_5\text{ and }u\in[1,2]$$

and

$$\delta_P(T,D)\ge\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\text{ for }P=E_4\cap E_5\text{ and }u\in[1,a]\\ \dfrac{15-3u^2}{11-u^3}\text{ for }P=E_4\cap E_5\text{ and }u\in[a,2]\end{cases}$$

where a is a root of $3u^3-9u^2+3u+5$ on $[1,2]$. Note that $a\in[1.355,1.356]$.
Step 2. Suppose $P\in E_3\cup E_5$. Without loss of generality we can assume that $P\in E_5$. In this case we set $\mathcal{C}=E_5$. Then $\tau(\mathcal{C})=2$. The Zariski Decomposition of the divisor $D-vE_5$ is given by:

$$P(v)=\begin{cases}-K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4\text{ for }v\in[0,1]\\ -K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4-(v-1)L_{45}\text{ for }v\in[1,u]\\ -K_T-(u-1)C_2-vE_5-\frac{v}{2}E_4-(v-1)L_{45}-(v-u)C_2\text{ for }v\in[u,2]\end{cases}$$

and

$$N(v)=\begin{cases}\frac{v}{2}E_4\text{ for }v\in[0,1]\\ \frac{v}{2}E_4+(v-1)L_{45}\text{ for }v\in[1,u]\\ \frac{v}{2}E_4+(v-1)L_{45}+(v-u)C_2\text{ for }v\in[u,2]\end{cases}$$

Moreover,

$$P(v)^2=\begin{cases}5-4v+2uv-u^2-\frac{v^2}{2}\text{ for }v\in[0,1]\\ 6-6v+\frac{v^2}{2}+2uv-u^2\text{ for }v\in[1,u]\\ \frac{3(2-v)^2}{2}\text{ for }v\in[u,2]\end{cases}$$

and

$$P(v)\cdot\mathcal{C}=\begin{cases}2-u+v/2\text{ for }v\in[0,1]\\ 3-u-v/2\text{ for }v\in[1,u]\\ 3-3v/2\text{ for }v\in[u,2]\end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^1 5 - 4v + 2uv - u^2 - \frac{v^2}{2}dv + \int_1^u 6 - 6v + \frac{v^2}{2} + 2uv - u^2 dv + \int_u^2 \frac{3(2-v)^2}{3}dv\Big) = \frac{11-u^3}{15-3u^2}$$

Thus, $\delta_P(T, D) \leq \frac{15-3u^2}{11-u^3}$ for $P \in E_5$. Note that we have:

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0, 1] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1, u] \\ \frac{(3-3v/2)^2}{2} \text{ for } v \in [u, 2] \end{cases}$$

- if $P = E_5 \cap C_2$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0, 1] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1, u] \\ \frac{3(2-v)(6-4u+v)}{8} \text{ for } v \in [u, 2] \end{cases}$$

- if $P = E_5 \cap L_{45}$ then

$$h_D(v) = \begin{cases} \frac{(2-u+v/2)^2}{2} \text{ for } v \in [0, 1] \\ \frac{(6-2u-v)(2-2u+3v)}{8} \text{ for } v \in [1, u] \\ \frac{3(2-v)(v+2)}{8} \text{ for } v \in [u, 2] \end{cases}$$

So we have

- if $P \in E_5 \backslash (E_4 \cup C_2 \cup L_{45})$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(3-u-v/2)^2}{2}dv + \int_u^2 \frac{(3-3v/2)^2}{2}dv\Big) = \frac{21+6u-18u^2+5u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}$$

- if $P = E_5 \cap C_2$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(3-u-v/2)^2}{2}dv + \int_u^2 \frac{3(2-v)(6-4u+v)}{8}dv\Big) = \frac{45-30u+2u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}$$

- if $P = E_5 \cap L_{45}$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^1 \frac{(2-u+v/2)^2}{2}dv + \int_1^u \frac{(6-2u-v)(2-2u+3v)}{8}dv + \frac{3(2-v)(v+2)}{8}dv\Big) = \frac{26-12u^2+3u^3}{2(15-3u^2)} \leq \frac{11-u^3}{15-3u^2}$$

We obtain that

$$\delta_P(T, D) = \frac{15-3u^2}{11-u^3} \text{ for } P \in (E_3 \cup E_5)\backslash(E_2 \cup E_4) \text{ and } u \in [1, 2].$$

Step 3. Suppose $P \in L_{24}$. In this case we set $\mathcal{C} = L_{24}$. Then $\tau(\mathcal{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{24}$ is given by:

$$P(v) = \begin{cases} D - vL_{24} - \frac{v}{2}(E_2+E_4) \text{ for } v \in [0, 4-2u] \\ D - vL_{24} - (u+v-2)(E_2+E_4) - (2u+v-4)(E_3+E_5) \\ \text{ for } v \in [4-2u, 3-u] \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}(E_2+E_4) \text{ for } v \in [0, 4-2u] \\ (u+v-2)(E_2+E_4) + (2u+v-4)(E_3+E_5) \text{ for } v \in [4-2u, 3-u] \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} -u^2 - 2v + 5 \text{ for } v \in [0, 4-2u] \\ (u+v-3)(3u+v-7) \text{ for } v \in [4-2u, 3-v] \end{cases}$$

and

$$P(v) \cdot \mathcal{C} = \begin{cases} 1 \text{ for } v \in [0, 4-2u] \\ 5-2u-v \text{ for } v \in [4-2u, 3-u] \end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^{4-2u} -u^2 - 2v + 5dv + \int_{4-2u}^{3-uu} (u+v-3)(3u+v-7)dv\Big) = \frac{4u^3-15u^2+6u+17}{15-3u^2}$$

Thus, $\delta_P(T, D) \leq \frac{15-3u^2}{4u^3-15u^2+6u+17}$ for $P \in L_{24}$. If $P \in L_{24}\backslash(E_2 \cup E_4)$ then

$$h_D(v) = \begin{cases} \frac{1}{2} \text{ for } v \in [0, 4-2u] \\ \frac{(5-2u-v)^2}{2} \text{ for } v \in [4-2u, 3-u] \end{cases}$$

So for $P \in L_{24}\backslash(E_2 \cup E_4)$ we have

$$\begin{aligned} S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) &= \frac{2}{5-u^2}\Big(\int_0^{4-2u} \frac{1}{2} dv + \int_{4-2u}^{3-u} \frac{(5-2u-v)^2}{2} dv \\ &= \frac{u^3-6u^2+6u+5}{15-3u^2} \leq \frac{4u^3-15u^2+6u+17}{15-3u^2} \end{aligned}$$

We obtain that

$$\delta_P(T, D) = \frac{15-3u^2}{u^3-6u^2+6u+5} \text{ for } P \in L_{24}\backslash(E_2 \cup E_4) \text{ and } u \in [1, 2].$$

Step 4.1. Suppose $P \in L_{12} \cup L_{14}$ and $u \in [1, 3/2]$. Without loss of generality, we can assume that $P \in L_{14}$. In this case we set $\mathcal{C} = L_{14}$. Then $\tau(\mathcal{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{14}$ is given by:

$$P(v) = \begin{cases} D - vL_{14} - \frac{v}{2}E_4 \text{ for } v \in [0, 2-u] \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 \text{ for } v \in [2-u, 1] \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 - (v-1)L_{23} \text{ for } v \in [1, 4-2u] \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (v-1)L_{23} - (2u+v-4)E_5 \\ \text{ for } v \in [4-2u, 3-u] \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_4 \text{ for } v \in [0, 2-u] \\ \frac{v}{2}E_4 + (u+v-2)E_1 \text{ for } v \in [2-u, 1] \\ \frac{v}{2}E_4 + (u+v-2)E_1 + (v-1)L_{23} \text{ for } v \in [1, 4-2u] \\ (u+v-2)(E_1+E_4) + (v-1)L_{23} + (2u+v-4)E_5 \\ \text{ for } v \in [4-2u, 3-u] \end{cases}$$

Moreover

$$P(v)^2 = \begin{cases} 5 - 2v - \frac{v^2}{2} - u^2 \text{ for } v \in [0, 2-u] \\ 9 - 4u - 6v + \frac{v^2}{2} + 2uv \text{ for } v \in [2-u, 1] \\ \frac{(v-2)(3v+4u-10)}{2} \text{ for } v \in [1, 4-2u] \\ 2(u+v-3)^2 \text{ for } v \in [4-2u, 3-u] \end{cases}$$

and

$$P(v) \cdot \mathcal{C} = \begin{cases} v/2 + 1 \text{ for } v \in [0, 2-u] \\ 3 - u - v/2 \text{ for } v \in [2-u, 1] \\ 4 - u - 3v/2 \text{ for } v \in [1, 4-2u] \\ 2(3 - u - v) \text{ for } v \in [4-2u, 3-u] \end{cases}$$

Thus,

$$\begin{aligned} S_D(\mathcal{C}) &= \frac{1}{5-u^2}\Big(\int_0^{2-u} 5 - 2v - \frac{v^2}{2} - u^2 dv + \int_{2-u}^{1} 9 - 4u - 6v + \frac{v^2}{2} + 2uvdv \\ &\quad + \int_1^{4-2u} \frac{(v-2)(3v+4u-10)}{2} dv + \int_{4-2u}^{3-u} 2(u+v-3)^2 dv\Big) \\ &= \frac{3u^3 - 12u^2 + 6u + 13}{15 - 3u^2} \end{aligned}$$

Thus, $\delta_P(T, D) \leq \frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P \in L_{14}$. Note that we have:

- if $P \in L_{14} \backslash (E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(4-u-3v/2)^2}{2} \text{ for } v \in [1, 4-2u] \\ 2(3-u-v)^2 \text{ for } v \in [4-2u, 3-u] \end{cases}$$

- if $P = L_{14} \cap E_1$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v \in [2-u, 1] \\ \frac{(8-2u-3v)(2u+v)}{8} \text{ for } v \in [1, 4-2u] \\ (3-u-v) \text{ for } v \in [4-2u, 3-u] \end{cases}$$

- if $P = L_{14} \cap L_{23}$ then

$$h_D(v) = \begin{cases} \frac{(v/2+1)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(8-2u-3v)(4-2u+v)}{8} \text{ for } v \in [1, 4-2u] \\ 2(2-u)(3-u-v) \text{ for } v \in [4-2u, 3-u] \end{cases}$$

So we have

- if $P \in L_{14} \backslash (E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2} dv + \int_{2-u}^{1} \frac{(3-u-v/2)^2}{2} dv + \int_1^{4-2u} \frac{(4-u-3v/2)^2}{2} dv + \int_{4-2u}^{3-u} 2(3-u-v)^2 dv \Big) = \frac{21-u^3-6u}{2(15-3u^2)} \leq \frac{3u^3-12u^2+6u+13}{15-3u^2}$$

- if $P = L_{14} \cap E_1$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2} dv + \int_{2-u}^{1} \frac{(6-2u-v)(2u+3v-2)}{8} dv + \int_1^{4-2u} \frac{(8-2u-3v)(2u+v)}{8} dv + \int_{4-2u}^{3-u} (3-u-v) dv \Big) = \frac{19-2u^3}{2(15-3u^2)}$$

- if $P = L_{14} \cap L_{23}$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2} dv + \int_{2-u}^{1} \frac{(3-u-v/2)^2}{2} dv + \int_1^{4-2u} \frac{(8-2u-3v)(4-2u+v)}{8} dv + \int_{4-2u}^{3-u} 2(2-u)(3-u-v) dv \Big) = \frac{26-12u^2+3u^3}{2(15-3u^2)} \leq \frac{3u^3-12u^2+6u+13}{15-3u^2}$$

We obtain that

$$\delta_P(T, D) = \frac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P \in L_{14} \backslash (E_4 \cup E_5 \cup E_1) \text{ and } u \in [1, 3/2]$$

and

$$\delta_P(T,D) \geq \begin{cases} \dfrac{15-3u^2}{3u^3-12u^2+6u+13} \text{ for } P = L_{14}\cap E_1 \text{ and } u\in[1,b] \\ \dfrac{2(15-3u^2)}{19-2u^3} \text{ for } P = L_{14}\cap E_1 \text{ and } u \in [b,3/2] \end{cases}$$

where b is a root of $8u^3-24u^2+12u+7$ on $[1, 3/2]$. Note that $b \in [1.261, 1.262]$.
Step 4.2. Suppose $P \in L_{12} \cup L_{14}$ and $u \in [3/2, 2]$. Without loss of generality, we can assume that $P \in L_{14}$. In this case we set $\mathcal{C} = L_{14}$. Then $\tau(\mathcal{C}) = 3-u$. The Zariski Decomposition of the divisor $D - vL_{14}$ is given by:

$$P(v) = \begin{cases} D - vL_{14} - \frac{v}{2}E_4 \text{ for } v \in [0, 2-u] \\ D - vL_{14} - \frac{v}{2}E_4 - (u+v-2)E_1 \text{ for } v \in [2-u, 4-2u] \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (2u+v-4)E_5 \text{ for } v \in [4-2u, 1] \\ D - vL_{14} - (u+v-2)(E_1+E_4) - (v-1)L_{23} - (2u+v-4)E_5 \\ \text{ for } v \in [1, 3-u] \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_4 \text{ for } v \in [0, 2-u] \\ \frac{v}{2}E_4 + (u+v-2)E_1 \text{ for } v \in [2-u, 4-2u] \\ (u+v-2)(E_1+E_4) + (2u+v-4)E_5 \text{ for } v \in [4-2u, 1] \\ (u+v-2)(E_1+E_4) + (v-1)L_{23} + (2u+v-4)E_5 \text{ for } v \in [1, 3-u] \end{cases}$$

Moreover

$$P(v)^2 = \begin{cases} 5 - 2v - \frac{v^2}{2} - u^2 \text{ for } v \in [0, 2-u] \\ 9 - 4u - 6v + \frac{v^2}{2} + 2uv \text{ for } v \in [2-u, 4-2u] \\ 2u^2 + 4uv + v^2 - 12u - 10v + 17 \text{ for } v \in [4-2u, 1] \\ 2(u+v-3)^2 \text{ for } v \in [1, 3-u] \end{cases}$$

and

$$P(v)\cdot\mathcal{C} = \begin{cases} 1 + v/2 \text{ for } v \in [0, 2-u] \\ 3 - u - v/2 \text{ for } v \in [2-u, 4-2u] \\ 5 - 2u - v \text{ for } v \in [4-2u, 1] \\ 2(3-u-v) \text{ for } v \in [1, 3-u] \end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5 - 2v - \frac{v^2}{2} - u^2 dv$$

$$+\int_{2-u}^{4-2u} 9-4u-6v+\frac{v^2}{2}+2uvdv$$
$$+\int_{4-2u}^{1} 2u^2+4uv+v^2-12u-10v+17dv$$
$$+\int_{1}^{3-u} 2(u+v-3)^2dv\Big) = \frac{3u^3-12u^2+6u+13}{15-3u^2}$$

Thus, $\delta_P(T,D) \leq \frac{15-3u^2}{3u^3-12u^2+6u+13}$ for $P \in L_{14}$. Note that we have:

- if $P \in L_{14}\backslash(E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$h_D(v) = \begin{cases} \frac{(1+v/2)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 4-2u] \\ \frac{(5-2u-v)^2}{2} \text{ for } v \in [4-2u, 1] \\ 2(3-u-v)^2 \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P = L_{14} \cap E_1$ then

$$h_D(v) = \begin{cases} \frac{(1+v/2)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(6-2u-v)(2u+3v-2)}{8} \text{ for } v \in [2-u, 4-2u] \\ \frac{(v+1)(5-2u-v)}{2} \text{ for } v \in [4-2u, 1] \\ 2(3-u-v) \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P = L_{14} \cap L_{23}$ then

$$h_D(v) = \begin{cases} \frac{(1+v/2)^2}{2} \text{ for } v \in [0, 2-u] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [2-u, 4-2u] \\ \frac{(5-2u-v)^2}{2} \text{ for } v \in [4-2u, 1] \\ 2(2-u)(3-u-v) \text{ for } v \in [1, 3-u] \end{cases}$$

- if $P \in L_{14}\backslash(E_4 \cup E_1 \cup L_{23} \cup E_5)$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2}dv + \int_{2-u}^{4-2u} \frac{(3-u-v/2)^2}{2}dv$$
$$+\int_{4-2u}^{1} \frac{(5-2u-v)^2}{2}dv + \int_1^{3-u} 2(3-u-v)^2dv\Big)$$
$$= \frac{7u^3-36u^2+48u-6}{2(15-3u^2)} \leq \frac{3u^3-12u^2+6u+13}{15-3u^2}$$

- if $P = L_{14} \cap E_1$ then

$$S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) = \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{(v/2+1)^2}{2}dv$$

$$
\begin{aligned}
&+\int_{2-u}^{4-2u}\frac{(6-2u-v)(2u+3v-2)}{8}dv\\
&+\int_{4-2u}^{1}\frac{(v+1)(5-2u-v)}{2}dv+\int_{1}^{3-u}2(3-u-v)dv\Big)\\
&=\frac{3u^3-18u^2+27u-4}{15-3u^2}
\end{aligned}
$$

- if $P=L_{14}\cap L_{23}$ then

$$
\begin{aligned}
S_D(W^{\mathcal{C}}_{\bullet,\bullet};P)=\frac{2}{5-u^2}\Big(&\int_0^{2-u}\frac{(v/2+1)^2}{2}dv+\int_{2-u}^{4-2u}\frac{(3-u-v/2)^2}{2}dv\\
&+\int_{4-2u}^{1}\frac{(5-2u-v)^2}{2}dv+\int_{1}^{3-u}2(2-u)(3-u-v)dv\Big)\\
&=\frac{3u^3-12u^2+26}{2(15-3u^2)}\leq\frac{3u^3-12u^2+6u+13}{15-3u^2}
\end{aligned}
$$

We obtain that

$$
\delta_P(T,D)=\frac{15-3u^2}{3u^3-12u^2+6u+13}\text{ for }P\in L_{14}\backslash(E_1\cup E_4\cup E_5)\text{ and }u\in[3/2,2]
$$

and

$$
\delta_P(T,D)\geq\frac{15-3u^2}{3u^3-18u^2+27u-4}\text{ for }P=L_{14}\cap E_1\text{ and }u\in[3/2,2]
$$

□

Corollary A.3 *Let P be a point in T that is contained in $L_{12}\cup L_{14}\cup L_{24}\cup E_2\cup E_3\cup E_4\cup E_4$ then*

$$
\delta_P(T,D)\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\ \textit{for } u\in[1,a],\\[2ex] \dfrac{15-3u^2}{11-u^3}\ \textit{for } u\in[a,2]\end{cases}
$$

Corollary A.4 *Suppose O is a point on a del Pezzo surface $\overline{T}$ with two $\mathbb{A}_1$ singularities and nine lines such that $\delta_O(T)\leq\frac{6}{5}$ then*

$$
\delta_O(\overline{T},\overline{D})\geq\begin{cases}\dfrac{15-3u^2}{16+3u-9u^2+2u^3}\ \textit{for } u\in[1,a],\\[2ex] \dfrac{15-3u^2}{11-u^3}\ \textit{for } u\in[a,2]\end{cases}
$$

A.3 Polarized δ-invariant on Del Pezzo surface of degree 4 with $\mathbb{A}_2$ singularity

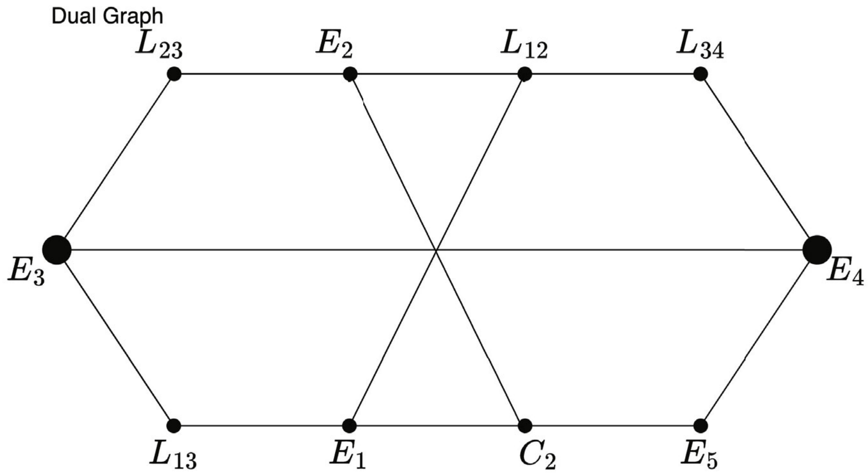
Now, let us use the notations and assumptions of Section 2 with a minor difference: we assume that $\overline{T}$ has a singular point of type $\mathbb{A}_2$. Let us show that in the case when O is the singular point of the surface $\overline{T}$ we have

$$\delta_O(\overline{T}, \overline{D}) = \frac{15 - 3u^2}{u^3 - 6u^2 + 19}$$

which immediately implies that $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) \leq \frac{80}{83}$. In this case, the morphism η is a blow up of $\mathbb{P}^2$ at points P_1, P_2, and P_3 in general position; after that blowing up a point P_4 which belongs to the exceptional divisor corresponding to P_3 and no other negative curve and after that a point P_5 which belongs to the exceptional divisor corresponding to P_4 and no other negative curve. By [6, Section 6.5] we have:

$$\delta_P(T) = \begin{cases} 6/7 \text{ if } P \in E_3 \cup E_4, \\ 8/7 \text{ if } P \in (L_{13} \cup L_{23} \cup L_{34} \cup E_5)\backslash(E_3 \cup E_4), \\ 4/3 \text{ if } P \in (L_{12} \cup C_2) \cap (E_1 \cup E_2), \\ 18/13 \text{ if } P \in (L_{12} \cup C_2 \cup E_1 \cup E_2)\backslash((L_{12} \cup C_2) \cap (E_1 \cup E_2)), \\ 3/2, \text{ otherwise} \end{cases}$$

where E_1, E_2, E_3, E_4, E_5 are exceptional divisors corresponding to P_1, P_2, P_3, P_4, P_5 respectively, C_2 is a strict transform of a (-1)-curve coming from the conic on $\mathbb{P}^2$, L_{ij} are strict transforms of the lines passing through P_i and P_j for $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ and L_{34} is a strict transform of a (-1)-curve coming from a line passing through P_3 on $\mathbb{P}^2$. The dual graph of (-1) and (-2)-curves is given in the following picture:



Now let's prove that:

Lemma A.3 *Suppose P is a point on T and $D = -K_T - (u-1)C_2$ with $D^2 = 5 - u^2$ then*

$$\delta_P(T, D) = \frac{15 - 3u^2}{u^3 - 6u^2 + 19} \textit{ for } P \in E_4 \backslash (L_{34} \cup E_5)$$

Proof Suppose $P \in E_4 \backslash (L_{34} \cup E_5)$. In this case we set $\mathcal{C} = E_4$. Then $\tau(\mathcal{C}) = 2$. The Zariski Decomposition of the divisor $D - vE_4$ is given by:

$$P(v) = \begin{cases} -K_T - (u-1)C_2 - vE_4 - \frac{v}{2}E_3 \text{ for } v \in [0, 2-u] \\ -K_T - (u-1)C_2 - vE_4 - \frac{v}{2}E_3 - (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ -K_T - (u-1)C_2 - vE_4 - \frac{v}{2}E_3 - (u+v-2)E_5 - (v-1)L_{34} \text{ for } v \in [1, 2] \end{cases}$$

and

$$N(v) = \begin{cases} \frac{v}{2}E_3 \text{ for } v \in [0, 2-u] \\ \frac{v}{2}E_3 + (u+v-2)E_5 \text{ for } v \in [2-u, 1] \\ \frac{v}{2}E_3 + (u+v-2)E_5 + (v-1)L_{34} \text{ for } v \in [1, 2] \end{cases}$$

Moreover,

$$P(v)^2 = \begin{cases} 5 - u^2 - \frac{3v^2}{2} \text{ for } v \in [0, 2-u] \\ 9 - 4u - 4v + 2uv - 1/2v^2 \text{ for } v \in [2-u, 1] \\ \frac{(v-2)(v+4u-10)}{2} \text{ for } v \in [1, 2] \end{cases}$$

and

$$P(v) \cdot \mathcal{C} = \begin{cases} 3v/2 \text{ for } v \in [0, 2-u] \\ 2 - u + v/2 \text{ for } v \in [2-u, 1] \\ 3 - u - v/2 \text{ for } v \in [1, 2] \end{cases}$$

Thus,

$$S_D(\mathcal{C}) = \frac{1}{5-u^2}\Big(\int_0^{2-u} 5 - u^2 - \frac{3v^2}{2} dv + \int_{2-u}^1 9 - 4u - 4v + 2uv - 1/2v^2 dv + \int_1^2 \frac{(v-2)(v+4u-10)}{2} dv\Big) = \frac{19 + u^3 - 6u^2}{15 - 3u^2}$$

Thus, $\delta_P(T, D) \le \frac{15-3u^2}{19+u^3-6u^2}$ for $P \in E_4$. Note that for $P \in E_4\backslash(E_5 \cup L_{34})$ we have:

$$h_D(v) = \begin{cases} \frac{9v^2}{8} \text{ for } v \in [0, 2-u] \\ \frac{(2-u+v/2)^2}{2} \text{ for } v \in [2-u, 1] \\ \frac{(3-u-v/2)^2}{2} \text{ for } v \in [1, 2] \end{cases}$$

So we have

$$\begin{aligned} S_D(W^{\mathcal{C}}_{\bullet,\bullet}; P) &= \frac{2}{5-u^2}\Big(\int_0^{2-u} \frac{9v^2}{8} dv + \int_{2-u}^1 \frac{(2-u+v/2)^2}{2} dv \\ &\quad + \int_1^2 \frac{(3-u-v/2)^2}{2} dv\Big) \\ &= \frac{21+6u-18u^2+5u^3}{2(15-3u^2)} \le \frac{19+u^3-6u^2}{15-3u^2} \end{aligned}$$

So we obtain that

$$\delta_P(T, D) = \frac{15-3u^2}{u^3-6u^2+19} \text{ for } P \in E_4\backslash(L_{34} \cup E_5).$$

□

Corollary A.5 *We have* $S(W^{\overline{T}}_{\bullet,\bullet}; E_4) = \frac{83}{80}$*, which implies that* $\delta_O(\overline{T}, W^{\overline{T}}_{\bullet,\bullet}) \le \frac{80}{83}$.

Proof

$$\begin{aligned} S\big(W^{\overline{T}}_{\bullet,\bullet}; E_4\big) &= \frac{3}{(-K_X)^3} \int_1^2 \int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vE_4\big) dv du \\ &\quad + \frac{3}{(-K_X)^3} \int_0^1 \int_0^\infty \mathrm{vol}\big(P(u)\big|_{\overline{T}} - vE_4\big) dv du \\ &= \frac{3}{20} \int_1^2 (5-u^2) S_D(E_4) du + \frac{3}{5} \\ &= \frac{3}{20} \int_1^2 (5-u^2) \frac{19+u^3-6u^2}{15-3u^2} du + \frac{3}{5} = \frac{83}{80} \end{aligned}$$

□

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