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Connecting probability for random bounded-range one-dimensional network

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Abstract
We consider a class of bounded-range $1D$ network models on a cycle and prove that, unlike the corresponding infinite-volume models, which never contain infinite components, they actually exhibit a phase transition for connectivity. We further show that depending on the specific choice of the edge probabilities, the last obstruction to connectivity can either be the existence of isolated vertices or the split of the cycle into two spatially separated components.

Keywords Random graphs · Connectivity · One-dimensional models · Spread-out percolation

1 Introduction

The class of $1D$ network models has been one of the most common and natural topics to study since the early stages of the development of percolation theory, especially motivated by the important role that they played in the context of Ising and Potts ferromagnetic models. The result that the $1D$ nearest-neighbour model has no phase transition by Ising himself in [1] is considered one of the foundation stones of statistical mechanics (see e.g. [2, 3] for more recent and precise results). In particular, translation-invariant network models on the integer line, in which the vertex set can be represented as the set of integer numbers $\mathbb{Z}$ and edge probabilities depend on the distance of the end-vertices, are known to differ significantly in their behaviour when compared to higher-dimensional models, as for the existence of an infinite component it is required the existence of connections of unbounded length, while already for $2D$ models, it is possible for nearest-neighbour percolation to have an infinite component when $p \geq 1/2$, as proved in a classic paper from Kesten [4]. In $1D$ translation-invariant

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percolation models instead, it was later proved by Aizenmann and Newman in [5] that for an infinite component to exist, it is necessary that the probability of an edge of length d to be present does not decay faster than $1/d^2$. The fact that bounded-range models, usually referred to as spread-out percolation [6], are always subcritical (i.e. have no infinite components, something that is way easier to prove than the main theorem from [5] and is actually a straightforward application of the second Borel-Cantelli lemma) makes their study rather uninteresting. We thus choose as the subject of our analysis of spread-out $1D$ percolation, not in the actually infinite volume model, but in the limit of the finite volume one, that is, spread-out percolation on a cycle of length n, as its size tends to infinity. Also in this case, if we keep the edge probabilities constant, the model is always subcritical (i.e. the largest connected component is much smaller than the entire network), but given that we can scale the probability of edges to be vacant as a function of n, we know that there has to be a supercritical phase where there is a large connected component and in particular even the entire system becomes fully connected. In particular it is easy to see that this happens if, for example choose the probability of nearest-neighbour edges to be $1 - o(1/n)$, so that with high probability (w.h.p., that is, with probability converging to 1) all nearest-neighbour edges are present. In this paper we study such a model around the connectivity threshold, showing that the nature of the last obstructions to connectivity depends on the specific properties of the edge probabilities.

The original impulse to the study of $1D$ models was mainly theoretical, but in more recent years, the interest in $1D$ model has resurfaced thanks to the growing research aimed at developing intelligent transportation systems and self-driving vehicles. In this setting, $1D$ models are used to represent vehicular ad-hoc networks (VANETs) [7], that is, a network of wireless real-time communication among vehicles and between vehicles and roadside units along a specific road. VANETs are fundamental to improve safety features, such as allowing automatic response (for self-driving vehicles) or immediate warning (for human drivers) in dangerous situations that might be out of vision, or early warning of the need to make space for emergency service. Here, as usual in the modelling of wireless networks (see e.g. [8]), vehicles and road-side units are represented by nodes, the existence of active wireless transmission by edges, and the $1D$ line is a good approximation of the local geometric structure of highways. Many problems related to wireless communication can thus be formulated in terms of graph-theoretic questions, such as what is the average number of hops needed between two vertices, or whether the entire network is connected.

Further, continuous space $1D$ models, known as an example of Random Geometric Graphs, have also been studied, as in the work of Han and Makowski [9] and Badiu and Coon [10]. Also in this case it is easy to prove that the bounded-range, infinite-volume models almost surely never contain infinite components, while in the finite-volume regime, there is a more interesting complex behaviour. In this case, the vertex set instead of corresponding to $\mathbb{Z}$ is the result of a Poisson Point Process, and edges are generated, in a deterministic or random way, based on the distance between their end-vertices. In this setting, Wilsher et al. showed in [11] results similar to those we present in this paper, with less mathematical rigour but greater generality in the connectivity probabilities.

The rest of the paper is structured as follows: in Sect. 2 we formally define the model and state the main theorem we prove. In Sects. 3 and 4 we compute the thresholds for the existence of isolated points and cut edges, which are the minimal obstructions to connectivity. In Sect. 5 we prove that all the other possible disconnections are less likely to happen. In Sect. 6 we use the results of the previous sections to prove the main theorem and finally in Sect. 7 we discuss the implications of our results and possible future extensions.

2 Model description and main theorem

We define a $1D$ spread-out percolation process $(C_n(t, \mathbf{q}))_{t\in[0,\infty)}$ on the n-cycle C_n as follows. We establish the vertex set as $V := \{v_i : i \in \mathbb{Z}/n\mathbb{Z}\}$, we fix a finite range of connections D, and sequence of rates of appearance of edges $\mathbf{q} = \{q_1, \ldots, q_D\}$. The edge e_{ij} between two vertices v_i and v_j at distance $d(i, j) \le D$ on the cycle (i.e. such that $d(i, j) = \min\{(i - j)_{\text{mod } n}, (j - i)_{\text{mod } n}\}$) is vacant at time 0 and becomes present independently of the others at a time $T(e_{ij})$ distributed as

$$T(e_{ij}) \sim \mathsf{Exp}(q_{d(i,j)}). \tag{2.1}$$

Edges of length greater than D are almost surely vacant at any time $t \in [0, \infty)$. Consequently at a given time $t \in [0, \infty)$ each edge e_{ij} between vertices v_i and v_j is present with probability

$$p_{d(i,j)}(t) = \begin{cases} 1 - \exp\big\{ - tq_{d(i,j)}\big\} & \text{if } d(i, j) \le D; \\ 0 & \text{if } d(i, j) > D. \end{cases} \tag{2.2}$$

This static definition of the network at time t is the one we will use in the proofs throughout the paper. We study the asymptotic behavior of the process $(C_n(t, \mathbf{q}))_{t\in[0,\infty)}$ as $n \to \infty$. In particular, we are interested in identifying the minimal obstruction for connectivity. We show that depending on the behaviour of $\mathbf{q}$, the minimal obstruction is either the existence of isolated vertices or the presence of *cut edges*, that is, gaps between two consecutive vertices which are not bridged by any edge (see Fig. 1 for examples).

Given $\mathbf{q}$ we define the following values which, as we will see, represent the rates at which isolated vertices are connected and cut edges are bridged:

$$c_{is}(\mathbf{q}) := \sum_{d=1}^{D} 2q_d; \quad c_{cut}(\mathbf{q}) := \sum_{d=1}^{D} dq_d. \tag{2.3}$$

We can now state the main theorem of the paper:

Theorem 2.1 *Consider the graph $C_n(t, \mathbf{q})$ defined above at time $t = t(n) = \alpha \log n$, then, for every $\mathbf{q} \in \mathbb{R}_+^D$. Define $c_0(\mathbf{q}) := \min\{c_{is}(\mathbf{q}), c_{cut}(\mathbf{q})\}$. Then*

$$\lim_{n\to\infty} \mathbb{P}(C_n(t, \mathbf{q}) \text{ is connected}) = \begin{cases} 0 & \text{if } \alpha < 1/c_0(\mathbf{q}); \\ 1 & \text{if } \alpha > 1/c_0(\mathbf{q}). \end{cases} \tag{2.4}$$

In all the following proofs, we will assume, without loss of generality, since we are interested in the behaviour of the model as $n \to \infty$, that $n \geq 5D$. This ensures that all the possible v_{i+k} for $-D \leq k \leq D$ are distinct and for every k_1, k_2 such that $-D \leq k_1 < k_2 \leq D$, $d(v_{i+k_1}, v_{i+k_2}) = k_2 - k_1$, thus avoiding us more complex computations that would be useful only for small n. Even if the model produces an undirected graph and is periodic, we find it useful to be able to distinguish the two end-vertices of an edge. Given an edge e_{ij} such that $d(i, j) = (j - i)_{\text{mod } n}$ we will define v_j as its right end-vertex and v_i as its left end-vertex, if instead $d(i, j) = (i - j)_{\text{mod } n}$, it will be the other way around.

It is interesting to note that not only the finite-volume spread-out percolation can reach full connectivity, unlike the equivalent infinite-volume model, which remains subcritical for every $t < \infty$, but that depending on the specific structure of $\mathbf{q}$, the last obstruction to connectivity can change significantly. In particular, we can compute

$$c_{is}(\mathbf{q}) - c_{cut}(\mathbf{q}) = q_1 - \sum_{k=3}^{D}(k-2)q_k. \tag{2.5}$$

We will see that if $c_{is}(\mathbf{q}) - c_{cut}(\mathbf{q}) > 0$ then at time $\alpha \log n$ such that $1/c_{is}(\mathbf{q}) < \alpha < 1/c_{cut}(\mathbf{q})$ and so w.h.p. $C_n(t, \mathbf{q})$ contains isolated vertices but not cut edges, while $c_{is}(\mathbf{q}) - c_{cut}(\mathbf{q}) < 0$, than at time $1/c_{is}(\mathbf{q}) < \alpha < 1/c_{cut}(\mathbf{q})$, $C_n(t, \mathbf{q})$ contains w.h.p. cut edges but not isolated vertices.

3 Isolated vertices

We now prove that the threshold for the existence of isolated vertices is given by $t = t(n) = \log n / c_{is}(\mathbf{q})$. We define the number of isolated vertices in $C_n(t, \mathbf{q})$ as Y_n.

Proposition 3.1 *Consider the graph $C_n(t, \mathbf{q})$ defined above at time $t = t(n) = \alpha \log n$, then, for every $\mathbf{q} \in \mathbb{R}_+^D$,*

$$\lim_{n\to\infty} \mathbb{P}(Y_n = 0) = \begin{cases} 0 & \text{if } \alpha < 1/c_{is}(\mathbf{q}); \\ 1 & \text{if } \alpha > 1/c_{is}(\mathbf{q}). \end{cases} \tag{3.1}$$

Proof We compute the expected number of isolated vertices, which, based on the results from [12], is everything we need to know whether there are actually any isolated vertices. Since $C_n(t, \mathbf{q})$ is translation-invariant, we know that for ever $i \in \mathbb{Z}/n\mathbb{Z}$, defining I_i as the event that v_i is isolated, $\mathbb{E}[Y_n] = n\mathbb{P}(I_i)$. We know, that each vertex

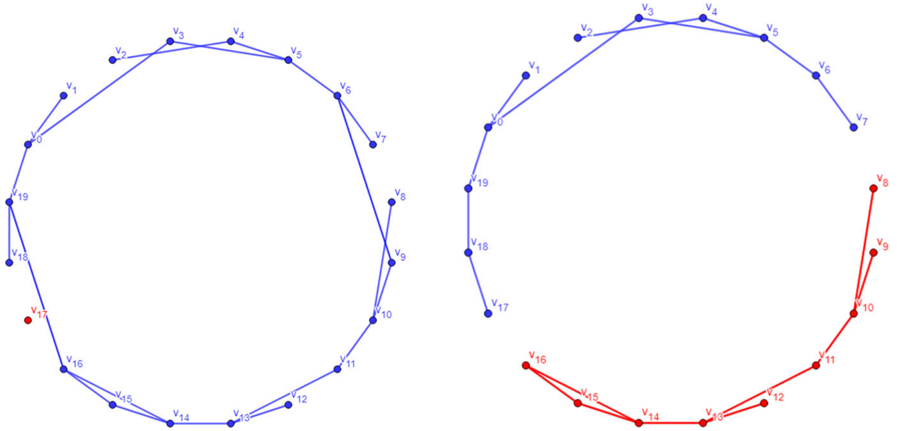

Fig. 1 Examples of two disconnected graphs with $n = 20$, $D = 3$. The one on the left has an isolated vertex, v_{17} (red), and a component containing all the rest of the vertices (blue). The one on the right has two cut edges, with middle points $m_{16.5}, m_{7.5}$, which separate it into two connected components $\mathscr{C}_1$ (blue) and $\mathscr{C}_2$ (red)

v_i is the end-vertex of two potential edges of length d to v_{i-d} and v_{i+d} for every $d \leq D$, so

$$\mathbb{P}(I_i) = \prod_{d \leq D} (1 - p_{d(i,j)}(t)) = \exp\Big\{ - t \sum_{d \leq D} q_{d(i,j)} \Big\} = \exp\big\{ - t c_{is}(\mathbf{q}) \big\}. \tag{3.2}$$

We compute

$$\mathbb{P}(I_i) = \exp\big\{ - c_{is}(\mathbf{q}) \alpha \log n \big\} = n^{-\alpha c_{is}(\mathbf{q})}, \tag{3.3}$$

so that

$$\mathbb{E}[Y_n] = n^{1 - \alpha c_{is}(\mathbf{q})}. \tag{3.4}$$

If $\alpha < c_{is}(\mathbf{q})$ this means that $\lim_{n\to\infty} \mathbb{E}[Y_n]/\log n = \infty$. We know from [12, Theorem 4.1] that this implies $\lim_{n\to\infty} \mathrm{Var}(Y_n)/\mathbb{E}[Y_n]^2 = 0$ and consequently $\lim_{n\to\infty} \mathbb{P}(Y_n = 0) = 0$ by the second moment method. On the other hand, we note that if $\alpha > 1/c_{is}(\mathbf{q})$, then

$$\lim_{n\to\infty} \mathbb{E}[Y_n] = n^{1 - \alpha c_{is}(\mathbf{q})} = 0. \tag{3.5}$$

This proves the lower bound on the threshold for the existence of isolated vertices by the first moment method. $\square$

4 Cut edges

We next compute the threshold for the existence of cut edges. We define the set of potential cut edges as the set of all middle points of the length-1 edges in the cycle C_n

as $(m_{i+1/2})_{i\in\mathbb{Z}/n\mathbb{Z}}$, where $m_{i+1/2}$ is the middle point of the edge $\{v_i, v_{i+1}\}$. For each middle point $m_{i+1/2}$ we define the set of edges *bridging over* $m_{i+1/2}$ as

$$B_{i+1/2} := \Big\{e_{k_1k_2} : k_1 \in \{i-D+1,\dots,i\}, k_2 \in \{i+1,\dots,i+D\}\Big\}, \tag{4.1}$$

and $H_{i+1/2}$ as the event that all edges in $B_{i+1/2}$ are vacant. If there exist two different $i, j \in \mathbb{Z}/n\mathbb{Z}$ such that both $H_{i+1/2}$ and $H_{l+1/2}$ happen, then $C_n(t,\mathbf{q})$ is disconnected. We define the number of cut edges as

$$Z_n := \sum_{i=0}^{n-1} I_{H_{i+1/2}}. \tag{4.2}$$

By translation invariance of $C_n(t,\mathbf{q})$ we know that $\mathbb{E}[Z_n] = n\mathbb{P}(H_{i+1/2})$ for all $i \in \mathbb{Z}/n\mathbb{Z}$.

We now prove the following proposition

Proposition 4.1 *Consider the graph $C_n(t,\mathbf{q})$ defined above, with $t = t(n) = \alpha \log n$, then, for every $\mathbf{q} \in \mathbb{R}_+^D$, and for every $k \in (0,\infty)$*

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le k) = \begin{cases} 0 & \text{if } \alpha < 1/c_{cut}(\mathbf{q}); \\ 1 & \text{if } \alpha > 1/c_{cut}(\mathbf{q}). \end{cases} \tag{4.3}$$

Proof We note that the set $B_{i+1/2}$ contains exactly k edges of length k for every $k \le D$. Thus we can write

$$\begin{aligned}\mathbb{P}(H_{i+1/2}) &= \prod_{e\in B_{i+1/2}} \mathbb{P}(e \text{ is vacant}) = \exp\Big\{-t\sum_{k=1}^{D} kq_k\Big\} \\ &= \exp\Big\{-\alpha c_{cut}(\mathbf{q})\log n\Big\}.\end{aligned} \tag{4.4}$$

So that

$$\mathbb{E}[Z_n] = n\mathbb{P}(H_{i+1/2}) = n^{1-\alpha c_{cut}(\mathbf{q})}. \tag{4.5}$$

Thus, if $\alpha < 1/c_{cut}(\mathbf{q})$, $\lim_{n\to\infty}\mathbb{E}[Z_n] = \infty$ while if $\alpha > 1/c_{cut}(\mathbf{q})$, $\lim_{n\to\infty}\mathbb{E}[Z_n] = 0$. By the first moment method, we prove that

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le k) = 1 \quad \text{if } \alpha > 1/c_{cut}(\mathbf{q})\ \forall k > 0. \tag{4.6}$$

Next, we prove the existence of cut point w.h.p. when $\alpha < 1/c_{cut}(\mathbf{q})$ by the second moment method. In particular, we need to prove that in such a regime

$$\lim_{n\to\infty} \frac{\mathrm{Var}(Z_n)}{\mathbb{E}[Z_n]^2} = 0. \tag{4.7}$$

We write

$$\begin{aligned}\mathrm{Var}(Z_n) &= \mathbb{E}[Z_n^2] - \mathbb{E}[Z_n]^2 \\ &= \sum_{i,j\in\mathbb{Z}/n\mathbb{Z}} \mathbb{P}(H_{i+1/2}H_{j+1/2}) - \sum_{i,j\in\mathbb{Z}/n\mathbb{Z}} \mathbb{P}(H_{i+1/2})\mathbb{P}(H_{j+1/2}) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \mathbb{P}(H_{i+1/2}) \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \big(\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2})\big). \end{aligned} \tag{4.8}$$

Due to the translation invariance of $C_n(t, \mathbf{q})$ we can rewrite

$$\mathrm{Var}(Z_n) = \mathbb{E}[Z_n] \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \big(\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2})\big), \tag{4.9}$$

where the value on the right-hand side does not depend on the specific choice of i. We thus have to prove that, for every choice of $\mathbf{q}$,

$$\begin{aligned}&\lim_{n\to\infty} \frac{\mathbb{E}[Z_n] \sum_{j\in\mathbb{Z}/n\mathbb{Z}} \big(\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2})\big)}{\mathbb{E}[Z_n]^2} \\ &\quad\le \lim_{n\to\infty} \frac{1 + \sum_{j\in\mathbb{Z}/n\mathbb{Z}\setminus\{i\}} \big(\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2})\big)}{\mathbb{E}[Z_n]} = 0. \end{aligned} \tag{4.10}$$

If $d(i, j) \ge D$, then $B_{i+1/2} \cap B_{j+1/2} = \varnothing$ thus $H_{i+1/2}$ and $H_{j+1/2}$ are independent and we write

$$\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2}) = 0. \tag{4.11}$$

If $d(i, j) = k \le D$, then, $B_{i+1/2} \cap B_{j+1/2}$ contains $l - k$ edges of length l for every l such that $k < l \le D$. We can thus write the following conditional expectation

$$\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) = \exp\Big\{ -t \sum_{d=1}^{D} (d \wedge k) q_d \Big\}. \tag{4.12}$$

Consequently, we can rewrite

$$\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2}) = \mathbb{P}(H_{j+1/2})\Big(\exp\Big\{ t \sum_{d=k+1}^{D} (d-k) q_d \Big\} - 1 \Big). \tag{4.13}$$

We obtain

$$\begin{aligned}&\frac{\sum_{j\in\mathbb{Z}/n\mathbb{Z}\setminus\{i\}} \big(\mathbb{P}(H_{j+1/2} \mid H_{i+1/2}) - \mathbb{P}(H_{j+1/2})\big)}{\mathbb{E}[Z_n]} \\ &\quad= \frac{2\mathbb{P}(H_{j+1/2}) \sum_{k=1}^{D-1} \Big(\exp\Big\{ t \sum_{d=k+1}^{D} (d-k) q_d \Big\} - 1 \Big)}{\mathbb{P}(H_{j+1/2}) n} \end{aligned}$$

$$= \frac{2\sum_{k=1}^{D-1}\left(\exp\left\{t\sum_{d=k+1}^{D}(d-k)q_d\right\}-1\right)}{n} \tag{4.14}$$

We write

$$\frac{\sum_{k=1}^{D-1}\left(\exp\left\{t\sum_{d=k+1}^{D}(d-k)q_d\right\}-1\right)}{n} \leq \frac{(D-1)\exp\left\{t(c_{cut}(\mathbf{q})-q_1)\right\}}{n}. \tag{4.15}$$

If $t = \alpha \log n$, $\alpha < 1/c_{cut}(\mathbf{q})$, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \frac{2(D-1)\exp\left\{t(c_{cut}(\mathbf{q})-q_1)\right\}}{n} \\ &\quad \leq \lim_{n\to\infty} \frac{2(D-1)\exp\left\{(1-q_1\alpha)\log n\right\}}{n} = 0. \end{aligned} \tag{4.16}$$

Substituting the bounds from (4.16) into (4.14) we obtain (4.10). From this, we can conclude by the second moment method that for every $k < \infty$

$$\lim_{n\to\infty} \mathbb{P}(Z_n \leq k) \leq \lim_{n\to\infty} \frac{\mathrm{Var}(Z_n)}{\mathbb{E}[Z_n - k]^2} = 0. \tag{4.17}$$

□

5 Other disconnections

What is left to prove is that, if $\alpha > 1/c_0(\mathbf{q})$, then w.h.p. there are no other disconnections in $C_n(t, \mathbf{q})$. This is the most delicate part of the paper since there are several ways in which such events can arise, and we will have to find a way to write them down in terms of combinations of a few events, which have to be decreasing since we want to prove that any other disconnection is unlikely for t large enough.

We start defining the concept of the *spanning interval* of a connected component $\mathscr{C}$:

Definition 5.1 Given a connected component $\mathscr{C}$, we define its spanning interval $S(\mathscr{C})$ as

$$S(\mathscr{C}) := \{m_{i+1/2} | B_{i+1/2} \cap \mathscr{C} \neq \varnothing\}. \tag{5.1}$$

From this we define its associated vertex set as $V(S(\mathscr{C}))$ as the set of all vertices adjacent to the midpoints in $S(\mathscr{C})$.

We can think of the sets $S(\mathscr{C})$ and $V(S(\mathscr{C}))$ as a convex envelopes of $\mathscr{C}$. We next define the notion of a *wrapping* component:

Definition 5.2 Given a connected component $\mathscr{C}$ in $C_n(t, \mathbf{q})$, we say that $\mathscr{C}$ is *wrapping* if $S(\mathscr{C}) = (m_{i+1/2})_{i\in\mathbb{Z}/n\mathbb{Z}}$.

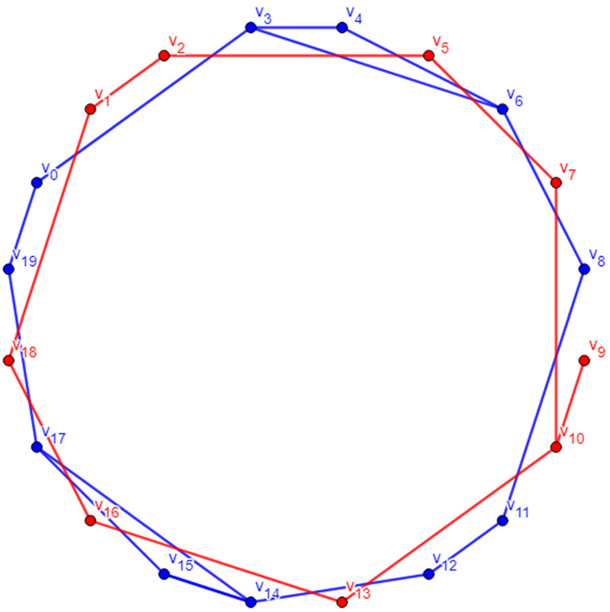

Fig. 2 Example of a graph with $n = 20$, $D = 3$ with two components $\mathscr{C}_1$ (red) and $\mathscr{C}_2$ (blue), both wrapping

Intuitively, a component $\mathscr{C}$ is wrapping if it is possible to complete an entire lap of $\mathbb{Z}/n\mathbb{Z}$ moving along the edges of $\mathscr{C}$ (see Fig. 2). It follows from the definition that, if $\mathscr{C}$ is not wrapping, $S(\mathscr{C})$ and $V(S(\mathscr{C}))$ are intervals on the circle $[n]$. The two extremal vertices $v_i, v_j \in V(S(\mathscr{C}))$ belong to $\mathscr{C}$ and the two nearest neighbour edges $\{v_{i-1}, v_i\}$ and $\{v_j, v_{j+1}\}$ are both vacant.

We next define the notion of *intertwining* components

Definition 5.3 We say that two components $\mathscr{C}_1, \mathscr{C}_2$ *intertwine* if at least one of the two is not wrapping and both $\mathscr{C}_1 \cap V(S(\mathscr{C}_2)) \neq \varnothing$ and $\mathscr{C}_2 \cap V(S(\mathscr{C}_1)) \neq \varnothing$.

Intuitively two components intertwine if there is a sector of $\mathbb{Z}/n\mathbb{Z}$ where vertices of the two components keep alternating (see Fig. 3).

From this, we can characterize 3 ways in which a disconnection can happen without the existence of either isolated vertices or cut edges:

- W_1={There are $\mathscr{C}_1, \mathscr{C}_2$ which are both *wrapping*}.
- W_2={There is $\mathscr{C}$ such that $|\mathscr{C}| \geq 2$, $\max_{v_i, v_j \in \mathscr{C}} d(i, j) < D$}.
- W_3={There are $\mathscr{C}_1, \mathscr{C}_2$ which are *intertwining*}.

We explain this better in the following proposition:

Proposition 5.4 *Consider the graph $C_n(t, \mathbf{q})$. Assume that $Y_n = 0$, $Z_n \leq 1$ and $C_n(t, \mathbf{q}) \in W_1^c \cap W_2^c \cap W_3^c$. Then $C_n(t, \mathbf{q})$ is connected.*

Proof We prove this by contradiction. Assume $C_n(t, \mathbf{q})$ is the union of $k > 1$ multiple connected components $(\mathscr{C}_i)_{i \leq k}$, but all the assumptions hold. If $V(S(\mathscr{C}_i))) \cap V(S(\mathscr{C}_j)) = \varnothing$ for all $i, j \leq k$, then all the midpoints $m_{i+1/2}$ such that v_i and

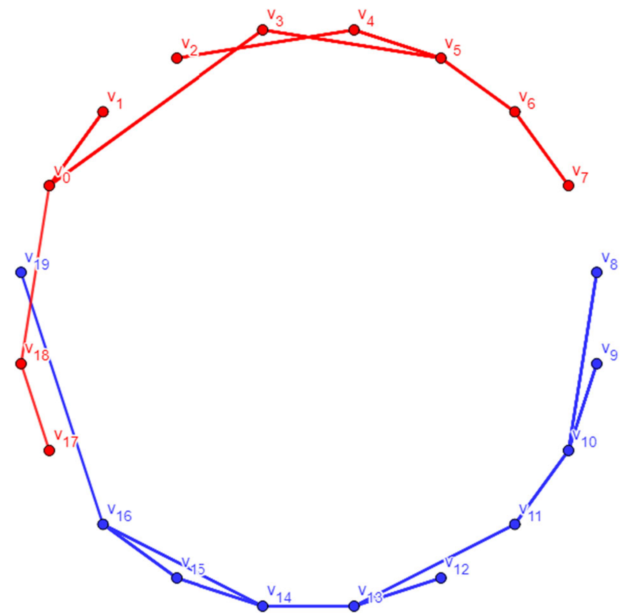

Fig. 3 Example of a graph with $n = 20$, $D = 3$ with two components $\mathscr{C}_1$ (blue) and $\mathscr{C}_2$ (red) which intertwine

v_{i+1} are in different spanning intervals identify cut edges and thus $Z_n \geq 2$. If there exists i, j such that $V(S(\mathscr{C}_i))) \cap V(S(\mathscr{C}_j)) \neq \varnothing$, then either $\mathscr{C}_i, \mathscr{C}_j$ are both wrapping (so that $C_n(t, \mathbf{q}) \in W_1$), or $\mathscr{C}_i, \mathscr{C}_j$ are intertwining (so that $C_n(t, \mathbf{q}) \in W_3$), or $V(S(\mathscr{C}_i))) \cap V(S(\mathscr{C}_j)) \neq \varnothing$ but $\mathscr{C}_i \cap V(S(\mathscr{C}_j)) = \varnothing$ (or vice versa). For the latter to happen, it is necessary that there is at least one edge of $\mathscr{C}_j$ that bridges over all midpoints in $S(\mathscr{C}_i)$ (see Fig. 4), and for this to be possible is necessary that $|\mathscr{C}_i| \leq |V(S(\mathscr{C}_i))| < D$. Consequently either $|\mathscr{C}_i| = 1$, and then $Y_n > 0$, or $|\mathscr{C}_i| \geq 2$, and then $C_n(t, \mathbf{q}) \in W_2$, thus we have reached a contradiction. $\square$

We have now to prove that the probability of each of these events converges to 0 as $n \to \infty$.

Lemma 5.5 *Consider the graph $C_n(t, \mathbf{q})$ defined above, with $t = t(n) = \alpha \log n$, $\alpha > c_0(\mathbf{q})$. Then for every $\mathbf{q} \in \mathbb{R}_+^D$*

$$\lim_{n \to \infty} \mathbb{P}(W_1) = 0. \tag{5.2}$$

Proof Consider if there are two wrapping components $\mathscr{C}_1$, $\mathscr{C}_2$ that means that every midpoint $m_{i+1/2}$ is bridged by both an edge of $\mathscr{C}_1$ and an edge of $\mathscr{C}_2$. This means that among the first $D-1$ vertices to the left of $m_{i+1/2}$ there must be both a vertex $v_i \in \mathscr{C}_1$ and a vertex $v_j \in \mathscr{C}_2$, and thus, there must be at least one vacant nearest-neighbour edge between them. The probability of the event W_1 that there are two components that both wrap around the circle is thus bounded from above by that of the event that

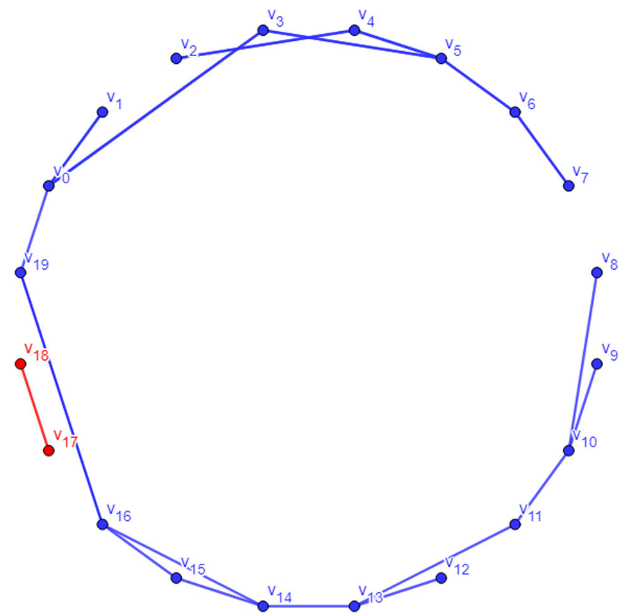

Fig. 4 Example of a graph with $n = 20$, $D = 3$ with two components such that W_2 happens. Cluster $\mathscr{C}$ (red) satisfies $|\mathscr{C}| = 2 > 1$, $\max_{v_i, v_j \in \mathscr{C}} d(i, j) = 1 < D$. In this case $v_l = v_{18}$, $v_k = v_{17}$

there are at least n/D vacant nearest-neighbour edges, that is, by the first moment method,

$$\lim_{n\to\infty} \mathbb{P}(W_1) \leq \lim_{n\to\infty} \frac{n e^{-td_1}}{nD} = \lim_{n\to\infty} \frac{n^{-\alpha d_1}}{D} = 0. \tag{5.3}$$

□

Lemma 5.6 *Consider the graph $C_n(t, \mathbf{q})$ defined above, with $t = t(n) = \alpha \log n$, $\alpha > c_0(\mathbf{q})$. Then for every $\mathbf{q} \in \mathbb{R}_+^D$*

$$\lim_{n\to\infty} \mathbb{P}(W_2) = 0. \tag{5.4}$$

Proof Let us assume that W_2 happens and pick a connected component $\mathscr{C}$ such that $1 < |\mathscr{C}|$, $\max_{v_i, v_j \in \mathscr{C}} d(i, j) < D$. Then $\mathscr{C}$ has a unique couple of vertices

$$(v_k, v_l) = \arg \max_{v_i, v_j \in \mathscr{C}} d(i, j), \tag{5.5}$$

and, assuming without loss of generality that v_l is $\max_{v_i, v_j \in \mathscr{C}} d(i, j)$ to the right of v_k by definition of W_2 all the edges which have v_k as left end-vertex and all those that have v_l as right end-vertex are vacant. We further know that there are $(D-1)n$ possible couples (v_k, v_l). So, we have

$$W_2 \subseteq \bigcup_{(v_k,v_l):(l-k)_{\mathrm{mod}\,n}<D} I_{v_k,v_l}. \tag{5.6}$$

$$I_{v_k,v_l} = \{e_{kk'} \text{ is vacant } \forall k' \in \{k-D,\ldots k-1\}\} \cap \{e_{ll'} \text{ is vacant } \forall l' \in \{l+1,\ldots l+D\}\}. \tag{5.7}$$

We write, for every $(v_k, v_l) : (l-k)_{\mathrm{mod}\,n} < D$, noting that the two events in the right-hand side of (5.7) are independent for n large enough,

$$\mathbb{P}(I_{v_k,v_l}) = \Big(\prod_{d=1}^{D}(1-p_d)\Big)^2 = \exp\Big\{-t\sum_{d=1}^{D}2q_d\Big\} = n^{-\alpha c_{is}(\mathbf{q})}. \tag{5.8}$$

As there are $(D-1)n$ valid choices for $(v_k, v_l) : (l-k)_{\mathrm{mod}\,n} < D$, we can bound, using (5.6), and, recalling that $c_{is}(\mathbf{q}) \geq c_0(\mathbf{q})$

$$\lim_{n\to\infty} \mathbb{P}(W_2) \leq \lim_{n\to\infty} (D-1)n^{1-\alpha c_{is}(\mathbf{q})} = 0. \tag{5.9}$$

□

Finally, we get to bound the probability of the event W_3:

Lemma 5.7 *Consider the graph $C_n(t,\mathbf{q})$ defined above, with $t = t(n) = \alpha \log n$, $\alpha \geq c_0(\mathbf{q}) \log n$, then, for every $\mathbf{q} \in \mathbb{R}_+^D$*

$$\lim_{n\to\infty} \mathbb{P}(W_3) = 0. \tag{5.10}$$

Proof To bound the probability of W_3, we start noting a few necessary events that need to happen for the existence of two intertwining components. Let us consider the two components $\mathscr{C}_1$ and $\mathscr{C}_2$. By definition the set $V(S(\mathscr{C}_1) \cap V(S(\mathscr{C}_2)$ contains at least a vertex of $\mathscr{C}_1$ and one of $\mathscr{C}_2$. Moreover, each midpoint in $S(\mathscr{C}_1) \cap S(\mathscr{C}_2)$ is bridged by both an edge of $\mathscr{C}_1$ and an edge of $\mathscr{C}_2$. This means that among the D vertices to the left of $m_{i+1/2}$ there must be both a vertex $v_i \in \mathscr{C}_1$ and a vertex $v_j \in \mathscr{C}_2$, and thus, there must be at least one vacant nearest-neighbour edge between them. Consequently, the set $S(\mathscr{C}_1) \cap S(\mathscr{C}_2)$ has to contain at least on vacant nearest-neighbour edge, and cannot contain D consecutive present nearest-neighbour edges. Finally, given extremal vertices v_i to the left and v_j to the right of $V(S(\mathscr{C}_1) \cap V(S(\mathscr{C}_2)$ we know that the edges $\{v_i, v_i - h\}$ and $\{v_j, v_j + h\}$ are vacant for all $h \leq D$. Given an edge e we call $r(e)$ its right end-vertex and $l(e)$ its left end-vertex. For W_3 to happen, there must be a sequence $\mathbf{E} = \{e_1, \ldots e_k\}$ of $k \geq 3$ edges such that fore each $i < k$, $l(e_{i+1}) = r(e_i) + h$ for some $h \leq D$ and

$$I_{\mathbf{E}} = \{\forall i \in \{2,\ldots,k-1\}, e_i \text{ is vacant} \cap \{\forall h \leq D, \{l(e_1), l(e_1)-h\}, \{r(e_k), r(e_k)+h\} \text{ are vacant}\}. \tag{5.11}$$

We note that $\mathbb{P}(W_3) \leq \sum_{\mathbf{E}\subset[n]} \mathbb{P}(I_{\mathbf{E}})$. For every $k \geq 3$, there are at most nD^{k-1} possible valid sequences $\mathbf{E}$ of length k, as once we have fixed e_1, every subsequent e_j

has to be within distance D to the right of e_{j-1}. From this we compute that

$$\begin{aligned}\mathbb{P}(I_{\mathbf{E}}) &= \exp\Big\{-c_0(\mathbf{q})\log n\Big((|\mathbf{E}|-2)q_1 + \sum_{d=1}^{D} 2q_h\Big)\Big\} \\ &= \exp\Big\{-(c_0(\mathbf{q})/c_{is}(\mathbf{q}))\log n - (|\mathbf{E}|-2)q_1c_0(\mathbf{q})\log n\Big\} \\ &\geq n^{-1}n^{-(|\mathbf{E}|-2)q_1c_0(\mathbf{q})}. \end{aligned} \tag{5.12}$$

So we can write

$$\begin{aligned}\mathbb{P}(W_3) \leq \sum_{\mathbf{E}\subset[n]} \mathbb{P}(I_{\mathbf{E}}) \leq \sum_{\mathbf{E}\subset[n]} \mathbb{P}(V_{\mathbf{E}}^{(1)} \cap V_{\mathbf{E}}^{(2)}) &\leq \sum_{k\geq 3} n^{-1}n^{-(|\mathbf{E}|-2)q_1c_0(\mathbf{q})}nD^{k-1} \\ &= \sum_{k\geq 3} n^{-(|\mathbf{E}|-2)q_1c_0(\mathbf{q})}D^{k-1}, \end{aligned} \tag{5.13}$$

so that

$$\lim_{n\to\infty}\mathbb{P}(W_3) \leq \lim_{n\to\infty}\sum_{k\geq 3} n^{-(|\mathbf{E}|-2)q_1c_0(\mathbf{q})}D^{k-1} = 0. \tag{5.14}$$

□

6 Proof of Theorem 2.1

We can now prove the main theorem by combining all the different lemmas we have proved on the probabilities of various disconnecting events:

Proof of Theorem 2.1 We have proved in Proposition 3.1 that if $\alpha < 1/c_{is}(\mathbf{q})$ then $\lim_{n\to\infty}\mathbb{P}(Y_n \geq 1) = 1$ and in Proposition 4.1 that if $\alpha < 1/c_{cut}(\mathbf{q})$ then $\lim_{n\to\infty}\mathbb{P}(Z_n \geq 2) = 1$, so

$$\begin{aligned}&\alpha < \max\{1/c_{is}(\mathbf{q}), 1/c_{cut}(\mathbf{q})\} = 1/c_0(\mathbf{q}) \Longrightarrow \\ &\qquad \lim_{n\to\infty}\mathbb{P}(C_n(t,\mathbf{q}) \text{ is connected}) \leq \lim_{n\to\infty}\mathbb{P}(\{Y_n = 0\}\cap\{Z_n \leq 1\}) = 0. \end{aligned} \tag{6.1}$$

On the other hand, from the same Propositions 3.1 and 4.1 we know that if $\alpha > \max\{1/c_{is}(\mathbf{q}), 1/c_{cut}(\mathbf{q})\} = 1/c_0(\mathbf{q})$, w.h.p. there are neither isolated vertices nor cut edges. By Proposition 5.4, using Lemmas 5.5, 5.6 and 5.7 we can then exclude w.h.p. the existence of all other disconnecting events so we can write

$$\mathbb{P}(C_n(t,\mathbf{q}) \text{ is connected}) \geq 1 - \mathbb{P}(Y_n \geq 1) - \mathbb{P}(Z_n \geq 2) - \mathbb{P}(W_1) - \mathbb{P}(W_2) - \mathbb{P}(W_3), \tag{6.2}$$

so that

$$\lim_{n\to\infty}\mathbb{P}(C_n(t,\mathbf{q}) \text{ is connected}) = 1. \tag{6.3}$$

□

7 Conclusion

We have studied the evolution and the structure of bounded-range finite-volume $1D$ random network models, highlighting how the way they reach full connectivity depends heavily on the specific structure of the connectivity probability, as the last obstruction to connectivity can be either an isolated vertex or two cut edges that split the network into two separate macroscopic components. Connectivity problems are significant in investigating the robustness of networks, and in our model, we thus find two different points of failure for bounded-range $1D$ networks, which are of interest in the modelling of VANETs. Our model depends only on a finite number of parameters, represented by the sequence $\mathbf{q}$, which are easy to fit to real-world data, as they are just the probability that units within a certain distance are directly connected. This can make it useful to approximate real one-dimensional systems and use the estimated parameters $\tilde{\mathbf{q}}$ to get an idea of which kinds of failures (local disconnections of individual units or global split in the network) are more likely to occur by comparing the values of $c_{cut}(\tilde{\mathbf{q}})$ and $c_{is}(\tilde{\mathbf{q}})$. We are still aware of the relatively simplistic nature of this model and of the utility of investigating more complex structures and dynamics, such as processes in which edges and vertices can appear and disappear at random, and in which long-range connections are possible, even if unlikely.

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Declarations

Conflict of interest Lorenzo Federico declares that he has no Conflict of interest.

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