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A minimal and non-alternative realisation of the Cayley plane

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Abstract
The compact 16-dimensional Moufang plane, also known as the Cayley plane, has traditionally been defined through the lens of octonionic geometry. In this study, we present a novel approach, demonstrating that the Cayley plane can be defined in an equally clean, straightforward and more economic way using two different division and composition algebras: the paraoctonions and the Okubo algebra. The result is quite surprising since paraoctonions and Okubo algebra possess a weaker algebraic structure than the octonions, since they are non-alternative and do not satisfy the Moufang identities. Intriguingly, the real Okubo algebra has SU (3) as automorphism group, which is a classical Lie group, while octonions and paraoctonions have an exceptional Lie group of type G_2. This is remarkable, given that the projective plane defined over the real Okubo algebra is nevertheless isomorphic and isometric to the octonionic projective plane which is at the very heart of the geometric realisations of all types of exceptional Lie groups. Despite its historical ties with octonionic geometry, our research underscores the real Okubo algebra as the weakest algebraic structure allowing the definition of the compact 16-dimensional Moufang plane.

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Introduction

A Moufang plane is a projective plane where every line is a translation line or, alternatively, where the "little Desargues theorem" holds (see in Sect. 6). Among the various characteristics of Moufang planes, a notable one is their dimensionality. Specifically, it is well-known that all compact, connected Moufang planes are of dimension 2, 4, 8 and 16 and isomorphic to precisely the projective planes over the Hurwitz division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. Of all these planes, the 16-dimensional Moufang plane stands out due to the historical obstacles in its definition arising from the lack of associativity of the octonions $\mathbb{O}$. This definitional challenge sparked significant interest in mathematical research during the early 20th century, culminating in one of the most fascinating interplays between projective geometry, algebra, and differential geometry. Indeed, one of the most remarkable achievements of the resulting mathematical research activity, mainly due to Cartan [8], Jordan, Wigner and von Neumann [43] and Freudenthal [30–33], is an interesting three-fold description of these planes: as a completion of the affine plane $\mathscr{A}^2(\mathbb{K})$, for every $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$; as the rank-1 idempotent elements of the rank-three Jordan algebra $\mathfrak{J}_3(\mathbb{K})$; as a coset manifold with a specific isometry and isotropy group. Furthermore, the investigation of octonionic geometry, particularly the study of the octonionic projective plane $\mathbb{O}P^2$, unraveled a deep connection between octonions and exceptional Lie groups [30–34, 60, 75, 76]. This connection, which was first envisaged by Cartan[9] and then explored by Chevalley and Schafer [10], is so deep that every known realization of compact exceptional Lie groups somehow involves the octonions $\mathbb{O}$ in one form or another [78]. Notably, each of these realizations of exceptional Lie groups has a geometrical aspect in which the 16-dimensional Moufang plane plays a pivotal role. Indeed, following Freudenthal [30–34], one can obtain all exceptional Lie groups of type F_4, E_6, E_7 and E_8 as transformation groups of the 16-dimensional Moufang plane preserving the features of elliptic geometry, projective geometry, symplectic geometry and metasymplectic geometry respectively [45].

Historically, the definition of the compact 16-dimensional Moufang plane arose out of octonionic geometry. However, in this work we show that this plane can be defined in an equally clean, straightforward and more minimal way by means of two different division composition algebras endowed with less algebraic structure than the octonions, and that the Moufang identities are not satisfied, historically associated with the Moufang property of the plane.

Clearly, in order to define a 16-dimensional plane that satisfies the affine and projective axioms of incidence geometry, an 8-dimensional division algebra is necessary. Hurwitz theorem [40] states the existence of only one 8-dimensional division composition algebra with a unit element, i.e. the algebra of octonions $\mathbb{O}$. Yet, when non-unital algebras are considered, three 8-dimensional division composition algebras emerge [20]: the aforementioned octonions $\mathbb{O}$, the para-octonions $p\mathbb{O}$ (not to be confused with the split-octonions that are not a division algebra) and the real Okubo algebra $\mathcal{O}$.

All three 8-dimensional algebras, being division and composition, allow independent and self-contained definitions of an affine and projective plane over them. Quite unexpectedly, despite the three different algebraic origins, the three definitions give rise to the same incidence plane: the compact 16-dimensional Moufang plane. This result is quite surprising because the three algebras, though deeply related, display very different properties. For instance, while octonions have a unit element and paraoctonions have a paraunit, the Okubo algebra merely contains idempotent elements. These differences apparently show up into the projective planes defined over these algebras: e.g., as a consequence of not having an identity element, the points on the Okubo plane $(0,0)$, (x,x) and (y,y) are not all three incident to the same Okubo line, nor does there exists an Okubo collineation that switches coordinates, i.e. $(x,y) \longrightarrow (y,x)$, as one has in octonionic case. Despite these apparent differences, in Sect. 5 we show

Table 1 Synoptic table of the algebraic properties of octonions $\mathbb{O}$, paraoctonions $p\mathbb{O}$ and the real Okubo algebra $\mathcal{O}$

Property	$\mathbb{O}$	$p\mathbb{O}$	$\mathcal{O}$
Unital	Yes	No	No
Paraunital	Yes	Yes	No
Alternative	Yes	No	No
Flexible	Yes	Yes	Yes
Composition	Yes	Yes	Yes
Automorphism	G_2	G_2	$\mathrm{SU}(3)$

that the projective planes, obtained directly from their corresponding foundational algebras, are all isomorphic and even all isometric one another.

The result is remarkable in itself. However, since the 16-dimensional Moufang plane is so deeply related with exceptional Jordan algebras, exceptional Lie Groups and symmetric spaces, it also paves the way for a novel, more minimal algebraic realization of these ubiquitous mathematical objects. A synoptic summary of the algebraic properties of octonions $\mathbb{O}$, paraoctonions $p\mathbb{O}$ and of the real Okubo algebra $\mathcal{O}$ is summarized in Table 1. It is worth noting that the minimal algebraic structure between such three algebras is the Okubo algebra $\mathcal{O}$, which is neither unital, nor para-unital; it is non-alternative and has the smallest automorphism group, i.e. $\mathrm{SU}(3)$ which has dimension 8 compared to G_2 that is a 14-dimensional group. Both paraoctonions $p\mathbb{O}$ and the real Okubo algebra $\mathcal{O}$ are non-alternative, but flexible algebras. Their relation to the Moufang plane is thus intriguing, because, notoriously, Moufang planes are associated to Moufang identities, that in turn imply the alternativity of the underlying algebra [39, 48]. In fact, all this does not give rise to any contradiction, since both the Okubo and paraoctonionic projective planes can be coordinatised by an alternative algebra, i.e. the octonions, through a non-linear planar ternary field as we show in Sect. 6.

An even more striking observation is that, while the octonions possess an automorphism group that is an exceptional group, the automorphism Lie group of the real Okubo algebra is not exceptional, nor has any immediate relation to G_2 itself. Nevertheless, the projective plane over the Okubo algebra gives rise to a geometric realisation of all types of exceptional Lie groups G_2, F_4, E_6, E_7 and E_8 as the transformation group respectively preserving: the non-degenerate quadrangles of the plane (type G_2); the distances of the plane (type F_4); the usual incidence relations between line and points (type E_6); extended incidence relations according to symplectic and metasymplectic geometry (type E_7 and E_8, for this last part see Freudenthal work [34, Sec. 4.13]). It is well known that all compact exceptional Lie groups have $\mathrm{SU}(3)$ as a subgroup, this work points out how the presence of a subgroup $\mathrm{SU}(3)$ is related with an Okubo structure underlying the 16-dimensional Moufang plane.

It is here worth recalling (see e.g. [72, 73]) that Lie groups of type E_6 are largely studied and are still viable candidates for GUT theories and that the real Okubo algebra was discovered by Susumo Okubo in his investigations on $\mathrm{SU}(3)$ as the gauge group for QCD [55]. Thus, we expect the Okubo formulation of the Cayley plane to find a physical application as a concrete alternative to its octonionic realisation and to the octonionic formulation of the rank-3 exceptional Jordan algebra, also known as Albert algebra. Additionally, it is known that M-theory may display an hidded Cayley-Moufang fibration [65]. Here it is worth noting that variations in the foundational algebra of this plane could potentially lead to novel physical theories.

The present work is thus structured as follows. In Sect. 1 we review the three algebras we are going to use: octonions $\mathbb{O}$, paraoctonions $p\mathbb{O}$ and the real Okubo algebra $\mathcal{O}$. In Sect. 3 we define the three affine and projective planes. Since the construction is formally very similar we develop only the details of the construction of Okubo affine and projective plane, pointing out the differences occurring in the other algebras. The main result is in Sect. 5 where we present the isomorphism between the three planes. Finally, in Sect. 6 we discuss our findings and introduce a software tool that facilitates direct and numerical verification of calculations involving octonionic, para-octonionic, and Okubo computations. This tool has been made publicly available and can be accessed on our GitHub repository at https://github.com/DCorradetti/OkuboAlgebras.

1 Composition algebras

Composition algebras are algebras endowed with a norm that enjoys the multiplicative property, i.e. $n(x \cdot y) = n(x)n(y)$. Composition algebras with multiplicative identity are called Hurwitz algebras and are fully classified [20]. On the other hand, composition algebras without multiplicative identity but with associative norm were discovered by Petersson [57] and indipendently by Okubo [49]; they are now called symmetric composition algebras [44] and are completely classified in para-Hurwitz and Okubo algebras [20]. Para-Hurwitz algebras are non-unital composition algebras strictly related to their unital companion, i.e. the corresponding Hurwitz algebra, while on the other hand Okubo algebras are somewhat more unique in feature appearing only as 8-dimensional algebras and with some very peculiar characteristics that distinguish them from both Hurwitz and para-Hurwitz algebras. It is worth noting that while it is possible to define an Okubo algebra over any field, here we will be focusing on the Okubo algebra over $\mathbb{R}$, which is a division composition algebra.

In this section we review some useful notions about composition algebras. Then we focus on Hurwitz algebras and, subsequently, we enter into the realm of symmetric composition algebras, specifically highlighting para-Hurwitz and Petersson algebras that in fact exhaust all algebras of this family. Even if this section is made of known results, we thought it might be worthwhile to collect them in a few pages of review content given their paramount importance in the understanding of the algebraic context of the subsequent sections.

1.1 Composition algebras

An *algebra*, denoted by A, is a vector space over a field $\mathbb{F}$ equipped with a bilinear multiplication. For our discussion, we will restrict our attention to algebras of finite dimension and the field $\mathbb{F}$ will be taken to be either the field of real $\mathbb{R}$ or complex numbers $\mathbb{C}$. The specific properties of the multiplication operation in an algebra lead to various classifications. Specifically, an algebra A is said to be *commutative* if $x \cdot y = y \cdot x$ for every $x, y \in A$; is *associative* if satisfies $x \cdot (y \cdot z) = (x \cdot y) \cdot z$; is *alternative* if $x \cdot (y \cdot y) = (x \cdot y) \cdot y$; and finally, *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$. It is worth noting that the last three proprieties can be seen as successive weakinings of associativity, i.e.

$$\text{associative} \Rightarrow \text{alternative} \Rightarrow \text{flexible}. \tag{1.1}$$

This observation stems from a nontrivial theorem proved by Artin (see [66]) who showed that all alternative algebras are flexible.

Since A must be a group with respect to addition, every algebra has a zero element $0 \in A$. Furthermore, if the algebra does not have zero divisors, it is referred to as a *division* algebra, i.e. an algebra for which $x \cdot y = 0$ implies $x = 0$ or $y = 0$. While the zero element is a universal feature in any algebra, the algebra is termed *unital* if there exists an element $1 \in X$ such that $1 \cdot x = x \cdot 1 = x$ for all $x \in A$.

Consider an algebra A. Then a quadratic form n on A over the field $\mathbb{F}$, i.e. a bilinear form of the type $n(x, x)$, is called *norm* and its polarization is given by

$$\langle x, y \rangle = n\left(x + y\right) - n\left(x\right) - n\left(y\right), \tag{1.2}$$

so that the norm can be explicitly given as

$$n\left(x\right) = \frac{1}{2} \langle x, x \rangle, \tag{1.3}$$

for every $x \in A$. An algebra A with a non-degenerate norm n that satisfies the following multiplicative property, i.e.

$$n\left(x \cdot y\right) = n\left(x\right) n\left(y\right), \tag{1.4}$$

for every $x, y \in A$, is called a *composition* algebra and is denoted with the triple $(A, \cdot, n)$ or simply as A if there are no reason for ambiguity.

Given a composition algebra A, applying equation (1.2) to the multiplicative property of the norm expressed in (1.4), we find that

$$\langle x \cdot y, x \cdot z \rangle = n\left(x\right) \langle y, z \rangle, \tag{1.5}$$

for every $x, y, z \in A$, which is an useful identity to be aware of.

1.2 Unital composition algebras

Composition algebras that possess a unit element are called *Hurwitz algebras*. The interplay between the multiplicative property of the norm in (1.4) and the existence of a unit element, is full of interesting implications. Indeed, every Hurwitz algebra is endowed with an order-two antiautomorphism called *conjugation*, defined by

$$\overline{x} = \langle x, 1 \rangle 1 - x. \tag{1.6}$$

Table 2 *On the left,* we have summarized the algebraic properties, i.e. totally ordered (O), commutative (C), associative (A), alternative (Alt), flexible (F), of all Hurwitz algebras, namely $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ along with their split counterparts $\mathbb{C}_s$, $\mathbb{H}_s$, $\mathbb{O}_s$

Hurwitz	O.	C.	A.	Alt.	F.	p-Hurwitz	O.	C.	A.	Alt.	F.
$\mathbb{R}$	Yes	Yes	Yes	Yes	Yes	$p\mathbb{R} \cong \mathbb{R}$	Yes	Yes	Yes	Yes	Yes
$\mathbb{C}$, $\mathbb{C}_s$	No	Yes	Yes	Yes	Yes	$p\mathbb{C}$, $p\mathbb{C}_s$	No	Yes	No	No	Yes
$\mathbb{H}$,$\mathbb{H}_s$	No	No	Yes	Yes	Yes	$p\mathbb{H}$,$p\mathbb{H}_s$	No	No	No	No	Yes
$\mathbb{O}$,$\mathbb{O}_s$	No	No	No	Yes	Yes	$p\mathbb{O}$,$p\mathbb{O}_s$	No	No	No	No	Yes

On the right, we have summarized the algebraic properties of all para-Hurwitz algebras, namely $p\mathbb{R}$, $p\mathbb{C}$, $p\mathbb{H}$ and $p\mathbb{O}$ accompanied by their split counterparts $p\mathbb{C}_s$, $p\mathbb{H}_s$, $p\mathbb{O}_s$

The linearization of the norm, when paired with the composition, results in the notable relation $\langle x \cdot y, z\rangle = \langle y, \overline{x} \cdot z\rangle$, that imply that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$ and

$$x \cdot \overline{x} = n\,(x)\,1. \tag{1.7}$$

Moreover, from the existence of a unit element in a composition algebra we have that elements with unit norm form a goup and, even more strikingly, that the whole algebra must be alternative (for a proof see [20, Prop. 2.2]).

Equation (1.7) can be rephrased in the well-known *Hamilton-Cayley equation,* $x^2 - \langle x, 1\rangle\, x - n\,(x)\,1 = 0$, which holds true for every unital composition algebra. Finally, a relation that is crucial for the Veronesean representation of the projective plane over a unital composition algebras, is the following

$$x \cdot (\overline{x} \cdot y) = (x \cdot \overline{x}) \cdot y = n\,(x)\,y, \tag{1.8}$$

which has a nice analogue in the case of *symmetric composition* algebras that we discuss in Sect. 1.3.

A major theorem by Hurwitz proves that the only unital composition algebras over the reals are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ accompanied by their split counterparts $\mathbb{C}_s$, $\mathbb{H}_s$, $\mathbb{O}_s$ (see [20, 40], Cor. 2.12). Consequently, there are seven Hurwitz algebras, each having real dimensions of 1, 2, 4, or 8. Out of these, four are also division algebras, i.e. $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, while three are split algebras and thus have zero divisors, i.e. $\mathbb{C}_s$, $\mathbb{H}_s$, $\mathbb{O}_s$. The properties of such algebras are quite different one another. More specifically, $\mathbb{R}$ is also totally ordered, commutative and associative; $\mathbb{C}$ is just commutative and associative; $\mathbb{H}$ is only associative and, finally, $\mathbb{O}$ is only alternative.

As shown by Table 2 all properties of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are valid also for the split companions with the only difference that the latter are not division algebras and do have zero divisors. Generalizations of Hurwitz Theorem can be done over arbitrary fields (see [81, p. 32]) but for our purposes this will not be needed.

1.3 Symmetric composition algebras

We now turn our attention to a special class of composition algebras, i.e. symmetric composition algebras, that are not unital but exhibit many properties analogous of Hurwitz algebras. Composition algebras with associative norms (see below) were independently studied by Petersson [57], Okubo [55], and Faulkner [29]. In [53], Okubo-Osborn showed that over an algebraically closed field the only two types of symmetric composition algebras are para-Hurwitz algebras and Okubo algebras, but a final classification was done by Elduque and Myung [24, 25].

A symmetric composition algebra $(A, *, n)$ is a composition algebra wherein the norm is associative, i.e. satisfies the identity

$$\langle x * y, z\rangle = \langle x, y * z\rangle \,, \tag{1.9}$$

where $x, y, z \in A$ and $\langle x, y\rangle = n(x + y) - n(x) - n(y)$, as stated in (1.2).

From Eq. (1.9), we extract a significant attribute of symmetric composition algebras. More precisely, considering:

$$\langle (x * y) * x, z\rangle = \langle x * y, x * z\rangle = n(x)\langle y, z\rangle \,, \tag{1.10}$$

and given that $n(x + y) = n(x) + n(y) + \langle x, y\rangle$, we can deduce

$$n((x * y) * x - n(x) y) = 2n^2(x) n(y) - n(x)\langle x * y, x * y\rangle = 0. \tag{1.11}$$

Thus, since the norm n is non singular we have the following important proposition

Proposition 1 *Let $(A, *, n)$ be symmetric composition algebra then*

$$(x * y) * x = n(x) * y, \tag{1.12}$$

for every $x, y \in A$.

In the realm of Hurwitz algebras, and similarly for symmetric composition algebras, all automorphisms are isometries. Indeed, it sufficies to consider that a map $\varphi : A \longrightarrow A$ such that $\varphi(x * y) = \varphi(x) * \varphi(y)$,implies that $\varphi((x * y) * x) = n(x) * \varphi(y)$, on one side, while on the other hand, $(\varphi(x) * \varphi(y)) * \varphi(x) = n(\varphi(x)) * \varphi(y)$, so that it must be

$$n(\varphi(x)) = n(x)\,, \tag{1.13}$$

for every $x \in A$.

In fact, symmetric composition algebras are deeply intertwined with Hurwitz algebras. Indeed, given a symmetric composition algebra $(A, *, n)$ and a norm 1 element $a \in A$, we can utilize Kaplansky's trick to define a new product

$$x \cdot y = (a * x) * (y * a)\,, \tag{1.14}$$

for every $x, y \in A$, resulting in a new composition algebra $(A, \cdot, n)$. Now, consider the element $e = a * a$. Since (1.12) and $n(a) = 1$ we then have that

$$e \cdot x = (a * (a * a)) * (x * a) = x, \tag{1.15}$$

$$x \cdot e = (a * x) * ((a * a) * a) = x, \tag{1.16}$$

for every $x \in A$. Consequently, $(A, \cdot, n)$ is a unital composition algebra, or equivalently, a Hurwitz algebra. As a direct implication of the Hurwitz theorem, symmetric composition algebras can only have dimensions of 1, 2, 4, or 8.

1.3.1 Para-Hurwitz algebras

An important class of symmetric composition algebras is that of para-Hurwitz algebras. Given any Hurwitz algebra $(A, \cdot, n)$ a conjugation is naturally defined as

$$\overline{x} = \langle x, 1 \rangle\, 1 - x, \tag{1.17}$$

for every $x \in A$. Then, consider the new product

$$x \bullet y = \overline{x} \cdot \overline{y}, \tag{1.18}$$

for every $x, y \in A$. Since $n(x) = n(\overline{x})$ we have that

$$n(x \bullet y) = n(\overline{x} \cdot \overline{y}) = n(x)\, n(y), \tag{1.19}$$

and thus the algebra $(A, \bullet, n)$ is again a composition algebra. On the other hand $(A, \bullet, n)$ is not an unital algebra since

$$x \bullet 1 = 1 \bullet x = \overline{x}. \tag{1.20}$$

Moreover, the algebra is a symmetric composition algebra since it can be shown to satisfy

$$\langle x \bullet y, z \rangle = \langle x, y \bullet z \rangle, \tag{1.21}$$

and it is then called a *para-Hurwitz* algebra [20]. For every Hurwitz algebra, i.e. unital composition algebra, of dimension > 1 we have a para-Hurwitz algebra that is a symmetric composition algebra that we denote as $p\mathbb{C}, p\mathbb{C}_s, p\mathbb{H}, p\mathbb{H}_s$, $p\mathbb{O}$ and $p\mathbb{O}_s$ respectively. It is worth noting that all para-Hurwitz algebras are non-alternative algebras, since

$$x \bullet (x \bullet y) = \overline{x} \cdot \left(\overline{\overline{x} \cdot \overline{y}}\right) = \overline{x} \cdot (y \cdot x), \tag{1.22}$$

$$(x \bullet x) \bullet y = (x \cdot x) \cdot \overline{y}, \tag{1.23}$$

thus, in general, $x \bullet (x \bullet y) \neq (x \bullet x) \bullet y$. Nevertheless, by Proposition 1 they are flexible and more specifically

$$x * y * x = n(x) * y, \tag{1.24}$$

for every $x, y \in A$. Moreover, if the the Hurwitz algebra $(A, \cdot, n)$ is a division algebra, then also the para-Hurwitz $(A, *, n)$ defined from (1.18) is a division algebra. Algebraic properties of the Hurwitz algebras are summarized in Table 2.

1.3.2 Petersson algebras

A generalisation of para-Hurwitz algebras was presented by Petersson in [57]. Starting with a Hurwitz algebra $(A, \cdot, n)$, he introduced a new algebra $(A, *, n)$ such that

$$x * y = \tau\left(\overline{x}\right) \cdot \tau^2\left(\overline{x}\right), \tag{1.25}$$

where τ is an order-three automorphisms, i.e. $\tau^3 = \mathrm{id}$. The new algebra, typically denoted as A_τ, becomes a composition algebra that is non-unital. Moreover, Petersson demonstrated that over an algebraically closed field like $\mathbb{C}$, there exists a specific automorphism that results in a non-para-Hurwitz algebra. This new algebra is a symmetric composition algebra containing idempotent elements.

Petersson algebras are crucial in characterizing symmetric composition algebras since we have the following

Theorem 2 (Elduque-Perez [27, Th. 2.5]) *An algebra A is a symmetric composition algebra with an nonzero idempotent if and only if there exists a Hurwitz algebra H and an automorphism τ of H such that A is isomorphic to the algebra H_τ.*

2 Octonions, paraoctonions and the real Okubo algebra

In this section, we delve into the three division algebras of primary interest: the octonions $\mathbb{O}$, the paraoctonions $p\mathbb{O}$ and the real Okubo algebra $\mathcal{O}$. The main objective of this section is summarized in Table 3 that synoptically illustrates the relationships between the product of the three algebras. It's crucial to note that although it is possible to switch from one algebra to another by altering the product definition, none of these algebras is isomorphic one of the others: e.g. the octonions $\mathbb{O}$ are alternative and unital, para-octonions $p\mathbb{O}$ are nor alternative nor unital but do have a para-unit, while the Okubo algebra $\mathcal{O}$ is non-alternative and only has idempotents elements. It's also worth highlighting that the Okubo algebra $\mathcal{O}$ is the least structured among these algebras.

2.1 The algebra of octonions

The algebra of octonions $\mathbb{O}$ is the only division Hurwitz algebra with a dimension of eight. We define the composition algebra of octonion $(\mathbb{O}, \cdot, n)$ as the eight dimensional real vector space with basis $\{\mathrm{i}_0 = 1, \mathrm{i}_1, ..., \mathrm{i}_7\}$ with a bilinear product encoded through the *Fano plane* and explained in Fig. 1.

Given an element $x \in \mathbb{O}$ with decomposition

$$x = x_0 + \sum_{k=1}^{7} x_k \mathrm{i}_k, \tag{2.1}$$

the norm n is the obvious Euclidean one defined by

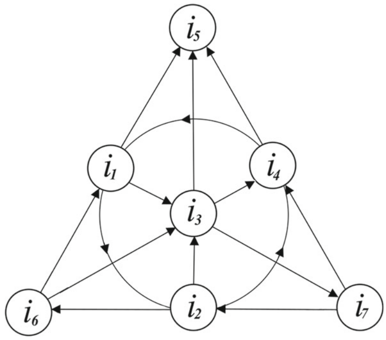

Fig. 1 Multiplication rules for octonions $\mathbb{O}$ as real vector space $\mathbb{R}^8$ in the basis $\{\mathrm{i}_0 = 1, \mathrm{i}_1, ..., \mathrm{i}_7\}$. Lines in the Fano plane identify associative triples of the product and the arrow indicates the sign (positive in the sense of the arrow and negative in the opposite sense). In addition to the previous rules it is intended that $\mathrm{i}_k^2 = -1$

$$n(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2, \tag{2.2}$$

for which the conjugation results

$$\overline{x} = x_0 - \sum_{k=1}^{7} x_k i_k, \tag{2.3}$$

and therefore

$$n(x) = \overline{x} \cdot x, \tag{2.4}$$

as it happens for every Hurwitz algebra. Then a look at (2.2) shows that $n(x) = 0$ if and only if $x = 0$ and thus the inverse of a non-zero element of the octonions is easily found as

$$x^{-1} = \frac{\overline{x}}{n(x)}. \tag{2.5}$$

Also, from (2.4) we have that the octonionic inner product is given by

$$\langle x, y \rangle = x\overline{y} + y\overline{x}, \tag{2.6}$$

so that $\langle x, x \rangle = 2n(x)$.

Straightforward calculations shows that the algebra of octonions is neither commutative nor associative, but it is alternative. But, since any two elements of an alternative algebra generate an associative subalgebra, it is then easy to see that $(\mathbb{O}, \cdot, n)$ is indeed an Hurwitz algebra since it is unital and

$$\begin{aligned} n(x \cdot y) &= (\overline{x \cdot y}) \cdot (x \cdot y) \\ &= (\overline{y} \cdot \overline{x}) \cdot (x \cdot y) \\ &= \overline{y} \cdot (\overline{x} \cdot x) \cdot y = n(x)\, n(y). \end{aligned} \tag{2.7}$$

Since the algebra is a composition algebra and any non-zero element has non-zero norm, i.e. $n(x) \neq 0$ then $(\mathbb{O}, \cdot, n)$ is also a division algebra since if $x \cdot y = 0$ then

$$n(x \cdot y) = n(x)\, n(y) = 0, \tag{2.8}$$

which implies that $x = 0$ or $y = 0$. Moreover, an important relation that will be used later on in the definition of the projective plane is the following consequence of alternativity, i.e.

$$\overline{x} \cdot (x \cdot y) = n(x)\, y, \tag{2.9}$$

for every $x, y \in \mathbb{O}$.

2.1.1 Moufang identities

While octonions are not a group under multiplication due to their lack of associativity, non-zero octonions form a *Moufang loop*, i.e. a loop that satisfy the following *Moufang identities*, i.e.

$$((x \cdot y) \cdot x) \cdot z = x \cdot (y \cdot (x \cdot z)), \tag{2.10}$$

$$((z \cdot x) \cdot y) \cdot x = z \cdot (x \cdot (y \cdot x)), \tag{2.11}$$

$$(x \cdot y) \cdot (z \cdot x) = x \cdot ((y \cdot z) \cdot x), \tag{2.12}$$

for every $x, y, z \in \mathbb{O}$. Moufang identities are particularly relevant since they are historically linked to geometrical properties of the Moufang plane (see Sect. 6). It is worth noting that any unital algebra satisfying Moufang identities is an alternative algebra. Indeed, setting $z = 1$ Moufang identities turn into the flexible identity, i.e.

$$(x \cdot y) \cdot x = x \cdot (y \cdot x), \tag{2.13}$$

while setting $y = 1$ we have the identity for the left and right alternativity, i.e.

$$(x \cdot x) \cdot z = x \cdot (x \cdot z), \tag{2.14}$$

$$(z \cdot x) \cdot x = z \cdot (x \cdot x). \tag{2.15}$$

Thus, non alternative algebras do not satisfy Moufang identities.

2.2 Okubo algebras

Symmetric composition algebras might have remained relatively unnoticed among algebraists had Petersson [57] not demonstrated that for every field $\mathbb{F}$ there exists a unique eight-dimensional algebra that is not a para-Hurwitz algebra. This result essentially broadened the reach of the Hurwitz classification theorem. On the other hand, Okubo algebras were independently developed by mathematical physicist Susumo Okubo in the course of his work on quarks and Gell-Mann matrices while pursuing an algebra that featured $\mathrm{SU}(3)$ as automorphism group instead of G_2 as in the case of Octonions[55]. Even more interestingly, Okubo discovered that such algebra is a division composition algebra and a deformation of its product would give back the octonions[49–51]. It was with more recent works [44], with the joint efforts of Osborn, Elduque and Myung [19, 24–26, 53, 54], that the context of Okubo algebras was fully elucidated.

Following [49] and [24], we define the real Okubo Algebra $\mathcal{O}$ as the set of three by three Hermitian traceless matrices over the complex numbers $\mathbb{C}$ with the following bilinear product

$$x * y = \mu \cdot xy + \overline{\mu} \cdot yx - \frac{1}{3}\mathrm{Tr}\,(xy)\,, \tag{2.16}$$

where $\mu = {}^{1}\!/\!{}_{6}\left(3 + \mathrm{i}\sqrt{3}\right)$ and the juxtaposition is the ordinary associative product between matrices. It is worth noting that (2.16) can be seen as a modification of the Jordanian product. Indeed, setting $\mu = {}^{1}\!/\!{}_{2}$ and negletting the last term, we retrieve the usual Jordan product over Hermitian traceless matrices, i.e.

$$x \circ y = \frac{1}{2}xy + \frac{1}{2}yx. \tag{2.17}$$

Nevertheless, Hermitian traceless matrices are not closed under such product, thus requiring the additional term $-{}^{1}\!/\!{}_{3}\mathrm{Tr}\,(xy)$ for the closure of the algebra. Indeed, setting in (2.16) $\mathrm{Im}\mu = 0$, one retrieves from the traceless part of the exceptional Jordan algebra $\mathfrak{J}_3\,(\mathbb{C})$, whose derivation Lie algebra is $\mathfrak{su}\,(3)$.

Analyzing (2.16), it becomes evident that the resulting algebra is neither unital, associative, nor alternative. Nonetheless, $\mathcal{O}$ is a *flexible* algebra, i.e.

$$x * (y * x) = (x * y) * x, \tag{2.18}$$

which will turn out to be an even more useful property than alternativity in the definition of the projective plane. Even though the Okubo algebra is not unital, it does have idempotents, i.e. $e * e = e$, such as

$$e = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{2.19}$$

that together with

$$\begin{aligned}
\mathrm{i}_1 &= \sqrt{3}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{i}_2 = \sqrt{3}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\mathrm{i}_3 &= \sqrt{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{i}_4 = \sqrt{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\mathrm{i}_5 &= \sqrt{3}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{i}_6 = \sqrt{3}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
\mathrm{i}_7 &= \sqrt{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
\end{aligned} \tag{2.20}$$

form a basis for $\mathcal{O}$ that has real dimension 8.[1] It is worth noting that the choice of the idempotent e as in (2.19) does not yield to any loss of generality for the subsequent development of our work since all idempotents are conjugate under the automorphism group (cfr. [19, Thm. 20]). The choice of this special basis is motivated on the fact that it will turn to be an orthonormal basis with respect to the norm in (2.21) and that through a special bijective map between Okubo algebra and octonions the elements of the basis $\{e, \mathrm{i}_1, ..., \mathrm{i}_7\}$ here defined will correspond to the octonionic one previously defined.

Let us consider the quadratic form n over Okubo algebra, given by

$$n(x) = \frac{1}{6}\mathrm{Tr}\left(x^2\right), \tag{2.21}$$

for every $x \in \mathcal{O}$. It is straightforward to see that this *norm* has signature $(8, 0)$, is associative and composition over the real Okubo algebra, i.e.

$$n(x * y) = n(x)\, n(y), \tag{2.22}$$

$$\langle x * y, z\rangle = \langle x, y * z\rangle, \tag{2.23}$$

where $\langle \cdot, \cdot \rangle$ is the *polar form* given by

$$\langle x, y\rangle = n(x + y) - n(x) - n(y). \tag{2.24}$$

Therefore, Okubo algebra is a *symmetric composition* algebra [44, Ch. VIII] and, thus, enjoying the notable relation

$$x * (y * x) = (x * y) * x = n(x)\, y. \tag{2.25}$$

For our purposes it will be of paramount importance to notice the following [52]

Proposition 3 *The Okubo Algebra is a division algebra.*

Proof Without any loss of generality, let us suppose that $d \neq 0$ is a left divisor of zero, i.e. $d * x = 0$, then

$$n(d * x) = n(d)\, n(x) = 0.$$

But, since the algebra is symmetric composition algebra, for the (2.25) we also have

$$(d * x) * d = 0 = n(d)\, x, \tag{2.26}$$

[1] Actually, the 8 matrices three by three (2.19)-(2.20) are, up to an overall factor $\sqrt{3}$, the Gell-Mann matrices. In particular, the idempotent (2.19) is $-\sqrt{3}$ times the eighth Gell-Mann matrix λ_8,with the first and third rows and columns exchanged.

and therefore $n(d) = 0$, i.e. $\mathrm{Tr}\left(d^2\right) = 0$. But, since the element d is of the form

$$d = \begin{pmatrix} \xi_1 & x_1 + \mathrm{i}y_1 & x_2 + \mathrm{i}y_2 \\ x_1 - \mathrm{i}y_1 & \xi_2 & x_3 + \mathrm{i}y_3 \\ x_2 - \mathrm{i}y_2 & x_3 - \mathrm{i}y_3 & -\xi_1 - \xi_2 \end{pmatrix}, \tag{2.27}$$

where $x_i, y_i, \xi_i \in \mathbb{R}$, the norm $n(d)$ is given by

$$n(d) = \frac{1}{3}\left(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 + \xi_1^2 + \xi_2^2 + \xi_1\xi_2\right), \tag{2.28}$$

which yields that $\mathrm{Tr}\left(d^2\right) \neq 0$ in case of $\xi_1, \xi_2 \in \mathbb{R}$ and $\xi_1, \xi_2 \neq 0$. □

Unfortunately, since $\mathcal{O}$ is not a unital algebra, an element x does not have an inverse. This implies that, concerning its product, the Okubo algebra is not a loop (as it was in the case of the octonions that were a Moufang loop) but only a quasigroup. Nevertheless, considering the existence of the idempotent e, and inspired by the identity

$$x * (e * x) = (x * e) * x = n(x)\, e,$$

we can define $(x)_L^{-1} = n(x)^{-1}(e * x)$ and $(x)_R^{-1} = n(x)^{-1}(x * e)$ so that, given a definite choice of the idempotent e, one has

$$(x)_L^{-1} * x = x * (x)_R^{-1} = e.$$

As an implication of the previous argument we have the following

Proposition 4 *An equation of the kind*

$$a * x = b, \quad \text{or} \quad x * a = b, \tag{2.29}$$

has a unique solution which is respectively given by

$$x = \frac{1}{n(a)} b * a, \quad \text{or} \quad x = \frac{1}{n(a)} a * b, \tag{2.30}$$

for every $a, b \in \mathcal{O}$, with $a \neq 0$.

Proof Let us consider the equation $a * x = b$. Since $\mathcal{O}$ is a division algebra and $a \neq 0$ we can multiply by a obtaining

$$(a * x) * a = b * a, \tag{2.31}$$

but since $(a * x) * a = n(a)\, x$ and $n(a) \in \mathbb{R}$, we then have $x = n(a)^{-1}\, b * a$. A similar argument is valid for the case of $x * a = b$. □

Although the above proposition is straightforward, it has profound geometrical implications, as it confirms the applicability of affine and projective axioms to planes over the Okubo algebra. This topic will be elaborated upon in subsequent sections.

2.3 Conjugation and the trivolution

In unital composition algebras, as noted earlier, there exists a canonical involution, an order-two antihomomorphism known as *conjugation*. This can be defined using the orthogonal projection of the unit element as

$$x \mapsto \overline{x} = \langle x, 1 \rangle \, 1 - x. \tag{2.32}$$

This canonical involution has the distinctive property of being an antihomomorphism with respect to the product, i.e., $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$, and the basic property with the norm of $x \cdot \overline{x} = n(x) \, 1$.

For non-unital composition algebras, the previous definition isn't applicable. However, if an idempotent element e is present in the algebra, one might be tempted to extend the previous definition

$$x \mapsto \widetilde{x} = \langle x, e \rangle \, e - x, \tag{2.33}$$

to investigate if similar properties remain valid.

In the case of a para-Hurwitz $p\mathbb{K}$ obtained from an Hurwitz algebra $(\mathbb{K}, \cdot, n)$ imposing the new product $x \bullet y = \overline{x} \cdot \overline{y}$,for every $x, y \in \mathbb{K}$ we have a special element, called para-unit, i.e. $1 \in p\mathbb{K}$ such that $1 \bullet x = \overline{x}$. Thus, we might want to have a look to the map L_1 given by left multiplication by the para-unit, i.e.

$$x \mapsto L_1(x) = 1 \bullet x. \tag{2.34}$$

Clearly, the same arguments apply to $R_1(x)$ since $x \bullet 1 = 1 \bullet x$. Indeed, we notice that $L_1^2(x) = x$ thus L_1 is an involution and, since

$$L_1(x \bullet y) = 1 \bullet (x \bullet y) = y \cdot x, \tag{2.35}$$

and

$$L_1(x) \bullet L_1(y) = \overline{x} \bullet \overline{y} = x \cdot y, \tag{2.36}$$

the map is also an anti-homomorphism, i.e. $L_1(x \bullet y) = L_1(y) \bullet L_1(x)$. Finally, since

$$x \bullet L_1(x) = x \bullet 1 \bullet x = n(x) \, 1. \tag{2.37}$$

Thus, the order-two anti-homomorphism $x \mapsto \widetilde{x} = L_1(x)$ realises for the para-Hurwitz algebra $p\mathbb{K}$ all the main features of the canonical involution or conjugation of the Hurwitz algebra $\mathbb{K}$.

Unfortunately, the situation within the Okubo algebra is less straightforward. Indeed, if we consider the idempotent e and define the map

$$x \mapsto \langle x, e \rangle \, e - x, \tag{2.38}$$

it exhibits order two but is neither a homomorphism nor an antihomomorphism. On the other hand, if we consider the maps

$$x \longrightarrow L_e(x) = e * x, \tag{2.39}$$
$$x \longrightarrow R_e(x) = x * e, \tag{2.40}$$

then we do have in both cases a nice relation with the norm, since

$$x * L_e(x) = n(x)\, e, \tag{2.41}$$
$$R_e(x) * x = n(x)\, e, \tag{2.42}$$

Yet, even if $R_e \circ L_e = \mathrm{id}$ holds true as in the para-Hurwitz case, neither L_e and R_e is an automorphism nor an antiautomorphism. On the other hand, if we generalise (2.32) with the following map

$$x \mapsto \langle x, e\rangle\, e - x * e, \tag{2.43}$$

we have indeed a special automorphism, that we call τ, which, nevertheless, is not of order two but of order three. Therefore, while it is not possible to have a involution over Okubo algebra that enjoys the same properties as the conjugation of Hurwitz algebras, it is possible to define something in a similar fashion such as an order-three automorphism τ, hereafter referred to as a *trivolution*, defined as

$$x \longrightarrow \tau(x) = \langle x, e\rangle\, e - x * e, \tag{2.44}$$

or, equivalently, as

$$x \longrightarrow \tau(x) = L_e(x)^2 = e * (e * x), \tag{2.45}$$
$$x \longrightarrow \tau^2(x) = R_e(x)^2 = (x * e) * e. \tag{2.46}$$

It is easy to see that the automorphism τ is of order 3 since, applying flexibility, we have $R_e^2 \circ L_e^2 = \mathrm{id}$. It is also worth noting the stunning analogy with the conjugation expressed for unital composition algebra in (2.32) and at the same time the analogy with the one expressed for para-Hurwitz algebras in (2.39).

Even more interesting, the order-three automorphism τ is also an order-three automorphism over the octonions $\mathbb{O}$. Indeed, if we consider the base given in (2.20) and set $e = \mathrm{i}_0$, we can define τ as the linear map given by

$$\begin{aligned}
\tau(\mathrm{i}_k) &= \mathrm{i}_k, k = 0, 1, 3, 7\\
\tau(\mathrm{i}_2) &= -\tfrac{1}{2}\left(\mathrm{i}_2 - \sqrt{3}\mathrm{i}_5\right),\\
\tau(\mathrm{i}_5) &= -\tfrac{1}{2}\left(\mathrm{i}_5 + \sqrt{3}\mathrm{i}_2\right),\\
\tau(\mathrm{i}_4) &= -\tfrac{1}{2}\left(\mathrm{i}_4 - \sqrt{3}\mathrm{i}_6\right),\\
\tau(\mathrm{i}_6) &= -\tfrac{1}{2}\left(\mathrm{i}_6 + \sqrt{3}\mathrm{i}_4\right),
\end{aligned} \tag{2.47}$$

This definition extends to an order-three homomorphism over octonions $\mathbb{O}$ once we consider $\{\mathrm{i}_0 = 1, \mathrm{i}_2, ..., \mathrm{i}_7\}$ as a basis for this algebra. It is interesting to note that in the octonions there are two complex planes, generated by $\{\mathrm{i}_2, \mathrm{i}_5\}$ and $\{\mathrm{i}_4, \mathrm{i}_6\}$ on which the automorphism τ acts as the cubic root of unity $\frac{1}{2}\left(1+\sqrt{3}\mathrm{i}\right)$. For completeness we give also the action of the inverse τ^{-1} over this basis, i.e.

$$\begin{aligned}
\tau^2(\mathrm{i}_k) &= \mathrm{i}_k, k = 0, 1, 3, 7\\
\tau^2(\mathrm{i}_2) &= -\tfrac{1}{2}\left(\mathrm{i}_2+\sqrt{3}\mathrm{i}_5\right),\\
\tau^2(\mathrm{i}_5) &= -\tfrac{1}{2}\left(\mathrm{i}_5-\sqrt{3}\mathrm{i}_2\right),\\
\tau^2(\mathrm{i}_4) &= -\tfrac{1}{2}\left(\mathrm{i}_4+\sqrt{3}\mathrm{i}_6\right),\\
\tau^2(\mathrm{i}_6) &= -\tfrac{1}{2}\left(\mathrm{i}_6-\sqrt{3}\mathrm{i}_4\right).
\end{aligned} \tag{2.48}$$

2.4 Okubo algebra, octonions and para-octonions

An important feature of the Okubo algebra $\mathcal{O}$ is its interplay with the algebra of octonions $\mathbb{O}$. Indeed, octonions and the Okubo algebra are linked one another in such a way that we can easily pass from one to the other simply changing the definition of the bilinear product over the vector space of the algebra. Let us consider the Kaplansky's trick we introduced earlier and let us define a new product over the Okubo algebra $\mathcal{O}$ as

$$x \cdot y = (e * x) * (y * e), \tag{2.49}$$

where $x, y \in \mathcal{O}$ and e is an idempotent of $\mathcal{O}$. Given that $e * e = e$ and $n(e) = 1$, the element e acts as a left and right identity, i.e.

$$x \cdot e = e * x * e = n(e)\, x = x, \tag{2.50}$$

$$e \cdot x = e * x * e = n(e)\, x = x. \tag{2.51}$$

Moreover, since the Okubo algebra is a composition algebra, the same norm n enjoys the following relation

$$n(x \cdot y) = n((e * x) * (y * e)) = n(x)\, n(y), \tag{2.52}$$

which means that $(\mathcal{O}, \cdot, n)$ is a unital composition algebra of real dimension 8. Since it is also a division algebra, then it must be isomorphic to that of octonions $\mathbb{O}$ as noted by Okubo himself [49, 51].

On the other hand, if we consider the order three automorphism of the octonions in (2.47), the Okubo algebra is then realised as a Petersson algebra from the octonions setting

$$x * y = \tau(\overline{x}) \cdot \tau^2(\overline{y}). \tag{2.53}$$

Table 3 In this table we see how to obtain the Okubo product $*$, the para-octonionic product $\bullet$ and the octonionic product $\cdot$ from Okubo algebra $(\mathcal{O}, *)$, para-octonions $(p\mathbb{O}, \bullet)$ and octonions $(\mathbb{O}, \cdot)$ respectively

Algebra	$(\mathcal{O}, *)$	$(p\mathbb{O}, \bullet)$	$(\mathbb{O}, \cdot)$
$x * y$	$x * y$	$\tau(x) \bullet \tau^2(y)$	$\tau(\overline{x}) \cdot \tau^2(\overline{y})$
$x \bullet y$	$\tau^2(x) * \tau(y)$	$x \bullet y$	$\overline{x} \cdot \overline{y}$
$x \cdot y$	$(e * x) * (y * e)$	$(\mathbf{1} \bullet x) \bullet (y \bullet \mathbf{1})$	$x \cdot y$

Note that (2.44) is formulated assuming the knowledge of the Okubo product. Reading the same maps as Okubo maps we then have the notable relation, i.e.

$$\overline{x} = R_e^3(x) = ((x * e) * e) * e, \tag{2.54}$$

$$\tau(x) = R_e^4(x) = (((x * e) * e) * e) * e, \tag{2.55}$$

so that, in fact, the two maps are linked one another, i.e.

$$\tau(x) = \overline{x} * e \tag{2.56}$$

$$\overline{x} = \tau(e * x). \tag{2.57}$$

While these maps are intertwined, it's important to highlight their distinct impacts on the algebra's structure. While τ is an automorphism for both Okubo algebra $\mathcal{O}$ and octonions $\mathbb{O}$, $\overline{x}$ do not respect the algebrical structure of the Okubo algebra $\mathcal{O}$, since it is not an automorphism nor an anti-automorphism with respect to the Okubo product, while it is an anti-homomorphism over octonions $\mathbb{O}$.

The scenario with para-octonions, $p\mathbb{O}$ is more straightforward. By definition, para-octonions are obtainable from octonions $\mathbb{O}$ through

$$x \bullet y = \overline{x} \cdot \overline{y}, \tag{2.58}$$

while, on the other hand, octonions $\mathbb{O}$ are obtainable from para-octonions $p\mathbb{O}$ through the aid of the para-unit $1 \in p\mathbb{O}$, such that

$$x \cdot y = (1 \bullet x) \bullet (y \bullet 1) \tag{2.59}$$

$$= \overline{x} \bullet \overline{y} = x \cdot y. \tag{2.60}$$

The new algebra $(p\mathbb{O}, \cdot, n)$ is again an eight-dimensional composition algebra which is also unital and division and thus, by the Hurwitz theorem, isomorphic to that of octonions $\mathbb{O}$. Moreover, since $\tau(\overline{x}) = \overline{\tau(x)}$, we also have that the Okubo algebra is obtainable from the para-Hurwitz algebra with the introduction of a Petersson-like product, i.e.

$$x * y = \tau(x) \bullet \tau^2(y). \tag{2.61}$$

We thus have that all algebras are obtainable one from the other as summarized in Table 3.

Nonetheless, it's vital to note that while transitioning from one algebra to another is feasible, these algebras are not isomorphic. For example, while the octonions $\mathbb{O}$ are alternative and unital, para-octonions $p\mathbb{O}$ are nor alternative nor unital but do have a para-unit. In contrast, the Okubo algebra $\mathcal{O}$ is non-alternative and only contains idempotent elements.

3 Affine and projective planes

An *incidence plane* P^2 is given by the triple $\{\mathscr{P}, \mathscr{L}, \mathscr{R}\}$ where $\mathscr{P}$ is the set of points of the plane, $\mathscr{L}$ is the set of lines and $\mathscr{R}$ are the incidence relations of poins and lines. The plane P^2 is called an *affine plane* if it satisfies the axioms of affine geometry, which means that the relations $\mathscr{R}$ are such that: two points are joined by a single line; any two non-parallel lines intersect in one point; finally, for each line and each point there is a unique line which passes through the point and it is parallel to the line. Instead, for P^2 to be projective $\mathscr{R}$ must satisfy the property for which:

(1) Any two distinct points are incident to a unique line.
(2) Any two distinct lines are incident with a unique point.
(3) (*not degenerate*) There exist four points such that no three are incident one another.

From definitions provided earlier, both affine and projective planes can be defined in very abstract terms. However, in this section, our focus is on the study of incidence planes defined in a natural way over algebras: we aim to generalize the construction of classical affine and projective planes [61]. Our interest lies in the definition wherein points of the affine plane are characterized by two coordinates, where lines are linear functions of the coordinates with respect to the a sum and a product that are those of the algebra itself. To achieve a projective completion, it becomes necessary to introduce an additional line at infinity and a few suitable relations. This foundational approach to constructing affine and projective planes proves effective -with minor yet meaningful modifications- for all three 8-dimensional division composition algebras: octonions $\mathbb{O}$, para-octonions $p\mathbb{O}$ and Okubo algebra $\mathcal{O}$. Thus, we outline the framework for defining the affine and projective planes for all three cases. However, in order to avoid repetitions we will develop in all the details only the Okubo case $\mathcal{O}$, highlighting in Sect. 3.4 the differences and variations needed in the other two cases.

3.1 The Okubo affine plane and its completion

A direct consequence of Proposition 4 is the feasibility of defining affine geometry over the Okubo algebra or, in other words, an Okubo affine plane $\mathscr{A}_2(\mathcal{O})$ that satisfies all axioms of affine geometry.

Indeed, we identify a *point* on the Okubo affine plane $\mathscr{A}_2(\mathcal{O})$ by two coordinates (x, y) with $x, y \in \mathcal{O}$, while a *line* of slope $s \in \mathcal{O}$ and offset $t \in \mathcal{O}$ is the set $[s, t] = \{(x, s * x + t) : x \in \mathcal{O}\}$. Thus, the x axis is represented by the line $[0, 0]$. On the other hand, *vertical lines* are identified by $[c]$ which stands for the set $\{c\} \times \mathcal{O}$. Here $c \in \mathcal{O}$ represents the intersection with the x axis, thus $[0]$ denotes the y axis.

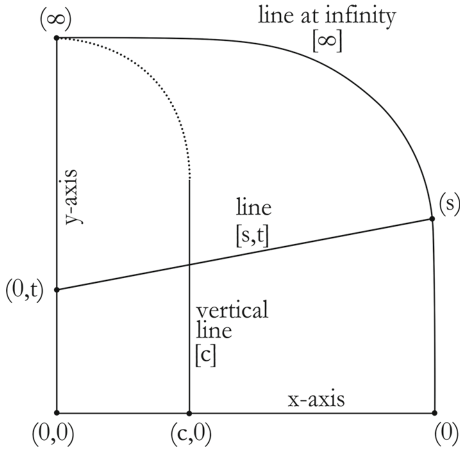

Fig. 2 Representation of the completion of the affine plane: $(0, 0)$ represents the origin, (0) the point at the infinity on the x-axis, (s) is the point at infinity of the line $[s, t]$ of slope s while (∞) is the point at the infinity on the y-axis and of vertical lines $[c]$

Finally, as for the incidence rules we say that a point $(x, y) \in \mathscr{A}_2(\mathcal{O})$ is *incident* to a line $[s, t] \subset \mathscr{A}_2(\mathcal{O})$ if belongs to such line, i.e. $(x, y) \in [s, t]$.

We now proceed to prove that the set of points, lines and incidence relations previously defined forms an affine plane.

Theorem 5 *The Okubo affine plane $\mathscr{A}_2(\mathcal{O})$ with the previous incidence rules satisfies the axioms of affine geometry.*

Proof First of all, we can straightforwardly see that given any two points (x_1, y_1) and (x_2, y_2) there is a unique line joining them. If $x_1 = x_2 = x$, the line is simply $[x]$. On the other hand, if $x_1 \neq x_2$, the line is given by $[s, y_1 - s * x_1]$, where s is determined by the linear equation

$$s * (x_1 - x_2) = (y_1 - y_2),$$

which has a unique solution given by

$$s = \frac{(x_1 - x_2) * (y_1 - y_2)}{n(x_1 - x_2)}. \tag{3.1}$$

Similarly, for two lines $[s_1, t_1]$ and $[s_2, t_2]$ with distinct slopes $s_1 \neq s_2$, a unique point of intersection exists, i.e. $\{(x, s_1 * x + t_1)\}$ where x is

$$x = \frac{(t_2 - t_1) * (s_1 - s_2)}{n(s_1 - s_2)}. \tag{3.2}$$

If two lines have the same slope, they are disjoint. Two such lines are called *parallel*. Finally, for each line $[s, t]$ and each point (x, y) there is a unique line, given by i.e. $[s, y - s * x]$, which passes through (x, y) and that is parallel to $[s, t]$. $\square$

The projective completion of the affine plane $\overline{\mathscr{A}_2}(\mathcal{O})$ is obtained adding a line at infinity $[\infty]$, i.e.

$$[\infty] = \{(s) : s \in \mathcal{O} \cup \{\infty\}\},$$

where (s) identifies the end point at infinity of a line with slope $s \in \mathcal{O} \cup \{\infty\}$. Finally, we define (∞) the point at infinity of $[\infty]$. We now proceed to verify that the plane $\overline{\mathscr{A}_2}(\mathcal{O})$ satisfies axioms of projective geometry: every two lines intersect in a unique point; for every two points passes a unique line; there are at least four points that form a non degenerate quadrangle. Indeed, have the following

Theorem 6 *The extended Okubo affine plane $\overline{\mathscr{A}_2}(\mathcal{O})$ is a projective plane.*

Proof First we need to show that for every two points of the extended plane there still passes a unique line. This is straightforward since if the points are of the affine plane the line was already determined; if are both of them at infinity, i.e. (s) and $\left(s'\right)$, then such line is $[\infty]$; finally, if one is on the affine plane (x, y) and the other is at infinity (s), the line that joins them is $[s, y - s * x]$. On the other hand, if two lines are not parallel the interstection was already determined; if they are parallel lines, such as $[s, t_1]$ and $[s, t_2]$, they now intersect in the point (s); finally, two vertical lines intersect in (∞). The only thing that is left is to verify that it exists a non-degenerate quadrangle where no three points are collinear which in this case can be found easily, e.g. the quadrangle $\diamond = \{(0, 0), (e, e), (0), (\infty)\} \subset \overline{\mathscr{A}_2}(\mathcal{O})$ is such that no three elements are incident to the same line. Indeed, the lines that join those points are $[0, 0], [0], [\infty], [0, e], [e, 0]$ and $[e]$ and none of those contains three elements of $\diamond$.

Remark 7 As in the standard projective plane over a field, we would like to point out to the reader the existence of a fundamental triangle also in the extended Okubo affine plane. More precisely, the entirety of the affine plane is encompassed by a triangle given by three special points: the *origin* $(0, 0)$; the *0-point at infinity*, i.e. the point (0) obtained prolonging the x axis to infinity; finally, the *∞-point at infinity*, i.e. the point (∞) obtained prolonging the y axis to infinity. We designate $\triangle$ the set made by those three points, i.e. $\triangle = \{(0, 0), (0), (\infty)\}$.

3.2 The Okubo projective plane

We will now define directly the projective plane $\mathcal{O}P^2$ and subsequently illustrate its correspondence with the completion of the affine plane $\overline{\mathscr{A}_2}(\mathcal{O})$. Historically, numerous tricks were used for defining projective planes over non-associative algebras such as octonions. Here we will use a variation of the one proposed by H. Salzmann [61] which is based on what he calls "Veronese coordinates". Let V be the 27-dimensional real vector space $V \cong \mathcal{O}^3 \times \mathbb{R}^3$, with elements of the form

$$(x_\nu; \lambda_\nu)_\nu = (x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3),$$

where $x_\nu \in \mathcal{O}$, $\lambda_\nu \in \mathbb{R}$ and $\nu = 1, 2, 3$. We then define the Veronese vectors to be those $w \in V$ that satisfy the following *Veronese conditions,*

$$\lambda_1 x_1 = x_2 * x_3,\ \ \lambda_2 x_2 = x_3 * x_1,\ \ \lambda_3 x_3 = x_1 * x_2, \tag{3.3}$$

$$n(x_1) = \lambda_2\lambda_3,\ n(x_2) = \lambda_3\lambda_1, n(x_3) = \lambda_1\lambda_2. \tag{3.4}$$

It is straightforward to see that if $w = (x_\nu; \lambda_\nu)_\nu$ is a Veronese vector then also $\mu w = (\mu x_\nu; \mu\lambda_\nu)_\nu$ is Veronese for every $\mu \in \mathbb{R}$. The set of Veronese vectors is therefore a subset that we will call H and for every w that is Veronese we indicate as $\mathbb{R}w \subset H$ the class of real multiples of w. The *Okubo projective plane* $\mathcal{O}P^2$ is then the geometry having the 1-dimensional subspaces $\mathbb{R}w$ as *points*, i.e.

$$\mathscr{P}_{\mathcal{O}} = \{\mathbb{R}w : w \in H \smallsetminus \{0\}\}. \tag{3.5}$$

The set of *lines* $\mathscr{L}_{\mathcal{O}}$ is formed by subspaces ℓ_w in the projective plane $\mathcal{O}P^2$ that are orthogonal to a Veronese vector $w \in H$, i.e.

$$\ell_w = w^\perp = \{z \in H : \beta(z, w) = 0\}, \tag{3.6}$$

where the bilinear form β is the extension to V of the polarisation of the Okubo norm. More specifically, defining $\langle x, y\rangle = n(x+y) - n(x) - n(y)$, for any two Okubo elements $x, y \in \mathcal{O}$, the bilinear form β is given by

$$\beta(v, w) = \sum_{\nu=1}^{3} \left(\langle x_\nu, y_\nu\rangle + \lambda_\nu \eta_\nu\right), \tag{3.7}$$

where $v = (x_\nu; \lambda_\nu)_\nu$ and $w = (y_\nu; \eta_\nu)_\nu$ are Veronese vectors in $H \subset V$.

Finally, the incidence relations are again given by inclusion $\subseteq$, i.e. we say that a point $\mathbb{R}w \in \mathcal{O}P^2$ is *incident* to the line ℓ_v iff $\mathbb{R}w \in v^\perp$, i.e. $\beta(w, v) = 0$.

Remark 8 Since all real multiples of a Veronese vectors $v = (x_\nu; \lambda_\nu)_\nu$ identify the same point on the projective plane, we will usually take as representative of the class the one such that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, such that an alternative definition of the set of points in (3.5) could be

$$\mathcal{O}P^2 = \left\{(x_\nu; \lambda_\nu)_\nu \in H \smallsetminus \{0\}, \lambda_1 + \lambda_2 + \lambda_3 = 1\right\}. \tag{3.8}$$

Remark 9 It is also worth noting how the norm n, defined over the symmetric composition algebra $\mathcal{O}$, is intertwined with the geometry of the plane. This relationship becomes evident when considering the quadratic form of the bilinear symmetric form β, i.e.

$$q(v) := \frac{1}{2}\beta(v, v) = n(x_1) + n(x_2) + n(x_3) + \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right), \tag{3.9}$$

where $v = (x_\nu; \lambda_\nu)_\nu$.

In the next section we will show that the triple $\mathcal{O}P^2 = \{\mathscr{P}_{\mathcal{O}}, \mathscr{L}_{\mathcal{O}}, \subseteq\}$ is indeed a projective plane and, even more, is equivalent to the completion of the affine plane $\overline{\mathscr{A}_2}\left(\mathcal{O}\right)$.

3.3 Correspondence between affine and projective plane

In establishing a one-to-one correspondence between the completion of the affine plane $\overline{\mathscr{A}_2}\left(\mathcal{O}\right)$ and the projective plane $\mathcal{O}P^2$, we must ensure that such a correspondence maintains the incidence relations. Specifically, a point incident to a line in $\overline{\mathscr{A}_2}\left(\mathcal{O}\right)$ should map to a point in $\mathcal{O}P^2$ that is incident to the image of the original line. As demonstrated in [16], the map which sends points and lines from $\overline{\mathscr{A}_2}\left(\mathcal{O}\right)$ to $\mathcal{O}P^2$ defined by

$$\begin{array}{rcc} (x, y) & \mapsto & \mathbb{R}\left(x, y, x * y; n\left(y\right), n\left(x\right), 1\right), \\ (x) & \mapsto & \mathbb{R}\left(0, 0, x; n\left(x\right), 1, 0\right), \\ (\infty) & \mapsto & \mathbb{R}\left(0, 0, 0; 1, 0, 0\right), \\ [s, t] & \mapsto & \left(t * s, -t, -s; 1, n\left(s\right), n\left(t\right)\right)^{\perp}, \\ [c] & \mapsto & \left(-c, 0, 0; 0, 1, n\left(c\right)\right)^{\perp}, \\ [\infty] & \mapsto & \left(0, 0, 0; 0, 0, 1\right)^{\perp}, \end{array} \tag{3.10}$$

is indeed well-defined and keeps the incidence relations.

Lemma 10 *The aforementioned correspondence* (3.10) *is well-defined and is a one-to-one correspondence between points and lines of the affine plane* $\overline{\mathscr{A}_2}\left(\mathcal{O}\right)$ *and points and lines of the projective plane* $\mathcal{O}P^2$

Proof In fact, this is just a trivial check that relies on the Veronese conditions and $\mathcal{O}$ being a symmetric composition algebra for which just (1.4) and (1.12) has to be used. For example, let (x, y) be a point of the affine plane, then the vector $\left(x, y, x * y; n\left(y\right), n\left(x\right), 1\right)$ is a Veronese vector since a direct check of (3.3) and (3.4) yields to

$$\begin{array}{ccc} n\left(y\right)x = y * \left(x * y\right), & n\left(x\right)y = \left(x * y\right) * x, & x * y = x * y, \\ n\left(x\right) = n\left(x\right), & n\left(y\right) = n\left(y\right), & n\left(x * y\right) = n\left(x\right)n\left(y\right), \end{array} \tag{3.11}$$

that are either identically true or obtainable from the fact that Okubo algebra is a composition algebra, i.e. $n\left(x * y\right) = n\left(x\right)n\left(y\right)$, or from the symmetric composition identity, i.e. $n\left(x\right)y = \left(x * y\right) * x$. On the other hand, for any Veronese vector $v = \left(x_\nu; \lambda_\nu\right)_\nu$ with $\lambda_3 \neq 0$ we have that subspace $\mathbb{R}v$ is the same of

$$\mathbb{R}v = \mathbb{R}\left(x, y, x * y; n\left(y\right), n\left(x\right), 1\right), \tag{3.12}$$

where $x = \lambda_3^{-1}x_1$ and $y = \lambda_3^{-1}x_2$ which is again a Veronese vector. The check with a generic line proceeds on the same way, but it might be interesting to explicitly check that

$$[\infty] \longrightarrow \left(0, 0, 0; 0, 0, 1\right)^{\perp}, \tag{3.13}$$

is indeed a line. First of all, we need to find the Veronese vectors orthogonal to $(0, 0, 0; 0, 0, 1)$. These are vectors with $\lambda_3 = 0$ and, therefore, with $n\left(x_1\right) = n\left(x_2\right) =$

0, and thus with $x_1 = x_2 = 0$. Then, elements orthogonal to $(0, 0, 0; 0, 0, 1)$ might take only two forms depending on x_3 being 0 or $x_3 \neq 0$, i.e.

$$(0, 0, 0; 0, 0, 1)^{\perp} = \{\mathbb{R}\,(0, 0, x; n\,(x)\,, 1, 0)\} \cup \{\mathbb{R}\,(0, 0, 0; 1, 0, 0)\}\,, \tag{3.14}$$

where $x \in \mathcal{O}$. In fact, these are in a trivial way elements of an Okubo line $\mathcal{O} \cup \{\infty\}$as one-point compactification of the Okubo algebra. □

While it was a trivial check that (3.10) is well defined and is a one-to-one correspondence between $\overline{\mathscr{A}_2}\,(\mathcal{O})$ and $\mathcal{O}P^2$, the proof that incidence rules are preserved by (3.10) is a little bit more involved and for this reason we will show it in a complete form with the following

Lemma 11 *The correspondence in* (3.10) *preserves the incidence relations between* $\overline{\mathscr{A}_2}\,(\mathcal{O})$ *and* $\mathcal{O}P^2$.

Proof We need to show that the image of a point (x, y) incident to the line $[s, t]$ is mapped by (3.10) into a point of the projective plane, i.e. $\mathbb{R}\,(x, y, x * y; n\,(y)\,, n\,(x)\,, 1)$, that is incident to the image of $[s, t]$, i.e. is incident to $(t * s, -t, -s; 1, n\,(s)\,, n\,(t))^{\perp}$. By definition of incidence on the projective plane and of (3.6), the image of (x, y) is incident to the image of $[s, t]$ if and only if the following condition is satisfied

$$\langle t * s, x\rangle - \langle t, y\rangle - \langle s, x * y\rangle + n\,(y) + n\,(s)\,n\,(x) + n\,(t) = 0. \tag{3.15}$$

Noting that

$$\langle s * x, t - y\rangle = n\,(s * x + t - y) - n\,(s * x) - n\,(t - y)\,, \tag{3.16}$$

and since (1.9), we then have

$$\begin{aligned} \langle s * x, t - y\rangle &= \langle s * x, t\rangle - \langle s * x, y\rangle \\ &= \langle t, s * x\rangle - \langle s, x * y\rangle \\ &= \langle t * s, x\rangle - \langle s, x * y\rangle\,, \end{aligned} \tag{3.17}$$

and, therefore,

$$\langle t * s, x\rangle - \langle s, x * y\rangle = n\,(s * x + t - y) - n\,(s * x) - n\,(t - y)\,. \tag{3.18}$$

Inserting the latter into (3.15) and noting that $n\,(s)\,n\,(x) = n\,(s * x)$, then (3.15) is equivalent to

$$n\,(s * x + t - y) = 0. \tag{3.19}$$

Since the Okubo algebra is a division composition algebra, and the only element of zero norm is zero, then (3.15) is satisfied iff $s * x + t - y = 0$, that is $(x, y) \in [s, t]$. The cases for the incidence of (s) and (∞) with $[\infty]$ can be proved in the same way. □

Once it is shown that the map (3.10) gives a one to one correspondence that keeps incidence relation we thus have the following

Theorem 12 *The Okubo plane given by the triple* $\mathcal{O}P^2 = \{\mathscr{P}_{\mathcal{O}}, \mathscr{L}_{\mathcal{O}}, \subseteq\}$ *is a projective plane and is isomorphic to the completion of the affine plane* $\overline{\mathscr{A}_2}(\mathcal{O})$.

3.4 Octonionic and para-octonionic planes

The previous definitions pertaining to the Okubo plane can be generalized to the para-octonionic case and, with minor variations, to the octonionic case as referenced in [13, 61]. Indeed, in the context of the paraoctonionic case, there is no need to alter the definitions formally, provided we replace the Okubonic product $*$ with the para-octonionic product $\bullet$. In a manner analogous to the Okubo case, a point of the para-octonionic affine plane $\mathscr{A}_2(p\mathbb{O})$ is given by a pair of elements (x, y) with x,$y \in \{p\mathbb{O}\}$, while a *line* of slope $s \in p\mathbb{O}$ and offset $t \in p\mathbb{O}$ is the set $[s, t] = \{(x, s \bullet x + t) : x \in p\mathbb{O}\}$ and, of course, we say that a point $(x, y) \in \mathscr{A}_2(p\mathbb{O})$ is *incident* to a line $[s, t] \subset \mathscr{A}_2(p\mathbb{O})$ if belongs to such line, i.e. $(x, y) \in [s, t]$. The octonionic case $\mathbb{O}$ with the product $\cdot$ follows the same definitions.

For the affine plane, distinctions primarily manifest in the octonionic equations that describe the slope s of the line passing through two points of the plane and coordinate x of the intersection of two generic lines as found in (3.1) and (3.2). In the para-octonionic scenario, the expressions remain as

$$s = \frac{(x_1 - x_2) \bullet (y_1 - y_2)}{n(x_1 - x_2)}, \quad x = \frac{(t_2 - t_1) \bullet (s_1 - s_2)}{n(s_1 - s_2)}. \tag{3.20}$$

However, the octonionic variant introduces a slight modification due to the unique properties of octonions as a unital composition algebra. Given that $x^{-1} = \overline{x}/n(x)$, the equations transform to

$$s = \frac{(y_1 - y_2) \cdot \overline{(x_1 - x_2)}}{n(x_1 - x_2)}, \quad x = \frac{\overline{(s_1 - s_2)} \cdot (t_2 - t_1)}{n(s_1 - s_2)}.$$

Similar modifications are observed in the projective planes' definitions, as seen in (3.3) and (3.4). For the para-octonionic case, given a vector $(x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3) \in p\mathbb{O}^3 \times \mathbb{R}^3$, one isolate the subset of Veronese vectors satisfying the following conditions

$$\lambda_1 x_1 = x_2 \bullet x_3, \;\; \lambda_2 x_2 = x_3 \bullet x_1, \;\; \lambda_3 x_3 = x_1 \bullet x_2, \tag{3.21}$$

$$n(x_1) = \lambda_2\lambda_3, \; n(x_2) = \lambda_3\lambda_1, n(x_3) = \lambda_1\lambda_2, \tag{3.22}$$

which closely resemble the Okubo conditions. In contrast, the octonionic variant relies on the Veronese conditions applicable to all Hurwitz algebras, i.e.,

$$\lambda_1 \overline{x_1} = x_2 \cdot x_3, \;\; \lambda_2 \overline{x_2} = x_3 \cdot x_1, \;\; \lambda_3 \overline{x_3} = x_1 \cdot x_2, \tag{3.23}$$

$$n(x_1) = \lambda_2\lambda_3, \; n(x_2) = \lambda_3\lambda_1, n(x_3) = \lambda_1\lambda_2. \tag{3.24}$$

To conclude, these differences in the Veronese conditions correspond to varied formulations of the one-to-one relationship between the affine and projective planes. Within this relationship, the para-octonions retain the formal mapping in (3.3), and more specifically one still has

$$(x, y) \mapsto \mathbb{R}\left(x, y, x \bullet y; n(y), n(x), 1\right), \tag{3.25}$$

$$[s, t] \mapsto \left(t \bullet s, -t, -s; 1, n(s), n(t)\right)^{\perp}, \tag{3.26}$$

while for the octonionic plane one has to modify them as follow

$$(x, y) \mapsto \mathbb{R}\left(x, y, \overline{y} \cdot x; n(y), n(x), 1\right), \tag{3.27}$$

$$[s, t] \mapsto \left(\overline{s} \cdot t, -t, -s; 1, n(s), n(t)\right)^{\perp}. \tag{3.28}$$

Interestingly, as inferred from the above equations, the Veronese conditions for para-octonions are simpler than those for octonions, as we do not need to use the conjugation. This leads to an intriguing observation: defining the Cayley plane appears more intuitive using para-octonions than octonions.

4 Collineations on the plane

In this section we study the collineations of the Okubo affine and projective plane. We start presenting explicit forms of elations, more specifically translations and shears, and of the triality collineation (see below). The direct study of the motion group is important since it might be an alternative way in proving the isomorphism of the Okubo plane with the Cayley plane. Indeed, it is well known that any 16-dimensional compact plane with a collineation group of dimension greater than 40 is isomorphic to the Cayley plane $\mathbb{O}P^2$ (see [61, Chap. 8]). In fact, this is not needed since we will write an explicit isomorphism between the Okubo plane $\mathcal{O}P^2$, the paraoctonionic plane $p\mathbb{O}P^2$ and the octonionic plane $\mathbb{O}P^2$ in the next section. As result, the collineation groups of the three planes coincide and is the exceptional Lie group $\mathrm{E}_{6(-26)}$. Nevertheless, it is noteworthy that a variation in the foundational algebra defining the plane, despite preserving the overall collineation group, alters the algebraic description of the collineations. Consequently, in the Okubo realization of the 16-dimensional Moufang plane, the reflection $(x, y) \longrightarrow (y, x)$, is not a collineation, whereas it is in its octonionic realisation.

4.1 Collineations

A *collineation* is a bijection φ of the set of points of the plane onto itself, such that lines map to lines. Since the identity map is a collineation, the inverse φ^{-1} and the composition $\varphi \circ \varphi'$ are collineations if φ, φ' are both collineations, then the set of collineations is in fact a group under composition that we will denote as $\mathrm{Aut}\left(\mathcal{O}P^2\right)$. A notable characteristic of collineations is that they keep incidence relations of both affine and projective planes. Indeed, given two points p_1 and p_2, there is only one line passing through them and, clearly, the image of such a line is the only one that passes through p_1^φ and p_2^φ (where we used the classical notation p^φ and ℓ^φ to indicate the image of the point p and the line ℓ through the collineation φ). We thus have the following

Proposition 13 *Collineations of the affine plane send parallel lines into parallel lines.*

As a consequence of the previous proposition we also have the following

Corollary 14 *Any affine collineation can be extended uniquely as a projective collineation.*

Proof Since an affine collineation sends parallel lines $[s, t]$ into parallel lines $[s, t]^\varphi$, so that in fact we have that all lines with slope s go in lines with slope s^φ. Clearly, we can extend the affine collineation to the projective plane if and only if we set that parallels lines go to the same point at infinity, i.e. setting $(s)^\varphi = (s^\varphi)$. □

A set of collineations $\triangle$ is called *transitive* on a set M if for every $x, y \in M$ it exists a collineation φ such that $x^\varphi = y$. On the other hand, a set of collineations $\triangle$ is called *doubly transitive* if for any quadruple of points $x, y, z, w \in M$, it exists a collineation φ such that $x^\varphi = y$ and $z^\varphi = w$.

4.2 Axial collineations

Given a collineation φ we say that φ is *axial* if it fixes every point of a line ℓ. In this case, the line ℓ is called an *axis* of φ. On the other hand we say that φ is *central* if it fixes every line passing through a point p, which is then called a *center* of φ.

It is known from a general setting of projective geometry that any collineation of a projective plane is axial if and only if is central (see [39, Thm. 4.9]). Moreover, it is easy to see that an axial collineation that has two centers or two axis is the identity. Indeed, let us suppose that φ has two center p and q, then any other point r outside the line joining p and q would be fixed since r is given as intersection of two fixed lines, one passing through p and the other through q. On the other hand, we could just replicate the argument for the point r with p and determine that the collineation must fix also the line joining p and q.

Given a point p and a line ℓ, we denote an axial collineation with center p and line ℓ as $\varphi_{[p,\ell]}$ and the group of such collineations as $\Gamma_{[p,\ell]}$. It is then easy to verify what is known as the *conjugation formula*, i.e.

Lemma 15 (Conjugation formula [39, Lemma. 4.11]) *For every collineation φ the group $\Gamma_{[p^\varphi,\ell^\varphi]}$ of collineations with center p^φ and axis ℓ^φ is just the conjugate of $\Gamma_{[p,\ell]}$ by φ*

$$\varphi^{-1} \circ \Gamma_{[p,\ell]} \circ \varphi = \Gamma_{[p^\varphi,\ell^\varphi]}. \tag{4.1}$$

Moreover, an axial collineation $\varphi_{[p,\ell]}$ that fixes a point q outside $p \cup \ell$ is the identity. Indeed if q is fixed by $\varphi_{[p,\ell]}$ then joining the points of ℓ with q we would see that also q is a center.

Since axial collineations fix only a point called center and a line called axis, they can be easily divided in two classes:

(1) those for which the center p is incident to the axis ℓ and that are called *elations*;
(2) those collineations for which the center p is not incident to the axis ℓ and that are called *homologies*.

Finally, a last lemma worth reviewing, since it is a standard argument that we will use.

Lemma 16 (see [61, sec. 23.9]) *Suppose that $\triangle$ is a set of collineations of center p and axis ℓ, and let m be a line through p with $m \neq \ell$. If $\triangle$ is transitive on the set of points of m that are not incident with the center or the axis, i.e., $m \setminus \{p, m \wedge \ell\}$ then $\triangle$ is the group of all collineations with center p and axis ℓ, i.e. $\Gamma_{[p,\ell]}$.*

4.3 Elations

We now focus on a special class of axial collineations called *elations*, i.e. that are those collineations in which the center is incident to the axis.

Corollary 17 *Collineations of $\overline{\mathscr{A}_2}(\mathcal{O})$ that have the line at infinity $[\infty]$ as axis and center incident with the axis are precisely the translations*

$$\begin{gathered} \tau_{a,b} : (x, y) \longrightarrow (x + a, y + b)\,, \\ \tau_{a,b} \mid_{[\infty]} = id. \end{gathered} \tag{4.2}$$

Proof First of all, we show that this are collineations that have axis $[\infty]$ and center incident to the axis. Indeed, given a line $[s, t]$ or $[c]$, then its image through $\tau_{a,b}$ is another line given by

$$[s, t]^{\tau_{a,b}} = [s, t - s * a + b]\,, \tag{4.3}$$

$$[c]^{\tau_{a,b}} = [c + a]\,. \tag{4.4}$$

Since this are collineations of the affine plane, they do extend in a unique way as collineations on the projective plane and since the slope (s) is unchanged, then the line at infinity is the axis of the collineation. Moreover, let us now consider (4.3). Clearly if $a \neq 0$ there is a unique slope s, namely $s = n(s)^{-1}(a * b)$, such that $[s, t]^{\tau_{a,b}} = [s, t]$ for every $t \in \mathcal{O}$. But the set $\{[s, t] : t \in \mathcal{O}\}$ is exactly the set of parallel lines that pass through the point (s), i.e. (s) is a center of the collineation and is incident to the axis $[\infty]$. The same reasoning can be applied when $a = 0$, since in that case all vertical lines $[c]$ with $c \in \mathcal{O}$ would be invariant and the center of the collineation would be (∞).

We now need to demonstrate that all elations with axis $[\infty]$ and center (p) are of the form of $\tau_{a,b}$. First of all since $[\infty]$ is the axis, which means that the collineation fixes pointwise the line $[\infty]$, then the image of a line of slope (p) will be a line of the same slope (p). Now let be q_1, q_2 any two points in the affine plane incident to a line of slope p. Then there exists a translation of the form $\tau_{a,b}$ that sends q_1 in q_2. The group of translations $\tau_{a,b}$ is thus transitive on the line M joining q_1 and q_2 which has slope (p). This means that the group of translations is that of all collineation with center (p) and axis $[\infty]$ by Lemma 16. □

We now focus on elations that have vertical axis [0] and center in (∞) which they too enjoy an easy and elegant characterization.

Theorem 18 *Collineations of* $\overline{\mathscr{A}_2}(\mathbb{O})$ *that have the vertical axis* [0] *as axis and center in* (∞) *are precisely the shears*

$$\begin{aligned} \sigma_a : (x, y) &\longrightarrow (x, y + ax)\,, \\ (s) &\longrightarrow (s + a)\,, \\ (\infty) &\longrightarrow (\infty)\,. \end{aligned} \tag{4.5}$$

Proof First of all we show that this are collineations that have axis [0] and center in (∞). Indeed, given a line $[s, t]$ or $[c]$, then its image through σ_a is another line given by

$$[s, t]^{\sigma_a} = [s + a, t]\,, \tag{4.6}$$

$$[c]^{\sigma_a} = [c]\,, \tag{4.7}$$

so that σ_a are indeed collineations. Since all lines of the form $[c]$ are invariant, therefore the point (∞) that joins them is the center of all σ_a. On the other hand, looking at (4.5) it is evident that all points of the for $(0, t)$ are fixed by all σ_a and thus [0] is the axis. Since $(\infty) \in [0]$, then σ_a are elations for every $a \in \mathbb{O}$.

Now we proceed with the same argument of the previous theorem to show that all the elations with axis [0] and center (∞) are of the previous form. Let M be the vertical line $[c]$ with $c \neq 0$ and let us consider two points $q_1, q_2 \in M \smallsetminus (\infty)$. Let us suppose $q_1 = (c, y)$ and $q_2 = \left(c, y'\right)$ then the shear σ_a with $a = n\,(c)^{-1}\, c * \left(y' - y\right)$, sends q_1 in q_2. Thus the group of shears is transitive over $M \smallsetminus (\infty)$ and thus coincides with the group of all elations with axis [0] and center (∞), i.e. i.e. $\Gamma_{[(\infty),[0]]}$. □

Translations and shears occur also in the octonionic realisation of the 16-dimensional Moufang plane. We now point out a transformation that is a collineation when formulated in the octonionic realisation, but is not a collineation on the Okubo projective plane.

Proposition 19 *The reflection of the coordinates over the Okubo plane given by* $(x, y) \longrightarrow (y, x)$, *is not collineation.*

Proof Let us consider the image of a line $[s, t]$ through the map that sends $(x, y) \longrightarrow (y, x)$. Let us suppose that

$$[s, t] = \{(x, s * x + t) : x \in \mathbb{O}\} \to \left[s', t'\right] = \{(s * x + t, x) : x \in \mathbb{O}\}\,, \tag{4.8}$$

and let us determine s' and t'. Since by definition $\left[s', t'\right] = \left\{\left(x', s' * x' + t'\right) : x' \in \mathbb{O}\right\}$ we then have that

$$\begin{cases} x' = s * x + t, \\ x = s' * x' + t', \end{cases} \tag{4.9}$$

which means

$$x' = s * \left(s' * x' + t'\right) + t, \tag{4.10}$$

and thus for the (1.12) after multiplying on the LHS for s, we obtain

$$\left(x' - t\right) * s = n\left(s\right)\left(s' * x' + t'\right), \tag{4.11}$$

and, finally,

$$n\left(s\right) t' + t * s = x' * s - n\left(s\right) s' * x', \tag{4.12}$$

which yields to a slope s' that varies with x' and thus it is not a line since the slope is not fixed for all x'. □

Remark 20 In the case of octonions $\mathbb{O}$ reflections over the affine and projective plane are collineations. In fact, the previous map can be defined over the octonionic projective plane as the collineation

$$\rho : \begin{cases} (x, y) \longrightarrow (y, x), \\ (s) \longrightarrow \left(s^{-1}\right), \\ (\infty) \longrightarrow (0), \\ (0) \longrightarrow (\infty), \end{cases} \tag{4.13}$$

with $x, y, s \in \mathbb{O}$ which is an axial collineation of axis $[1, 0]$ and center (-1) that sends

$$\rho : \begin{cases} [s, t] \longrightarrow \left[s^{-1}, -s^{-1}t\right], \quad s \neq 0 \\ [0, t] \longrightarrow [t], \\ [t] \longrightarrow [0, t], \\ [0] \longrightarrow [\infty], \\ [\infty] \longrightarrow [0]. \end{cases} \tag{4.14}$$

From a heuristic point of view the previous theorem is clear since, for this reflection to be a collineation, it would require the existence of an inverse at the infinity line, i.e. $(s) \longrightarrow \left(s^{-1}\right)$. Also, note that once we try to define such collineation reading it from the octonions from Tab. 3, i.e. defining implicitly $s^{-1} = x$ such that read in the octonionic algebra we would have $x \cdot s = 1$, we then have two choices for the implicit definition of x, i.e.

$$(x * e) * (e * s) = e, \text{ or } (s * e) * (e * x) = e, \tag{4.15}$$

that yield to different, even though τ-conjugated, definitions of x which thus would violate the uniqueness of the extension to the projective plane of an affine collineation. Another heuristic reason for the lack of such collineation is that the axis of such reflection, if it would exists as in the octonionic case $\overline{\mathscr{A}_2}\left(\mathbb{O}\right)$ would be the line $[1, 0]$ containing all elements of the form (x, x) with $x \in \mathbb{O}$. In our case, it is easy to verify that points (x, x) are not all collinear, e.g. the line joining the point $(0, 0)$ with (x, x) is given by $\left[n\left(x\right)^{-1} x * x, 0\right]$ for every $x \in \mathcal{O}$.

Remark 21 The previous proposition does not mean that the set of collineations is not transitive over $\mathbb{O}P^2$ since for every pair of points $p_1 = (x, y)$ and $p_2 = \left(x', y'\right)$ we can find a collineation such that $(x, y)^{\varphi} = \left(x', y'\right)$ for example the translation $\tau_{a,b}$ with $a = x' - x$ and $b = y' - y$. Even more the Okubo projective plane, as a Corollary of Theorem 24, is transitive on quadrangles.

4.4 Triality collineations

Through the use of the Okubo-Veronese coordinates a special set of collineations can be easily spotted, i.e. the *triality collineation* [61] given by a cyclic permutation of the coordinates

$$\widetilde{t} : (x_1, x_2, x_3; \lambda_1, \lambda_2, \lambda_3) \longrightarrow (x_2, x_3, x_1; \lambda_2, \lambda_3, \lambda_1) . \tag{4.16}$$

Proposition 22 *The triality collineation can be read on the affine plane in the following way:*

$$\widetilde{t} : \begin{cases} (x, y) & \longrightarrow \frac{1}{n(y)} (y, x * y) , \quad y \neq 0 \\ (x) & \longrightarrow \frac{1}{n(x)} (0, x) , x \neq 0 \\ (x, 0) & \longrightarrow (x) , \\ (0) & \longrightarrow (\infty) , \\ (\infty) & \longrightarrow (0, 0) . \end{cases} \tag{4.17}$$

In particular it induces a collineation $t : \mathscr{A}_2 (\mathbb{O}) \rightarrow \mathscr{A}_2 (\mathbb{O})$ *on the affine plane.*

Proof If $y \neq 0$, the image of $t\,(x, y)$ by the bijection (3.10) in the projective plane is given by

$$\frac{1}{n\,(y)} (y, x * y) \longrightarrow \frac{1}{n\,(y)} \left(y, x * y, \frac{y * x * y}{n\,(y)}; \frac{n\,(x * y)}{n\,(y)}, 1, n\,(y) \right), \tag{4.18}$$

and since $y * x * y = n\,(y)\,x$ and $n\,(x * y) = n\,(x) * n\,(y)$, then the image of $t\,(x, y)$ is in $\mathbb{R}\,(y, x * y, x; n\,(x)\,, 1, n\,(y))$ which is the image of the triality collineation $\widetilde{t}$ of the projective point $\mathbb{R}\,(x, y, x * y; n\,(y)\,, n\,(x)\,, 1)$. With the same procedure we find the other correspondences. □

Remark 23 As shown in Fig. 3 the triality collineation t sends the line at infinity $[\infty]$ into the line $[0]$, while the y axis $[0]$ is sent into the x axis $[0, 0]$; finally the x axis $[0, 0]$ is sent into the line at infinity $[\infty]$. This phenomenon is the dual, of what happens, in the reverse order, for the three points $(0, 0)$,(0) and (∞).

5 Three realizations of the 16-dimensional Moufang plane

In Sect. 3, we have constructed three projective planes coming from three division algebras, by modifications of Veronese-type formulas. In the preceding Sect. 4.1, we

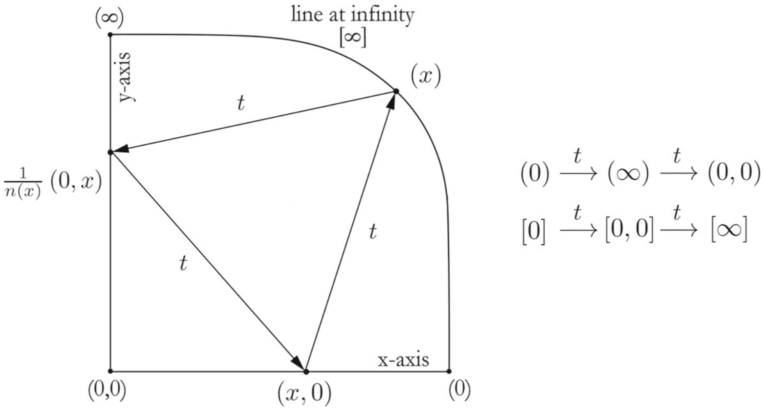

Fig. 3 Action on the affine plane $\mathscr{A}_2(\mathcal{O})$ of the triality collineation defined in (4.17)

explicitly constructed the primary families of collineations for the Okubonic plane and highlighted certain distinctive features of the plane. More specifically, we observed that the points $(0, 0)$, (x, x), and (y, y) are not collinear -as it happens in projective planes obtained over Hurwitz algebras-, and the transformation from $(x, y) \longrightarrow (y, x)$ does not constitute a collineation. Despite these distinctions, in Theorem 24, we construct two collineations, i.e. (5.4) and (5.16) that prove the three planes to be projectively isomorphic. Furthermore, we will show that such collineations are isometries. As a consequence, there exists a complete equivalence among the Okubonic, octonionic, and paraoctonic planes that we will extensively discuss in Sect. 6.

5.1 Isomorphism between Okubo and the Cayley plane

In the context of projective spaces, an isomorphism refers to a bijection between the points of the spaces that preserves the incidence relations. We have the following

Theorem 24 *The Okubo projective $\mathcal{O}P^2$ plane is isomorphic to the octonionic projective plane $\mathbb{O}P^2$.*

Proof Consider the following bijective map (see Sect. 2.3) defined over the real Okubo algebra given by

$$x \longrightarrow \overline{x} = \langle x, e \rangle \, e - x, \tag{5.1}$$

$$x \longrightarrow \tau(x) = \langle x, e \rangle \, e - x * e, \tag{5.2}$$

where $*$ is the Okubo product. Notice that τ, as bijective maps over the octonions, is an order three automorphism that realizes the Okubo algebra as a Petersson algebra since

$$x * y = \tau(\overline{x}) \cdot \tau^2(\overline{y}), \tag{5.3}$$

for every $x, y \in \mathbb{O}$. Let the Okubo projective plane be $\mathcal{O}P^2 = \{\mathscr{P}_{\mathcal{O}}, \mathscr{L}_{\mathcal{O}}, \mathscr{R}_{\mathcal{O}}\}$ and consider the bijective map $\Phi : \mathcal{O}P^2 \longrightarrow \mathbb{O}P^2$ given by

$$\Phi : \begin{cases} (x, y) & \longrightarrow \left(\tau^2 (\overline{x}), y\right), \\ (s) & \longrightarrow (\tau (\overline{s})), \\ (\infty) & \longrightarrow (\infty), \\ [s, t] & \longrightarrow [\tau (\overline{s}), t], \\ [c] & \longrightarrow \left[\tau^2 (\overline{c})\right], \\ [\infty] & \longrightarrow [\infty]. \end{cases} \tag{5.4}$$

If we call the image incidence plane $\Phi\left(\mathcal{O}P^2\right) = \left\{\mathscr{R}_{\mathcal{O}}^{\Phi}, \mathscr{L}_{\mathcal{O}}^{\Phi}, \mathscr{R}_{\mathcal{O}}^{\Phi}\right\}$, then we notice that the octonionic projective plane is given by $\mathbb{O}P^2 = \left\{\mathscr{P}_{\mathcal{O}}^{\Phi}, \mathscr{L}_{\mathcal{O}}^{\Phi}, \mathscr{R}_{\mathbb{O}}\right\}$. To show the projective isomorphism and complete the theorem we need to show that $\mathscr{R}_{\mathbb{O}} \cong \mathscr{R}_{\mathcal{O}}^{\Phi}$, in other words that every point in the Okubo plane (x, y) is incident to an Okubo line ℓ if and only if the image point $(x, y)^{\Phi}$ is incident to the image of the octonion line ℓ^{Φ}, i.e. $\Phi((x, y)) \in \Phi(\ell)$. By definition of the Okubo projective plane

$$(x, y) \in [s, t] = \{y = s * x + t\}, \tag{5.5}$$

$$(s) \in [s, t], \tag{5.6}$$

$$(\infty) \in [\infty], \tag{5.7}$$

for every $x, y, s \in \mathcal{O}$. But, since the image of the line $[s, t]$ is

$$[\tau (\overline{s}), t] = \left\{(x, y) \in \mathbb{O} : y = \tau (\overline{s}) \cdot \tau^2 (\overline{x}) + t\right\}, \tag{5.8}$$

and since

$$s * x = \tau (\overline{s}) \cdot \tau^2 (\overline{x}), \tag{5.9}$$

we then have that

$$(x, y)^{\Phi} = \left(\tau^2 (\overline{x}), y\right) \in [\tau (\overline{s}), t] = [s, t]^{\Phi}, \tag{5.10}$$

$$(s)^{\Phi} = (\tau (\overline{s})) \in [\tau (\overline{s}), t] = [s, t]^{\Phi}, \tag{5.11}$$

$$(\infty) \in [\infty], \tag{5.12}$$

which thus concludes the proof of the theorem. $\square$

In the previous theorem we explicitly found an isomorphism between the completion of the affine plane over the Okubo algebra and that over the octonions. For practical reason it is also useful to have the isomorphism $\widetilde{\Phi} : \mathcal{O}P^2 \longrightarrow \mathbb{O}P^2$ developed for the Veronese formalism, i.e. between Veronese Okubo and octonionic vectors. The isomorphism between the Okubo Veronese vectors and octonionic Veronese vectors is given by

$$\begin{cases} (x, y, x * y; n(y), n(x), 1) \longrightarrow \left(\tau^2(\overline{x}), y, y \cdot \overline{\tau^2(\overline{x})}; n(y), n(x), 1\right), \\ (0, 0, x; n(x), 1, 0) \longrightarrow \left(0, 0, \tau^2(\overline{x}); n(x), 1, 0\right), \\ (0, 0, 0; 1, 0, 0) \longrightarrow (0, 0, 0; 1, 0, 0), \end{cases} \tag{5.13}$$

where the first vectors are Veronese under conditions (3.3) and (3.4), while the image vectors are Veronese under conditions (3.23) and (3.24) that involves conjugation and octonionic product.

5.2 Isomorphism with the para-octonionic plane

Recall that a point of the paraoctonionic affine plane $\mathscr{A}_2(p\mathbb{O})$ plane is given by a pair of elements (x, y) with x,$y \in \{p\mathbb{O}\}$, while a *line* of slope $s \in p\mathbb{O}$ and offset $t \in p\mathbb{O}$ is the set $[s, t] = \{(x, s \bullet x + t) : x \in p\mathbb{O}\}$ and, of course, we say that a point $(x, y) \in \mathscr{A}_2(p\mathbb{O})$ is *incident* to a line $[s, t] \subset \mathscr{A}_2(p\mathbb{O})$ if belongs to such line, i.e. $(x, y) \in [s, t]$. The previous definitions define a para-octonionic affine plane, a projective plane can be directly defined through the Veronese conditions (3.21) and (3.22), i.e.,

$$\lambda_1 x_1 = x_2 \bullet x_3, \ \ \lambda_2 x_2 = x_3 \bullet x_1, \ \ \lambda_3 x_3 = x_1 \bullet x_2 \tag{5.14}$$

$$n(x_1) = \lambda_2\lambda_3, \ n(x_2) = \lambda_3\lambda_1, n(x_3) = \lambda_1\lambda_2. \tag{5.15}$$

An explicit isomorphism between the Okubo projective plane $\mathcal{O}P^2$ and the para-octonionic projective plane $p\mathbb{O}P^2$ is obtained considering the bijective map $p\Phi : \mathcal{O}P^2 \longrightarrow p\mathbb{O}P^2$ given by

$$p\Phi : \begin{cases} (x, y) & \longrightarrow \left(\tau^2(x), y\right), \\ (s) & \longrightarrow (\tau(s)), \\ (\infty) & \longrightarrow (\infty), \\ [s, t] & \longrightarrow [\tau(s), t], \\ [c] & \longrightarrow \left[\tau^2(c)\right], \\ [\infty] & \longrightarrow [\infty]. \end{cases} \tag{5.16}$$

The proof that the map given in (5.16) is a collineation adheres closely to the steps outlined in Theorem 24. This is expected, given the similarity between the maps. The sole distinction between para-octonions $p\mathbb{O}$ and octonions $\mathbb{O}$ is the presence of a paraunit in the former, as opposed to a unit in the latter, and the fact that while the former is merely flexible, the latter is alternative.

5.3 Isometries

Theorem 24 and its para-octonionic counterpart ensures projective isomorphism between the three planes $\mathcal{O}P^2$, $\mathbb{O}P^2$ and $p\mathbb{O}P^2$. If projective spaces are isomorphic, they share the same incidence relations between points and lines. However, this does not imply that the distances or angles between points are preserved and this is of high importance in our case since, according to Lie theory, it is well-known that the collineation group of the octonionic plane is the minimally non-compact real form of the exceptional Lie group E_6, namely $\mathrm{E}_{6(-26)}$, while the group of elliptic motions, i.e. isometries, over the octonionic projective plane is its maximal compact subgroup, namely $\mathrm{F}_{4(-52)}$ (see [13]). By the map in (5.4) we have an Okubo realization of both the Lie groups $\mathrm{F}_{4(-52)}$ and $\mathrm{E}_{6(-26)}$. Thus, we can write the homogeneous space presentation of the compact Cayley-Moufang plane $\mathcal{O}P^2$ as $\mathrm{F}_{4(-52)}/\mathrm{Spin}(9)$, a 16-dimensional symmetric coset. In fact, we have the following

Theorem 25 *The map Φ defined in* (5.4) *is an isometry.*

Proof By construction the Okubo plane comes equipped with the following distance:

$$d_{\mathcal{O}}(p_1, p_2) = n(x_1 - x_2)^2 + n(y_1 - y_2)^2, \tag{5.17}$$

with $p_1 = (x_1, y_1) \in \mathscr{A}^2(\mathcal{O})$ and $p_2 = (x_2, y_2) \in \mathscr{A}^2(\mathcal{O})$. By its very construction the octonionic plane comes with the following distance:

$$d_{\mathbb{O}} = n(x_1 - x_2)^2 + n(y_1 - y_2)^2, \tag{5.18}$$

with $p_1 = (x_1, y_1) \in \mathscr{A}^2(\mathbb{O})$ and $p_2 = (x_2, y_2) \in \mathscr{A}^2(\mathbb{O})$.Then, the images by Φ of the two points p_1 and p_2 are given by

$$p_1^{\Phi} = \left(\tau^2(\overline{x}_1), y_1\right), p_2^{\Phi} = \left(\tau^2(\overline{x}_2), y_2\right). \tag{5.19}$$

Therefore the octonionic distance between p_1^{Φ} and p_2^{Φ} is given by

$$d_{\mathbb{O}}\left(p_1^{\Phi}, p_2^{\Phi}\right) = n\left(\tau^2(\overline{x}_1) - \tau^2(\overline{x}_2)\right)^2 + n(y_1 - y_2)^2, \tag{5.20}$$

but since all automorphism of Hurwitz and para-Hurwitz algebras as isometries and τ^2 is an automorphism we have

$$\begin{aligned} d_{\mathbb{O}}\left(p_1^{\Phi}, p_2^{\Phi}\right) &= n\left(\tau^2(\overline{x}_1 - \overline{x}_2)\right)^2 + n(y_1 - y_2)^2 && (5.21)\\ &= n(\overline{x}_1 - \overline{x}_2)^2 + n(y_1 - y_2)^2 && (5.22)\\ &= n(x_1 - x_2)^2 + n(y_1 - y_2)^2 && (5.23)\\ &= d_{\mathcal{O}}(p_1, p_2), && (5.24)\end{aligned}$$

thus completing the proof. □

Similar and straightforward proof can be given for the map $p\Phi$ for the para-octonionic case.

5.4 Collineation groups

As a corollary of Theorem 24 we have that the Lie group of collineations of the Okubo projective plane is the following

Corollary 26 *The group of collineations of the Okubo projective plane is* $E_{6(-26)}$.

Proof Let $\Phi : \mathcal{O}P^2 \longrightarrow \mathbb{O}P^2$ be the isomorphism in (5.4) and $\Phi^{-1} : \mathbb{O}P^2 \longrightarrow \mathcal{O}P^2$ its inverse given by

$$\Phi^{-1} : \begin{cases} (x, y) & \longrightarrow (\tau(\overline{x}), y) \\ (s) & \longrightarrow \left(\tau^2(\overline{s})\right) \\ (\infty) & \longrightarrow (\infty) \\ [s, t] & \longrightarrow \left[\tau^2(\overline{s}), t\right] \\ [c] & \longrightarrow [\tau(\overline{c})] \\ [\infty] & \longrightarrow [\infty] \end{cases}, \tag{5.25}$$

where $x, y, s, t, c \in \mathbb{O}$. Then, since both Φ and Φ^{-1} send lines to lines, for every collineation of the octonionic projective plane $\gamma \in \text{Aut}\left(\mathbb{O}P^2\right)$, then the composition of collineations

$$\widetilde{\gamma} = \Phi^{-1}\gamma\Phi, \tag{5.26}$$

is a collineation of the Okubo projective plane. Conversely any collineation $\widetilde{\gamma} \in \text{Aut}\left(\mathcal{O}P^2\right)$ induces a collineation

$$\gamma = \Phi\widetilde{\gamma}\Phi^{-1}, \tag{5.27}$$

in $\text{Aut}\left(\mathbb{O}P^2\right)$. Moreover, it is clear that

$$\widetilde{\gamma} \circ \widetilde{\delta} = \left(\Phi^{-1}\gamma\Phi\right)\left(\Phi^{-1}\delta\Phi\right) = \Phi^{-1}(\gamma\delta)\Phi, \tag{5.28}$$

so that the two collineation groups are identical $\text{Aut}\left(\mathbb{O}P^2\right) \cong \text{Aut}\left(\mathcal{O}P^2\right)$, i.e. $\text{Aut}\left(\mathcal{O}P^2\right) \cong \text{E}_{6(-26)}$. □

For the sake of completeness, we now recover the collineation $\widetilde{\gamma} = \Phi^{-1}\gamma\Phi$ corresponding to the octonionic collineation γ given by the reflection $(x, y) \longrightarrow (y, x)$. By (5.26), we have that the octonionic reflection given by switching coordinates is on the Okubo plane given by the collineation

$$\widetilde{\gamma} : (x, y) \longrightarrow \left(\tau(\overline{y}), \tau^2(\overline{x})\right). \tag{5.29}$$

Moreover, given that the group of elliptic motion of the octonionic plane is $\mathrm{F}_{4(-52)}$, we have the following corollary of Theorem 25

Corollary 27 *The group of elliptic motion of the Okubo projective plane is* $F_{4(-52)}$.

Corollary 26 and 27 state the existence of an Okubonic geometric realization of exceptional Lie groups $\mathrm{E}_{6(-26)}$ and $\mathrm{F}_{4(-52)}$ which, to our knowledge was never pointed out.

5.5 G_2 as stabilizer of a quadrangle

Another exceptional Lie group with direct geometrical significance in the octonionic plane is $G_{2(-14)}$. This exceptional Lie group is recognized as the group of automorphisms of the octonions, denoted as $\mathrm{Aut}(\mathbb{O}) = G_{2(-14)}$. However, this is also the group $\Gamma(\diamondsuit, \mathbb{O})$ of collineations that fix each of the three points of a quadrangle [13, 61] of the projective octonionic plane. Given Theorems 24 and 25 our objective is to identify an Okubo realization of $G_{2(-14)}$. To this end, we examine the subgroups of collineations $\Gamma(\triangle, \mathcal{O})$ and $\Gamma(\diamondsuit, \mathcal{O})$. Specifically, the former represents the group of collineations that preserve every point of the triangle $\triangle = \{(0,0), (0), (\infty)\}$, while the latter is the group that maintains every point of the quadrangle $\diamondsuit = \{(0,0), (e,e), (0), (\infty)\}$.

Proposition 28 *The group* $\Gamma(\triangle, \mathcal{O})$ *of collineations that fix every point of* $\triangle$ *are transformations of this form*

$$(x, y) \mapsto (A(x), B(y)) \tag{5.30}$$

$$(s) \mapsto (C(s)) \tag{5.31}$$

$$(\infty) \mapsto (\infty) \tag{5.32}$$

where A, B *and* C *are invertible linear maps over* $\mathcal{O}$ *and in respect to multiplication they satisfy*

$$B(s * x) = C(s) * A(x). \tag{5.33}$$

Proof A collineation that fixes $(0,0)$, (0) and (∞), also leaves invariant the x-axis and y-axis. Moreover, since the incidence relations must be preserved, it maps all lines parallel to the x-axis and the y-axis to lines parallel to the x-axis and the y-axis. Then, the first coordinate is the image of a function that does not depend on y and the second coordinate is image of a fuction that does not depend by x, i.e. $(x, y) \mapsto (A(x), B(y))$ and $(s) \mapsto (C(s))$. Now consider the image of a point on the line $[s, t]$. The point is of the form $(x, s * x + t)$ and its image goes to

$$(x, s * x + t) \mapsto (A(x), B(s * x + t)). \tag{5.34}$$

In order this to be a collineation, the points of $[s, t]$ must all belong to a line that, setting $x = 0$, passes through the points $p_1 = (0, B(t))$ and $p_2 = (C(s))$, e.g. $[C(s), B(t)]$. Every line $(A(x), B(s * x + t))$ passing through p_1 and p_2 must satisfy the condition

$$B(s * x + t) = C(s) * A(x) + B(t). \tag{5.35}$$

Given (5.35), if B is an automorphism with respect to the sum over $\mathcal{O}$, then $B(s * x) = C(s) * A(x)$. Conversely if $B(s * x) = C(s) * A(x)$ is true than $B(s * x + t) = B(s * x) + B(t)$ and B is an automorphism with respect to the sum. □

A further corollary of Theorem 24 is that the Okubo projective plane, being the 16-dimensional Moufang plane, has a collineation group that is transitive on quadrangles. Without loss of generality we can thus consider the quadrangle $\diamondsuit$ given by the points $(0, 0)$, (e, e), (0) and (∞), that is $\diamondsuit = \triangle \cup \{(e, e)\}$, and consider the collineations that fix the set $\diamondsuit$.

For this purpose, it is important to note that a relation analogous to (5.33) holds in the case of both para-octonions and octonions. Specifically, in the octonionic case, we have

$$B(s \cdot x) = C(s) \cdot A(x), \tag{5.36}$$

for all $x, s \in \mathbb{O}$. This is particularly relevant since, in the case of octonions, the relation simplifies if we require the collineations to also stabilize the point (e, e), i.e. to stabilize the non-degenerate quadrangle $\diamondsuit = \triangle \cup \{(e, e)\}$. In this scenario, the previous formula transforms into

$$A(s \cdot x) = A(s) \cdot A(x), \tag{5.37}$$

which implies that the stabilizer of the non-degenerate quadrangle is isomorphic to the automorphism group of the octonions, denoted as $G_{2(-14)}$. Switching back to the Okubo algebra from the previous result, we then arrive at the following statement:

Theorem 29 *The exceptional Lie group $G_{2(-14)}$ has the following realisation*

$$G_{2(-14)} \cong \left\{(A, B, C) \in Spin(8)^3 : \begin{array}{l} B(s * x) = C(s) * A(x), \\ A(e) = B(e) = C(e) = e \end{array}\right\}, \tag{5.38}$$

or, equivalently

$$G_{2(-14)} \cong \left\{(A, B, C) \in Tri(\mathcal{O}) : \begin{array}{l} B(e * x) = e * A(x), \\ B(x * e) = C(x) * e \end{array}\right\}, \tag{5.39}$$

where $x, s \in \mathcal{O}$ and $e * e = e \in \mathcal{O}$.

We give here two independent proofs of the same statement. The first is a Corollary of the isomorphism in Theorem 24, relying on the fact that the stabilizer of a non-degenerate quadrangle on the octonionic projective plane is $\mathrm{G}_{2(-14)}$. The second is a proof based on Lie theory, that does not rely on the knowledge of the geometry of the octonionic plane.

Proof From the previous proposition the stabilizer of the triangle $\triangle$ is given by the group of triples $A, B, C \in \mathrm{Spin}(8)$ such that

$$B(s * x) = C(s) * A(x), \tag{5.40}$$

with $x, s \in \mathcal{O}$. To stabilize $\diamond = \triangle \cup \{(e, e)\}$, we have to impose

$$(e, e) \mapsto (A(e), B(e)) = (e, e), \tag{5.41}$$

and since $e * e = e$, then $C(e) = e$ and $A(e) = B(e) = C(e) = e$, obtaining the RHS of (5.38). Knowing that the stabilizer of a non-degenerate quadrangle of the compact 16-dimensional Moufang plane is $G_{2(-14)}$, we then obtain the identification with the LHS of (5.38). ◻

We now proceed with the second proof of Theorem 29.

Proof Recall that the triality group Tri($\mathcal{O}$) of the real Okubo algebra $\mathcal{O}$ is defined as

$$\mathrm{Tri}(\mathcal{O}) := \left\{(A, B, C) \in \mathrm{Spin}(8)^3 : B(s * x) = C(s) * A(x), \ \forall s, x \in \mathcal{O}\right\}, \tag{5.42}$$

and that, as proved in [44], $\mathrm{Tri}(\mathcal{O}) \simeq \mathrm{Spin}(8) \simeq \mathrm{Spin}(\mathcal{O})$. Let us consider the action of triality (5.42) in three cases: the first where $s = x = e$, for which

$$B(e) = C(e) * A(e), \tag{5.43}$$

the second with $s \in \mathcal{O}$ and fixed $x = e$, i.e.,

$$B(s * e) = C(s) * A(e), \tag{5.44}$$

finally, the case $x \in \mathcal{O}$ and $s = e$ for which

$$B(e * x) = C(e) * A(x). \tag{5.45}$$

Now, we want to determine the subgroup of Tri($\mathcal{O}$) defined by the following constraints

$$B(e * x) = e * A(x), \tag{5.46}$$

$$B(x * e) = C(x) * e. \tag{5.47}$$

Since $\mathcal{O}$ is a division algebra, (5.45) and the constraint (5.46), as well as (5.44) and the constraint (5.47), imply

$$C(e) = e = A(e), \tag{5.48}$$

which in turn imply, by (5.43), that $B(e) = e * e = e$. Thus one can reformulate the constraints (5.46) and (5.47), for any subgroup of $\mathrm{Tri}(\mathcal{O}) \simeq \mathrm{Spin}(8)$, with (5.48).

A well-known theorem by Dynkin (see Th. 1.5 of [18]) states that a maximal (and non-symmetric) embedding of SU_3 =Aut($\mathcal{O}$) into Spin (8) exists such that all 8-dimensional irreducible representations of Spin (8) stay irreducible in SU_3, all reducing to the same adjoint representation. By using the Dynkin labels to identify the representations, it holds that

$$\mathrm{Spin}(8) \underset{\max,\mathrm{ns}}{\supset} SU_3,$$

$$\begin{array}{c} (1,0,0,0) = (1,1)\,, \\ (0,0,0,1) = (1,1)\,, \\ (0,0,1,0) = (1,1)\,, \end{array} \tag{5.49}$$

where we adopted the conventions of [77], and the subscripts "max", "s" and "ns" respectively stand for maximal, symmetric and non-symmetric. Thus, for the triality of Spin(8), the adjoint irrepr. $(1,1)$ of SU_3, for which the basis (2.20) of the Okubo algebra $\mathcal{O}$ provides a realization

$$\mathcal{O} \simeq < e, \mathrm{i}_1, .., \mathrm{i}_7 > \simeq (1,1) \text{ of } SU_3, \tag{5.50}$$

can be mapped to any of the three 8-dimensional irreducible representations of Spin (8). Let $\mathcal{B}$ the set of elements $\{e, \mathrm{i}_1, .., \mathrm{i}_7\}$, then, with no loss of generality, up to triality of Spin (8), one can identify

$$\begin{array}{l} C(\mathcal{B}) := \{C(e)\,, C(\mathrm{i}_1)\,, .., C(\mathrm{i}_7)\} \simeq (1,0,0,0) \text{ of Spin}\,(8)\,, \\ A(\mathcal{B}) := \{A(e)\,, A(\mathrm{i}_1)\,, .., A(\mathrm{i}_7)\} \simeq (0,0,0,1) \text{ of Spin}\,(8)\,, \\ B(\mathcal{B}) := \{B(e)\,, B(\mathrm{i}_1)\,, .., B(\mathrm{i}_7)\} \simeq (0,0,1,0) \text{ of Spin}\,(8)\,. \end{array} \tag{5.51}$$

We now implement the first constraint of (5.48), namely of $C(e) = e$. The largest subgroup of Spin (8) allowing $C(e) = e$ is its maximal (and symmetric) subgroup Spin(7), for which it holds that

$$\text{Spin}(8) \underset{\text{max,s}}{\supset} \text{Spin}\,(7) \ni A, B, C : \begin{cases} (1,0,0,0) = \underset{C(e)=e}{(0,0,0)} \oplus \underset{\{C(\mathrm{i}_1),..,C(\mathrm{i}_7)\}}{(1,0,0)}\,, \\ (0,0,0,1) = \underset{\{A(e),A(\mathrm{i}_1),..,A(\mathrm{i}_7)\}}{(0,0,1)}\,, \\ (0,0,1,0) = \underset{\{B(e),B(\mathrm{i}_1),..,B(\mathrm{i}_7)\}}{(0,0,1)}\,, \end{cases} \tag{5.52}$$

where $(1,0,0)$ is the 7-dimensional fundamental (*vector*) irrepr. of Spin (7) and $(0,0,1)$ denotes its 8-dimensional (*spinor*) irrepr.. Next, one must impose the second constraint of (5.48). This can be implemented by a further symmetry breaking implying $A(e) = e$, which, together to $C(e) = e$, implies also that $B(e) = e$. The largest subgroup of Spin (7) allowing for an action of this kind is $G_{2(-14)}$, which is a maximal and non-symmetric subgroup of Spin (7) itself:

$$\text{Spin}\,(7) \underset{\text{max,ns}}{\supset} G_{2(-14)} \ni A, B, C : \tag{5.53}$$

$$\begin{cases} (0,0,0) = \underset{C(e)=e}{(0,0)}\,, \\ (1,0,0) = \underset{\{C(\mathrm{i}_1),..,C(\mathrm{i}_7)\}}{(0,1)}\,, \\ (0,0,1) = \underset{A(e)=e}{(0,0)} \oplus \underset{\{A(\mathrm{i}_1),..,A(\mathrm{i}_7)\}}{(0,1)}\,, \\ (0,0,1) = \underset{B(e)=e}{(0,0)} \oplus \underset{\{B(\mathrm{i}_1),..,B(\mathrm{i}_7)\}}{(0,1)}\,. \end{cases} \tag{5.54}$$

Thus, the (largest) subgroup of $\mathrm{Tri}(\mathcal{O}) \simeq$Spin (8) defined by the constraints (5.48) (or, equivalently, by the constraints (5.46) and (5.47)) is $G_{2(-14)}$, which is next-to-maximal (and symmetric) in Spin (8), being determined by the chain of two maximal embeddings:

$$\mathrm{Spin}\ (8) \supset_{\max,\mathrm{s}} \mathrm{Spin}\ (7) \supset_{\max,\mathrm{ns}} G_{2(-14)}. \tag{5.55}$$

□

6 Discussions and verifications

Let A be an algebra, and let x, y, z be elements of this algebra. It is well known that the validity of Moufang identities (2.12) in the algebra, i.e.

$$((x \cdot y) \cdot x) \cdot z = x \cdot (y \cdot (x \cdot z)), \tag{6.1}$$

$$((z \cdot x) \cdot y) \cdot x = z \cdot (x \cdot (y \cdot x)), \tag{6.2}$$

$$(x \cdot y) \cdot (z \cdot x) = x \cdot ((y \cdot z) \cdot x), \tag{6.3}$$

are linked with the Moufang properties of projective plane over the algebra [48, 61, Sec. 12.15]. Furthermore, as an immediate corollary, if the algebra is unital, then, setting $y = 1$ the Moufang identities imply alternativity, resulting in

$$(x \cdot x) \cdot z = x \cdot (x \cdot z), \tag{6.4}$$

$$(z \cdot x) \cdot x = z \cdot (x \cdot x), \tag{6.5}$$

for every $x, z \in A$. In fact, the relation between alternative rings and Moufang planes is a one-to-one correspondence [56, p. 160], [39, p. 143] and is so deep that Moufang planes are also called "alternative planes" [74].

Considering this background, the isomorphism between the Okubo projective plane and the Cayley plane appears counterintuitive. The Okubo algebra is non-alternative and non-unital, making it markedly different from the often-considered octonionic algebra used for realizing the 16-dimensional Moufang plane. Notably, the Moufang identities in (6.3) do not hold in the Okubo algebra. Hence, the emergence of a Moufang plane from a projective plane over the Okubo algebra demands a thorough explanation.

In fact, the Moufang property of a plane is tied to the alternativity of its associated planar ternary ring (PTR). Typically, this ternary ring is linear (see below) and thus isomorphic to the algebra from which the plane is defined. However, as we will show in this section, the Okubo case is not so simple. Since the Okubo algebra lacks an identity, it is not a ring and therefore cannot be employed to coordinatize the Okubo projective plane. Clearly the best candidate for such a coordinatization are the octonions $\mathbb{O}$.

6.1 Coordinatizing the Okubo plane with Octonions

In incidence geometry, relationships between incidence planes and algebraic structures are developed after a process of relabeling called coordinatisation that involves a set

$\mathscr{C}$ containing the symbols $0, 1 \in \mathscr{C}$ and not containing the symbol ∞. In most of cases, dealing with unital algebras the set $\mathscr{C}$ is just the original algebra used for the definition of the plane, but since the Okubo algebra is not unital the set of symbol $\mathscr{C}$ cannot be $\mathcal{O}$. In our case, a natural candidate for the set $\mathscr{C}$ is clearly the ring of octonions $\mathbb{O}$. To coordinatise the Okubo plane with octonionic coordinates we consider the non-degenerate quadrangle $\diamondsuit = \{(0,0), (e,e), (0), (\infty)\}$ of the Okubo projective plane $\mathcal{O}P^2$ and use the ring of octonions $\mathbb{O}$ for its coordinatisation so that, in the new coordinates, the quadrangle is $\{(0,0), (1,1), (0), (\infty)\}$. From the general theory we know [29, Cor. 3.4] that up to isomorphism there is a unique standard coordinatisation that maps $\diamondsuit$ into $\{(0,0), (1,1), (0), (\infty)\}$. In fact, such coordinatisation is easily obtained sending the Okubo elements with their respective octonionic representative following the previous deformation of the product (see Tab. 3), so that $e \in \mathcal{O}$ is sent to $1 \in \mathbb{O}$ and the other elements of the base $\{\mathrm{i}_1, ..., \mathrm{i}_7\}$ in (2.20) are sent into a multiple of the imaginary unit of the octonions $\mathbb{O}$.

Once the relabeling process called coordinatization is done, we can now define a unique ternary ring with ternary operation θ that encodes algebraically the geometrical properties of the incidence plane (see [39, Ch. 5]). Indeed, we define the planar ternary ring (PTR) by the incidence rules of the plane so that

$$\theta(s, x, t) = y, \quad \text{iff} \quad (x, y) \in [s, t], \tag{6.6}$$

for all $x, y, s, t \in \mathscr{C}$. For this ternary ring we then define an *associated product* and an *associated sum*, i.e.

$$sx := \theta(s, x, 0), \tag{6.7}$$

$$x + t := \theta(1, x, t). \tag{6.8}$$

Then algebraic properties of the associated product and of the associated sum are then studied in order to deduce geometrical properties of the coordinatized projective plane.

Note that in case of the octonionic projective plane $\mathbb{O}P^2$ we have that the product and the sum of θ coincided with those defined over the algebra of octonions $(\mathbb{O}, +, \cdot)$ so that

$$\theta(s, x, t) = sx + t = s \cdot x + t. \tag{6.9}$$

When this happen, the planar ternary ring is called *linear* [61, Sec. 22.4]. Unfortunately, this is not the case for the Okubo projective plane so that the ternary ring derived coordinatising the Okubo projective plane with the ring of the octonions $\mathbb{O}$ is not linear since

$$\theta(s, x, t) = sx + t \neq s * x + t, \tag{6.10}$$

as one easily might expect since the octonionic product is not the Okubo product.

In summary, while the planar ternary ring is alternative (thereby not contradicting the one-to-one relationship between Moufang planes and alternative rings), the originating algebra is non-alternative. This distinction arises because the ternary ring derived from coordinatization is nonlinear.

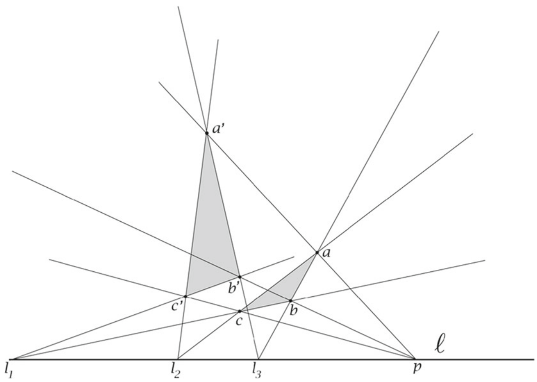

Fig. 4 Little Desargues configuration: two triangles a, b, c and a', b', c' are perspective, i.e. lines $\overline{aa'}$, $\overline{bb'}$ and $\overline{cc'}$ interesect on the same point p that is the origin of the perspectivity, then the points of intersection of corresponding sides all lie on one line ℓ that is the axis of the perspectivity. In the Little Desargues configuration the perspectivity that relates the two triangles is also an elation, thus the center of the perspectivity p is incident to the axis ℓ

Remark 30 It is worth noting that while we are relabeling the Okubo coordinates with octonions in order to obtain a planar ternary ring, this process does not constitutes at all a projective isomorphism between the Okubo plane $\mathcal{O}P^2$ and the octonionic projective plane $\mathbb{O}P^2$. Indeed, consider any map $\phi : \mathcal{O} \longrightarrow \mathbb{O}$ is such that

$$\begin{aligned} 0_{\mathcal{O}} &\longrightarrow 0_{\mathbb{O}}, \\ e_{\mathcal{O}} &\longrightarrow 1_{\mathbb{O}}, \\ x_{\mathcal{O}} &\longrightarrow \phi(x). \end{aligned} \tag{6.11}$$

Then consider on the octonionic plane $\mathbb{O}P^2$ the three points $(0_{\mathbb{O}}, 0_{\mathbb{O}})$, $(1_{\mathbb{O}}, 1_{\mathbb{O}})$, $(\phi(x), \phi(x))$. These three points are collinear, belonging to the same line $[1, 0] = \{x \in \mathbb{O} : (x, 1 \cdot x)\}$, i.e. the line passing through the origin with slope $s = 1$. Nevertheless, in the Okubo plane $\mathcal{O}P^2$ the three points $(0_{\mathcal{O}}, 0_{\mathcal{O}})$, $(e_{\mathcal{O}}, e_{\mathcal{O}})$, (x, x) are not collinear. Indeed the only line joining $(0_{\mathcal{O}}, 0_{\mathcal{O}})$, $(e_{\mathcal{O}}, e_{\mathcal{O}})$ is $[e_{\mathcal{O}}, 0]$,i.e. $(0_{\mathcal{O}}, 0_{\mathcal{O}})$, $(e_{\mathcal{O}}, e_{\mathcal{O}}) \in [e_{\mathcal{O}}, 0] = \{x \in \mathcal{O} : (x, e * x)\}$, while the point $(x, x) \notin [e_{\mathcal{O}}, 0]$ since $x \neq e * x$ for any $x \neq e$. We thus have that any relabeling process of the Okubo algebra with the octonionic algebra does not yield to a collineation and, in fact, changes the incidence rules of the plane.

6.2 Direct verification of the "Little Desargues Theorem"

It is well-known that the "*Little Desargues Theorem*" is valid in every Moufang plane. This theorem is a weaker version of the *Desargues Theorem*, which holds true for every Moufang plane over an associative algebra.

The Deasargues theorem states that if two triangles a, b, c and a', b', c', are perspective, i.e. lines $\overline{aa'}$, $\overline{bb'}$ and $\overline{cc'}$ interesect on the same point p that is the origin of the perspectivity, then the points of intersection of corresponding sides all lie on one line ℓ, termed the axis of the perspectivity. However, this theorem is not valid in non-associative Moufang planes. A special case arises when the point p also lies on the axis ℓ, making the perspectivity an elation (see Sect. 4.3); this case is valid in all Moufang planes and is known as the "Little Desargues Theorem" (see Fig. 4).

Rather than presenting a formal proof of the theorem's validity (which is already established for any Moufang plane), we opted for a numerical verification using a Wolfram Mathematica notebook now available at the repository https://github.com/DCorradetti/OkuboAlgebras. The notebook is fully documented with a notation coherent to that used in this article, so that it can be easily used to verify all calculations of this article involving octonions, para-octonions and the real Okubo algebra. To validate the Little Desargues Theorem, we first represented the real Okubo algebra three-by-three complex Hermitian matrices endowed with the Okubo product in (2.16). Next, we developed the following Mathematica functions:

- `sLine`[x_1,y_1,x_2,y_2]: Computes the slope of the line connecting points (x_1, y_1) and (x_2, y_2);
- `tLine`[x_1,y_1,x_2,y_2]: Determines the offset of the line connecting points (x_1, y_1) and (x_2, y_2);
- `xPoint`[s_1,t_1,s_2,t_2]:Calculates the x-coordinate of the intersection point of lines $[s_1, t_1]$ and $[s_2, t_2]$;
- `yPoint`[s_1,t_1,s_2,t_2]:Calculates the y-coordinate of the intersection point of lines $[s_1, t_1]$ and $[s_2, t_2]$;
- `incidence`[x,y,s,t]:Determines the y-coordinate of the intersection point of lines $[s_1, t_1]$ and $[s_2, t_2]$;

All function arguments are elements of the real Okubo algebra. Then, to set up the configuration for the Little Desargues Theorem, we:

(1) Defined the center of the perspectivity p and the axis ℓ so that $p \in \ell$.
(2) Picked a point a not belonging to ℓ and a point a' incident to the line $\overrightarrow{ap}$.
(3) Picked a point b not belonging to ℓ nor $\overrightarrow{ap}$ and defined the line $\overrightarrow{ab}$.
(4) Found the intersection $l_3$of $\overrightarrow{ab}$ with ℓ.
(5) Found the point b' given by the intersection of the lines $\overrightarrow{l_3 a'}$ and $\overrightarrow{bp}$.
(6) Picked a point c not belonging to ℓ nor $\overrightarrow{ap}$ nor $\overrightarrow{bp}$ and thus defined the line $\overrightarrow{ac}$.
(7) Found the intersection $l_2$of $\overrightarrow{ac}$ with ℓ.
(8) Found the point c' given by the intersection of lines $\overrightarrow{l_2 a'}$ and $\overrightarrow{cp}$.

Finally, to check the validity of the "little Desargues theorem" we verified that the point l_1, given by the intersection of $\overrightarrow{cb}$ with $\overrightarrow{c'b'}$, was incident to ℓ. As shown in the Wolfram notebook, we verified the validity of the "little Desargues theorem", and, choosing $p \notin \ell$, that the full Desargues theorem is not valid.

Table 4 Summary of the algebraic properties of the three division and composition algebras that allows a straightforward and natural definition of the Cayley plane with the mathematical setup described in this thesis

	$\mathbb{O}$	$p\mathbb{O}$	$\mathcal{O}$
Unital	Yes	No	No
Paraunital	No	Yes	No
Alternative	Yes	No	No
Flexible	Yes	Yes	Yes
Composition	Yes	Yes	Yes
Automorphism	G_2	G_2	SU (3)

7 Conclusions

This work provides a novel construction of the 16-dimensional Cayley plane using two flexible algebras: the para-octonions and the real Okubo algebra. Despite lacking many algebraic properties of octonions, including alternativity and identity element, both algebras nonetheless give the same projective plane up to isometries. Through two explicit collineations, we established an equivalence between the Okubo, octonionic, and para-octonionic planes, i.e., $\mathcal{O}P^2$, $\mathbb{O}P^2$ and $p\mathbb{O}P^2$. Moreover, numerical computations directly confirmed foundational projective properties like the Little Desargues Theorem.

Among the three constructions of the 16-dimensional Moufang plane, the one based on the Okubo algebra $\mathcal{O}$ is the simplest possible for the definition of such a plane, requiring only an 8-dimensional algebra that is neither alternative nor unital and that has an automorphism group of real dimension 8, compared to that of the octonions, and para-octonios that has dimension 14 (see Table 4). Surprisingly, even if historically octonions $\mathbb{O}$ were the first algebra used for the definition of the compact 16-dimensional Moufang plane, they are the less economic algebra that can be used for the definition such plane. For the sake of completeness we should say that, in order to have an affine plane correctly defined with a natural setup as that defined above, the 8-dimensional algebra used for its definition must be a division algebra. Moreover, to have a completion of the affine plane in correspondence with a Veronese-type of condition as those above, one needs to have a composition algebra. Thus, for the generalised Hurwitz theorem [20], the only three algebras for which this setup can exist are those recalled in this paper.

As a corollary of the construction presented in this work, concrete geometric realizations of the real forms of the exceptional Lie groups $E_{6(-26)}$, $F_{4(-52)}$ and $G_{2(-14)}$ emerge without recurring to the uses of octonions. This challenges the conventional thinking that links exceptional Lie and Jordan structures to octonions and emphasizes the role of symmetric composition algebras. Future work should further explore, algebraic and physical interpretations of this new realization.

From the algebraic point of view, the existence of an Okubo Jordan algebra [28] is known, that we expect to be linked with 16-dimensional Moufang plane as the exceptional Jordan algebra is [43]. An Okubo construction of Tits-Freudenthal Magic Square was already considered for symmetric composition algebras [22, 23], so we do expect to find a geometrical interpretation of those construction as Freudenthal and

Rosenfeld did for the Hurwitz version[34, 58]. From the physical side, given the connections between M-theory and the octonionic Cayley plane, we expect that alternative constructions like the Okubo algebra that has SU (3) as automorphism group instead of G_2 may unravel novel phases of the theory. In particular, the investigation of the physical consequences of the lack of unity of Okubo algebra may turn out to be rather intriguing. We also have provided a valuable reference implementation of the algebras with examples in a Wolfram Mathematica notebook in order to help researchers in practical calculations.

All in all, by going beyond the deeply rooted connections between octonions and exceptional mathematics, this study paves the way to reimagining non-associative geometry. Symmetric composition algebras, once considered pathological, may encapsulate geometric worlds equally rich as their unital cousins. Much work remains to fully chart the landscape.

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Declarations

Conflict of interest The authors have no conflict of interest.

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