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On the modelling of thermal convection in porous media through rate-type equations

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Abstract
The paper investigates current models of flows in porous media from the viewpoint of the mixture theory. The constitutive equations are investigated for compressible, viscous, heat-conducting fluids subject to relaxation phenomena. The thermodynamic analysis is performed via the Clausius-Duhem inequality based directly on the peculiar fields of the mixture. The detailed analysis so developed involves the peculiar heat fluxes and stresses per se while the balance equations for energy and entropy of the whole body would involve also diffusion effects. Following the objectivity principle, the constitutive equations for stresses and heat fluxes are taken to be governed by objective rate equations.

Keywords Porous media · Theory of mixtures · Thermodynamic consistency · Objectivity

Mathematics Subject Classification 76S05 · 76R50 · 76T30 · 74D10

1 Introduction

The modelling of fluid flow in porous media shows interesting problems relative to both balance equations and constitutive properties. A variety of models occurs mainly because of the nature of the fluid and the distributed contact between the fluid and the skeletal solid.

The first model governing the fluid flow in a porous medium is due to Darcy [1]. In local form and stationary conditions, Darcy's equation associates the seepage velocity

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$\mathbf{v}$ to the pressure gradient ∇p in the form

$$\mathbf{v} = -\frac{k}{\mu}\nabla p, \tag{1}$$

where k is the permeability of the skeletal solid and μ the shear viscosity of the fluid.

Yet the interest in the dynamics of porous media indicates some suitable properties that are in order for a realistic model. In this regard viscosity, compressibility, and relaxation properties are needed for the fluid along with a proper model for heat conduction. Furthermore the motion of the fluid within the porous medium requires an appropriate description of fluid–solid interaction. References [2–6] and references therein give an exhaustive scenario of the models adopted in the literature. On the side of simplifications, often the fluid is assumed to be incompressible. Instead, on the side of generalizations, relaxation properties are modelled for the stress and the heat flux.

This paper is based on the view that a convenient approach should be grounded on the theory of mixtures. In this sense next section reviews the main point of the mixture theory and next quite general models are established for mixtures of fluids and fluid–solid mixtures.

Now, the physical relevance of a model depends on the verified thermodynamic consistency. Hence attention is addressed to the formulation of the second law of thermodynamics for mixtures. Though the mixture theory is by now well established [7–11], the application to solid–fluid mixtures is still of interest mainly in connection with the constitutive properties. Furthermore, the modelling of relaxation properties through rate equations involves objective time derivatives [12], which is a further interesting topic in the thermodynamic analysis.

Some features denote the originality of the present approach. The fluid is allowed to be compressible, besides being viscous. The consistency with the entropy principle is stated and developed in detail with reference to quantities pertaining to the single constituents rather than to the mixture as a whole. Results are derived for mixtures with several temperatures and for mixtures with a single temperature.

Notation. The body under consideration is a mixture of n constituents occupying a time-dependent region of the three-dimensional space. The subscript $\alpha = 1, 2, \ldots, n$ labels the fields pertaining to the α-th constituent and $\sum_\alpha$ is a shorthand for $\sum_{\alpha=1}^{n}$. Sym is the set of symmetric tensors and tr denotes the trace. The compact notation is used; for any pair of vectors $\mathbf{u}$, $\mathbf{v}$ the symbol $\mathbf{u}\cdot\mathbf{v}$ denotes the inner product, $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, and likewise for tensors, $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij} = \mathrm{tr}\,(\mathbf{A}\mathbf{B}^T)$. The symbol ∇ denotes the gradient, $\nabla\cdot$ the divergence, and $\otimes$ the dyadic product.

2 A setting from the theory of mixtures

The natural setting for the fluid flow in a porous medium is that of mixtures. Since we are interested in porous media we restrict attention to non-reacting mixtures. The mixture consists of n constituents though the application to porous media involves 2 constituents.

Denote by the subscript $\alpha = 1, 2, .., n$ the quantities pertaining to the α-th constituent. For any function f_α the dashed symbol $\grave{f}_\alpha$ denotes the material derivative relative to the pertinent constituent, viz.

$$\grave{f}_\alpha = \partial_t f_\alpha + (\mathbf{v}_\alpha \cdot \nabla) f_\alpha.$$

The conservation of mass of single constituents results into the n continuity equations

$$\grave{\rho}_\alpha + \rho_\alpha \nabla \cdot \mathbf{v}_\alpha = 0. \tag{2}$$

The equations of motion are written in the form

$$\rho_\alpha \grave{\mathbf{v}}_\alpha = \nabla \cdot \mathbf{T}_\alpha + \rho_\alpha \mathbf{b}_\alpha + \mathbf{m}_\alpha, \tag{3}$$

where $\mathbf{T}_\alpha$ is the (Cauchy) stress tensor, $\mathbf{b}_\alpha$ the body force, $\mathbf{m}_\alpha$ the interaction force, or growth, between constituents. The growths are subject to

$$\textstyle\sum_\alpha \mathbf{m}_\alpha = \mathbf{0}.$$

No body couples are considered and then the balance of angular momentum results in

$$\mathbf{T}_\alpha = \mathbf{T}_\alpha^T.$$

Let ε_α be the specific internal energy. We write the local version of the balance of energy in the form

$$\rho_\alpha (\varepsilon_\alpha + \tfrac{1}{2} \mathbf{v}_\alpha^2)\grave{} = \nabla \cdot (\mathbf{v}_\alpha \mathbf{T}_\alpha) - \nabla \cdot \mathbf{q}_\alpha + \rho_\alpha r_\alpha + e_\alpha, \tag{4}$$

where r_α is the energy supply and e_α the energy growth, so that

$$\textstyle\sum_\alpha e_\alpha = 0.$$

We notice that $(\frac{1}{2}\mathbf{v}_\alpha^2)\grave{} = \mathbf{v}_\alpha \cdot \grave{\mathbf{v}}_\alpha$ and use (3) to obtain

$$\rho_\alpha \grave{\varepsilon}_\alpha = \mathbf{T}_\alpha \cdot \mathbf{D}_\alpha - \nabla \cdot \mathbf{q}_\alpha + \rho_\alpha r_\alpha + \hat{\varepsilon}_\alpha, \tag{5}$$

where

$$\hat{\varepsilon}_\alpha = e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha.$$

Lastly we look at the second law of thermodynamics which, also for mixtures, places restrictions on the admissible constitutive equations. For any α-th constituent let θ_α be the absolute temperature and η_α the specific entropy. The balance of entropy is derived by the general view ([13], §6.5) that
entropy change = *entropy transfer* + *entropy production*

is made formal by letting $\mathbf{j}_\alpha$ be the entropy flux, $\rho_\alpha r_\alpha/\theta_\alpha$ the entropy supply and $\rho_\alpha\gamma_\alpha$ the entropy production so that

$$\rho_\alpha \grave{\eta}_\alpha + \nabla\cdot\mathbf{j}_\alpha - \frac{\rho_\alpha r_\alpha}{\theta_\alpha} = \rho_\alpha\gamma_\alpha. \tag{6}$$

The set of functions

$$\{\rho_\alpha, \mathbf{v}_\alpha, \mathbf{T}_\alpha, \mathbf{b}_\alpha, \varepsilon_\alpha, \mathbf{q}_\alpha, r_\alpha, \theta_\alpha, \eta_\alpha, \mathbf{j}_\alpha, \gamma_\alpha\}$$

constitutes a thermodynamic process. The axiom, known as entropy principle or second law of thermodynamics, about the increase of entropy in a closed system is stated by saying that the entropy production is non-negative for any thermodynamic process consistent with the balance equations. Formally, for mixtures *the second law of thermodynamics requires that*

$$\textstyle\sum_\alpha \rho_\alpha\gamma_\alpha \geq 0 \tag{7}$$

for any thermodynamic process.

This statement is based on Refs. [7–10]. However, following [12], §9.3, we let the entropy productions $\{\gamma_\alpha\}$ be given by constitutive equations, as is done for the entropy fluxes $\{\mathbf{j}_\alpha\}$ after [8].

If the constitutive equations make the inequality non-valid then those constitutive equations are not admissible. That is why we can see the second law as the selection of physically admissible constitutive models.

For technical convenience we put

$$\mathbf{j}_\alpha = \frac{\mathbf{q}_\alpha}{\theta_\alpha} + \mathbf{k}_\alpha,$$

$\mathbf{k}_\alpha$ being referred to as extra-entropy flux. Hence we can write Eq. (6) as

$$\frac{1}{\theta_\alpha}\{\rho_\alpha\theta_\alpha\grave{\eta}_\alpha + \nabla\cdot\mathbf{q}_\alpha - \rho_\alpha r_\alpha - \frac{1}{\theta_\alpha}\mathbf{q}_\alpha\cdot\nabla\theta_\alpha + \theta_\alpha\nabla\cdot\mathbf{k}_\alpha\} = \rho_\alpha\gamma_\alpha.$$

Substitution of $\nabla\cdot\mathbf{q}_\alpha - \rho_\alpha r_\alpha$ from (5) results in

$$\frac{1}{\theta_\alpha}\{\theta_\alpha\rho_\alpha\grave{\eta}_\alpha + \mathbf{T}_\alpha\cdot\mathbf{D}_\alpha - \rho_\alpha\grave{\varepsilon}_\alpha + \hat{\varepsilon}_\alpha - \frac{1}{\theta_\alpha}\mathbf{q}_\alpha\cdot\nabla\theta_\alpha + \theta_\alpha\nabla\cdot\mathbf{k}_\alpha\} = \rho_\alpha\gamma_\alpha.$$

Using the Helmholtz free energy $\psi_\alpha = \varepsilon_\alpha - \theta_\alpha\eta_\alpha$ we have

$$\frac{1}{\theta_\alpha}\{-\rho_\alpha(\grave{\psi}_\alpha + \eta_\alpha\grave{\theta}_\alpha) + \mathbf{T}_\alpha\cdot\mathbf{D}_\alpha + \hat{\varepsilon}_\alpha - \frac{1}{\theta_\alpha}\mathbf{q}_\alpha\cdot\nabla\theta_\alpha + \theta_\alpha\nabla\cdot\mathbf{k}_\alpha\} = \rho_\alpha\gamma_\alpha.$$

Hence the second law is expressed by the Clausius-Duhem (CD) inequality

$$\textstyle\sum_\alpha \frac{1}{\theta_\alpha}\{-\rho_\alpha(\grave{\psi}_\alpha+\eta_\alpha\grave{\theta}_\alpha)+\mathbf{T}_\alpha\cdot\mathbf{D}_\alpha+\hat{\varepsilon}_\alpha-\frac{1}{\theta_\alpha}\mathbf{q}_\alpha\cdot\nabla\theta_\alpha+\theta_\alpha\nabla\cdot\mathbf{k}_\alpha\} = \sum_\alpha\rho_\alpha\gamma_\alpha \geq 0. \tag{8}$$

Before investigating the thermodynamic requirements on the pertinent constitutive equations, we look at properties of a mixture as a single continuum.

3 Properties of a mixture as a whole

It might be of interest to derive equations for mixtures viewed as a single continuum. Yet we show that different models arise depending on the conditions required to the constituents. Define

$$\rho = \sum_\alpha \rho_\alpha, \qquad \rho \mathbf{v} = \sum_\alpha \rho_\alpha \mathbf{v}_\alpha. \tag{9}$$

Hence letting

$$\mathbf{u}_\alpha = \mathbf{v}_\alpha - \mathbf{v}, \qquad \mathbf{h}_\alpha = \rho_\alpha \mathbf{u}_\alpha$$

it follows that

$$\sum_\alpha \mathbf{h}_\alpha = \mathbf{0}, \quad \sum_\alpha (\rho_\alpha \mathbf{v}_\alpha \otimes \mathbf{v}_\alpha) = \rho \mathbf{v} \otimes \mathbf{v} + \sum_\alpha \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha. \tag{10}$$

The barycentric velocity $\mathbf{v}$ is viewed as the velocity of the continuum. Furthermore, for any (scalar or vector) function $\mathbf{f}(t, \mathbf{x})$ we let

$$\dot{\mathbf{f}} = \partial_t \mathbf{f} + (\mathbf{v} \cdot \nabla)\mathbf{f}.$$

For technical purposes the use of the partial time derivative $\partial_t \mathbf{f}_\alpha$ is now more convenient than that of the peculiar time derivative $\grave{\mathbf{f}}_\alpha$. Hence we start with the continuity Eq. (2) and observe that

$$\grave{\rho}_\alpha + \rho_\alpha \nabla \cdot \mathbf{v}_\alpha = \partial_t \rho_\alpha + \mathbf{v}_\alpha \cdot \nabla \rho_\alpha + \rho_\alpha \nabla \cdot \mathbf{v}_\alpha = \partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha)$$

whence (2) can be written in the form

$$\partial_t \rho_\alpha + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) = 0. \tag{11}$$

Summation over α results in the continuity equation for the whole body,

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{12}$$

For any peculiar quantity $\mathbf{f}_\alpha$, using (11) we have

$$\begin{aligned} \rho_\alpha \grave{\mathbf{f}}_\alpha &= \rho_\alpha [\partial_t \mathbf{f}_\alpha + (\mathbf{v}_\alpha \cdot \nabla)\mathbf{f}_\alpha)] = \partial_t(\rho_\alpha \mathbf{f}_\alpha) + \mathbf{f}_\alpha \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha) + \rho_\alpha (\mathbf{v}_\alpha \cdot \nabla)\mathbf{f}_\alpha \\ &= \partial_t(\rho_\alpha \mathbf{f}_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{f}_\alpha \otimes \mathbf{v}_\alpha), \end{aligned}$$

where $\nabla \cdot (\rho_\alpha \mathbf{f}_\alpha \otimes \mathbf{v}_\alpha) = \partial_{x_j}(\rho_\alpha \mathbf{f}_\alpha v_j^\alpha)$. To split peculiar terms in barycentric parts and relative (or diffusive) parts we replace $\mathbf{v}_\alpha$ with $\mathbf{v} + \mathbf{u}_\alpha$ and observe that

$$\rho_\alpha \grave{\mathbf{f}}_\alpha = \partial_t(\rho_\alpha \mathbf{f}_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{f}_\alpha \mathbf{v}) + \nabla \cdot (\rho_\alpha \mathbf{f}_\alpha \mathbf{u}_\alpha). \tag{13}$$

Letting $\mathbf{f}_\alpha = \mathbf{v}_\alpha$ it follows that

$$\rho_\alpha \dot{\mathbf{v}}_\alpha = \partial_t(\rho_\alpha \mathbf{v}_\alpha) + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \mathbf{v}) + \nabla \cdot (\rho_\alpha \mathbf{v}_\alpha \mathbf{u}_\alpha).$$

Summation on α and use of (12) yield

$$\sum_\alpha \rho_\alpha \dot{\mathbf{v}}_\alpha = \partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla \cdot (\sum_\alpha \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) = \rho \dot{\mathbf{v}} + \nabla \cdot (\sum_\alpha \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha).$$

Hence summing (3) on α we obtain the expected equation

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{b} \tag{14}$$

where

$$\mathbf{T} = \sum_\alpha (\mathbf{T}_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha), \qquad \rho \mathbf{b} = \sum_\alpha \rho_\alpha \mathbf{b}_\alpha. \tag{15}$$

With a view to the balance of energy (4) we consider $\rho_\alpha(\varepsilon_\alpha + \frac{1}{2}\mathbf{v}_\alpha^2)\grave{}$ and use (13) with $\mathbf{f}_\alpha$ replaced by $\varepsilon_\alpha + \frac{1}{2}\mathbf{v}_\alpha^2$; we put $\mathbf{v}_\alpha = \mathbf{v} + \mathbf{u}_\alpha$ and compute

$$\begin{aligned}\rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{v}_\alpha^2)\grave{} &= \partial_t[\rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{v}^2 + \mathbf{v} \cdot \mathbf{u}_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)] + \nabla \cdot [\rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{v}^2 + \mathbf{v} \cdot \mathbf{u}_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{v}] \\ &+ \nabla \cdot [\rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{v}^2 + \mathbf{v} \cdot \mathbf{u}_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{u}_\alpha].\end{aligned}$$

The summation on α yields

$$\begin{aligned}\rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{v}_\alpha^2)\grave{} &= \partial_t \sum_\alpha \rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2) + \partial_t(\tfrac{1}{2}\rho \mathbf{v}^2) + \nabla \cdot \sum_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{v} + \nabla \cdot [(\tfrac{1}{2}\rho \mathbf{v}^2)\mathbf{v}] \\ &+ \nabla \cdot \sum_\alpha \rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{u}_\alpha + \nabla \cdot \sum_\alpha \rho_\alpha(\mathbf{v} \cdot \mathbf{u}_\alpha)\mathbf{u}_\alpha.\end{aligned}$$

This suggests that we let

$$\rho\varepsilon = \sum_\alpha \rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2) \tag{16}$$

and notice that

$$\begin{aligned}&\partial_t(\rho\varepsilon) + \nabla \cdot (\rho\varepsilon\mathbf{v}) = \rho\partial_t\varepsilon + \varepsilon\partial_t\rho + \rho(\mathbf{v} \cdot \nabla)\varepsilon + \varepsilon\nabla \cdot (\rho\mathbf{v}) = \rho\dot{\varepsilon}, \\ &\partial_t(\tfrac{1}{2}\rho\mathbf{v}^2) + \nabla \cdot [(\tfrac{1}{2}\rho\mathbf{v}^2)\mathbf{v}] = \rho(\tfrac{1}{2}\mathbf{v}^2)\dot{} = \rho\mathbf{v} \cdot \dot{\mathbf{v}}, \\ &\nabla \cdot \sum_\alpha \rho_\alpha(\mathbf{v} \cdot \mathbf{u}_\alpha)\mathbf{u}_\alpha = \nabla \cdot [\mathbf{v}\sum_\alpha \rho_\alpha \otimes \mathbf{u}_\alpha].\end{aligned}$$

Hence the summation of (4) on α leads to

$$\rho\dot{\varepsilon} + \rho\mathbf{v} \cdot \dot{\mathbf{v}} = \nabla \cdot (\mathbf{v}\mathbf{T}) - \nabla \cdot \mathbf{q} + \rho r,$$

where

$$\mathbf{q} = \sum_\alpha[\mathbf{q}_\alpha - \mathbf{v}_\alpha\mathbf{T}_\alpha + \rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{u}_\alpha], \qquad \rho r = \sum_\alpha \rho_\alpha(r_\alpha + \mathbf{b}_\alpha \cdot \mathbf{v}_\alpha). \tag{17}$$

Then in view of (14) we obtain

$$\rho\dot{\varepsilon} = \mathbf{T} \cdot \mathbf{D} - \nabla \cdot \mathbf{q} + \rho r. \tag{18}$$

Lastly, consider the balance of entropy (6) and use (13) to obtain

$$\rho_\alpha \dot{\eta}_\alpha = \partial_t(\rho_\alpha \eta_\alpha) + \nabla \cdot (\rho_\alpha \eta_\alpha \mathbf{v}) + \nabla \cdot (\rho_\alpha \eta_\alpha \mathbf{u}_\alpha) \tag{19}$$

Letting

$$\rho\eta = \sum_\alpha \rho_\alpha \eta_\alpha, \qquad \mathbf{j} = \sum_\alpha (\mathbf{j}_\alpha + \rho_\alpha \eta_\alpha \mathbf{u}_\alpha), \qquad \rho s = \sum_\alpha \frac{\rho_\alpha r_\alpha}{\theta_\alpha}, \qquad \rho\gamma = \sum_\alpha \rho_\alpha \gamma_\alpha \tag{20}$$

and summing (19) with respect to α we find

$$\partial_t(\rho\eta) + \nabla \cdot (\rho\eta\mathbf{v}) + \nabla \cdot \sum_\alpha \rho_\alpha \eta_\alpha \mathbf{u}_\alpha = \rho\dot{\eta} + \nabla \cdot \sum_\alpha \rho_\alpha \eta_\alpha \mathbf{u}_\alpha .$$

Thus the sum of (6) yields

$$\rho\dot{\eta} + \nabla \cdot \mathbf{j} - \rho s = \rho\gamma. \tag{21}$$

Notice that $\rho\gamma$ is just the entropy production, per unit volume, occurring in the second law for mixtures (7).

Equations (12), (14), (18), and (21) have the standard form of balance equations for continua; they are so provided the relations between peculiar quantities $\rho_\alpha, \mathbf{v}_\alpha, \dots, \gamma_\alpha$ and single-body quantities $\rho, \mathbf{v}, \dots, \gamma$ are given by (9), (15), 16), (17), (20). Some equations define properties of the mixture as simple averages like e.g.

$$\mathbf{v} = \sum_\alpha \omega_\alpha \mathbf{v}_\alpha, \quad \eta = \sum_\alpha \omega_\alpha \eta_\alpha, \quad \gamma = \sum_\alpha \omega_\alpha \gamma_\alpha,$$

where

$$\omega_\alpha = \frac{\rho_\alpha}{\rho}$$

is the mass fraction (or concentration) of the α-th constituent. Other quantities instead involve diffusive effects like e.g.

$$\varepsilon = \sum_\alpha \omega_\alpha (\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2),$$
$$\mathbf{T} = \sum_\alpha (\mathbf{T}_\alpha - \rho_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha), \quad \mathbf{q} = [\mathbf{q}_\alpha - \mathbf{v}_\alpha \mathbf{T}_\alpha + \rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{u}_\alpha].$$

As is exemplified in § 5, seemingly different conclusions arise according as we apply the CD inequality or the equations for the mixture as a whole.

4 Remarks on mixtures with a single temperature

Quite often mixtures are assumed to be subject to a single temperature. It is worth inspecting the differences among mixtures with several temperatures and those with a single temperature.

Mixtures with a single temperature are viewed as subject to the constraint

$$\theta_\alpha = \theta, \quad \alpha = 1, 2, \dots, n.$$

Look at (8) and, following [11], consider

$$\rho_\alpha \eta_\alpha \grave{\theta}_\alpha + \frac{1}{\theta_\alpha} \cdot \nabla\theta_\alpha.$$

Now $\theta_\alpha = \theta$ implies that

$$\grave{\theta}_\alpha = \partial_t \theta + \mathbf{v}_\alpha \cdot \nabla\theta = \dot{\theta} + \mathbf{u}_\alpha \cdot \nabla\theta.$$

Consequently, since $\sum_\alpha \rho_\alpha \eta_\alpha = \rho\eta$, we might replace

$$-\sum_\alpha \frac{1}{\theta_\alpha}[\rho_\alpha \eta_\alpha \grave{\theta}_\alpha + \frac{1}{\theta_\alpha}\mathbf{q}_\alpha \cdot \nabla\theta_\alpha]$$

with

$$-\frac{1}{\theta}\{\rho\eta\dot{\theta} + \sum_\alpha [\mathbf{q}_\alpha + \rho_\alpha \eta_\alpha \theta \mathbf{u}_\alpha] \cdot \nabla\theta\}.$$

This seems to indicate that a single temperature induces a heat flux $\rho_\alpha \eta_\alpha \theta \mathbf{u}_\alpha$ in any α-th constituent. Yet we follow a different view.

5 Exploitation of the CD inequality

The CD inequality (8) involves the detailed, distinct components of the entropy production while (21) gives an average description of the peculiar entropy productions. To derive a more accurate description of the mixture it is then natural to apply inequality (8). This is accomplished as follows for fluid mixtures and fluid–solid mixtures.

5.1 Mixtures of fluids

Consider a mixture of two viscous fluids and let

$$\rho_\alpha, \theta_\alpha, \mathbf{D}_\alpha, \mathbf{q}_\alpha, \nabla\theta_\alpha, \quad \alpha = 1, 2,$$

be the variables for the constitutive functions $\psi_\alpha, \eta_\alpha, \mathbf{T}_\alpha, \grave{\mathbf{q}}_\alpha, \mathbf{k}_\alpha$ of constituent α. Instead the interaction terms $\hat{\varepsilon}_\alpha$ and γ_α are allowed to depend on all of the variables ($\alpha = 1, 2$). The function $\grave{\mathbf{q}}_\alpha$, for the rate of $\mathbf{q}_\alpha$, has to be consistent with the objectivity principle. The simplest way is to consider the corotational derivative $\overset{\circ}{\mathbf{q}}_\alpha$ and to assume

$$\overset{\circ}{\mathbf{q}}_\alpha := \dot{\mathbf{q}}_\alpha - \mathbf{W}_\alpha \mathbf{q}_\alpha = -\frac{1}{\tau_\alpha}(\mathbf{q}_\alpha + \kappa_\alpha \nabla\theta_\alpha), \quad \tau_\alpha > 0. \tag{22}$$

For definiteness the stress $\mathbf{T}_\alpha$ is assumed in the Newtonian form

$$\mathbf{T}_\alpha = -p_\alpha(\rho_\alpha, \theta_\alpha)\mathbf{1} + \mu_\alpha \mathbf{D}_\alpha + \lambda_\alpha(\mathrm{tr}\, \mathbf{D}_\alpha)\mathbf{1}. \tag{23}$$

Computation of $\grave{\psi}_\alpha$ and substitution in (8) yields

$$\begin{aligned}\textstyle\sum_\alpha \frac{1}{\theta_\alpha}\{&-\rho_\alpha(\partial_{\theta_\alpha}\psi_\alpha + \eta_\alpha)\grave{\theta}_\alpha - \rho_\alpha \partial_{\rho_\alpha}\psi_\alpha \grave{\rho}_\alpha - \rho_\alpha \partial_{\mathbf{D}_\alpha}\psi_\alpha \cdot \grave{\mathbf{D}}_\alpha \\ &- \rho_\alpha \partial_{\mathbf{q}_\alpha}\psi_\alpha \cdot \grave{\mathbf{q}}_\alpha - \rho_\alpha \partial_{\nabla\theta_\alpha}\psi_\alpha \cdot (\nabla\theta_\alpha)\grave{} + \mathbf{T}_\alpha \cdot \mathbf{D}_\alpha \\ &- \frac{1}{\theta_\alpha}\mathbf{q}_\alpha \cdot \nabla\theta_\alpha + \theta_\alpha \nabla \cdot \mathbf{k}_\alpha + e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha\} = \textstyle\sum_\alpha \rho_\alpha \gamma_\alpha.\end{aligned} \tag{24}$$

The linearity and arbitrariness of $\grave{\mathbf{D}}_\alpha, (\nabla\theta_\alpha)\grave{}, \grave{\theta}_\alpha$ imply that

$$\partial_{\mathbf{D}_\alpha}\psi_\alpha = \mathbf{0}, \quad \partial_{\nabla\theta_\alpha}\psi_\alpha = \mathbf{0}, \quad \eta_\alpha = -\partial_{\theta_\alpha}\psi_\alpha.$$

To exploit the remaining inequality we use (2), (22), and (23) to replace $\grave{\rho}_\alpha, \grave{\mathbf{q}}_\alpha$, and $\mathbf{T}_\alpha$ in (24) and obtain

$$\begin{aligned}\textstyle\sum_\alpha \frac{1}{\theta_\alpha}\{&(\rho_\alpha^2 \partial_{\rho_\alpha}\psi_\alpha - p_\alpha)\mathrm{tr}\, \mathbf{D}_\alpha + 2\mu_\alpha \mathbf{D}_\alpha \cdot \mathbf{D}_\alpha + \lambda_\alpha(\mathrm{tr}\, \mathbf{D}_\alpha)^2 \\ &- \rho_\alpha(\partial_{\mathbf{q}_\alpha}\psi_\alpha \otimes \mathbf{q}_\alpha) \cdot \mathbf{W}_\alpha + \frac{\rho_\alpha}{\tau_\alpha}\partial_{\mathbf{q}_\alpha}\psi_\alpha \cdot \mathbf{q}_\alpha + (\frac{\rho_\alpha \kappa_\alpha}{\tau_\alpha}\partial_{\mathbf{q}_\alpha}\psi_\alpha - \frac{1}{\theta_\alpha}\mathbf{q}_\alpha) \cdot \nabla\theta_\alpha \\ &+ \theta_\alpha \nabla \cdot \mathbf{k}_\alpha + e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha\} = \textstyle\sum_\alpha \rho_\alpha \gamma_\alpha.\end{aligned}$$

The arbitrariness of the skew tensor $\mathbf{W}_\alpha$ implies

$$\partial_{\mathbf{q}_\alpha}\psi_\alpha \otimes \mathbf{q}_\alpha \in \mathrm{Sym}.$$

We find that $\mathbf{k}_\alpha$ can depend only on θ_α so that

$$\nabla \cdot \mathbf{k}_\alpha = \partial_{\theta_\alpha}\mathbf{k}_\alpha \cdot \nabla\theta_\alpha.$$

Yet, by the isotropy of the constituent, $\mathbf{k}_\alpha$ cannot depend only on θ_α and hence it follows that $\mathbf{k}_\alpha = \mathbf{0}$. Now if $\mathbf{q}_\alpha, \nabla\theta_\alpha$ vanish and the remaining constituents make $e_\alpha = 0$, $\mathbf{m}_\alpha = \mathbf{0}$, we get

$$\textstyle\sum_\alpha \frac{1}{\theta_\alpha}\{(\rho_\alpha^2 \partial_{\rho_\alpha}\psi_\alpha - p_\alpha)\mathrm{tr}\, \mathbf{D}_\alpha + 2\mu_\alpha \mathbf{D}_\alpha \cdot \mathbf{D}_\alpha + \lambda_\alpha(\mathrm{tr}\, \mathbf{D}_\alpha)^2\} \geq 0$$

for any stretching tensor $\mathbf{D}_\alpha$. It follows that

$$p_\alpha = \rho_\alpha^2 \partial_{\rho_\alpha}\psi_\alpha, \quad \mu_\alpha \geq 0, \quad 2\mu_\alpha + 3\lambda_\alpha \geq 0.$$

Again we let $e_\alpha = 0, \mathbf{m}_\alpha = \mathbf{0}$ and observe that by the arbitrariness of $\mathbf{q}_\alpha$, $\nabla\theta_\alpha$ implies

$$\frac{\rho_\alpha \kappa_\alpha}{\tau_\alpha}\partial_{\mathbf{q}_\alpha}\psi_\alpha - \frac{1}{\theta_\alpha}\mathbf{q}_\alpha = \mathbf{0}, \quad \partial_{\mathbf{q}_\alpha}\psi_\alpha \cdot \mathbf{q}_\alpha \geq 0.$$

Hence it follows that

$$\psi_\alpha = \Psi_\alpha(\rho_\alpha, \theta_\alpha) + \frac{\tau_\alpha}{2\rho_\alpha\theta_\alpha\kappa_\alpha}\mathbf{q}_\alpha^2,$$

which satisfies both conditions

$$\partial_{\mathbf{q}_\alpha}\psi_\alpha \otimes \mathbf{q}_\alpha \in \mathrm{Sym}, \qquad \partial_{\mathbf{q}_\alpha}\psi_\alpha \cdot \mathbf{q}_\alpha \geq 0.$$

The conclusions attained so far hold for any mixture. For definiteness we now restrict attention to binary mixtures and then consider the quantity

$$\sum\nolimits_\alpha \frac{1}{\theta_\alpha}\{e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha\}$$

as $n = 2$. It is reasonable to assume that e_α and $\mathbf{m}_\alpha$ depend on differences of temperature and velocity. Hence we can write

$$\sum\nolimits_\alpha \frac{1}{\theta_\alpha}\{e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha\} = \sum\nolimits_\alpha \rho_\alpha\hat{\gamma}_\alpha \geq 0,$$

where $\hat{\gamma}_\alpha$ is the entropy production when $\mathbf{D}_\alpha = \mathbf{0}$, $\mathbf{q}_\alpha = \mathbf{0}$, $\nabla\theta_\alpha = \mathbf{0}$. Let

$$\begin{aligned} e_1 &= N(\theta_2 - \theta_1) + M(\mathbf{v}_2 - \mathbf{v}_1)\cdot\mathbf{v} = -e_2, \\ \mathbf{m}_1 &= M(\mathbf{v}_2 - \mathbf{v}_1) = -\mathbf{m}_2, \end{aligned}$$

possibly with N, M dependent on ρ_1, ρ_2. We have

$$\sum\nolimits_\alpha \frac{1}{\theta_\alpha}\{e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha\} = N\frac{(\theta_2 - \theta_1)^2}{\theta_1\theta_2} + M(\mathbf{v}_2 - \mathbf{v}_1)\cdot(\frac{\mathbf{u}_2}{\theta_2} - \frac{\mathbf{u}_1}{\theta_1}).$$

Now observe that

$$\mathbf{u}_2 = \frac{\rho_2\mathbf{u}_2}{\rho_2} = -\frac{\rho_1\mathbf{u}_1}{\rho_2}.$$

Hence it follows that

$$M(\mathbf{v}_2 - \mathbf{v}_1)\cdot(\frac{\mathbf{u}_2}{\theta_2} - \frac{\mathbf{u}_1}{\theta_1}) = M(\frac{\rho_1}{\rho_2} + 1)(\frac{\rho_1}{\rho_2\theta_2} + \frac{1}{\theta_1})\mathbf{u}_1^2. \tag{25}$$

Consequently the assumption on e_α and $\mathbf{m}_\alpha$ makes the entropy production $\sum_\alpha \rho_\alpha\hat{\gamma}_\alpha$ positive definite with respect to the differences $\theta_1 - \theta_2, \mathbf{u}_1 - \mathbf{u}_2$. We might notice the seeming unboundedness of $\sum_\alpha \rho_\alpha \hat{\gamma}_\alpha = \sum_\alpha (1/\theta_\alpha)\{e_\alpha - \mathbf{m}_\alpha \cdot \mathbf{v}_\alpha\}$ as ρ_1 (or ρ_2) approaches zero. This is avoided by letting, e.g., $M = c\,\rho_1^2\rho_2^2$, and likewise $N = d\,\rho_1^2\rho_2^2$.

5.2 Mixtures with a single temperature

The local contact between the constituents makes it reasonable to assume that the constituents be at the same temperature θ. A remarkable example in this sense is often given by plasma models where electrons and ions might be viewed as fluids with different temperatures.

Let $\theta_\alpha = \theta, \alpha = 1, 2, \dots, n$. If $\theta_\alpha = \theta$ then we find that

$$\sum_\alpha \frac{1}{\theta_\alpha}[\rho_\alpha \eta_\alpha \grave{\theta}_\alpha + \frac{1}{\theta}_\alpha \mathbf{q}_\alpha \cdot \nabla \theta_\alpha] = \frac{1}{\theta}\{\rho\eta\dot{\theta} + \sum_\alpha [\mathbf{q}_\alpha + \rho_\alpha \eta_\alpha \theta(\mathbf{v}_\alpha - \mathbf{v})] \cdot \nabla\theta\}.$$

This might indicate that the effective α-th heat flux would be

$$\mathbf{q}_\alpha + \rho_\alpha \eta_\alpha \theta(\mathbf{v}_\alpha - \mathbf{v}),$$

which is not consistent with the result

$$\mathbf{q} = [\mathbf{q}_\alpha - \mathbf{v}_\alpha \mathbf{T}_\alpha + \rho_\alpha(\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{u}_\alpha]$$

for the mixture as a single body.

Instead, the CD inequality (24) reads

$$\begin{aligned}&\sum_\alpha \frac{1}{\theta_\alpha}\{-\rho_\alpha(\partial_{\theta_\alpha}\psi_\alpha + \eta_\alpha)[\dot{\theta} + (\mathbf{v}_\alpha - \mathbf{v}) \cdot \nabla\theta] + \dots \\ &+ \frac{\rho_\alpha}{\tau_\alpha}\partial_{\mathbf{q}_\alpha}\psi_\alpha \cdot \mathbf{q}_\alpha + (\frac{\rho_\alpha \kappa_\alpha}{\tau_\alpha}\partial_{\mathbf{q}_\alpha}\psi_\alpha - \frac{1}{\theta_\alpha}\mathbf{q}_\alpha) \cdot \nabla\theta = \sum_\alpha \rho_\alpha \gamma_\alpha.\end{aligned}$$

By (24) it follows that

$$\eta_\alpha = -\partial_{\theta_\alpha}\psi_\alpha.$$

This relation holds for each constituent and any value of θ_α. Hence it holds even if $\theta_\alpha = \theta$,

$$\eta_\alpha = -\partial_\theta \psi_\alpha.$$

The analogous conclusion is valid for each $\mathbf{q}_\alpha$ and then

$$\frac{\rho_\alpha \kappa_\alpha}{\tau_\alpha}\partial_{\mathbf{q}_\alpha}\psi_\alpha = \frac{1}{\theta}\mathbf{q}_\alpha.$$

6 Solid–fluid mixtures

Also with a view to the modelling of porous media, we consider a binary mixture with a solid and a fluid. We first examine the possibility of modelling a fluid of the Kelvin–Voigt type [2]. The classical Kelvin–Voigt solid is characterized by letting the stress be a superposition of a strain term and a strain-rate term ([12], §6.1.7). A Kelvin–Voigt fluid is characterized by letting

$$\mathbf{T} = \mu \mathbf{D} + \nu \dot{\mathbf{D}}.$$

Yet $\dot{\mathbf{D}}$ is not objective and hence it cannot enter a constitutive equation. We then assume $\mathbf{T}$ depends on $\mathbf{D}$ and an objective derivative of $\mathbf{D}$. For definiteness we consider the corotational derivative

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} - \mathbf{W}\mathbf{D} - \mathbf{D}\mathbf{W}^T,$$

where $\mathbf{W}$ is the spin tensor. Likewise we use the corotational derivative $\overset{\circ}{\mathbf{q}} = \dot{\mathbf{q}} - \mathbf{W}\mathbf{q}$ to represent the rate equation of the heat flux $\mathbf{q}$.

Denote by the subscripts f, s the quantities pertaining to the fluid and solid constituents. Hence we let

$$\Gamma_f = (\rho_f, \theta, \mathbf{D}_f, \overset{\circ}{\mathbf{D}}_f, \mathbf{q}_f, \nabla\theta), \qquad \Gamma_s = (\mathbf{E}_s, \theta, \mathbf{q}_s, \nabla\theta)$$

be the set of variables for the fluid and the solid. Since $\theta_f = \theta_s = \theta$ then we observe that

$$\textstyle\sum_\alpha \hat{\varepsilon}_\alpha = -\sum_\alpha \mathbf{m}_\alpha \cdot \mathbf{u}_\alpha;$$

we put $\mathbf{m}_f = -\beta(|\mathbf{u}|)\mathbf{u}$, $\mathbf{u} = \mathbf{u}_f - \mathbf{u}_s$, and hence

$$\textstyle\sum_\alpha \hat{\varepsilon}_\alpha = \beta(|\mathbf{u}|)\, |\mathbf{u}|^2.$$

Furthermore we let

$$\mathbf{T}_f = -p(\rho_f, \theta)\mathbf{1} + \hat{\mathbf{T}}_f, \quad \hat{\mathbf{T}}_f = \mu_V \mathbf{D}_f + \sigma_K \overset{\circ}{\mathbf{D}}_f, \tag{26}$$

$$\overset{\circ}{\mathbf{q}}_f = -\frac{1}{\tau_f}(\mathbf{q}_f + \kappa_f \nabla\theta_f), \quad \overset{\circ}{\mathbf{q}}_s = -\frac{1}{\tau_s}(\mathbf{q}_s + \mathbf{K}_s \nabla\theta_s), \quad \tau_f, \tau_s > 0, \tag{27}$$

where $\mathbf{K}_s \in \mathrm{Sym}$ is non-singular. With these assumptions the extra-entropy fluxes $\mathbf{k}_f, \mathbf{k}_s$ turn out to be zero; to save writing we omit them. Hence the CD inequality takes the form

$$-\rho_f(\dot{\psi}_f + \eta_f \dot{\theta}_f) - \rho_s(\dot{\psi}_s + \eta_s \dot{\theta}_s) - p_f \mathrm{tr}\, \mathbf{D}_f + \hat{\mathbf{T}}_f \cdot \mathbf{D}_f + \mathbf{T}_s \cdot \mathbf{D}_s$$
$$-\frac{1}{\theta_f}\mathbf{q}_f \cdot \nabla\theta_f - \frac{1}{\theta_s}\mathbf{q}_s \cdot \nabla\theta_s + g(|\mathbf{u}|)|\mathbf{u}|^2 = \theta(\rho_f \gamma_f + \rho_s \gamma_s).$$

Compute $\dot{\psi}_f(\Gamma_f)$, $\dot{\psi}_s(\Gamma_s$ and observe that the relations

$$\begin{aligned} &\eta_f = -\partial_\theta \psi_f, \qquad \eta_s = -\partial_\theta \psi_s, \qquad p_f = \rho_f^2 \partial_{\rho_f} \psi_f, \\ &\partial_{\nabla\theta} \psi_f = \mathbf{0}, \qquad \partial_{\nabla\theta} \psi_s = \mathbf{0}, \end{aligned}$$

hold as particular cases when $\theta_f = \theta_s = \theta$. We now recall the identity $\dot{\mathbf{E}}_s = \mathbf{F}_s^T \mathbf{D}_s \mathbf{F}_s$ and notice that, by (27),

$$\dot{\mathbf{q}}_f = \overset{\circ}{\mathbf{q}}_f + \mathbf{W}\mathbf{q}_f = -\frac{1}{\tau_f}(\mathbf{q}_f + \kappa_f \nabla\theta_f) + \mathbf{W}_f \mathbf{q}_f$$

and the like for $\mathbf{q}_s$. Furthermore, by (26), we have

$$\hat{\mathbf{T}}_f = \mu \mathbf{D}_f + \nu(\dot{\mathbf{D}}_f - \mathbf{W}_f \mathbf{D}_f - \mathbf{D}_f \mathbf{W}_f^T).$$

Thus we can write the remaining terms of the CD inequality as

$$\begin{aligned} &(-\rho_f \partial_{\mathbf{D}_f} \psi_f + \nu \mathbf{D}_f) \cdot \dot{\mathbf{D}}_f - \rho_s \partial_{\mathbf{D}_s} \psi_s \cdot \dot{\mathbf{D}}_s + \mathbf{T}_s \cdot \mathbf{D}_s - \rho_s \partial_{\mathbf{E}_s} \psi_s \cdot (\mathbf{F}_s^T \mathbf{D}_s \mathbf{F}_s) \\ &\quad + \rho_f \partial_{\mathbf{q}_f} \psi_f \cdot [\mathbf{W}_f \mathbf{q}_f + \frac{1}{\tau_f}(\mathbf{q}_f + \kappa_f \nabla\theta_f)] - \frac{1}{\theta_f} \mathbf{q}_f \cdot \nabla\theta_f \\ &\quad + \rho_s \partial_{\mathbf{q}_s} \psi_s \cdot [\mathbf{W}_s \mathbf{q}_s + \frac{1}{\tau_s}(\mathbf{q}_s + \mathbf{K}_s \nabla\theta_s)] - \frac{1}{\theta_s} \mathbf{q}_s \cdot \nabla\theta_s \\ &\quad + \mu_V \mathbf{D}_f \cdot \mathbf{D}_f + \sigma_K(-\mathbf{W}_f \mathbf{D}_f - \mathbf{D}_f \mathbf{W}_f^T) \cdot \mathbf{D}_f + \beta(|\mathbf{u}|)|\mathbf{u}|^2 = \theta(\rho_f \gamma_f + \rho_s \gamma_s). \end{aligned}$$

The linearity and arbitrariness of $\dot{\mathbf{D}}_f, \dot{\mathbf{D}}_s, \nabla\theta_f, \nabla\theta_s, \mathbf{W}_f, \mathbf{W}_s$ imply the following consequences,

$$\rho_f \partial_{\mathbf{D}_f} \psi_f = \sigma_K \mathbf{D}_f, \quad \mathbf{T}_s = \rho_s \mathbf{F}_s \partial_{\mathbf{E}_s} \psi_s \mathbf{F}_s^T \tag{28}$$

$$\frac{\rho_f \kappa_f}{\tau_f} \partial_{\mathbf{q}_f} \psi_f = \frac{1}{\theta_s} \mathbf{q}_s, \quad \frac{\rho_s}{\tau_s} \partial_{\mathbf{q}_s} \psi_s \mathbf{K}_s = \frac{1}{\theta_s} \mathbf{q}_s, \tag{29}$$

$$\partial_{\mathbf{q}_f} \psi_f \otimes \mathbf{q}_f \in \mathrm{Sym}, \quad \partial_{\mathbf{q}_s} \psi_s \otimes \mathbf{q}_s \in \mathrm{Sym}. \tag{30}$$

By (28) and (29) we have the results

$$\begin{aligned} \psi_f &= \frac{\tau_f}{2\rho_f \theta_f \kappa_f} |\mathbf{q}_f|^2 + \frac{\sigma_K}{2\rho_f} |\mathbf{D}_f|^2 + \Psi(\theta_f, \rho_f) \\ \psi_s &= \frac{\tau_s}{2\rho_s \theta_s} \mathbf{q}_s \cdot \mathbf{K}_s^{-1} \mathbf{q}_s + \Psi(\theta_s, \mathbf{E}_s), \end{aligned}$$

which are valid also if $\theta_f = \theta_s$.

Furthermore, since $\mathbf{D}_f \mathbf{D}_f \in \mathrm{Sym}$ then

$$\mathbf{D}_f \cdot (\mathbf{W}_f \mathbf{D}_f) = 0, \quad (\mathbf{D}_f \mathbf{W}_f) \cdot \mathbf{D}_f = 0.$$

Thus the CD inequality reduces to

$$\frac{1}{\theta\kappa_f}|\mathbf{q}_f|^2 + \frac{1}{\theta}\mathbf{q}_s \cdot \mathbf{K}_s^{-1}\mathbf{q}_s + \mu_V \mathbf{D}_f \cdot \mathbf{D}_f + \beta(|\mathbf{u}|)|\mathbf{u}|^2 = \theta(\rho_f\gamma_f + \rho_s\gamma_s) \geq 0.$$

The arbitrariness of $\mathbf{q}_f, \mathbf{q}_s, \mathbf{D}_f, \mathbf{u}$ implies that

$$\kappa_f > 0, \quad \mathbf{K}_s > \mathbf{0}, \quad \mu_V \geq 0, \quad \beta \geq 0. \tag{31}$$

7 Consequences on models for viscous fluids in porous media

The flow of a viscous fluid in a porous medium has been widely investigated mainly in connection with rate-type models of heat conduction(see, e.g., [2–6]). Here we inspect the thermodynamic consistency of a model often involved to describe thermal convection.

For definiteness here the model is considered in a form recently investigated (see [2] and refs therein) though with attention restricted to incompressible fluids. The equation of motion is taken in the form

$$\dot{\mathbf{v}} - \lambda_K \Delta \partial_t \mathbf{v} = -\frac{1}{\rho}\nabla p + \nu_V \Delta \mathbf{v} + \alpha\theta\mathbf{g} - \mu_D \mathbf{v}, \tag{32}$$

with $\mathbf{v}$ subject to the incompressibility constraint

$$\nabla \cdot \mathbf{v} = 0,$$

while $\mathbf{g}$ is the gravity acceleration vector and $\alpha < 0$ is the coefficient of thermal expansion. The balance of energy is taken in the form

$$\dot{\theta} = -\nabla \cdot \mathbf{q} + \zeta \Delta\theta, \tag{33}$$

while the heat flux is subject to a Maxwell-Cattaneo like equation

$$\tau \mathcal{D}\mathbf{q} = -\mathbf{q} - \kappa \nabla\theta + \xi_1 \Delta \mathbf{q} + \xi_2 \nabla(\nabla \cdot \mathbf{q}). \tag{34}$$

Here the fields of interest are those pertaining to the fluid and, for ease of notation, we omit the subscript f. Accordingly, e.g., $\mathbf{v}$, $\mathbf{D}$, p stand for $\mathbf{v}_f$, $\mathbf{D}_f$, p_f while $\mathbf{u} = \mathbf{u}_f - \mathbf{u}_s$ stands for $\mathbf{v}_f$ in case $\mathbf{v}_s = \mathbf{0}$.

We now compare the scheme (32)–(34) of the literature with the analogous one of § 6. We begin with Eq. (32) and observe that, within the scheme of § 6, the equation of motion reads

$$\rho\dot{\mathbf{v}} = -\nabla p(\rho, \theta) + \mu_V \nabla \cdot \mathbf{D} + \sigma_K \nabla \cdot \overset{\circ}{\mathbf{D}} - \beta\mathbf{u}. \tag{35}$$

The last term $\beta\mathbf{u}$ is the generalization of the Darcy term $\mu_D\mathbf{v}$, in that β is a positive-valued function of $\mathbf{u}$. The non-linear function $\beta(|\mathbf{u}|)\mathbf{u}$ is just the representation of

Forchheimer's law [14]; thermodynamics merely requires that $\beta \geq 0$. If β is constant then the classical Darcy term $\mu_D \mathbf{v}$ is recovered.

The viscous term $\mu_V \nabla \cdot \mathbf{D}$ simplifies to $\rho \nu_V \Delta \mathbf{v}$ in case the fluid is incompressible, namely $\nabla \cdot \mathbf{v} = 0$.

The analog of the Kelvin–Voigt term, $\sigma_K \nabla \cdot \overset{\circ}{\mathbf{D}}$, involves the objective derivative $\overset{\circ}{\mathbf{D}}$. If $\mathbf{WD}$ and $\mathbf{DW}$ are neglected then

$$\sigma_K \nabla \cdot \overset{\circ}{\mathbf{D}} \sim \sigma_K \nabla \cdot \dot{\mathbf{D}} \sim \sigma_K \nabla \cdot \partial_t \mathbf{D} = \sigma_K \Delta \partial_t \mathbf{v}, \quad \text{if} \quad \nabla \cdot \mathbf{v} = 0,$$

and we get the term $\lambda_K \Delta \partial_t \mathbf{v}$ of (32). The positiveness of $\lambda_K = \sigma_K / \rho$ is obtained if the free energy ψ_f is assumed to attain a minimum at equilibrium ($\mathbf{D} = \mathbf{0}$).

The pressure p in (35) is a function of ρ, θ) while p is undetermined in (32) as a consequence of incompressibility. In (35)

$$\nabla p = \partial_\rho p \nabla \rho + \partial_\theta p \nabla \theta.$$

Now, to model the variation of p we consider the identity

$$-\nabla p + \rho \mathbf{g} = -\nabla p + \rho_0 \mathbf{g} + (\rho - \rho_0) \mathbf{g},$$

where $\mathbf{g}$ is the gravity acceleration vector. At the rest state (ρ_0, θ_0, p_0) we have

$$-\nabla p_0 + \rho_0 \mathbf{g} = \mathbf{0}.$$

Based on a linear approximation of $\rho(\theta)$ we put

$$\rho - \rho_0 = \alpha \rho_0 \vartheta, \qquad \vartheta := \theta - \theta_0, \qquad \alpha < 0.$$

Hence we have

$$-\nabla p + \rho_0 \mathbf{g} = -\nabla \mathscr{P} + \alpha \rho_0 \vartheta \mathbf{g},$$

where $\mathscr{P} = p - p_0$; this is the Oberbeck–Boussinesq approximation [15, 16]. Accordingly the vector $-(1/\rho) \nabla p + \alpha \theta \mathbf{g}$ has to be meant as $-(1/\rho) \nabla \mathscr{P} + \alpha \vartheta \mathbf{g}$. The viscous term $\nu_V \Delta \mathbf{v}$ traces back to Brinkman [3, 17, 18] and is merely given by the viscous stress in Newtonian fluids.

The balance of energy (33) arises from an internal energy that depends only on θ possibly in a composite form $\varepsilon(\rho(\theta), \theta)$. The term $\zeta \Delta \theta$ is quite unusual in that it is not the classical term provided by $\nabla \cdot \mathbf{q}$. This term is suggested by Payne and Song [19] and the motivation is not immediate [2, 20]. We observe that $\zeta \Delta \theta$ cannot be an energy growth within $\hat{\varepsilon}_f$ because it does not seem to provide a (positive) entropy production. We might view $\zeta \Delta \theta$ as an energy supply but the physical mechanism leading to the supply is not familiar for a mixture solid–fluid or merely a fluid.

The rate equation for the heat flux is assumed in the form (34). The higher-order terms $\xi_1 \Delta \mathbf{q}$, $\xi_2 \nabla(\nabla \cdot \mathbf{q})$ have not been considered in this paper: they are investigated in [21] and found to be thermodynamically consistent in that they are framed as extra-entropy fluxes. Instead the objective derivative allows the rate equation to be consistent with the objectivity principle: the constitutive equations must be invariant under the group of Euclidean transformations. Yet the thermodynamic restrictions depend on the chosen derivative. In the present mixture theory the chosen derivative is the corotational one and hence $\mathbf{q}_f, \mathbf{q}_s$ are not cross-coupled with other (stress) terms; cross-coupling happens with other derivatives [22]. Instead choosing the Truesdell derivative, as is done in [2], gives the possibility of a Lagrangian formulation with the material time derivative.

8 Conclusions

This paper investigates current models of flows in porous media. From the viewpoint of a continuum theory, the natural setting of porous media is that of mixture theory, chiefly solid–fluid mixtures. Accordingly, some aspects of the modelling are considered and hence the thermodynamic consistency of constitutive equations are examined for compressible, viscous, heat-conducting fluids subject to relaxation phenomena expressed by suitable rate equations.

The thermodynamic analysis is performed via the CD inequality based directly on the peculiar (α-th) fields of the mixture. This analysis is more detailed and hence more significant than analogous procedures based on equations for the mixture as a whole. As an example, the detailed analysis involves the peculiar heat fluxes $\mathbf{q}_\alpha$ per se while the balance equations for energy and entropy of the whole body involve, e.g., the resulting heat flux

$$\mathbf{q} = \sum_\alpha [\mathbf{q}_\alpha - \mathbf{v}_\alpha \mathbf{T}_\alpha + \rho_\alpha (\varepsilon_\alpha + \tfrac{1}{2}\mathbf{u}_\alpha^2)\mathbf{u}_\alpha],$$

which is affected by the diffusion velocities $\mathbf{u}_\alpha$.

As to the Kelvin–Voigt term, it is pointed out that objectivity requires the rate of the stretching $\mathbf{D}$ be objective. Instead usually the rate is merely expressed by the partial time derivative.

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Conflicts of interest The author declares no conflicts of interest.

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