Space curves, X -ranks and cuspidal projections

Edoardo Ballico¹

Received: 10 February 2021 / Accepted: 21 June 2021 / Published online: 1 July 2021
© The Author(s) 2021

Abstract

Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. We say that $q \in \mathbb{P}^3 \setminus X$ has X -rank 3 if there is no line $L \subset \mathbb{P}^3$ such that $q \in L$ and $\#(L \cap X) \geq 2$. We prove that for all hyperelliptic curves of genus $g \geq 5$ there is a degree $g + 3$ embedding $X \subset \mathbb{P}^3$ with exactly $2g + 2$ points with X -rank 3 and another embedding without points with X -rank 3 but with exactly $2g + 2$ points $q \in \mathbb{P}^3$ such that there is a unique pair of points of X spanning a line containing q . We also prove the non-existence of points of X -rank 3 for general curves of bidegree (a, b) in a smooth quadric (except in known exceptional cases) and we give lower bounds for the number of pairs of points of X spanning a line containing a fixed $q \in \mathbb{P}^3 \setminus X$. For all integers $g \geq 5$, $x \geq 0$ we prove the existence of a nodal hyperelliptic curve X with geometric genus g , exactly x nodes, $\deg(X) = x + g + 3$ and having at least $x + 2g + 2$ points of X -rank 3.

Keywords X -Rank · Space curve · Cuspidal projection · Hyperelliptic curve · Quadric surface

Mathematics Subject Classification 14H50 · 14N05

1 Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field with characteristic 0. For any $q \in \mathbb{P}^r$ the X -rank $r_X(q)$ of q is the minimal cardinality of a set $S \subset X$ such that $q \in \langle S \rangle$, where $\langle \rangle$ denote the linear span ([7,8]). Let $\mathcal{S}(X, q)$ denote the set of all $S \subset X$ such that $\#(S) = r_X(q)$ and $q \in \langle S \rangle$. Set

$$W_x^0(X) := \{q \in \mathbb{P}^r \mid r_X(q) = x\}.$$

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

✉ Edoardo Ballico
edoardo.ballico@unitn.it

¹ Department of Mathematics, University of Trento, 38123 Povo TN, Italy

In this paper we focus on the set $\mathcal{S}(X, q)$. The “generic rank of X ” or “generic X -rank” is the integer $r_X(q)$ for a general $q \in \mathbb{P}^r$. If $r = 3$ and $\dim X = 1$ (as in our paper) the generic X -rank is 2 and we study triples $(X, q, \mathcal{S}(X, q))$ such that $r_X(q) \in \{2, 3\}$. A detailed study of the dimensions of the algebraic sets $W_x^0(X)$ and their closures $W_x(X)$ in $\mathbb{P}^r$ for ranks higher than the generic one was done in [3, Theorem 3.1].

Fix an integer $t \geq 2$ and suppose you are interested in a set $\Sigma \subset \mathbb{P}^r$ such that $r_X(q) \geq t$ for all $q \in \Sigma$. In our situation Σ is irreducible and we also know that $r_X(q) = t$ for a general $q \in \Sigma$. For any $q \in \Sigma$ such that $r_X(q) = t$ set $\mathcal{S}(X, q)' := \mathcal{S}(X, q)$. For all $q \in \Sigma$ such that $r_X(q) > t$ set $\mathcal{S}(X, q)' := \emptyset$. Thus for all $q \in \Sigma_t$ the set $\mathcal{S}(X, q)'$ is the set of all $S \subset X$ such that $\#(S) = t$ and $q \in \langle S \rangle$. Set $\Sigma'_t := \{q \in \Sigma \mid r_X(q) = t\}$ and call $\mu_{\min}(\Sigma)$ or $\mu_{\min}(\Sigma)$ the minimal cardinality of the sets $\mathcal{S}(X, q)$, $q \in \Sigma'_t$. By its definition the integer $\mu_{\min}(\Sigma)$ is a positive integer. Set $\Sigma'' := \{q \in \Sigma' \mid \mathcal{S}(X, q) \text{ is finite}\}$. In our set up Σ' will be irreducible and Σ'' dense in Σ . In our paper we only need the case $t = 2$. Using Chevalley's theorem ([6, Ex. II.3.18, II.3.19]) it is easy to see that Σ'' is a constructible subset of $\mathbb{P}^r$. Set $\mu_{\max}(\Sigma'')$ the supremum of all $\mathcal{S}(X, q)$, $q \in \Sigma''$. Using again Chevalley's theorem it is easy to see that the set of all pairs $(\mathcal{S}(X, q), q)$, $q \in \Sigma''$, is constructible. This implies $\mu_{\max}(\Sigma'') < +\infty$ and hence this supremum is a maximum. The sets $\Sigma_1 := \{q \in \Sigma'' \mid \#(\mathcal{S}(X, q)) = \mu_{\max}(\Sigma'')\}$ and $\Sigma_2 := \{q \in \Sigma'' \mid \#(\mathcal{S}(X, q)) = \mu_{\min}(\Sigma'')\}$ are constructible. Their definitions imply $\Sigma_1 \neq \emptyset$ and $\Sigma_2 \neq \emptyset$.

It is known that $r_X(q) \leq r - \dim X + 1$ for any $q \in \mathbb{P}^r$ and any non-degenerate variety $X \subset \mathbb{P}^r$ ([9, Proposition 5.1]). All the result proven in this paper are with X a curve and $r = 3$. In this case we always have $r_X(q) \leq 3$ ([9, Proposition 5.1]). We have $r_X(q) = 1$ if and only if $q \in X$. Thus our first query is to see if the set $W_3^0(X) = \{q \in \mathbb{P}^3 \mid r_X(q) = 3\}$ is empty or not. Note that $q \in \mathbb{P}^3 \setminus X$ has $r_X(q) = 3$ if and only if the restriction to X of the linear projection $\ell_q : \mathbb{P}^3 \setminus \{q\} \rightarrow \mathbb{P}^2$ is injective. When X is smooth $\ell_{q|X}$ is injective if and only if $\ell_{q|X}$ is birational onto its image and $\ell_q(X)$ has only unibranch singularities (i.e. it only has “cusps”, possibly non-ordinary ones). This is the set-up of R. Piene paper ([12]) in which she gives several examples of smooth curves X with $W_3^0(X) \neq \emptyset$ and several examples with $W_3^0(X) = \emptyset$. We recall that the geometric genus of an integral projective curve is the genus of its normalization. As in [2, 3] our first aim is to check if $W_3^0(X) = \emptyset$ or not, but our second aim is to describe (giving upper bound or lower bounds or the exact values) the minimum/maximum cardinality of some set $\mathcal{S}(X, q)$ with $r_X(q) = 2$.

We prove the following results.

Theorem 1.1 *Fix an integer $g \geq 5$ and a genus g hyperelliptic curve Y . Then there exists an embedding $\varphi : Y \rightarrow \mathbb{P}^3$ such that $X := \varphi(Y)$ has degree $g + 3$, X is linearly normal, $W_3^0(X)$ finite and $\#(W_3^0(X)) = 2g + 2$.*

Proposition 1.2 *Fix an integer $g \geq 5$ and a genus g hyperelliptic curve Y . Then there exists an embedding $\varphi : Y \rightarrow \mathbb{P}^3$ such that $X := \varphi(Y)$ has degree $g + 3$, X is linearly normal, $W_3^0(X) = \emptyset$, X is contained in a smooth quadric Q and the following properties hold:*

- (1) take $\Sigma := \mathbb{P}^3 \setminus Q$; then $\mu_{\min}(\Sigma) \geq (g + 2)(g + 1)/2 - g - 2(g + 3)$;

- (2) *there are exactly $2g + 2$ points $q \in Q \setminus X$ such that $\#(\mathcal{S}(X, q)) = 1$;*
 (3) *$\#(\mathcal{S}(X, q)) \in \{1, \binom{g}{2} + 1, \binom{g+1}{2}, \binom{g+1}{2} + 1\}$ for every $q \in Q \setminus X$;*
 (4) *$\#(\mathcal{S}(X, q)) = \binom{g+1}{2} + 1$ for a general $q \in Q$.*

Proposition 1.3 *Fix integers d, g such that $0 \leq g \leq d - 3$. Then there is an integral and nodal curve $X \subset \mathbb{P}^3$ such that $\deg(X) = d$, $h^1(\mathcal{O}_X(1)) = 0$, X has geometric genus g , exactly $d - 3 - g$ nodes and $\#(W_3^0(X)) \geq g + d - 1$.*

In the preliminaries we will explain the notion of hyperosculating plane and its weight appearing in part (a) of the next proposition (see [11] for more details).

Proposition 1.4 *Fix integers $a \geq b > 0$ with $a \geq 3$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Let X be a general element of $|\mathcal{O}_Q(a, b)|$. Then:*

- (a) *X has no inflection tangents and it has only weight 1 hyperosculating planes.*
 (b) *For any $q \in \mathbb{P}^3 \setminus Q$ we have $r_X(q) = 2$, $\mathcal{S}(X, q)$ is finite and $\#(\mathcal{S}(X, q)) \geq (a + b - 1)(a + b - 2)/2 - ab + a + b - 2$.*
 (c) *$W_3^0(X) = \emptyset$;*
 (d) *For any $q \in Q \setminus X$ call $L_q \in |\mathcal{O}_Q(1, 0)|$ and $R_q \in |\mathcal{O}_Q(0, 1)|$ the lines of Q containing q . There are $y := 2ab - 2a$ lines $R_1, \dots, R_y \in |\mathcal{O}_Q(0, 1)|$ and $x := 2ab - 2b$ lines $L_1, \dots, L_x \in |\mathcal{O}_Q(1, 0)|$ with the following properties (with the convention $\binom{x}{2} = 0$ if $x = 0, 1$):*

- (1) *If $q \notin L_1 \cup \dots \cup L_x \cup R_1 \cup \dots \cup R_y$, then $\mathcal{S}(X, q) = \binom{a}{2} + \binom{b}{2}$.*
 (2) *If $q \in R_1 \cup \dots \cup R_y$ and $q \notin L_1 \cup \dots \cup L_x$, then $\mathcal{S}(X, q) = \binom{a-1}{2} + \binom{b}{2}$.*
 (3) *If $q \in L_1 \cup \dots \cup L_x$ and $q \notin R_1 \cup \dots \cup R_y$, then $\mathcal{S}(X, q) = \binom{a}{2} + \binom{b-1}{2}$.*
 (4) *If $q \in (L_1 \cup \dots \cup L_x) \cap (R_1 \cup \dots \cup R_y)$, then $\mathcal{S}(X, q) = \binom{a-1}{2} + \binom{b-1}{2}$.*

Sylvester's theorem ([5, 7, 8], [9, Theorem 4.1]) and the case $a = b = 2$ described in [12, Case 2 of Theorem 1] show the reasons for the exclusion of the cases $(a, b) = (2, 1)$ and $(a, b) = (2, 2)$ in Proposition 1.4. The case $(a, b) = (3, 1)$ is studied in [12, Case 3] of Theorem 1].

Proposition 1.5 *Fix integers $d \geq 4$ and x such that $1 \leq x \leq d - 1$. There is a pair (X, L) with the following properties:*

- (1) *$X \subset \mathbb{P}^3$ is a smooth rational curve and $\deg(X) = d$;*
 (2) *$L \subset \mathbb{P}^3$ is a line, $\deg(L \cap X) = d - 1$ and $\#((X \cap L)_{\text{red}}) = x$;*
 (3) *fix any $q \in L \setminus X \cap L$; if $x = 1$ we have $r_X(q) = 3$ and $\dim \mathcal{S}(X, q) = 2$; if $2 \leq x \leq d - 1$ we have $r_X(q) = 2$ and $\#(\mathcal{S}(X, q)) = \binom{x-1}{2}$.*

2 Preliminary observations

Notation 1 For any $q \in \mathbb{P}^3 \setminus \{q\}$ let $\ell_q : \mathbb{P}^3 \setminus \{q\} \rightarrow \mathbb{P}^2$ denote the linear projection from q .

Remark 2.1 Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Fix $q \in \mathbb{P}^3 \setminus X$. Since $q \notin X$, we have $r_X(q) > 1$. By [9, Proposition 5.1] we have $r_X(q) \leq 3$. By the

definition of X -rank we have $r_X(q) > 2$ (and hence $r_X(q) = 2$) if and only if $\ell_{q|X}$ is injective, i.e. if and only if $\#(L \cap X) \leq 1$ for every line L through q . Now suppose that X is contained in a smooth quadric surface Q and that $q \in Q \setminus X$. Let L_q and R_q denote the 2 lines containing q and contained in Q . For any line $D \not\subset Q$ such that $q \in D$ we have $\deg(D \cap (Q \setminus \{q\})) \leq 1$ by Bezout's theorem and in particular $\deg(D \cap X) \leq 1$. Thus $r_X(q) = 3$ if and only if both L_q and R_q contain a unique point of X .

Remark 2.2 Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Fix $o \in \mathbb{P}^3 \setminus X$. We have $r_X(o) = 2$ and $\mathcal{S}(X, o)$ is infinite if and only if $\ell_{o|X}$ is not birational onto its image. Thus $r_X(o) = 2$ and $\mathcal{S}(X, o)$ is infinite if and only if o is a Segre point in the sense of [4]. By [4] the set of all Segre points of X is finite (usually it is empty). If $\mathcal{S}(X, o)$ is infinite, then $\dim \mathcal{S}(X, o) = 1$.

In this paper we will apply the next remark and notation with $M \cong Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ a smooth quadric surface.

Remark 2.3 Let M be a projective variety, $D \subset M$ an effective Cartier divisor of M and $\mathcal{L}$ a line bundle on M . Let $Z \subset M$ be a zero-dimensional scheme. Let $\mathcal{I}_{Z,M}$ (resp. $\mathcal{I}_{D,M}$) denote the ideal sheaf of Z (resp. D) in M . We have $\mathcal{I}_{D,M} \cong \mathcal{O}_M(-D)$, because D is a Cartier divisor. The **residual scheme** $\text{Res}_D(Z)$ of Z with respect to D is the closed subscheme of M with $\mathcal{I}_{Z,M} : \mathcal{I}_{D,M}$ as its ideal sheaf. We have $\text{Res}_D(Z) \subseteq Z$ and in particular $\text{Res}_D(Z)$ is zero-dimensional. We have

$$\deg(Z) = \deg(Z \cap D) + \deg(\text{Res}_D(Z)).$$

For any line bundle $\mathcal{L}$ on M we have the **residual exact sequence**

$$0 \rightarrow \mathcal{I}_{\text{Res}_D, M} \otimes \mathcal{L}(-D) \rightarrow \mathcal{I}_{Z, M} \otimes \mathcal{L} \rightarrow \mathcal{I}_{Z \cap D, D} \otimes \mathcal{L}|_D \rightarrow 0 \quad (1)$$

Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. Set $d := \deg(X)$. Let g be the geometric genus (not the arithmetic genus) of X . For any $p \in X_{\text{reg}}$ let $T_p X \subset \mathbb{P}^3$ denote the tangent line of X at p . We recall that $T_p X$ is said to be an **inflection tangent** of X if the connected component W of the scheme-theoretic intersection $T_p X \cap X$ having $W_{\text{red}} = \{p\}$ has degree at least 3. Let $O_p(X)$ denote the **osculating plane** of X at p , i.e. the only plane $O_p(X) \subset \mathbb{P}^3$ such that the connected component Z of the scheme-theoretic intersection $O_p X \cap X$ having $Z_{\text{red}} = \{p\}$ has degree at least 3. The plane $O_p(X)$ is said to be a **hyperosculating plane** if $\deg(Z) \geq 4$. See [11] for the modern theory of osculating spaces. A hyperosculating plane $O_p(X)$ is said to have **weight** 1 if $\deg(Z) = 4$ and the tangent line $T_p X$ is not an inflection tangent. Usually a space curve has no inflection tangents, but it must have hyperosculating planes, except the rational normal curve. To explain this fact (and since we need it for singular curves, too), we explain the notion of osculating bundles with the rigorous foundations given by R. Piene ([11]). Let $f : Y \rightarrow X$ denote the normalization map. Using osculating bundles R. Piene defined the tangent line and the osculating plane at each $p \in Y$ (thus if $o \in \text{Sing}(X)$ and $\#(f^{-1}(o)) \geq 2$ there may be more than one tangent line or osculating plane associated to o). There is a notion of weight for each osculating plane of Y , with weight 0 if and only if $O_p(X)$ is not hyperosculating. The

sum of the weights of all osculating planes of (Y, f) is $4d + 12g - 12$ ([11, case $m = 3$ of the formula at p. 482]). Obviously this sum is zero if and only if $g = 0$ and $d = 3$, i.e. if and only if X is a rational normal curve. The tangent line $T_p X$ is called a **stall** if it is not an inflectional tangent and $O_p(X)$ is a hyperosculating plane.

Let $X \subset \mathbb{P}^3$ be a rational normal curve. The structure of $W_3^0(X)$ and $W_3(X)$ is known (also for rational normal curves in higher dimensional projective spaces) by Sylvester's theorem ([5], [7, §1.5], [8], [9, Theorem 4.1]). We have $W_3^0(X) = \tau(X) \setminus X$ and hence $r_X(q) = 2$ if and only if $q \in \mathbb{P}^3 \setminus \tau(X)$. We have $\#(\mathcal{S}(X, q)) = 1$ for all $q \in \mathbb{P}^3 \setminus \tau(X)$ ([7, Theorem 1.5.3]). For the case $d = 4$, $g = 1$ and X smooth (i.e. for linearly normal elliptic curves) see [12, Case 2) of Theorem 1]. For the different possibilities for $d = 4$ and $g = 0$ see [12, Case 3) of Theorem 1].

Remark 2.4 Fix an integral and non-degenerate curve $X \subset \mathbb{P}^3$. Set $d := \deg(X)$ and let g be the geometric genus of X . For any integral plane curve $D \subset \mathbb{P}^3$ and any $u \in D$ it is well-defined the arithmetic genus $\gamma_u(D)$ of the pair (D, u) . We have $\gamma_u(D) = 0$ if and only if D is smooth at u , $\gamma_u(D) = 1$ if and only if D has an ordinary node or an ordinary cusp and $\sum_{u \in \text{Sing}(D)} \gamma_u(D) = (t-1)(t-2)/2 - \gamma$, where $t := \deg(D)$ and γ is the geometric genus of D . For any $u \in \text{Sing}(X)$ set $\gamma_u(X) := \gamma_{\ell_q(X)}(\ell_q(X), \ell_q(u))$, where q is a general $q \in \mathbb{P}^3$. The positive integer $\gamma_u(X)$ is at least the arithmetic genus of the singularity (X, u) and equality holds if and only if u is a planar point of X (we are using that by Remark 2.2 for a general $q \in \mathbb{P}^3$ the morphism $\ell_{q|X}$ is birational onto its image). For a general $q \in \mathbb{P}^3$ we have $r_X(q) = 2$ and $\#(\mathcal{S}(X, q)) = (d-1)(d-2)/2 - g - \sum_{u \in \text{Sing}(X)} \gamma_u(X)$.

Now assume that X is smooth and hence $g = p_a(X)$. Assume that X is not a rational normal curve. Set $\Sigma_1 := W_2^0(X)$ and call Σ the complement in Σ_1 of the Segre points of X in the sense of Remark 2.2. The set $\Sigma_1 \setminus \Sigma$ is finite (Remark 2.2) and in most cases we have $\Sigma_1 = \Sigma$. We want to give here some upper bound for the integer $\mu_{\min}(\Sigma)$. Since $d \neq 3$, there exists at least one tangent line, say $L = T_u X$, which is either an inflection tangent or a stall. In both cases we see that there are points q with $r_X(q) = 2$ and $\#(\mathcal{S}(X, q)) \leq (d-1)(d-2)/2 - g - 2$. If L meets another tangent line of X or it is we have $\#(\mathcal{S}(X, q)) \leq (d-1)(d-2)/2 - g - 3$ for some $q \in L \setminus L \cap X$ (assuming $r_X(q) = 2$). Assume that L is a stall tangent but not an inflection tangent and that L meets X only at u . Since X is smooth at u , the linear projection from L induces a morphism $u : X \rightarrow \mathbb{P}^1$ such that $\deg(u) = d - 2$. Since $d > 3$, there is at least one ramification point of u , i.e. there is another tangent line of X meeting L . Better upper bounds for the integer $\mu_{\min}(\Sigma)$ are obtained looking at multiseccant lines of X . We exclude also the case $(d, g) = (4, 1)$ completely described in [12, Case 2 of Theorem 1]. Since $(d, g) \notin \{(3, 0), (4, 1)\}$ the genus formula for plane curves shows that X is contained in a surface T ruled by lines such that $\mu := \deg(X \cap L) \geq 3$ for any line L of this ruling of T and $\#((X \cap L)_{\text{red}}) = \deg(X \cap L)$ for a general line L of this ruling of T . Unless X is an elliptic curve and T is a ruled surface with as a base an elliptic curve isomorphic to X , there is at least one line L of this ruling of T such that $\#((L \cap X)_{\text{red}}) \leq \mu - 1$. Since the arithmetic genus of a planar singularity with multiplicity μ is $\geq \mu(\mu - 1)/2$, we have $\mu_{\min}(\Sigma) \leq (d-1)(d-2)/2 - g - \mu(\mu - 1)/2 + \binom{\mu-1}{2} = (d-1)(d-2)/2 + 1 - \mu$. Set

$$\alpha(d, g) = \frac{(d-2)(d-3)^2(d-4)}{12} - \frac{g(d^2 - 7d + 13 - g)}{2}.$$

By a formula of G. Castelnuovo and L. Berzolari $\alpha(d, g)$ is the expected number of quadrisecant lines of a smooth space curve of degree d and genus g . In the last century this formula (and several other enumerative formulas) were rigorously established in the following sense. Let $X \subset \mathbb{P}^3$ be a smooth, connected and non-degenerate curve with degree d and genus g . If $\alpha(d, g) \neq 0$, there is at least one line L such that $\deg(L \cap X) \geq 4$ (there are infinitely many L 's if $\alpha(d, g) < 0$). If $L \not\subset W_3(X)$, we may use this line with $\mu := \deg(X \cap L)$ for a general $q \in L$. If $\#(L \cap X)_{\text{red}} \geq 2$, then the line L shows that $r_X(q) = 2$ for all $q \in L \setminus L \cap X$. Now assume that $(L \cap X)_{\text{red}}$ is a single point, o . Since X is smooth at o , the linear projection from L induces a degree $d - \mu$ morphism $\varphi : X \rightarrow \mathbb{P}^1$. Note that $\mu = d - 1$ implies $g = 0$. Now assume $d \geq \mu + 2$. The map φ gives $r_X(q) = 2$ for a general $q \in L$. The Riemann-Hurwitz formula gives the existence of at least one $u \in X \setminus \{o\}$ such that $T_u X \cap L \neq \emptyset$. Thus we expect that for a huge number of (d, g, X) we have $\mu_{\min}(\mathbb{P}^3 \setminus X) = (d - 1)(d - 2)/2 - g - 4$ (take $\mu = 4$ and consider the contribution of the tangent line, expected to be unique for a general X).

3 The proofs

Proof of Theorem 1.1 Fix $L \in |\mathcal{O}_Q(1, 0)|$ and $o \in L$. Let $E \subset L$ be zero-dimensional scheme such that $E_{\text{red}} = \{o\}$ and $\deg(E) = g + 1$. Each $X \in |\mathcal{O}_Q(2, g + 1)|$ has degree $g + 3$ and (by the adjunction formula) arithmetic genus g . Since $h^1(\mathcal{O}_Q(-1, -g)) = h^1(\mathcal{O}_Q(-2, -g - 1)) = 0$ by the Künneth's formula, standard exact sequences give $h^0(\mathcal{O}_X) = 1$, $\langle X \rangle = \mathbb{P}^3$ and $h^0(\mathcal{O}_X(1)) = 4$ (i.e. X is linearly normal). The residual exact sequence (1) of $L \subset Q$ gives $h^1(\mathcal{I}_E(2, g + 1)) = 0$ because $h^1(\mathcal{O}_Q(1, g + 1)) = 0$, $\mathcal{I}_{E,L}(2, g + 1) \cong \mathcal{O}_L$ and $h^1(L, \mathcal{O}_L) = 0$. $\square$

Claim 1 A general $X \in |\mathcal{I}_E(1, g + 1)|$ is smooth and irreducible.

Proof of Claim 1 Since $h^0(\mathcal{O}_X) = 1$, it is sufficient to prove that X is smooth. Let $\mathcal{B}$ be the base locus of $|\mathcal{I}_E(2, g + 1)|$. For any $A \in |\mathcal{O}_Q(1, g + 1)|$, $L \cup A \in |\mathcal{I}_E(2, g + 1)|$. Since $|\mathcal{O}_Q(1, g + 1)|$ has no base points, we get $\mathcal{B} \subseteq L$. If $o \notin A$, the curve $L \cup A$ is smooth at o . Thus by Bertini's theorem to prove Claim 1 it is sufficient to prove that $\mathcal{B}_{\text{red}} = \{o\}$. Let R be the element of $|\mathcal{O}_Q(0, 1)|$ containing o . For any $B \in |\mathcal{O}_Q(2, 0)|$ we have $B \cup (g + 1)E \in |\mathcal{I}_E(2, g + 1)|$. Varying B we get $\mathcal{B} \subset (g + 1)E$. Thus $\mathcal{B} \subseteq L \cap (g + 1)R = E$. $\square$

Observation 1 Fix any integral $X \in |\mathcal{I}_E(2, g + 1)|$. We have $L \cap X = E$, because $\mathcal{O}_Q(2, g + 1) \cdot \mathcal{O}_Q(1, 0) = g + 1 = \deg(E)$ and L is not an irreducible component of X . Thus $r_X(q) > 1$ for all $q \in L \setminus \{o\}$. Fix $q \in L \setminus \{o\}$ and let R_q be the only element of $|\mathcal{O}_Q(0, 1)|$ containing q . By Remark 2.1 we have $r_X(q) = 2$ if R_q is not tangent to X and $r_X(q) = 3$ if R_q is tangent to X at some point. Let $u : X \rightarrow \mathbb{P}^1$ be the degree 2 morphism given by the restriction to X of the projection $Q \rightarrow \mathbb{P}^1$ induced by the linear system $|\mathcal{O}_Q(0, 1)|$. By the Riemann-Hurwitz formula u has exactly $2g + 2$ ramification points. For a general $X \in |\mathcal{I}_E(2, g + 1)|$ the point o is not one of these ramification points. Thus for a general $X \in |\mathcal{I}_E(2, g + 1)|$ there are exactly $2g + 2$ points $q \in L$ such that $r_X(q) = 3$.

Take $q \in Q \setminus (X \cup L)$. Let L_q be the element of $|\mathcal{O}_Q(1, 0)|$ containing q . We have $R_q \cap L_q = \{q\}$ if R_q is not one of the $2g + 2$ lines (call them $R_1, \dots, R_{2g+2}$) considered in Observation 1. The line R_q shows that $r_X(q) = 2$. Now assume $q \in (R_1 \cup \dots \cup R_{2g+2}) \setminus L \cap (R_1 \cup \dots \cup R_{2g+2})$. Let $E_q \subset L_q$ be the degree $g + 1$ divisor having q as its reduction. By Remark 2.1 $r_X(q) = 3$ if and only if $\#((L_q \cap X)_{\text{red}}) = 1$. This is true for all $q \in (R_1 \cup \dots \cup R_{2g+2}) \setminus L \cap (R_1 \cup \dots \cup R_{2g+2})$ if o is the only total ramification point of the degree $g + 1$ morphism $v : X \rightarrow \mathbb{P}^1$ obtained restricting to X the projection $Q \rightarrow \mathbb{P}^1$ induced by $|\mathcal{O}_Q(1, 0)|$. Since $\dim Q = 2$ and $\deg(E_q) = g + 1$, to prove it for all $q \in Q \setminus L$ and a general $X \in |\mathcal{I}_E(2, g + 1)|$ it is sufficient to prove that $h^1(\mathcal{I}_{E \cup E_q}(2, g + 1)) \leq g - 2$, which is done as in the proof of Claim 1.

For any smooth $C \in |\mathcal{O}_Q(1, 1)|$ and any $p \in C$ let $Z(5, C, p)$ denote the degree 5 zero-dimensional subscheme of C having p as its reduction.

Claim 2 We have $h^1(\mathcal{I}_{E \cup Z(5, C, p)}(2, g + 1)) = 0$ for all smooth $C \in |\mathcal{O}_Q(1, 1)|$ and all $p \in C \setminus C \cap L$.

Proof of Claim 2 Taking the residual exact sequence (1) of L we see that it is sufficient to prove that $h^1(\mathcal{I}_{Z(5, C, p)}(1, g + 1)) = 0$. Since C is a smooth conic, we have $\deg(\mathcal{O}_C(1, g + 1)) = g + 2$. Since $C \cong \mathbb{P}^1$, $\deg(Z(5, C, p)) = 5$ and $g \geq 2$, we have $h^1(C, \mathcal{I}_{Z(5, C, p)}(1, g + 1)) = 0$. The residual exact sequence (1) of C in Q proves the claim, because $h^1(\mathcal{O}_Q(0, g)) = 0$. $\square$

Observation 2 We have $(g + 2)(g + 1)/2 - g - 2(g + 3) > 0$ for all $g \geq 5$.

Now we take $q \in \mathbb{P}^3 \setminus Q$. To conclude the proof of the theorem it is sufficient to prove the following claim.

Claim 3 Fix $q \in \mathbb{P}^3 \setminus Q$. Then $r_X(q) = 2$ and $\#(\mathcal{S}(X, q)) \geq (g + 2)(g + 1)/2 - g - 2(g + 3)$.

Proof of Claim 3 If $\ell_{q|X}$ is not birational onto its image, then $r_X(q) = 2$ and $\mathcal{S}(X, q)$ is infinite (Remark 2.2); we may rule out this case for a general X , but it is not necessary to prove Claim 3. Thus we may assume that $\ell_q(X)$ is a degree $g + 3$ plane curve. Hence $\ell_q(X)$ has arithmetic genus $(g + 2)(g + 1)/2$ and geometric genus g . Assume $\ell_{q|X}$ injective. By Remark 2.1 no inflection tangent of X contains q and hence in the set-up of [12, §2] to compute the arithmetic genus of the singular point $\ell_q(u)$ of $\ell_q(X)$ it is sufficient to check the order of contact at u of the osculating plane. By Claim 2 this order of contact is at most 4 and hence the arithmetic genus of this singularity is at most 2. Thus $\ell_q(X)$ has only cuspidal singularities of arithmetic genus 1 or 2. By Observation 2 to get a contradiction it is sufficient to prove that there are at most $g + 3$ tangent lines of X passing through q . Let $H_q \subset \mathbb{P}^3$ be the polar plane of Q with respect to p . Since Q is a smooth quadric $C_q := H_q \cap Q$ is a smooth conic. For any $u \in Q$ we have $q \in T_u Q$ if and only if $u \in C_q$. Since $X \in |\mathcal{O}_Q(2, g + 1)|$ and $C_q \in |\mathcal{O}_Q(1, 1)|$ we have $\#(C_q \cap X_{\text{reg}}) \leq g + 3$. Thus at most $g + 3$ tangent lines of X contain q . Observation 2 concludes the proof. $\square$

Proof of Proposition 1.3 Let $Q = \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ be a smooth quadric surface. Let $T \in |\mathcal{O}_Q(2, d - 2)|$ be a union of 2 general elements of $|\mathcal{O}_Q(1, 0)|$ and $d - 1$

general elements of $|\mathcal{O}_Q(0, 1)|$. The curve T is a nodal and connected curve having $\deg(T) = d$, $p_a(T) = d - 3$, d irreducible components, $2d + 2$ singular points and it spans $\mathbb{P}^3$. Fix a set $A \subset \text{Sing}(X)$ such that $\#(A) = d - 3 - g$ and $T \setminus A$ is connected, i.e. no element of $|\mathcal{O}_Q(1, 0)|$ contains $d - 1$ elements of A and no element of $|\mathcal{O}_Q(0, 1)|$ contains 2 elements of A . We call A the set of the assigned nodes of T . By [13, Lemma 2.2 and Theorem 2.13] there is an integral nodal curve $Y \in |\mathcal{O}_Q(2, d - 2)|$ (thus $p_a(Y) = d - 3$) with exactly x nodes and hence with geometric genus g . To prove the theorem we will use a different degree d embedding of Y in $\mathbb{P}^3$. Set $\mathcal{R} := \mathcal{O}_Q(0, 1)|_Y$. Since $Y \in |\mathcal{O}_Q(2, d - 2)|$, $\mathcal{R}$ is a degree 2 spanned line bundle on Y . Since $h^1(Q, \mathcal{O}_Q(-2, -d + 3)) = 0$ by the Künneth's formula, we have $h^0(\mathcal{R}) = 2$. Note that if $a \in \text{Sing}(Y)$ and $b \in Y \setminus \{a\}$, then $u(a) \neq u(b)$. Thus $H^0(\mathcal{R})$ induces a degree 2 morphism $u : Y \rightarrow \mathbb{P}^1$. Since $p_a(Y) = d - 3$, for a general $o \in Y_{\text{reg}}$ we have $h^1(\mathcal{O}_Y((d - 3)o) = 0$, i.e. $h^0(\mathcal{O}_Y((d - 3)o)) = 1$. Thus the line bundle $\mathcal{L} := \mathcal{O}_Y((d - 2)o)$ is non-special, $h^0(\mathcal{L}) = 2$ and $\mathcal{L}$ is spanned. $\square$

Claim 1: For a general $o \in Y_{\text{reg}}$ the linear systems $|\mathcal{L}|$ and $|\mathcal{R}|$ induce an embedding $\varphi : Y \rightarrow Q$ such that $X := \varphi(Y) \in |\mathcal{O}_Q(2, d - 2)|$.

Proof of Claim 1 Since $\mathcal{R}$ and $\mathcal{L}$ are spanned, φ is a morphism. Since any $W \in |\mathcal{O}_Q(2, d - 2)|$ has arithmetic genus $p_a(Y)$, it is sufficient to prove that φ is birational onto its image. To get this condition it is sufficient to take as o a smooth point of Y which is not a ramification point of u . Since $X \in |\mathcal{O}_Q(2, d - 2)|$, we have $p_a(X) = d - 3$ and $h^1(\mathcal{O}_X(1)) = 0$. Since $h^0(\mathcal{L}) = 2$, the definition of $\mathcal{L}$ gives the existence of an $L \in |\mathcal{O}_Q(1, 0)|$ such that the scheme $L \cap X$ is the effective divisor $(d + 1)\varphi(o)$. Let $m : Y' \rightarrow Y$ denote the normalization map. By assumption Y' has genus g . Since $u \circ m : Y' \rightarrow \mathbb{P}^1$ is a degree 2 morphism, it has exactly $2g + 2$ ramification points, $o_1, \dots, o_{2g+2}$, by the Riemann-Hurwitz formula. Let R_i , $1 \leq i \leq 2g + 2$, be the element of $|\mathcal{O}_Q(0, 1)|$ containing $\varphi(o_i)$. By construction the degree 2 scheme $X \cap R_i$ contains the point $\varphi(o_i)$ having multiplicity at least 2 and hence it contains no other point of X . Note that $\varphi(o_i) \neq \varphi(o)$, because o is not a ramification point of u . Set $q_i := L \cap R_i$, $1 \leq i \leq 2g + 2$. $\square$

Observation 1: Fix $q \in Q \setminus X$ and call L' and L'' the two lines of Q containing q . Fix a line $D \not\subset Q$ such that $q \in D$. Since $q \in Q \setminus X$ and $D \not\subset Q$, Bezout's theorem gives $\deg(D \cap X) \leq 1$. Thus $r_X(q) > 2$ (and hence $r_X(q) = 3$ by [9, Proposition 5.1]) if and only if $\#((L' \cap X)_{\text{red}}) = \#((L'' \cap X)_{\text{red}}) = 1$.

Claim 2: We have $r_X(q_i) = 3$ for all $i = 1, \dots, 2g + 2$.

Proof of Claim 2 Since o is not a ramification point of u , we have $o \neq o_i$. Since $\{\varphi(o)\} = (X \cap L)_{\text{red}}$, and $q_i \in L$, we have $q_i \notin X$ and hence $r_X(q_i) > 0$. By construction o_i (resp. o) is the only point of X contained in R_i (resp. L). Thus no line through q_i contained in Q contains at least 2 points of X . By Observation 1 we have $r_X(q_i) = 3$.

Call $v_1, \dots, v_{d-3-g}$ the points of L such that the elements of $|\mathcal{O}_Q(0, 1)|$ containing v_i contains a point of $\text{Sing}(X)$. We saw that $v_i \notin \{\varphi(o), q_1, \dots, q_{2g+2}\}$. Let $R(v_i)$ be the element of $|\mathcal{O}_Q(0, 1)|$. By construction $\#((X \cap L)_{\text{red}}) = 1$. By Observation 1 we have $r_X(v_i) = 3$ for all i . Thus $\#(W_3^0(X)) \geq d - 3 - g + 2g + 2 = d + g - 1$. $\square$

Proof of Proposition 1.2 Take L and o as in the proof of Theorem 1.1. Call E' (resp. E) the zero-dimensional subscheme of L such that $\deg(E') = g$ (resp. $\deg(E) = g + 1$) and $E'_{\text{red}} = E_{\text{red}} = \{o\}$. We take as X a general element of $|\mathcal{I}_{E'}(2, g + 1)|$. Note that $E' \subset E$. Since $\deg(E') = g$, we have $\#((L \cap X)_{\text{reg}}) \leq 2$. By Claim 1 in the proof of Theorem 1.1 we have $h^1(\mathcal{I}_E(2, g + 1)) = 0$. Thus $h^1(\mathcal{I}_{E'}(2, g + 1)) = 0$. Hence $|\mathcal{I}_{E'}(2, g + 1)| \supsetneq |\mathcal{I}_E(2, g + 1)|$. Using E' instead of E and working as we did in the proof of Theorem 1.1 we get that X is smooth, $r_X(q) = 2$ for all $q \in \mathbb{P}^3 \setminus (L \cup X)$ and $\#(\mathcal{S}(X, q)) \geq (g + 2)(g + 1)/2 - g - 2(g + 3)$ for all $q \in \mathbb{P}^3 \setminus Q$.

Let $u : X \rightarrow \mathbb{P}^1$ be the degree 2 morphism obtained restricting to X the projection $Q \rightarrow \mathbb{P}^1$ induced by the complete linear system $|\mathcal{O}_Q(0, 1)|$. Let R_i , $1 \leq i \leq 2g + 2$, be the elements of $|\mathcal{O}_Q(0, 1)|$ containing one of the ramification points of u . Set $v_i := R_i \cap L$. We have $\#\{v_1, \dots, v_{2g+2}\} = 2g + 2$. To conclude the proof it is sufficient to show that $r_X(v_i) = 2$ for all i . We have $o \in L \cap X$. Since X is irreducible, we have $L \not\subset X$. Thus $\deg(X \cap L) = g + 1$ and $E' \subseteq E$, we have $\#((X \cap L)_{\text{red}}) \leq 2$ and $\#((X \cap L)_{\text{red}}) = 1$ if and only if $X \cap L = E$. Since $|\mathcal{I}_{E'}(2, g + 1)| \supsetneq |\mathcal{I}_E(2, g + 1)|$ and X is general in $|\mathcal{I}_{E'}(2, g + 1)|$, we have $(X \cap L)_{\text{red}} = \{o, o'\}$ for some $o' \neq o$. The line L shows that $r_X(v_i) \leq 2$ and that $r_X(v_i) = 2$ and $\{o, o'\} \in \mathcal{S}(X, q)$ if and only if $v_i \notin \{o, o'\}$. To prove that $v_i \notin \{o, o'\}$ for all i it is sufficient to prove that for a general $X \in |\mathcal{I}_{E'}(2, g + 1)|$ neither o nor o' is a ramification point of the morphism $u : X \rightarrow \mathbb{P}^1$. Call R the element of $|\mathcal{O}_Q(0, 1)|$ containing o and $w \subset R$ the degree 2 effective divisor of R having $\{o\}$ as its reduction. Note that $\deg(E' \cup w) = g + 1$ and that o is a ramification divisor of u if and only if $w \subset X$. As in the proof of Claim 1 of Theorem 1.3 we see that $h^1(\mathcal{I}_{E' \cup w}(2, g + 1)) = 0$, i.e. $|\mathcal{I}_{E' \cup w}(1, g + 1)|$ has codimension 1 in $|\mathcal{I}_{E'}(1, g + 1)|$. Thus $w \not\subset X$ for a general X . Fix any $o_1 \in L \setminus \{o\}$ and call R_1 the element of $|\mathcal{O}_Q(0, 1)|$ containing o_1 . Let $w_1 \subset R_1$ be the degree 2 effective divisor of R_1 having $\{o_1\}$ as its reduction. As in the proof of Claim 1 of Theorem 1.1 we see that $h^1(\mathcal{I}_{E' \cup w_1}(2, g + 1)) = 0$. Instead of X we take a general $X' \in |\mathcal{I}_{E' \cup w_1}(2, g + 1)|$. For this X' we have $(X' \cap L)_{\text{red}} = \{o, o_1\}$ and neither o nor o_1 is a ramification point of the degree 2 morphism $X' \rightarrow \mathbb{P}^1$ induced by $|\mathcal{O}_Q(0, 1)|$. Once we know this statement, even the original X has the same property for o, o' and thus from now on we may again consider a general $X \in |\mathcal{O}_{E'}(2, g + 1)|$. To conclude the proof we need to prove that $r_X(q) = 2$ and $\#(\mathcal{S}(X, q)) \geq 2$ for all $q \in Q \setminus (L \cup X)$.

For any $q \in Q \setminus L$ let L_q be the element of $|\mathcal{O}_Q(1, 0)|$ containing q . To prove that $r_X(q) = 2$ and $\#(\mathcal{S}(X, q)) \geq 2$ for all $q \in Q \setminus (L \cup X)$ it is sufficient to prove that $\#((L_q \cap X)_{\text{red}}) \geq 2$. Since $g + 1 \geq 5$, to get this inequality for all $q \in Q \setminus L$ it is sufficient to prove that $v : X \rightarrow \mathbb{P}^1$ has only ordinary ramification points. Since $\dim |\mathcal{O}_Q(1, 0)| = 1$ and each line has ∞^1 points, it is sufficient to prove that $h^1(\mathcal{I}_{E' \cup Z'}(2, g + 1)) = 0$, where Z' is a connected degree 3 zero-dimensional scheme contained in some $L' \in |\mathcal{O}_Q(1, 0)| \setminus \{L\}$. This is done using first the residual exact sequence of L and then the residual exact sequence of L' . $\square$

Proof of Proposition 1.4 By Bertini's theorem X is a smooth curve. Since $a > 0, b > 0$ and $a + b \geq 3$, X is connected and non-degenerate.

- (a) In this step we prove that X has no inflection tangents. Since $\deg(Q) = 2$, Bezout's theorem implies that each inflection tangent is contained in Q . For any $o \in Q$ let $L_1(o)$ (resp. $L_2(o)$) be the element of $|\mathcal{O}_Q(1, 0)|$ (resp. $|\mathcal{O}_Q(0, 1)|$)

containing o . Let $Z_i(o)$, $i = 1, 2$, be the degree 3 connected zero-dimensional subschemes of $L_i(o)$ having o as its reduction. Since $\dim Q = 2$, it is sufficient to prove that $|\mathcal{O}_Q(a, b)(-Z_i(o))|$ has codimension 3 in $|\mathcal{O}_Q(a, b)|$, i.e. $h^1(\mathcal{I}_{Z_i(o)}(a, b)) = 0$. This is always true for $i = 2$, because $a \geq 2$ and it is true for $i = 1$ if and only if $b \geq 2$. Now assume $b = 1$. In this case it is sufficient to notice that each irreducible $X \in |\mathcal{O}_Y(a, 1)|$ is smooth and that for each smooth X we have $\deg(X \cap L_1(o)) = 1$.

- (b) In this step we prove that X has no osculating plane with order of contact at least 5, i.e. it has only weight 1 hyperosculating planes (and hence there are exactly $4d + 12g - 12$ such planes by [11, case $m = 3$ of the formula at p. 482] (who credits [1, p. 200])).

For each smooth $H \in |\mathcal{O}_Q(1, 1)|$ and any $o \in H$, let $Z(H, o)$ be the degree 5 zero-dimensional subscheme of H having o as its reduction. Since X has no inflection tangents (step (a)) Q has ∞^3 plane sections and each of them has dimension 1, it is sufficient to prove that for each smooth $H \in |\mathcal{O}_Q(1, 1)|$ and each $o \in H$, we have $h^0(Q, \mathcal{I}_{Z(H, o)}(a, b)) = (a+1)(b+1) - 5$. Set $E := L_2(o) \cap Z(H, o)$ and $F := L_1(o) \cap Z(H, o)$. For any zero-dimensional scheme $W \subset X$ and all $(x, y) \in \mathbb{Z}$ we consider the residual exact sequences (1) of $L := L_1(o)$ and $R := L_2(o)$:

$$0 \rightarrow \mathcal{I}_{\text{Res}_L(W)}(x-1, y) \rightarrow \mathcal{I}_W(x, y) \rightarrow \mathcal{I}_{W \cap L, L}(x, y) \rightarrow 0 \quad (2)$$

$$0 \rightarrow \mathcal{I}_{\text{Res}_R(W)}(x, y-1) \rightarrow \mathcal{I}_W(x, y) \rightarrow \mathcal{I}_{W \cap R, LR}(x, y) \rightarrow 0 \quad (3)$$

- (b1) Assume $\deg(E) = \deg(F) = 1$. We use 3 times (2) always with $y = b$, first with $W = Z(H, o)$ and $x = a$, then with $W = \text{Res}_L(Z(H, o))$ and $x = a-1$ and then with $W = \text{Res}_L(\text{Res}_L(Z(H, o)))$ and $x = a-2$. Then we apply (3) with $y = b$, $x = a-3$ and as W the last residual scheme $\text{Res}_L(\text{Res}_L(\text{Res}_L(Z(H, o))))$. As a residual scheme we get a scheme W' having $\deg(W') = 1$, i.e. a point. It is sufficient to use that $h^1(\mathcal{I}_{W'}(a-3, b-1)) = 0$ if $a \geq 3$ and $b \geq 1$.
- (b2) Assume $\deg(F) > 1$. We saw in step (a) that $\deg(F) = 2$. We use (3) once with $(x, y) = (a, b)$ and then we apply (2) twice, first with $(x, y) = (a, b-1)$ and then with $(x, y) = (a-1, b-1)$.
- (b3) Assume $\deg(E) > 1$ and $\deg(F) = 1$. We saw that $\deg(E) = 2$. We apply once (2) with $(x, y) = (a, b)$ and $W = Z(H, o)$ and then apply twice (3).
- (c) In this step we prove that no stall tangent is contained in Q . Note that if $b = 1$ no element of $|\mathcal{O}_Q(0, 1)|$ is tangent to X . Thus we may assume $b \geq 2$ and $a \geq 3$. Recall that we assumed $a \geq 3$. We prove that for a general X no stall tangent is an element of $|\mathcal{O}_X(0, 1)|$, since the exclusion of elements of $|\mathcal{O}_X(0, 1)|$ is similar (when $b \geq 2$). Fix $L \in |\mathcal{O}_X(1, 0)|$ and $o \in L$. Set $M := T_o Q$. We have $L \cup R = M \cap Q$, where R is the element of $|\mathcal{O}_Q(0, 1)|$ containing o . Let $Z \subset L \cup R$ be a connected and curvilinear zero-dimensional scheme such that $\deg(Z) = 4$, $Z \cap L$ contains the degree 2 subscheme of L having o as its reduction, but not the one of degree 3. Since no inflection tangent of a general X is contained in Q , the possible o are ∞^2 and (for a fixed o) the possible Z are ∞^1 , it is sufficient to prove that $h^1(\mathcal{I}_Z(a, b)) = 0$. Set $W := \text{Res}_L(Z)$. Since

- $\deg(Z \cap L) = 2$, by assumption, the residual exact sequence (1) of $L \subset Q$ shows that it is sufficient to prove that $h^1(\mathcal{I}_W(a-1, b)) = 0$. Since $\deg(W) = 2$, we have $h^1(\mathcal{I}_W(a-1, b)) = 0$.
- (d) Now we prove that there is no $q \in \mathbb{P}^3 \setminus X$ such that q contains 4 distinct tangent lines to X . Since each $q \in Q \setminus X$ is only contained in two lines meeting X , it is sufficient to prove that no $q \in \mathbb{P}^3 \setminus X$ is contained in at least 4 distinct tangent lines. For any $q \in \mathbb{P}^3 \setminus Q$ the contact points of the tangent planes of Q at q are the intersection of Q with the first polar of Q with respect to q and hence it is a smooth conic D . Fix $o_1, o_2 \in Q$ such that $o_1 \neq o_2$ and $\{o_1, o_2\} \not\subset Q$. Set $L := T_{o_1}Q \cap T_{o_2}(Q)$ and take $q \in L$. Set $L_i := \langle q, o_i \rangle$, $i = 1, 2$. Let H be the polar of Q with respect to q . The set H is a plane such that $D := H \cap Q$ is a smooth conic (because Q is smooth and $q \notin Q$), with the property that for any $u \in Q$ we have $q \in T_uQ$ if and only if $u \in D$. Thus (for any $X \subset Q$) to get 4 tangent lines through q , the first two being L_1 and L_2 , we need to fix $o_3, o_4 \in D \setminus \{o_1, o_2\}$. Set $L_i := \langle q, o_i \rangle$, $i = 3, 4$. Let $Z_i \subset L_i$ be the degree 2 zero-dimensional scheme having o_i as its reduction. Set $Z := Z_1 \cup Z_2 \cup Z_3 \cup Z_4$. A smooth curve $X \supset \{o_1, \dots, o_k\}$ has L_i as its tangent line at o_i , $1 \leq i \leq 4$, if and only if $X \supset Z$. The choice of o_1 and o_2 depends on 4 parameters (two points of Q). Then we fixed a point of L (another parameter) and $o_3, o_4 \in D$ (two parameters, because $\dim D = 1$). Thus to prove that a general $X \in |\mathcal{O}_Q(a, b)|$ has no 4 distinct tangents meeting at one point it is sufficient to prove that $h^1(\mathcal{I}_Z(a, b)) = 0$. We have $Z \cap D = \{o_1, o_2, o_3, o_4\}$ and $\text{Res}_D(Z) = \{o_1, o_2, o_3, o_4\}$. Since $a + b \geq 5$, we have $h^1(D, \mathcal{I}_{Z \cap D, D}(a, b)) = 0$ and $h^1(D, \mathcal{I}_{Z \cap D, D}(a-1, b-1)) = 0$. Thus using twice the residual exact sequence (1) of D we get $h^1(\mathcal{I}_Z(a, b)) = 0$.
- (e) In this step we prove that no two stall tangent meets. No stall tangent is contained in Q (step (c)). Fix $o_1, o_2 \in Q$ such that no line of Q contains $\{o_1, o_2\}$. Fix a line L_i tangent to Q at o_i and assume $L_1 \cap L_2 \neq \emptyset$. Fix a plane $H_i \supset L_i$. Let $Z_i \subset C_i := H_i \cap Q$ be a connected zero-dimensional scheme with o_i as its reduction, curvilinear and containing the degree 2 subscheme of L_i with o_i as its reduction. Let C be any smooth curve containing o_i , $C \not\subset H_i$, with $T_{o_i}C = L_i$ and $H_i = \mathcal{O}_{o_i}C$. H_i is a hyperosculating plane of C if and only if the connected component of $H_i \cap C$ with o_i as its reduction (which contains $Z_i \cap L_i$, because $L_i = T_{o_i}C$) has degree at least 4. Since $\deg(Z_1 \cup Z_2) = 8$, a dimension count shows that to exclude this situation for a general $X \in |\mathcal{O}_Q(a, b)|$ it is sufficient to prove that $h^1(\mathcal{I}_{Z_1 \cup Z_2}(a, b)) = 0$. Each C_i is a smooth conic. We have $\deg(\mathcal{O}_{C_i}(x, y)) = a + b$ and $\deg(C_i \cap (Z_i \cup Z_2)) = 4$. Recall that $a + b \geq 5$. We have $h^1(C_2, \mathcal{I}_{Z_2}(a, b)) = 0$, $h^1(C_1, \mathcal{I}_{Z_1}(a-1, b-1)) = 0$ and $h^1(\mathcal{O}_Q(a-2, b-2)) = 0$. The long cohomology exact sequences of the residual sequences first of C_2 with respect to $\mathcal{O}_Q(a, b)$ and then of C_1 with respect to $\mathcal{O}_Q(a-1, b-1)$ give $h^1(\mathcal{I}_{Z_1 \cup Z_2}(a, b)) = 0$.
- (f) In this step we prove that there is no point q contained in a stall tangent and at least two other tangent lines. This is done mixing the proofs of steps (d) and (e). In this case we have a degree 8 zero-dimensional scheme Z with 3 connected components, two of degree 2 (corresponding to the two tangents) and one, W , curvilinear of degree 4 (corresponding to the stall tangent). We need to prove that $h^1(\mathcal{I}_Z(a, b)) = 0$. The curve $D := \langle W \rangle \cap Q$ is a smooth

conic. As in step (e) using the residual exact sequence of D in Q we see that $h^1(\mathcal{I}_Z(a, b)) \leq h^1(\mathcal{I}_{Z \setminus W}(a-1, b-1)) = 0$. The scheme $Z \setminus W$ is a degree 4 zero-dimensional scheme with 2 connected components, none of them being contained in a line of Q by step (c). Thus $h^1(\mathcal{I}_{Z \setminus W}(a-1, b-1)) = 0$ when $a+b \geq 5$.

- (g) In this step we prove that $\mathcal{S}(X, q)$ is finite for all $q \in \mathbb{P}^3 \setminus Q$. Fix $q \in \mathbb{P}^3 \setminus Q$ and assume that $\mathcal{S}(X, q)$ is infinite, i.e. assume that the morphism $\ell_{q|X}$ has degree $x \geq 2$ (Remark 2.2). Since $\deg(D \cap X) \leq 2$ for each line $D \not\subset Q$, we have $x = 2$. Thus $a+b$ is even and $\deg(\ell_q(X)) = (a+b)/2$. Thus X is contained in an irreducible cone of degree $(a+b)/2$ with vertex q . Since $X \subset Q \cap T$ and $\deg(X) = \deg(Q) \deg(T)$, X is the complete intersection of Q and T . Thus $a = b$. Since X is a complete intersection of a quadric and a surface of degree a , we have $h^0(\mathcal{I}_X(a)) = 1 + \binom{a+1}{3}$. The set of all integral $B \in |\mathcal{O}_{\mathbb{P}^3}(a)|$ which are cones with vertex q has dimension $\binom{a+2}{2} - 1$. Since $\dim \mathbb{P}^3 = 3$, the set of all integral $B \in |\mathcal{O}_{\mathbb{P}^3}(a)|$ which are cones with vertex some point of $\mathbb{P}^3$ has codimension at least $\binom{a+3}{3} - \binom{a+2}{2} - 3 = \binom{a+2}{3} - 3$ in $|\mathcal{O}_{\mathbb{P}^3}(a)|$. Since $\binom{a+2}{3} - 3 > 1 + \binom{a+1}{3}$ for all $a \geq 3$, we get that a general $X \in |\mathcal{O}_Q(a, a)|$ is not contained in a degree a cone.
- (h) Take $q \in \mathbb{P}^3 \setminus Q$. By step (g) q is not a Segre point of X . Hence $\ell_q(X)$ is a plane curve with degree $a+b$ (and hence with arithmetic genus $(a+b-1)(a+b-2)/2$) and with geometric genus $p_a(X) = ab - a - b + 1$. Since $(a+b-1)(a+b-2)/2 - ab + a + b - 2 > 0$ for all $a \geq 3, b \geq 1$, steps (b), (c), (d) and (e) show that $\ell_{q|X}$ is not injective (i.e. $r_X(q) = 2$). Since $\deg(D \cap X) \leq 2$ for any line D containing q , $\ell_q(X)$ has only singular points with multiplicities 2 and any line D containing q contains at most one element of $\mathcal{S}(X, q)$.

□

Proof of Proposition 1.5 Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Fix $L \in |\mathcal{O}_Q(0, 1)|$ and $o \in L$. Let $Z \subset L$ be the degree $d-x$ zero-dimensional subscheme of L with $Z_{\text{red}} = \{o\}$. Fix a general $X \in |\mathcal{I}_Z(d-1, 1)|$.

Claim 1: X is a smooth rational curve, $\deg(X) = d$, $\deg(X \cap L) = d-1$ and $\#((X \cap L)_{\text{red}}) = x$.

Proof of Claim 1: Among the elements of $|\mathcal{I}_Z(d-1, 1)|$ there are all curves $Y \cup L$ with $Y \in |\mathcal{O}_Q(d-1, 0)|$. Thus the base locus $\mathcal{B}$ of $|\mathcal{I}_Z(d-1, 1)|$ is contained in L . Let R be the element of $|\mathcal{O}_Q(1, 0)|$ containing o . Among the elements of $|\mathcal{I}_Z(d-1, 1)|$ there are all curves $U \cup (d-x)R$ with $U \in |\mathcal{O}_Q(x-1, 1)|$. Thus $\mathcal{B} \subseteq R \cap L = \{o\}$. By Bertini's theorem X is smooth outside o . Since there is a curve $Y \cup L \in |\mathcal{I}_Z(d-1, 1)|$ with $o \notin Y$, there are elements of $|\mathcal{I}_Z(d-1, 1)|$ smooth at o . Since smoothness is an open condition in the Zariski topology, X is smooth. Hence X is a smooth rational curve of degree d . Let $W \subset L$ be the degree $d-1$ zero-dimensional subscheme of L such that $W_{\text{red}} = \{o\}$. Since Z is contained in the scheme-theoretic intersection, we have $\deg(L \cap X) = d-1$ and $\#((X \cap L)_{\text{red}}) \leq x$. To prove that $\#((X \cap L)_{\text{red}}) = x$ we need to prove that (for a general X) we have $W \not\subset X$ and that X intersects transversally L outside o . We have $W \not\subset X$, because $W \not\subset U \cup (d-x)R$ for all $U \in |\mathcal{O}_Q(x-1, 1)|$.

such that $o \notin U$. For a general U we also see that $U \cup (d-x)R$ intersects transversally L outside o .

Fix any $q \in L \setminus X \cap L$. Fix a line $D \subset \mathbb{P}^3$ such that $D \neq L$. If $D \subset Q$ we have $D \in |\mathcal{O}_Q(1, 0)|$ and hence $\deg(D \cap X) = 1$. If $D \not\subset Q$ we have $\deg(D \cap X) \leq 1$ by Bezout, because $q \in Q \setminus X$ and $\deg(D \cap X) = 2$. Thus the restriction to $X \setminus L \cap X$ of the linear projection from q is injective. The last assertion of Claim 1 proves part (3) of the proposition. $\square$

Funding Open access funding provided by Università degli Studi di Trento within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Baker, H.F.: Principle of Geometry, Vol. V, analytical principles of the theory of curves, Cambridge University Press, Cambridge, (1933)
2. Ballico, E.L.: A. Bernardi and E. Ventura, Strict inclusions of high rank loci, [arXiv:1907.06203](https://arxiv.org/abs/1907.06203), J. Symbolic Computation (to appear), <https://doi.org/10.1016/j.jsc.2020.07.004>
3. Buczyński, J., Han, K., Mella, M., Teitler, Z.: On the locus of points of high rank. Eur. J. Math. **4**, 113–136 (2018)
4. Calabri, A., Ciliberto, C.: On special projections of varieties: epitome to a theorem of Beniamino Segre. Adv. Geom. **1**, 97–106 (2001)
5. Comas, G., Seiguer, M.: On the rank of a binary form. Found. Comp. Math. **11**(1), 65–78 (2011)
6. Hartshorne, R.: Algebraic Geometry. Springer-Verlag, Berlin-Heidelberg-New York (1977)
7. Iarrobino, A., Kanev, V.: Power sums, Gorenstein algebras, and determinantal loci. Lecture Notes in Mathematics, vol. 1721. Springer-Verlag, Berlin (1999)
8. Landsberg, J.M.: Tensors: Geometry and Applications, Graduate Studies in Mathematics. Amer. Math. Soc. Providence **128**, (2012)
9. Landsberg, J.M., Teitler, Z.: On the ranks and border ranks of symmetric tensors. Found. Comput. Math. **10**, 339–366 (2010)
10. Le Barz, P.: Sur les espaces multisécants aux courbes algébriques. Manuscripta Math. **119**(4), 433–452 (2006)
11. Piene, R.: Numerical characters of a curve in projective n -space, in: Real and Complex Singularities, Oslo 1976, P. Holm (ed.), pp. 475–496, Groningen: Sijthoff and Noordhoff, (1978)
12. Piene, R.: Cuspidal projections of space curves. Math. Ann. **256**(1), 95–119 (1981)
13. Tannenbaum, A.: Families of algebraic curves with nodes. Composition Math. **41**(1), 107–126 (1980)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.