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PERSPECTIVES

Mentoring Undergraduate Research in Mathematical Modeling

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Abstract
In writing about undergraduate research in mathematical modeling, I draw on my 31 years as a mathematics professor at the University of Nebraska–Lincoln, where I mentored students in honors' theses, REU groups, and research done in a classroom setting, as well as my prior experience. I share my views on the differences between research at the undergraduate and professional levels, offer advice for undergraduate mentoring, provide suggestions for a variety of ways that students can disseminate their research, offer some thoughts on mathematical modeling and how to explain it to undergraduates, and discuss the challenges involved in broadening research participation to include early career students and mid-tier students and how to deliver a research experience in a classroom setting. While different situations pose different challenges, different problems require different approaches, and different experiences lead to different conclusions, it is my hope that my experiences will be of broad value to a wide audience.

Keywords Undergraduate research · Mathematical modeling · Mathematics education

1 Introduction

Undergraduate research has been a large part of my academic career from beginning to end. I had my first experience of research as a college freshman on a self-directed project. My first mentoring experience came while I was still a graduate student and working as a visiting instructor at Union College (NY). In my career in the Department of Mathematics at the University of Nebraska–Lincoln (UNL) from 1989 through 2020, I mentored approximately 100 undergraduate students in research, including a mix of individual student projects, groups of 3–5 advanced undergraduates, and large

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groups of students in a classroom setting. All of these experiences have given me insights into how to think about and mentor undergraduate research, insights that I have shared in a variety of settings with junior faculty members from a variety of institutions. The purpose of this essay is to offer some general principles to help newcomers succeed at undergraduate research mentoring. I will focus on the characteristics of undergraduate research as contrasted with professional research, the features of mathematical modeling that make it different from mathematics per se, and bits of advice distilled from my experience. Not everything in my experience will apply to all settings, but hopefully any reader will find something useful for their own situation.

1.1 Undergraduate Research from a Student Perspective

My first exposure to undergraduate research was as a freshman at the University of Denver (DU). I was an avid gamer, and I got the idea of making myself a strategy guide for the game of Monopoly. Good trading in Monopoly requires an understanding of the expected value of the game's properties, which in turn depends on knowing how likely each space is to be landed on in the latter part of the game. I set about to determine these probabilities. First, I wrote formulas for the conditional probabilities of getting to each space from each other space. I accounted for everything, such as die rolls to get out of jail and the Community Chest card that sends a token back three spaces. I created a probability map and wrote a computer program to iterate it to steady state. Not many students had computer programming experience in 1973. I had been fortunate to be in an experimental version of my high school's Algebra 2 course in the prior year, in which I learned programming in Basic on a remote terminal connected to the main frame computer of my school district. The only storage mechanism for programs back then was a yellow paper tape that had holes punched in it by a machine attached to the remote terminal. As a freshman at DU, I had access to the same setup: a remote terminal connected to the university's mainframe and the ability to run Basic.

Undergraduate research was not a thing in 1973, so it never occurred to me to tell a math professor what I had done. If I had, my professor would probably not have described what I had done as research. After all, what I did could have been a group project for an operations research class. In mathematical terms, what I did doesn't sound like much; however, I had never heard of Markov chains, or even matrices, and had never seen a math problem solved by iteration.

As professional mathematicians, we generally use the word 'research' as a noun: it is our collective body of scholarship, measured by publications. In academia, 'unpublishable research' is an oxymoron, and by the 'research is publications' principle, my project would not count. From the perspective of a seasoned undergraduate research mentor, however, it is clear that my project should be counted as research. I may have been doing something that had been done before, but I did it by developing my own research tools. For undergraduates, we should think of 'research' as a verb, not a noun: it is the processes of generating and communicating knowledge.

1.2 Undergraduate Research in a Twenty-First Century Curriculum

Traditionally, research has been the province of the best and most advanced students. Of course it is important to provide research experiences for such students, but it is also important to broaden our idea about who should be doing research and in what setting. There are strong arguments for the value of research for students who are not at the very top of their class and for students who are very early in their careers.

In 2008, the University of Nebraska–Lincoln (UNL) adopted a new general education program called Achievement-Centered Education (ACE). This program identified four desired general education objectives, associated ten specific outcomes with these objectives, and specified guidelines that a course would have to meet to achieve a specific outcome (University of Nebraska–Lincoln 2008). The last of the objectives was 'Integrate these abilities and capacities [referring to the first three objectives], adapting them to new settings, questions, and responsibilities,' and its associated outcome was 'Generate a creative or scholarly product that requires broad knowledge, appropriate technical proficiency, information collection, synthesis, interpretation, presentation, and reflection.' Adoption of ACE made UNL one of the first institutions to identify undergraduate research as an essential component of undergraduate education. The argument that undergraduate research should be extended to all undergraduates is simple: the knowledge base in the Twenty-First Century changes so quickly that learning about what is currently known must be supplemented by learning to generate new knowledge.

There is also an argument to be made for introducing talented students to research at an early point in their careers. When I was crafting my university's research proposal for the NSF UBM (Interdisciplinary Training for Undergraduates in Biological and Mathematical Sciences) program in 2005, I felt that it would be beneficial for students who were going to be doing a serious interdisciplinary research project as juniors or seniors to have a preliminary experience with a limited research project as freshmen or sophomores. This idea resulted in the RUTE Summer Scholars component of the NSF-funded Research for Undergraduates in Theoretical Ecology (RUTE) program, for which I was the lead PI and program director (Ledder et al. 2013; University of Nebraska–Lincoln Department of Mathematics 2010). The program consisted of the equivalent of a 3-credit course, taught in the summer to talented students who had just finished their freshman year or just graduated from high school. What was novel about this course was that it was organized around a research agenda rather than a body of knowledge. A class of 20 students worked in groups of 5 to study the population dynamics of aphids and predation by coccinellids (lady beetles); their work consisted of laboratory experiments and mathematical modeling to explain the results of the experiments. One of our students presented his group's work in an undergraduate poster session at the National Academy of Sciences, where it was displayed alongside work done at other institutions by senior undergraduates (Larrieu 2010). Early research experiences can inspire students to do more research and to take courses that help them prepare for research.

When the RUTE program was funded in 2005, research for early career undergraduates was rare. Recently, interest in this idea has become more widespread, leading to the preparation of a forthcoming volume of contributed papers on research projects for students at the community college level (Goldwyn et al. 2022), including a paper I and a colleague contributed on mathematical epidemiology (Ledder and Homp 2022). There are special challenges in providing a research experience to students with limited mathematical course work, and this is one topic that will be discussed in this essay.

1.3 Mathematical Modeling for Mathematicians and Mathematics Students

While readers of this journal who are trained in mathematical modeling understand the differences between mathematics and mathematical modeling, people trained in areas other than mathematical modeling and undergraduate students might need an overview of these differences. This is particularly important because it has become fashionable to use the word ‘modeling’ in titles of textbooks that feature so-called ‘applications’ of mathematics, but do not actually include anything that I would describe as modeling. I will explain this claim and suggest ways to explain the differences to students in order to help them do serious research in modeling.

2 Some Thoughts on Undergraduate Research

As noted in the introduction, one essential difference between professional and undergraduate research is the ultimate goal: in professional research it is the product, whereas *in undergraduate research it is the process*. This is not to say that undergraduate research should not be publishable, but merely that publishability is not the primary goal. A second essential difference is the time frame. Professional research has no deadlines. Undergraduate research is restricted by the calendar.

Professional research in mathematical modeling for biology has four crucial characteristics, each of which is necessary for publishability:

1. It is *open-ended*; that is, the work goes where the problem leads rather than being scripted like a project in a textbook.
2. It is *original*, meaning that the essential elements have not previously been published.
3. It is *thorough*, meaning that problems are completely finished, with no loose ends that could have been tied up with a little more work.
4. It is *important*, meaning that it is judged worthy by the mathematics and/or biology communities.

Because of differences in the goal and the time frame, these characteristics need to be reconsidered in the context of undergraduate research.

1. Undergraduate research must be open-ended in the same way as professional research. We can give our students guidance, but we must not give them a script. I often use scripted projects in my classes (see Ledder and Homp 2021, for example), but these do not count as research.
2. In the context of undergraduate research, ‘original’ does not have to mean that nobody has solved the problem before, but merely that the students did not have

access to that work when they did their own research. My Markov chain work was original in this sense.
3. For undergraduates, we have to give up the requirement of being thorough. Graduate students who haven't completed a problem stay for another year, but undergraduates are done when their time runs out. Undergraduates can do single cases for a multi-case problem or they can make simplifying assumptions that we would not make in our own work.
4. While 'important' is a valuable attribute for undergraduate research to have, it can be replaced by 'interesting.' It is acceptable for an undergraduate to do research on a zombie model (Smith 2014) or on Blue Itchy Toes Syndrome (described below).

2.1 Advice on Mentoring Undergraduates

Research at any level requires ability, but other attributes are critical for undergraduates that new mentors might not appreciate and may need to help students develop.

1. *Commitment* Many undergraduate projects fail because students have not committed enough time for the job. In the academic year, there will always be a paper due soon or an exam coming up that will take precedence over a research project with no external deadlines. Ideally, a student's total credit load and work load should be light enough to leave time for research, but this can be problematic for students who need to work to stay in school. Summer is an ideal time for undergraduate research, provided that a stipend can be found so that the student can earn money for research rather than nonacademic work. During the academic year, it is best to have the research work done as part of an independent study course. The course could be structured like a regular course in that it would have specific due dates and grades for assignments, and it would take a slot in the student's schedule in place of another regular academic course.
2. *Preparation* One of the reasons biological modeling is a great area for undergraduate research is that many projects require no background knowledge and skills beyond a first course in calculus. Of course students with more background can do more, but, for example, some work in dynamical systems requires only algebra and calculus. As such, students with limited background can still contribute to modeling and research in this area. Whether we are working with advanced students or early students, we need to recognize that their coursework background often has gaps that we, as educators and faculty, must help fill in. Calculus courses generally don't do any dynamical systems analysis, and even differential equations courses often do not. Most mathematics curricula fall short on the most basic principles of mathematical modeling. I always start an REU with a one-week crash course on modeling and dynamical systems before helping my students choose a project. Depending on the project, it may also be necessary to do a crash course on scientific computation.
3. *Ownership* One of my first undergraduate research students wanted to extend her REU work into an Honors' thesis, but she could not give me a talk on that project because she had been assigned only a small part of a big project. This is a valuable lesson for mentors. I expect members of my undergraduate research teams to be

able to give a complete presentation on their project even if they focused their efforts on just a portion of it. I expect their talk to convey the impression that the problem was their idea rather than mine. If you attend student talks at a conference, you will see clear differences between students who acquired ownership of their project and those who did not. This is not something we should expect students to do on their own; they need our encouragement to shape their projects with their own ideas.

4. *Maturity* I found my first professional research experience as a graduate student to be very difficult. As a strong student, I seldom had any difficulty with a textbook problem. When I did, I was motivated by the idea that I should be able to solve any problem that the textbook author could solve. The first time I got stuck on a research problem, I could not fall back on this idea and I quickly lost confidence; maybe the problem was insoluble. We should not expect our research students to have adequate maturity at the beginning of their project, and we should be watchful for moments where they need a little encouragement, guidance, and/or moral support. Regular meetings between student and mentor give us an opportunity to identify critical moments and provide the needed support. One question raised by the need for most students to develop more maturity is whether they should work singly or in groups. At their best, groups offer a chance for students to learn from each other and for some to take the lead in moving the project forward and assist when others are stuck. My last REU group of 5 students was ideal. They were inseparable outside of work; at work, I could not distinguish the contributions of each individual because they were so in tune with each other. On the other hand, we have all seen groups in our classes where one student ended up carrying the rest. This can be avoided for the sake of everyone's learning experience through regular monitoring of the progress and learning of each team member.

2.2 Dissemination and Writing

Undergraduate research, like professional research, requires dissemination. Professional journals are one option, but by no means the only one. For an undergraduate, giving a talk or presenting a poster, even just to the local math club, is adequate communication of one's work. Other alternatives that should be considered are a journal for undergraduates only, such as the online Rose–Hulman journal (see (Flake et al. 2003) for an example of my students' work) and SIAM Undergraduate Research Online (SIURO), or a journal that is intended primarily for undergraduate work with faculty coauthors allowed, such as Spora. These journals are more flexible about thoroughness and importance than professional journals while still requiring papers to pass a serious professional review, and we can expect students to do the lion's share of the writing. Papers published in professional journals may look better on a vita for both the mentor and the students, but they are likely to be written primarily by the mentor, with minimal contribution to the writing by the students.

Many professional mathematicians are hesitant about trying to teach students to write, but it is something all of us can do. We may not have experience teaching writing, but we all have experience reviewing papers and know the difference between good writing and bad. A few guidelines can help us with this task.

Most of us have had the experience of reading bad student writing for our classes, which may give us the mistaken impression that our students generally cannot write well. While student writing ability obviously varies, few of our students have any experience with technical writing; consequently, they are likely to be far less good at it than they are at writing papers for an English or history class. In the RUTE Summer Scholars program described earlier, we expected our very strong class of students to be good writers and were surprised in the first year to find that hardly any of them were able to write a good scientific research paper. In our second year, we hired an English Department graduate student, whose specialty was teaching technical writing, to help us learn to teach the subject. This has been described elsewhere (Ledder et al. 2013), but a few points are worth noting here.

1. We can provide our students a set of guidelines for writing mathematics papers (see (Crannell 1994), for example). Alternatively, we can assign students to read a set of two well-written modeling papers and one poorly written one and ask students to identify which is the poorly written one (the flaws should be in structure and presentation, not spelling and grammar) and to make their own list of guidelines for mathematical modeling papers. We can specifically ask students for guidelines on structure and presentation, including best practices for writing a problem statement, labeling and referencing graphs, and presenting simulations or analysis.
2. Frequent small assignments are much better than one complete draft. We can ask students to submit a draft of the problem statement early in their research work rather than doing all the writing at the end, and continue to ask for individual sections as appropriate.
3. We are all more likely to benefit from criticism that we seek out than that which arrives unsolicited. Students should be asked to write out a list of questions for their reviewers, as this primes them to benefit from the feedback. Peer review allows students to benefit from seeing what is good and bad in each others' writing, just as our writing improves when we referee papers written by others.

3 Elements of Mathematical Modeling

My own modeling expertise is primarily in dynamical system and agent-based models in ecology and epidemiology. My students work on mechanistic models in these areas, and my views on modeling are obviously based on my own research and teaching experiences. Other areas and other types of modeling may not necessarily match the material presented here. In particular, empirical modeling is a rich area for student research, but one in which I have only limited experience. Nevertheless, the ideas outlined here should be of broad value, albeit not universal.

3.1 Mechanistic Models in Biology

I define a mechanistic model as *a collection of one or more variables, together with a self-contained set of rules that prescribe the values of those variables according to assumptions about the scientific principles that underlie the phenomena being modeled*. This definition has three important corollaries:

1. Mechanistic models serve as an approximate quantitative description of some actual or hypothetical real-world scenario.
2. Mechanistic models are created in the hope that the behavior they predict will capture enough of the features of that scenario to be useful.
3. The value of a mechanistic model depends on the setting to which it is applied and the questions it is used to address.

As noted earlier, many textbooks use the term ‘modeling’ in their title without engaging in anything I would describe as modeling. Typical non-modeling ‘application’ problems provide a narrative that consists of mathematical assumptions stated as though they were biological facts and asks students to show something or calculate a result. These problems fall short of modeling because they fail to address issues of fit between model and scenario. Sometimes they lead students to counter-factual predictions, such as the result in the Lotka–Volterra model that hunting of predators ultimately increases the average prey population without actually decreasing the average predator population. Leading to this prediction is not bad in itself, as long as the ultimate point is to call attention to how model predictions must be checked against reality before the model is accepted. You can see how rarely this is done by doing an internet search for ‘Lotka–Volterra’; you will have to scroll down a considerable distance before finding anything about the model’s flaws.

Of course it is well known that model predictions are not always correct for a biological scenario. Among the various ways this point is communicated is the commonly quoted aphorism ‘*All models are wrong, but some are useful.*’ I consider the propagation of this statement to be unfortunate, as it misdirects more than it enlightens. The emphasis in the statement is on ‘wrong,’ with ‘useful’ serving as a partial mitigation. This is not correct. ‘Useful’ is the whole point, not a mitigating factor. Judging a model to be ‘wrong’ implies that the purpose of a model is to replicate reality. It isn’t. The purpose of a model is to capture enough of the features of reality to help us understand a setting or make predictions. Judging a model to be ‘wrong’ because it isn’t the same as reality is like judging a painting to be ‘wrong’ because it isn’t the same as the subject of the painting. A model can give us insight into a biological scenario just as a portrait can give us insight into the character of the subject. We have no reason to expect anything more. Models should be classified on the continuum from ‘good’ to ‘bad,’ not ‘right’ to ‘wrong.’ The Lotka–Volterra model is certainly bad, but it is not wrong. A far better aphorism for students is ‘*Only good models are useful.*’

If an undergraduate research project turns out to have focused on a model that doesn’t capture essential features of reality, this is not a disaster, provided the discovery is acknowledged and the focus placed on what has been learned. The outcome should be reflected upon and either the model modified (if there is time available for this in the course curriculum or research program) or the possible reasons for its failing explained. Remember that learning about models and the modeling process must be at the heart of undergraduate research. Excellent models are seldom produced without critical missteps.

3.2 Some Differences Between Modeling and Mathematics

The three attributes of models listed above result in fundamental differences between the processes of doing modeling and doing mathematics. In mathematics, the assumptions define the setting of the problem and the conclusions that follow are correct for the setting; hence, the focus of mathematics is on proof. In modeling, the assumptions define a *conceptual model* of a real setting, not the real setting itself (Ledder 2013, 2022). The conclusions *for the model* follow from mathematical logic, but the conclusions *for the setting* depend on the validity of the conceptual model. This uncertainty for the setting changes the focus of the work. In modeling, proofs of theorems should be seen as mathematical exercises; the real focus should be on checking results against known outcomes. The Lotka–Volterra predator–prey model serves as a canonical example. Mathematical analysis shows that there is no prey-only equilibrium and that the solutions exhibit oscillations whose amplitude depends on the initial state because they orbit a neutrally stable equilibrium point. In the real world, however, local extinction of predators is common, and natural oscillations are stable limit cycles with initial transients. Proof of the properties of the Lotka–Volterra model has no biological value because the model fails to meet the most basic validity requirement of having the right qualitative behavior.

We mathematicians work hard to teach our students to think in terms of functions, beginning with their introduction in a pre-calculus course. While mathematical thinking is essential, in modeling it is often better to think in terms of variables rather than functions. The most stark example of this is in chemical thermodynamics, where there are a number of state variables that describe a system and there are two degrees of freedom. Any two of the state variables can be selected as the independent variables for derivatives. Thus, while both $(\partial H/\partial T)_V$ and $(\partial H/\partial T)_P$ are partial derivatives of the enthalpy H with respect to temperature, the two quantities are different because the second independent variable is volume in one case and pressure in the other. Being used to thinking in terms of functions makes it more difficult to grasp the idea. While this specific issue is in physical chemistry rather than biology, the same idea applies in more universal situations, such as scaling. Suppose we have a model for a population P as a function of time T.[1] We define dimensionless variables $p = P/N$ and $t = rT$, where N is a reference population and r a reference rate constant with dimension 1/time. To replace the derivative $\mathrm{d}P/\mathrm{d}T$ with $rN\mathrm{d}p/\mathrm{d}t$, we can differentiate the functional equation $P(T) = P(p(t(T)))$, but it is easier to simply replace the variable P with Np and the operator $d/\mathrm{d}T$ with $rd/\mathrm{d}t$.

[1] There are a variety of notation systems for linking dimensional and dimensionless variables. In most contexts, I prefer to use capital letters for one and lowercase for the other. See (Ledder 2017).

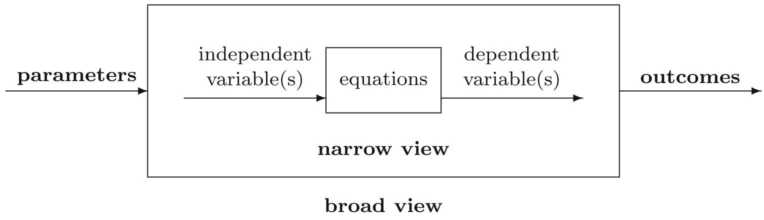

Fig. 1 Narrow and broad views of mathematical models (from (Ledder 2013, 2022))

3.3 Models as Mappings in the Parameter Space

One idea that is useful in many modeling contexts is that of models as mappings in the parameter space. To explain this to students, I present the concept of narrow and broad views of mathematical models (Ledder 2013, 2022), as illustrated in Fig. 1. For a dynamical system model, the narrow view is the more obvious view in which the variables are the independent and dependent variables in the model and the parameters are thought of as fixed constants, regardless of whether they have assigned values. In the broad view, we think of models as mappings from the parameter space to the space of outcomes; here, the model variables are merely a part of the rules that determine the mapping. When possible, it is good to address questions that reside in the broad view. For example, we can learn a little from narrow view plots of the susceptible population fraction versus time in an epidemic disease scenario, with various values given for the basic reproduction number and other parameters, but we can learn much more from a broad view plot of the final susceptible fraction versus the basic reproduction number, especially since such a plot is independent of the other model parameters.

For simple models, it is sometimes possible to do a complete characterization of the outcomes in the parameter space; for example, in many epidemiology models the boundary between scenarios where the disease-free equilibrium is globally asymptotically stable and those where the endemic disease equilibrium is globally asymptotically stable is the equality $\mathcal{R}_0 = 1$. For more complicated models, there are far too many parameters to study them all. In these cases, it is often helpful to subdivide the parameters into those that are within human control and those that are not. For example, in a scenario for the omicron variant of COVID-19, there are a number of disease-related parameters, such as the basic reproduction number and the probability of breakthrough infections for people who are immune to earlier strains, and there are also public health and behavioral parameters, such as the extent of mask use in public transportation and the fraction of people who will not be vaccinated, for whatever reason. An undergraduate research project could assume fixed values for all of the disease-related parameters and study the effects of one or two of the public health and behavioral parameters.

4 Some Thoughts on Broadening Research Participation

In the introduction, I argued for broader participation in undergraduate research. We turn now to some practical suggestions for how this can be accomplished.

Institutional resources for undergraduate research can make a huge difference. Funding can broaden participation by paying students to do research rather than work at a nonacademic job. The University of Nebraska–Lincoln has an undergraduate research program with enough money to fund a large share of student proposals (University of Nebraska–Lincoln 2022). Some other institutions do as well. Undergraduate faculty whose institutions lack such support can press administrators to consider funding for undergraduate research, using the argument that undergraduate research is a vital component of a Twenty-First Century general education.

4.1 Using a Classroom Setting for Research in Modeling

Even with generous funding for undergraduates who do research, there remains the problem of compensating faculty for the time they spend mentoring undergraduate research. We cannot significantly broaden research participation without finding a way to deliver research experiences at less of a cost in faculty time. The solution to this problem is to deliver research experiences in a classroom setting. Our RUTE Summer Scholars program took the form of a course with 20 students and two instructors, one in mathematics and one in biology. By the third year of the program, we were able to use advanced graduate students as the instructors rather than faculty. A ratio of 10 students per instructor is still too expensive to be broadly applicable, but it is important to note that we needed two instructors only because the research work was highly interdisciplinary, with a modeling component that could not be taught by a biologist and a laboratory component that could not be taught by a mathematician.[2] A more suitable model is that provided by Math in the City,[3] a UNL course created in 2006 by Dr. Petronela Radu (Radu 2013). This course has students working in teams to investigate research questions on a common theme. Sometimes the themes involve consultation with a local official or business, an example being the three projects of the 2010 class on route optimization for recycling drop-off sites, route optimization for city and county buildings, and cost versus environmental benefits, all done in consultation with the Lincoln city recycling coordinator and a local recycling business. Recently, this course has run every semester, with an average of 21 students divided into 4 or 5 project teams in each class. The course is taught by one faculty member or post doc, along with a graduate assistant. Starting in Fall 2020, the theme has been COVID-19 every semester, with me as the outside consultant. The changing situation has led to different projects over time, some focused on statistical analysis and others on modeling.

While it is not easy to teach research in a classroom setting, there are two key principles. *First, the course should be driven by a research agenda, not by a body of material to be learned as in a standard course.* This is not to say that there should be no instruction, but just that any instruction should be at the service of the research agenda. In the RUTE Summer Scholars course, I taught stage-structured population models, not because that was on a list of course topics but because this was the appropriate type of model for aphid population dynamics. In teaching this material,

[2] Although the latter does suggest an amusing mental picture.

[3] Named after the then-popular television show ‘Sex in the City.’

I focused on what the students needed; for example, we did material on eigenvalues and eigenvectors without doing the solution of linear systems that usually precedes material on eigenspaces in a matrix algebra course and without considering systems that were deficient in real eigenvalues. *Second, the mathematical tools needed for the research should focus as much as possible on relatively elementary material that most students will have had, and the treatment of them should be self-contained.* This will offer the opportunity for the students to apply what they have learned in a different and broader context while expanding their knowledge base to answer a real world or open-ended question or set of questions. A partial differential equations problem is not a good choice for undergraduate research by students with mixed mathematical interests, but dynamical systems analysis, curve-fitting, and agent-based modeling can be taught in a few class sessions to students who have not had the courses where those topics occur.

4.2 Research in Modeling for Early Career Students

As noted earlier, it is the common practice in mathematics to offer research opportunities only to students who have had extensive course work. Research in mathematical modeling for epidemiology and ecology can be done by students with a minimal background. Our RUTE Summer Scholars research project on aphid population dynamics was largely based on a professional research project done by my biologist colleague earlier in her career. The laboratory component consisted primarily of experiments to measure the vital rates of aphids (birth rates, death rates, and development rates from one larval stage to the next and on to pupa and adult) and a count of population versus time for an aphid colony (Ledder et al. 2013). The corresponding mathematical research consisted of developing and parameterizing a stage-structured matrix model, determining the dominant eigenvalue for the model numerically, and comparing the predictions to the population growth experiment. The students in the course were familiar with matrix multiplication, but none had taken a matrix algebra course. They were able to succeed in spite of this limited background because we found ways to teach the necessary mathematics efficiently. The idea of eigenvalues and eigenvectors can be discovered from exploration of population models, and it is possible to derive the equations for eigenvalues without using determinants (Ledder 2013, 2022), although we did later introduce determinants.

The idea that some advanced topics can be taught without all of the standard prerequisites is critical if we want to provide useful research tools to students who have had minimal coursework. Very little background in calculus is required to do dynamical system analysis for a 2D system. Students need to know that the derivative is the instantaneous rate of change, that differential equations are rules that prescribe rates of change in terms of the current system state, that long-term stable solutions can only occur at points in the state space where all the rates of change are zero, and that a variable is increasing if and only if its rate of change is positive (Ledder and Homp 2022). This limited background is sufficient for students to learn nullcline analysis. A little more basic material on derivative computation and solutions of homogeneous linear systems of differential equations suffices for them to be able to do linearized stability analysis with the Routh–Hurwitz conditions (Ledder 2013, 2022). Partial derivatives are in principle no different from ordinary derivatives: if you can say that the derivative of πy is π, then you can say that the derivative of xy with respect to y is x. Note that a complete treatment of solution methods for homogeneous linear systems, as would be found in any differential equations book, is unnecessary, as we only need to know the signs of the real part of the dominant eigenvalue. Of course it is necessary to present results such as the Routh–Hurwitz conditions without proof.

In addition to dynamical system models, agent-based models are an excellent area for talented early career students, many of whom have or can develop strong programming skills, as was my case. Aside from programming skills, the only background needed is some basic material on probability distributions, and the models are very intuitive because of being based on individual behavior. There are several good sources of information on agent-based modeling for beginning students (Gammack et al. 2013; Laubenbacher et al. 2013; Ledder and Homp 2022; Railsback and Grimm 2019) and how to use agent-based modeling with students (Ballow et al. 2020; Bañuelos et al. 2020; Bodine et al. 2020; Ledder and Homp 2022; Miller Neilan et al. 2021).

4.3 Research in Modeling for Mid-Tier Students

It is possible to provide successful research experiences for mid-tier students, provided expectations are reasonable; specifically, one should have low expectations for the amount of mathematics and programming, but high expectations for the amount of effort and creativity. It is important to limit the scope of the research areas. One possibility is to use a simple agent-based model that can be implemented as a physical activity as well as a computer program. The base program should be made available to the students and designed so that it is easy to make modifications. An example is the Blue Itchy Toes Syndrome disease model created by Michelle Homp and me (Homp and Ledder 2020; Ledder and Homp 2022). The model is easily implemented as a classroom activity, an online simulation, and a Matlab program. The base epidemiological model for BITS is discrete-time 'HPSR' (Healthy-Presymptomatic-Symptomatic-Removed), a structure which allows for exploration of interventions such as vaccination and isolation while being as simple as possible. Students can do simple parameter studies or explore variants with slightly more complicated etiology. We also have modules for SIR, SEIR, and the original strain of COVID-19, both program-based (Matlab and R versions) and spreadsheet-based. These modules are freely available (Ledder and Homp 2020; Ledder 2020) and come with extensive sets of scripted research questions that can be modified into unscripted questions. They have been used in a variety of settings, including high school classes, liberal arts math classes, classes for practicing secondary school teachers, and differential equations classes.

5 Final Thoughts

I have touched on a variety of topics and discussed some in detail. The main ideas on mentoring undergraduate research in mathematical modeling can be summed up in a few simple points.

1. The purpose of a model is to provide insight or make predictions, not to replicate reality. We should not expect too much from a model, nor should we accept its results without question. For example, a model whose assumptions clearly underestimate or overestimate a consequence by a small amount can serve a useful purpose as a lower or upper bound for expected outcomes.
2. Conclusions for a model follow from the model definition by mathematical inevitability; whether those conclusions hold for the biological setting depends on the quality of the model. Only good models are useful.
3. While the product is the goal of professional research, the process should be the primary goal of undergraduate research. A project that is not completely finished can make a successful talk or undergraduate paper and a model that is not a complete success can be an invaluable research experience if its limitations are explored in detail.
4. Undergraduate research students should not be expected to have mastered all of the mathematical tools needed for a project. The research plan must include the time and opportunity for students to fill in gaps in their background.
5. Research can be taught in a classroom setting if the course is driven by a research agenda rather than a body of material. Although the mathematical tools needed for the research should focus as much as possible on relatively elementary material that most students will have had, they should still be introduced in self-contained presentations.
6. Early career and mid-tier students can have very positive research experiences, provide the projects they are given are a good match for their background and ability and presented as opportunities to learn by self-discovery rather than literature review.

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