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ORIGINAL ARTICLE

On a Nonautonomous SEIRS Model in Epidemiology

Tailei Zhang*, Zhidong Teng

College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, People's Republic of China

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Abstract In this paper, we derive some threshold conditions for permanence and extinction of diseases that can be described by a nonautonomous SEIRS epidemic model. Under the quite weak assumptions, we establish some sufficient conditions to prove the permanence and extinction of disease. Some new threshold values are determined.

Keywords Nonautonomous SEIRS epidemic model · Disease · Permanence · Extinction

1. Introduction

To understand how to control and eradicate infectious disease is one of the main goals of mathematical epidemiology. From the study of autonomous models (Thieme, 2003; Anderson and May, 1978, 1979, 1992; Kermark and Mckendrick, 1927; Diekmann and Heesterbeek, 2000), we know that a disease can cause an epidemic when and only when the basic reproduction number R_0 is greater than 1. Thus to eradicate a disease, we need to reduce R_0 to less than 1. However, as we well know, the nonautonomous phenomenon often occurs in many realistic epidemic systems. Many diseases show seasonal behavior (London and Yorke, 1973; Dowell, 2001; Earn et al., 2002). Seasonality may come from various sources, e.g., varying transmission rates, fluctuations in birth rates, etc. Particularly, when we consider the long-term dynamical behaviors of an epidemic system, the parameters of the system usually will arise change with time.

In recent years, epidemiological models of ordinary differential equations have been studied by a number of authors (see, for example, Thieme, 2003; Anderson and May, 1992; Kermark and Mckendrick, 1927; Diekmann and Heesterbeek, 2000; Capasso, 1993; Ma et al., 2004; Brauer and Castillo-Chavez, 2001; Mena-Lorca and Hethcote, 1992). The basic and important research subjects for these systems are the existence of the threshold value which distinguishes whether the infectious disease will die out, the local and global stability of the disease-free equilibrium and the endemic equilibrium, the existence of periodic solutions, the persistence and extinction of the disease, etc. But most of them are concerned with local stability of equilibria. Stability, persistence and permanence have

*Corresponding author.
E-mail addresses: t.l.zhang@126.com (Tailei Zhang), zhidong@xju.edu.cn (Zhidong Teng).

been researched in a lot of papers in population biology (Cull, 1981; Takeuchi et al., 2006a, 2006b; Teng and Li, 2000; Teng and Yu, 1999; Teng and Chen, 2003). Hence, as the part of the population biology, permanence of disease plays an important role in epidemiology.

Recently, we see that there have been some research works on the nonautonomous epidemic dynamical systems, for example (Thieme, 1999, 2000; Zhang et al., 2005; Herzog and Redheffer, 2004; Li et al., 1999). Thieme (1999, 2000) studied the persistence and extinction for the following nonautonomous SIRS epidemic dynamical system

$$\begin{cases} N = S + I + R, \\ \dfrac{dI}{dt} = \alpha(t)SI - \big(\mu(t) + \gamma(t)\big)I, \\ \dfrac{dR}{dt} = \gamma(t)I - \mu(t)R - \xi(t)R, \end{cases}$$

where $N = N(t)$ is a given known continuous, bounded and nonnegative function defined on $R_+ = [0, +\infty)$ and expresses the size of the total population at time t. N is divided in three parts: susceptible S, infective I and recovered R. By applying the theory of the persistence and permanence for nonautonomous semiflows in the population biology which is developed in the same paper (Thieme, 1999, 2000), the authors obtained the sufficient conditions for the persistence and extinction of the disease.

In 2003, SARS began in the Guangdong province of China, however, it broke out at last in almost all parts of China. Zhang proposes a compartmental model (Zhang et al., 2005) that mimics the SARS control strategies implemented by the Chinese government after the middle of April 2003. In this paper, they obtain the following nonautonomous subsystem (Zhang et al., 2005, Eq. (3.2))

$$\begin{cases} E' = \lambda_f(t)(\delta_f E + I) - (\varepsilon + d_{ep})E, \\ I' = \varepsilon E - d_{iq} I - \alpha I, \\ P' = d_{ep} E + d_{sp} D - b_{sp} P - d_{pq} P, \\ D' = \lambda_q(t)(\delta_q P + D) + d_{pq} P + d_{ip} I - (\alpha + \gamma)D. \end{cases}$$

This subsystem is obviously of fundamental importance for the prevention and control of SARS outbreak. Therefore, the research on the nonautonomous epidemic dynamical systems also is very important and significant like on the autonomous epidemic dynamical systems.

The autonomous case is well studied (Liu et al., 1987; Hethcote, 2000) and in particular the references in (Liu et al., 1987). Liu and Hethcote (1987) studied the following SEIRS equations

$$\begin{cases} \dfrac{dS}{dt} = \mu - \lambda S^p I^q - \mu S + \delta R, \\ \dfrac{dE}{dt} = \lambda S^p I^q - (\mu + \varepsilon)E, \\ \dfrac{dI}{dt} = \varepsilon E - (\mu + \gamma)I, \\ \dfrac{dR}{dt} = \gamma I - (\mu + \delta)R. \end{cases} \tag{1}$$

Traditionally p and q are constants with $p > 0$ and $q > 0$ and $\mu, \lambda, \varepsilon, \gamma, \delta$ are nonnegative constants. When $\mu, \lambda, \varepsilon, \gamma, \delta$ in system (1) are replaced by nonnegative continuous functions of t and $p = q = 1$, we get a nonautonomous equation. For this equation, Herzog and Redheffer (2004) did some researches on positivity of solutions and the extinction of disease.

In (Li et al., 1999) a partially nonautonomous system is considered allowing varying total population size. From the ideas of these literatures (Thieme, 1999, 2000; Zhang et al., 2005; Liu et al., 1987; Herzog and Redheffer, 2004), we refer to the following nonautonomous SEIRS system

$$\begin{cases} \dfrac{dS}{dt} = \Lambda(t) - \beta(t)SI - \mu(t)S + \delta(t)R, \\ \dfrac{dE}{dt} = \beta(t)SI - \big(\mu(t) + \varepsilon(t)\big)E, \\ \dfrac{dI}{dt} = \varepsilon(t)E - \big(\mu(t) + \gamma(t)\big)I, \\ \dfrac{dR}{dt} = \gamma(t)I - \big(\mu(t) + \delta(t)\big)R, \\ N(t) = S(t) + E(t) + I(t) + R(t). \end{cases} \tag{2}$$

The letters S, E, I, R stand, respectively, for susceptible, exposed, infectious and recovered. $\Lambda(t)$ is the growth rate of population, function $\mu(t)$ is the instantaneous per capita mortality rate, function $\beta(t)$ is the daily contact rate, that is, the average number of contacts per day, functions $\varepsilon(t)$, $\gamma(t)$ and $\delta(t)$ are the instantaneous per capita rates of leaving the latent stage, infected stage and recovered stage, respectively.

In this paper, our main purpose is to look for permanence conditions for diseases modeled by SEIRS (susceptible-exposed-infectious-recovered-susceptible). The paper is organized as follows. Section 2 contains some basic preliminaries including initial conditions, some hypotheses, the definition of permanence, extinction of disease. Section 3 deals with the global existence and positivity of solutions of system (2). Permanence of solutions of system (2) is settled in Section 4. In Section 5, we will establish some sufficient conditions

on the extinction of the disease. We will give some corollaries in Section 6. Section 7 will give some examples to illustrate these theorems.

2. Notation and preliminaries

For any solution $(S(t), E(t), I(t), R(t))$ of system (2) with initial value

$$S(0) > 0, \qquad E(0) \geq 0, \qquad I(0) > 0, \qquad R(0) \geq 0. \tag{3}$$

On the persistence and extinction for the infectives I in system (2) we have the following definitions.

If $\liminf_{t\to\infty} I(t) > 0$, then we say that the infectives I are strongly persistent.

If there are positive constants v_1, v_2 such that

$$v_1 \leq \liminf_{t\to\infty} I(t) \leq \limsup_{t\to\infty} I(t) \leq v_2,$$

then we say that the infectives I are permanent.

If $\lim_{t\to\infty} I(t) = 0$, then we say that the infectives I go extinct.

For system (2), we introduce the following assumptions.

(H_1) Functions $\Lambda(t)$, $\beta(t)$, $\mu(t)$, $\varepsilon(t)$, $\gamma(t)$ and $\delta(t)$ are nonnegative, continuous and bounded on $R_+ = [0, +\infty)$ and $\beta(0) > 0$.

(H_2) There exist positive constants $\omega_i > 0$ $(i = 1, 2, 3)$ such that

$$\liminf_{t\to\infty} \int_t^{t+\omega_1} \beta(s)\,ds > 0, \quad \liminf_{t\to\infty} \int_t^{t+\omega_2} \mu(s)\,ds > 0$$

and

$$\liminf_{t\to\infty} \int_t^{t+\omega_3} \Lambda(s)\,ds > 0.$$

Remark 2.1. It is easy to prove that assumption (H_2) is equivalent to

$$\liminf_{t,s\to\infty} \frac{1}{t} \int_0^t \beta(r+s)\,ds > 0, \qquad \liminf_{t,s\to\infty} \frac{1}{t} \int_0^t \mu(r+s)\,ds > 0$$

and

$$\liminf_{t,s\to\infty} \frac{1}{t} \int_0^t \Lambda(r+s)\,ds > 0.$$

In particularly, when system (2) degenerates into ω-periodic system, that is, $\Lambda(t)$, $\beta(t)$, $\mu(t)$, $\varepsilon(t)$, $\gamma(t)$ and $\delta(t)$ are all nonnegative continuous periodic functions with period $\omega > 0$, then assumptions (H_2) is equivalent to the following cases

$$\overline{\beta} > 0, \qquad \overline{\mu} > 0 \quad \text{and} \quad \overline{\Lambda} > 0,$$

where for any continuous periodic function f with period $\omega > 0$, we denote by $\overline{f}$ the average value of $f(t)$, i.e. $\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)dt$.

When system (2) degenerates into almost periodic system, that is $\Lambda(t)$, $\beta(t)$, $\mu(t)$, $\varepsilon(t)$, $\gamma(t)$ and $\delta(t)$ are all nonnegative continuous almost periodic functions, then assumption (H_2) is equivalent to the following cases

$$m(\beta) > 0, \qquad m(\mu) > 0 \quad \text{and} \quad m(\Lambda) > 0,$$

where for any continuous almost periodic function f, we denote by $m(f)$ the average value of $f(t)$, i.e. $m(f) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(t)dt$.

Consider the following nonautonomous linear equation

$$\frac{dz}{dt} = \Lambda(t) - \mu(t)z. \tag{4}$$

We have the following result.

Lemma 2.1. *Suppose that assumptions* (H_1) *and* (H_2) *hold. Then*

(a) *Each fixed solution $z^*(t)$ of Eq. (4) with initial value $z^*(0) > 0$ is bounded and globally uniformly attractive on R_+.*

(b) *Let $z(t)$ be a solution of Eq. (4) and $\bar{z}(t)$ be a solution obtained in Eq. (4) when $\Lambda(t)$ is replaced by another continuous function $\bar{\Lambda}(t)$ and $z(0) = \bar{z}(0)$, then there is a constant $L > 0$ only depending on $\mu(t)$ such that*

$$\sup_{t\geq 0}\left|z(t) - \bar{z}(t)\right| \leq L \sup_{t\geq 0}\left|\Lambda(t) - \bar{\Lambda}(t)\right|.$$

(c) *There exist m, $M > 0$, such that*

$$m < \liminf_{t\to\infty} z(t) \leq \limsup_{t\to\infty} z(t) < M.$$

(d) *When Eq. (4) is ω-periodic, then Eq. (4) has a unique nonnegative ω-periodic solution $z^*(t)$ which is globally uniformly attractive.*

(e) *When Eq. (4) is almost periodic, then Eq. (4) has a unique nonnegative almost periodic solution $z^*(t)$ which is globally uniformly attractive.*

(f) *If $\mu(t) > 0$ for all $t \geq 0$ and $0 < \liminf_{t\to\infty} \frac{\Lambda(t)}{\mu(t)} \leq \limsup_{t\to\infty} \frac{\Lambda(t)}{\mu(t)} < \infty$, then for any solution $z(t)$ of Eq. (4) with the initial value $z(0) > 0$, we have*

$$\left(\frac{\Lambda}{\mu}\right)^m \leq \liminf_{t\to\infty} z(t) \leq \limsup_{t\to\infty} z(t) \leq \left(\frac{\Lambda}{\mu}\right)^M,$$

where

$$\left(\frac{\Lambda}{\mu}\right)^m = \liminf_{t\to\infty} \frac{\Lambda(t)}{\mu(t)}, \qquad \left(\frac{\Lambda}{\mu}\right)^M = \limsup_{t\to\infty} \frac{\Lambda(t)}{\mu(t)}.$$

Using the variation of constants formula and comparison theorem and the method of Liapunov function, we can prove this lemma very easily. Here, we omit it.

For convenience, we denote

$$a = \sup_{t\geq 0} \beta(t), \qquad b = \sup_{t\geq 0} \gamma(t), \qquad c = \sup_{t\geq 0} \mu(t)$$

and

$$d = \sup_{t\geq 0} \varepsilon(t), \qquad f = \sup_{t\geq 0} \delta(t)$$

3. Positivity

In this section, we will give conditions under which the solutions exist on $[0, +\infty)$ and are positive. The main result is as follows.

Theorem 3.1. *Suppose that assumptions* (H_1) *and* (H_2) *hold. The solution* $(S(t), E(t), I(t), R(t))$ *with initial condition* (3) *of system* (2) *is nonnegative and uniformly bounded on* $[0, +\infty)$.

Using (Thieme, 2003, Theorem A.4), we can easily proof this theorem. So, we omit it.

Remark 3.2. For any nonnegative initial value (3), we can show that the following (i), (ii), (iii) and (iv) are true.

(i) The solution $(S(t), E(t), I(t), R(t))$ of (2) exists on R_+ and $S(t) > 0 (t \geq 0)$, $E(t) > 0 (t > 0)$, $I(t) > 0 (t \geq 0)$ and $R(t) \geq 0 (t \geq 0)$.
(ii) If $E(0) > 0$ and $R(0) > 0$, then the solution $(S(t), E(t), I(t), R(t))$ of (2) exists on R_+ and $S(t) > 0 (t \geq 0)$, $E(t) > 0 (t \geq 0)$, $I(t) > 0 (t \geq 0)$ and $R(t) > 0 (t \geq 0)$.
(iii) If $E(0) = 0$, $I(0) = 0$ and $R(0) = 0$, then the solution $(S(t), E(t), I(t), R(t))$ of (2) exists on R_+ and $S(t) > 0 (t \geq 0)$, $E(t) = 0 (t \geq 0)$, $I(t) = 0 (t \geq 0)$ and $R(t) = 0 (t \geq 0)$.
(iv) If $E(0) > 0$ and $\beta(0) = 0$, then the solution $(S(t), E(t), I(t), R(t))$ of (2) exists on R_+ and $S(t) > 0 (t \geq 0)$, $E(t) > 0 (t \geq 0)$, $I(t) > 0 (t \geq 0)$ and $R(t) \geq 0 (t \geq 0)$.

4. Permanence

In this section, we wish to discuss the permanence of the disease in system (2), demonstrate how the disease in system (2) will be permanent under what conditions. Let the function

$$b(t, u) = 2\sqrt{\beta(t)\varepsilon(t)u(t)} - \big[\big(\mu(t) + \varepsilon(t)\big) + \big(\mu(t) + \gamma(t)\big)\big]$$

and $z^*(t)$ be some fixed solution of Eq. (4) with initial value $z^*(0) > 0$. We have the following theorem.

Theorem 4.1. *Suppose that assumptions* (H_1), (H_2) *hold and there is a constant* $\lambda > 0$ *such that*

$$R_0^* = \liminf_{t\to\infty} \int_t^{t+\lambda} b\big(s, z^*(s)\big)\, ds > 0. \tag{5}$$

Then the infective I *is permanent.*

Proof: Let $N^*(t) = S(t) + E(t) + I(t) + R(t)$ with the initial value $N^*(0) = S(0) + E(0) + I(0) + R(0)$, then $N^*(t)$ is a solution of Eq. (4). The system (2) is equivalent to the following system

$$\begin{cases} \dfrac{dE(t)}{dt} = \beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big)I(t) - \big(\mu(t) + \varepsilon(t)\big)E(t), \\ \dfrac{dI(t)}{dt} = \varepsilon(t)E(t) - \big(\mu(t) + \gamma(t)\big)I(t), \\ \dfrac{dR(t)}{dt} = \gamma(t)I(t) - \big(\mu(t) + \delta(t)\big)R(t). \end{cases} \tag{6}$$

Firstly, we prove that the number R_0^* is independent of the choice of $z^*(t)$. In fact, Lemma 2.1 implies that for any sufficiently small $\epsilon > 0$ and any solution $z(t)$ of Eq. (4) with initial value $z(0) > 0$, there exists $T > 0$ such that as $t \geq T$,

$$z^*(t) - \epsilon \leq z(t) \leq z^*(t) + \epsilon, \quad z^*(t) \geq m.$$

Hence,

$$b\big(t, z^*(t) - \epsilon\big) \leq b\big(t, z(t)\big) \leq b\big(t, z^*(t) + \epsilon\big).$$

For $t \geq T$, we obtain

$$\liminf_{t\to\infty} \int_t^{t+\lambda} b\big(s, z^*(s) + \epsilon\big)\, ds \leq R_0^* + 2\lambda\sqrt{\epsilon} \sup_{t\geq 0} \sqrt{\varepsilon(t)\beta(t)}$$

and

$$\liminf_{t\to\infty} \int_t^{t+\lambda} b\big(s, z^*(s) - \epsilon\big)\, ds \geq R_0^* - \frac{2}{\sqrt{m}}\lambda\epsilon \sup_{t\geq 0} \sqrt{\varepsilon(t)\beta(t)}.$$

By the arbitrariness of ϵ, we finally obtain

$$\liminf_{t\to\infty} \int_t^{t+\lambda} b\big(s, z(s)\big)\, ds = R_0^*.$$

This shows that R_0^* is independent of the choice of $z^*(t)$. Therefore,

$$\liminf_{t\to\infty} \int_t^{t+\lambda} b\big(s, N^*(s)\big)\, ds > 0. \tag{7}$$

Thus, by assumptions (H_1), (H_2) and (7), we can choose small enough positive constants $\epsilon_1, \epsilon_2, \epsilon_3$, then there exist $T_1 > 0$ and $\eta_1 > 0$ satisfying

$$\int_t^{t+\omega_2} \beta(\theta)M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1 \, d\theta < -\eta_1, \tag{8}$$

$$\int_t^{t+\omega_2} \gamma(\theta)\epsilon_2 - \big(\mu(\theta) + \gamma(\theta)\big)\epsilon_3 \, d\theta < -\eta_1, \tag{9}$$

$$\int_t^{t+\lambda} b\big(s, N^*(s) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big) \, ds > \eta_1, \tag{10}$$

and

$$N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3 \geq m, \quad N^*(t) \leq M \tag{11}$$

for all $t \geq T_1$, where $k = 1 + (aM + b)\omega_2$.

Firstly, we will prove

$$\limsup_{t\to\infty} I(t) > \epsilon_2. \tag{12}$$

for any solution of (6). Suppose that (12) is not true, then there exists a solution $(E(t), I(t), R(t))$ of (6) and $T_2 > T_1$ such that $I(t) \leq \epsilon_2$ for all $t \geq T_2$.

If $E(t) \geq \epsilon_1$ for all $t \geq T_2$, then from the first equation of system (6), we have

$$\begin{aligned} E(t) - E(T_2) &= \int_{T_2}^t \beta(\theta)\big(N^*(\theta) - E(\theta) - I(\theta) - R(\theta)\big)I(\theta) \\ &\quad - \big(\mu(\theta) + \varepsilon(\theta)\big)E(\theta) \, d\theta \\ &\leq \int_{T_2}^t \beta(\theta)M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1 \, d\theta \end{aligned}$$

for all $t \geq T_2$. Then $E(t) \to -\infty$ as $t \to \infty$ by (8). This is a contradiction. Hence, there is a $\tau_1 \geq T_2$ such that $E(\tau_1) < \epsilon_1$. Next, we will prove

$$E(t) \leq \epsilon_1 + aM\omega_2\epsilon_2. \tag{13}$$

for all $t \geq \tau_1$. Otherwise, there is a $\tau_2 > \tau_1$ such that $E(\tau_2) > \epsilon_1 + aM\omega_2\epsilon_2$. Hence, there must be a $\tau_3 \in (\tau_1, \tau_2)$ such that $E(\tau_3) = \epsilon_1$ and $E(t) > \epsilon_1$ for all $t \in (\tau_3, \tau_2)$. Choose an integer $p \geq 0$ such that $\tau_2 \in [\tau_3 + p\omega_2, \tau_3 + (p+1)\omega_2)$. Integrating the first equation of system (6) from τ_3 to τ_2, we obtain

$$\begin{aligned} \epsilon_1 + aM\omega_2\epsilon_2 &< E(\tau_2) \\ &= E(\tau_3) + \int_{\tau_3}^{\tau_2} \beta(\theta)\big(N^*(\theta) - E(\theta) - I(\theta) - R(\theta)\big)I(\theta) \\ &\quad - \big(\mu(\theta) + \varepsilon(\theta)\big)E(\theta) \, d\theta \\ &\leq \epsilon_1 + \int_{\tau_3}^{\tau_2} \beta(\theta)M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1 \, d\theta \\ &\leq \epsilon_1 + \int_{\tau_3+p\omega_2}^{\tau_2} \beta(\theta)M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1 \, d\theta \leq \epsilon_1 + aM\omega_2\epsilon_2. \end{aligned}$$

This is a contradiction. Hence, (13) is valid.

If $R(t) \geq \epsilon_3$ for all $t \geq T_2$, then from the third equation of system (6), we have

$$\begin{aligned} R(t) - R(T_2) &= \int_{T_2}^{t} \gamma(\theta) I(\theta) - \big(\mu(\theta) + \delta(\theta)\big) R(\theta)\, d\theta \\ &\leq \int_{T_2}^{t} \gamma(\theta)\epsilon_2 - \big(\mu(\theta) + \delta(\theta)\big)\epsilon_3\, d\theta \end{aligned}$$

for all $t \geq T_2$. By (9), if follows that $R(t) \to -\infty$ as $t \to \infty$. This is a contradiction. Hence, there is a $\tau_1 \geq T_2$ such that $R(\tau_1) < \epsilon_3$. In the following, we prove

$$R(t) \leq \epsilon_3 + b\omega_2\epsilon_2 \tag{14}$$

for all $t \geq \tau_1$. If it is not true, then there is a $\tau_2 > \tau_1$ satisfying $R(\tau_2) > \epsilon_3 + b\omega_2\epsilon_2$. Hence, there must be a $\tau_3 \in (\tau_1, \tau_2)$ such that $R(\tau_3) = \epsilon_3$ and $R(t) > \epsilon_3$ for all $t \in (\tau_3, \tau_2)$. Choose an integer $p \geq 0$ such that $\tau_2 \in [\tau_3 + p\omega_2, \tau_3 + (p+1)\omega_2)$. Integrating the third equation of system (6) from τ_3 to τ_2, we obtain

$$\begin{aligned} \epsilon_3 + b\omega_2\epsilon_2 &< R(\tau_2) \\ &= R(\tau_3) + \int_{\tau_3}^{\tau_2} \gamma(\theta) I(\theta) - \big(\mu(\theta) + \delta(\theta)\big) R(\theta)\, d\theta \\ &\leq \epsilon_3 + \int_{\tau_3}^{\tau_2} \gamma(\theta)\epsilon_2 - \big(\mu(\theta) + \delta(\theta)\big)\epsilon_3\, d\theta \\ &\leq \epsilon_3 + \int_{\tau_3 + p\omega_2}^{\tau_2} \gamma(\theta)\epsilon_2 - \big(\mu(\theta) + \delta(\theta)\big)\epsilon_3\, d\theta \leq \epsilon_3 + b\omega_2\epsilon_2. \end{aligned}$$

This is a contradiction. Hence, (14) is valid. From this, we conclude that there exists $T_0 > T_2$ such that (13) and (14) are both true for all $t \geq T_0$.

For $t \geq 0$, we define a differentiable function $V(t) = E(t)I(t)$. When $t \geq T_0$,

$$\begin{aligned} \dot{V}|_{(6)}(t) &= \beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big) I(t)^2 + \varepsilon(t) E(t)^2 \\ &\quad - \big[\big(\mu(t) + \varepsilon(t)\big) + \big(\mu(t) + \gamma(t)\big)\big] E(t) I(t) \\ &\geq b\big(t, N^*(t) - E(t) - I(t) - R(t)\big) V(t) \\ &\geq b\big(t, N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big) V(t). \end{aligned}$$

Integrating the above inequality from T_0 to t, we have

$$V(t) \geq V(T_0) \exp\left(\int_{T_0}^{t} b\big(s, N^*(s) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big)\, ds \right).$$

By (10), we obtain $\limsup_{t\to\infty} V(t) = \infty$. This contradicts with the boundedness of $E(t)$ and $I(t)$. From this contradiction, we finally conclude $\limsup_{t\to\infty} I(t) > \epsilon_2$.

Secondly, we will prove that there is a constant $v_1 > 0$ such that

$$\liminf_{t\to\infty} I(t) \geq v_1. \tag{15}$$

From (8–11) and (H_2), we have that there exist $T \geq T_1$, $P > 0$ and $\eta > 0$ such that

$$\int_t^{t+\alpha} \beta(\theta) M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1 \, d\theta < -M, \tag{16}$$

$$\int_t^{t+\alpha} \gamma(\theta)\epsilon_2 - \big(\mu(\theta) + \gamma(\theta)\big)\epsilon_3 \, d\theta < -M, \tag{17}$$

$$\int_t^{t+\alpha} b\big(s, N^*(s) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big) \, ds > \eta, \tag{18}$$

and

$$\int_t^{t+\alpha} \beta(\theta) \, d\theta > \eta \tag{19}$$

for every $\alpha \geq P$, $t \geq T$. Choose an integer $K_0 > 0$ such that

$$e^{-(c+d)P} m v_2 \eta e^{K_0 \eta} > \epsilon_1 + aM\omega_2\epsilon_2, \tag{20}$$

where $v_2 = \epsilon_2 e^{-(b+c)2P}$. By (12), for any $t_0 \geq 0$, we claim that it is impossible that $I(t) \leq \epsilon_2$ for all $t \geq t_0$. From this claim, we will discuss the following two possibilities.

(i) $I(t) \geq \epsilon_2$ for all large t.
(ii) $I(t)$ oscillates about ϵ_2 for all large t.

Finally, we will show that $I(t) \geq \epsilon_2 e^{-(b+c)(K_0+2)P} \triangleq v_1$ as t is large sufficiently. Evidently, we only need consider the case (ii). Let t_1 and t_2 be large sufficiently times satisfying

$$I(t_1) = I(t_2) = \epsilon_2,$$
$$I(t) < \epsilon_2 \quad \text{for all } t \in (t_1, t_2).$$

If $t_2 - t_1 \leq (K_0 + 2)P$, then

$$\dot{I}(t) = \varepsilon(t)E(t) - \big(\mu(t) + \gamma(t)\big)I(t) \geq -(b+c)I(t) \quad \text{and} \quad I(t_1) = \epsilon_2,$$

which implies $I(t) \geq \epsilon_2 e^{-(b+c)(K_0+2)P}$ for all $t \in [t_1, t_2]$.

If $t_2 - t_1 > (K_0 + 2)P$, then it is clear that $I(t) \geq \epsilon_2 e^{-(b+c)(K_0+2)P}$ for all $t \in [t_1, t_1 + (K_0 + 2)P]$.

If $E(t) \geq \epsilon_1$ for all $t \in [t_1, t_1 + P]$, then

$$\begin{aligned} E(t_1 + P) &= E(t_1) + \int_{t_1}^{t_1+P} \beta(\theta)S(\theta)I(\theta) - \big(\mu(\theta) + \varepsilon(\theta)\big)E(\theta) \, d\theta \\ &\leq M + \int_{t_1}^{t_1+P} \beta(\theta)M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1 \, d\theta < 0. \end{aligned}$$

This is a contradiction. Hence, there is a $\bar{t} \in [t_1, t_1 + P]$ such that $E(\bar{t}) < \epsilon_1$. From (13), we can obtain

$$E(t) \leq \epsilon_1 + aM\omega_2\epsilon_2 \quad \text{for all } t \in [\bar{t}, t_2]. \tag{21}$$

Similarly, there is a $\tilde{t} \in [t_1, t_1 + P]$ such that $R(\tilde{t}) < \epsilon_3$ and

$$R(t) \le \epsilon_3 + b\omega_2\epsilon_2 \quad \text{for all } t \in [\tilde{t}, t_2]. \tag{22}$$

Obviously, as $t \in [t_1, t_1 + 2P]$,

$$I(t) \ge \epsilon_2 e^{-(b+c)2P} \triangleq v_2 > v_1. \tag{23}$$

Therefore, from the first equation of system (6), (11), (21) and (22), we have

$$\begin{aligned}\dot{E}(t) &= \beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big)I(t) - \big(\mu(t) + \varepsilon(t)\big)E(t)\\ &\ge \beta(t)\big(N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big)I(t) - \big(\mu(t) + \varepsilon(t)\big)E(t)\\ &\ge m v_2\beta(t) - (c+d)E(t)\end{aligned}$$

for all $t \in [t_1 + P, t_1 + 2P]$. Integrating the above inequality from $t_1 + P$ to $t_1 + 2P$, by (19) we have

$$\begin{aligned}E(t_1 + 2P) &\ge e^{-(c+d)(t_1+2P)}\left[E(t_1 + P)e^{-(c+d)(t_1+P)} + \int_{t_1+P}^{t_1+2P} m v_2\beta(u)e^{(c+d)u}\,du\right]\\ &\ge e^{-(c+d)(t_1+2P)}\int_{t_1+P}^{t_1+2P} m v_2\beta(u)e^{(c+d)u}\,du\\ &\ge e^{-(c+d)P} m v_2\int_{t_1+P}^{t_1+2P}\beta(u)\,du > e^{-(c+d)P} m v_2\eta > 0.\end{aligned}$$

We claim that $I(t) \ge v_1$ for all $t \in [t_1 + (K_0 + 2)P, t_2]$. If it is not true, then there is a $T_0 \ge 0$ such that $I(t_1 + (K_0 + 2)P + T_0) = v_1$ and $I(t) \ge v_1$ on $[t_1, t_1 + (K_0 + 2)P + T_0]$. Let $V(t) = E(t)I(t)$ and $t_0 = t_1 + (K_0 + 2)P + T_0$. The derivative of $V(t)$ along solutions of (6) satisfies

$$\begin{aligned}\dot{V}|_{(6)}(t) &= \beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big)I(t)^2 + \varepsilon(t)E(t)^2\\ &\quad - \big[\big(\mu(t) + \varepsilon(t)\big) + \big(\mu(t) + \gamma(t)\big)\big]E(t)I(t)\\ &\ge b\big(t, N^*(t) - E(t) - I(t) - R(t)\big)V(t)\\ &\ge b\big(t, N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big)V(t)\end{aligned}$$

for all $t \in [t_1 + 2P, t_2]$. Integrating the above inequality from $t_1 + 2P$ to t_0, we further have

$$\begin{aligned}E(t_0)v_1 &\ge E(t_1 + 2P)I(t_1 + 2P)\exp\left(\int_{t_1+2P}^{t_0} b\big(t, N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big)\,dt\right)\\ &\ge e^{-(c+d)P} m v_2\eta v_1 e^{K_0\eta} > (\epsilon_1 + aM\omega_2\epsilon_2)v_1.\end{aligned}$$

Thus, $E(t_0) > \epsilon_1 + aM\omega_2\epsilon_2$ which contradicts with (21). So $I(t) \ge v_1$ is valid for any $t \in [t_1, t_2]$. Hence, we have

$$\liminf_{t\to+\infty} I(t) \ge v_1 > 0.$$

According to Theorem 3.1, we have that the infective I is permanent. The proof is completed. □

Remark 4.1. In system (2), $\Lambda(t)$, $\beta(t)$, $\mu(t)$, $\delta(t)$, $\varepsilon(t)$ and $\gamma(t)$ are replaced by nonnegative constants, i.e., system (2) becomes an autonomous SEIRS system. The basic reproduction number of the resulting system is given by

$$R_0 = \frac{\beta\Lambda\varepsilon}{\mu(\mu+\varepsilon)(\mu+\gamma)}.$$

When $R_0 > 1$, the corresponding autonomous system of system (2) is uniformly persistent (Liu et al., 1987).

On the other hand, the condition $R_0^* > 0$ is equivalent to

$$\tilde{R}_0 = \frac{\left(2\sqrt{\beta\varepsilon\frac{\Lambda}{\mu}}\right)}{((\mu+\varepsilon)+(\mu+\gamma))} > 1.$$

Evidently, $\tilde{R}_0 > 1$ implies $R_0 > 1$.

Remark 4.2. From Remark 4.1, we have the following problem, when the condition $R_0^* > 0$ is replaced by

$$R_0^{**} = \liminf_{t\to\infty} \int_t^{t+\lambda} \left[\beta(u)\varepsilon(u)z^*(u) - \big(\mu(u)+\varepsilon(u)\big)\big(\mu(u)+\gamma(u)\big)\right] du > 0, \quad (24)$$

whether we can obtain the permanent of disease of system (2) or not.

Theorem 4.2. *Suppose that assumptions* (H_1), (H_2) *hold and there exist two positive constants* r_1 *and* r_2 *such that*

$$\begin{aligned} &\liminf_{t\to\infty}\left[r_1\beta(t)N^*(t) - r_2\big(\mu(t)+\gamma(t)\big)\right] > 0,\\ &\liminf_{t\to\infty}\left[r_2\varepsilon(t) - r_1\big(\mu(t)+\varepsilon(t)\big)\right] > 0. \end{aligned} \quad (25)$$

Then the infective I *is permanent.*

Proof: By (H_1), (H_2) and (25), we can choose $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ which are small enough, then there exist $\eta > 0$ and $T_1 > 0$ such that

$$\int_t^{t+\omega_2} \beta(\theta)M\epsilon_2 - \big(\mu(\theta)+\varepsilon(\theta)\big)\epsilon_1\, d\theta < -\eta, \quad (26)$$

$$\int_t^{t+\omega_2} \gamma(\theta)\epsilon_2 - \big(\mu(\theta)+\gamma(\theta)\big)\epsilon_3\, d\theta < -\eta, \quad (27)$$

$$r_1\beta(t)\big(N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big) - r_2\big(\mu(t)+\gamma(t)\big) \geq \eta, \quad (28)$$

$$r_2\varepsilon(t) - r_1\big(\mu(t)+\varepsilon(t)\big) \geq \eta \quad (29)$$

for all $t \geq T_1$, where $k = 1 + (aM+b)\omega_2$.

Firstly, we claim that it is impossible that $I(t) \le \epsilon_2$ for all $t \ge T_1$. Suppose the contrary, being similar to the proof in Theorem 4.1, we know that there exists a $T_2 \ge T_1$ satisfying

$$E(t) \le \epsilon_1 + aM\omega_2\epsilon_2,$$
$$R(t) \le \epsilon_3 + b\omega_2\epsilon_2,$$

for all $t \ge T_2$. Construct a continuous differential function $V(t) = r_1E(t) + r_2I(t)$. When $t \ge T_2$, we have

$$\begin{aligned}\dot{V}|_{(6)}(t) &= \big[r_1\beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big) - r_2\big(\mu(t) + \gamma(t)\big)\big]I(t)\\ &\quad + \big[r_2\varepsilon(t) - r_1\big(\mu(t) + \varepsilon(t)\big)\big]E(t)\\ &\ge \big[r_1\beta(t)\big(N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big) - r_2\big(\mu(t) + \gamma(t)\big)\big]I(t)\\ &\quad + \big[r_2\varepsilon(t) - r_1\big(\mu(t) + \varepsilon(t)\big)\big]E(t) \ge \rho V(t).\end{aligned}$$

Here $\rho = \min\{\frac{\eta}{r_1}, \frac{\eta}{r_2}\} > 0$. Hence,

$$V(t) \ge V(T_2)e^{\rho(t-T_2)} \to +\infty (t \to \infty).$$

This is contrary to the boundedness of $V(t)$. Hence, the claim is proved. From this claim, we will discuss the following two possibilities.

(i) $I(t) \ge \epsilon_2$ for all large t.
(ii) $I(t)$ oscillates about ϵ_2 for all large t.

By (26) and (27), there exist $P > 0$ and $T \ge T_1$ such that

$$\int_t^{t+\alpha} \beta(\theta)M\epsilon_2 - \big(\mu(\theta) + \varepsilon(\theta)\big)\epsilon_1\, d\theta < -M, \tag{30}$$
$$\int_t^{t+\alpha} \gamma(\theta)\epsilon_2 - \big(\mu(\theta) + \gamma(\theta)\big)\epsilon_3\, d\theta < -M, \tag{31}$$

and $N^*(t) \le M$ for any $\alpha \ge P$, $t \ge T$. Choose a positive integer K_0 such that

$$\frac{1}{r_1}\big[r_1\epsilon_2 e^{-(b+c)P}e^{\rho K_0 P} - r_2\epsilon_2 e^{-(b+c)(K_0+1)P}\big] > \epsilon_1 + aM\omega_2\epsilon_2.$$

where $\rho = \min\{\frac{\eta}{r_1}, \frac{\eta}{r_1}\}$. In the following, we will prove $I(t) \ge \epsilon_2 e^{-(b+c)(K_0+1)P} \triangleq v_1$ for large sufficiently t. Evidently, we only need consider the case (ii). Let t_1 and t_2 be large sufficiently times satisfying

$$I(t_1) = I(t_2) = \epsilon_2,$$
$$I(t) < \epsilon_2 \quad \text{for all } t \in (t_1, t_2).$$

If $t_2 - t_1 \le (K_0 + 1)P$, then

$$\dot{I}(t) = \varepsilon(t)E(t) - \big(\mu(t) + \gamma(t)\big)I(t) \ge -(b + c)I(t),$$

which implies $I(t) \ge v_1$ for all $t \in [t_1, t_2]$.

If $t_2 - t_1 > (K_0 + 1)P$, then it is clear that $I(t) \geq v_1$ for all $t \in [t_1, t_1 + (K_0 + 1)P]$. It is easy to prove that

$$E(t) \leq \epsilon_1 + aM\omega_2\epsilon_2, \qquad R(t) \leq \epsilon_3 + b\omega_2\epsilon_2 \tag{32}$$

for all $t \in [t_1 + P, t_2]$ and $I(t_1 + P) \geq \epsilon_2 e^{-(b+c)P}$. We claim that $I(t) \geq v_1$ on $[t_1 + (K_0 + 1)P, t_2]$. If it is not true, then there is a $T_0 \geq 0$ such that $I(t_1 + (K_0 + 1)P + T_0) = v_1$ and $I(t) \geq v_1$ for all $t \in [t_1, t_1 + (K_0 + 1)P + T_0]$. Set $V(t) = r_1 E(t) + r_2 I(t)$ and $t_0 = t_1 + (K_0 + 1)P + T_0$. Hence, as $t \in [t_1 + P, t_2]$, one has

$$\begin{aligned}
\dot{V}|_{(6)}(t) &= \big[r_1\beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big) - r_2\big(\mu(t) + \gamma(t)\big)\big]I(t) \\
&\quad + \big[r_2\varepsilon(t) - r_1\big(\mu(t) + \varepsilon(t)\big)\big]E(t) \\
&\geq \big[r_1\beta(t)\big(N^*(t) - \epsilon_1 - k\epsilon_2 - \epsilon_3\big) - r_2\big(\mu(t) + \gamma(t)\big)\big]I(t) \\
&\quad + \big[r_2\varepsilon(t) - r_1\big(\mu(t) + \varepsilon(t)\big)\big]E(t) \geq \rho V(t).
\end{aligned}$$

Integrating the above inequality from $t_1 + P$ to t_0, we obtain

$$r_1 E(t_0) + r_2 I(t_0) \geq r_2 I(t_1 + P)e^{\rho K_0 P}.$$

This implies that

$$E(t_0) \geq \frac{1}{r_1}\big[r_1\epsilon_2 e^{-(b+c)P} e^{\rho K_0 P} - r_2\epsilon_2 e^{-(b+c)(K_0+1)P}\big] > \epsilon_1 + aM\omega_2\epsilon_2.$$

This is a contradiction with (32). Therefore, $I(t) \geq v_1$ for all $t \in [t_1, t_2]$. Thus, we have

$$\liminf_{t\to+\infty} I(t) \geq v_1 > 0.$$

The proof is completed. □

Remark 4.3. When system (2) transform into the autonomous system, the condition (25) is as follows

$$\begin{aligned}
&r_1\beta\frac{\Lambda}{\mu} - r_2(\mu + \gamma) > 0, \\
&r_2\varepsilon - r_1(\mu + \varepsilon) > 0,
\end{aligned}$$

which is equivalent to $R_0 > 1$.

Remark 4.4. In Theorem 4.2, if we are replaced the condition (25) by

$$\liminf_{t\to\infty} \frac{\beta(t)N^*(t)}{\mu(t) + \gamma(t)} \cdot \liminf_{t\to\infty} \frac{\varepsilon(t)}{\mu(t) + \varepsilon(t)} > 1 \tag{33}$$

then the infective I is still permanent. In fact, the condition (33) can derive the condition (25).

5. Extinction

On the extinction of infective I in system (6), we have the following results.

Theorem 5.1. *Suppose that assumptions* (H_1), (H_2) *hold. If there is a constant* $\lambda > 0$ *such that*

$$R_1^* = \limsup_{t\to\infty} \int_t^{t+\lambda} \big[\beta(\theta)N^*(\theta) - \mu(\theta)\big]\,d\theta \le 0 \tag{34}$$

or

$$R_2^* = \limsup_{t\to\infty} \frac{1}{t}\int_0^t \big[\beta(\theta)N^*(\theta) - \mu(\theta)\big]\,d\theta < 0 \tag{35}$$

then infective I *in system* (6) *is extinct i.e.* $\lim_{t\to\infty} I(t) = 0$.

Proof: From assumption (H_2), we can choose $\eta > 0$ small enough and $T_1 > 0$ big enough satisfying

$$\int_t^{t+\omega_2} \beta(\theta)\,d\theta \ge \eta$$

for all $t \ge T_1$.

For any constant $0 < \epsilon < 1$, we set $\epsilon_0 = \min\{\frac{\lambda\eta\epsilon}{2\omega_2}, \frac{1}{2}\eta\epsilon\} > 0$. If (34) holds, then there exists $T_2 \ge T_1$ such that

$$\int_t^{t+\lambda} \beta(\theta)N^*(\theta) - \mu(\theta)\,d\theta \le \epsilon_0$$

for all $t \ge T_2$. Choose an integer n_0 satisfying $\frac{2\omega_2}{\lambda} \le n_0 < \frac{2\omega_2}{\lambda} + 1$. Set $\lambda_0 = n_0\lambda$, then

$$\begin{aligned}
&\int_t^{t+\lambda_0} \beta(\theta)N^*(\theta) - \mu(\theta) - \beta(\theta)\epsilon\,d\theta \\
&\quad\le \int_t^{t+n_0\lambda} \beta(\theta)N^*(\theta) - \mu(\theta)\,d\theta - \int_t^{t+2\omega_2} \beta(\theta)\epsilon\,d\theta \\
&\quad\le n_0\epsilon_0 - 2\eta\epsilon \le -\frac{1}{2}\eta\epsilon \triangleq -\eta_0 < 0.
\end{aligned} \tag{36}$$

Construct a continuous function $V(t) = E(t) + I(t)$, differentiate V along a solution of (6) obtaining

$$\begin{aligned}
V'(t) &= \beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big)I(t) - \mu(t)V(t) - \gamma(t)I(t) \\
&\le \beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big)V(t) - \mu(t)V(t) \\
&\le \big(\beta(t)N^*(t) - \mu(t) - \beta(t)V(t)\big)V(t).
\end{aligned} \tag{37}$$

If $V(t) \ge \epsilon$ for all $t \ge T_2$, then from (5) we obtain

$$V(t) \le V(T_2) \exp \int_{T_2}^{t} \big[\beta(s)N^*(s) - \mu(s) - \beta(s)\epsilon\big]\, ds. \tag{38}$$

By (5), it follows that $V(t) \to 0$ as $t \to \infty$. This is a contradiction with $V(t) \ge \epsilon$. Hence, there must be a $t_1 \ge T_2$ such that $V(t_1) < \epsilon$. Let $N(\epsilon) = \sup_{t \ge T_2}\{|\beta(t)N^*(t) - \mu(t)| + \beta(t)\epsilon\}$, $N(\epsilon)$ is bounded for each $\epsilon \in (0, 1)$. Finally, we will prove

$$V(t) \le \epsilon \exp\big(N(\epsilon)\lambda_0\big) \tag{39}$$

for all $t \ge t_1$. If it is not true, then there exists a $t_2 > t_1$, such that $V(t_2) > \epsilon \exp(N(\epsilon)\lambda_0)$. Hence, there exists a $t_3 \in (t_1, t_2)$ such that $V(t_3) = \epsilon$ and $V(t) > \epsilon$ for all $t \in (t_3, t_2)$. Let p be a nonnegative integer such that $t_2 \in (t_3 + p\lambda_0, t_3 + (p+1)\lambda_0]$, then from (5) and (5) we have

$$\begin{aligned} \epsilon \exp\big(N(\epsilon)\lambda_0\big) < V(t_2) &\le V(t_3) \exp \int_{t_3}^{t_2} \beta(t)N^*(t) - \mu(t) - \beta(t)\epsilon\, dt \\ &\le \epsilon \exp\big(N(\epsilon)\lambda_0\big). \end{aligned}$$

This leads to a contradiction. Hence, inequality (39) holds. Furthermore, since ϵ can be arbitrarily small and $0 \le I(t) \le V(t)$, we conclude that $I(t) \to 0$ as $t \to \infty$. Suppose that (35) holds. There exist $\delta > 0$ and $T_0 > 0$ such that

$$\frac{1}{t} \int_0^t \beta(\theta)N^*(\theta) - \mu(\theta)\, d\theta < -\delta \tag{40}$$

for all $t \ge T_0$. From (38) we directly obtain

$$V(t) \le V(T_0) \exp \int_{T_2}^{t} \big[\beta(s)N^*(s) - \mu(s)\big]\, ds$$

for all $t \ge T_0$. By (40), $V(t) \to 0$ as $t \to \infty$. Therefore, we finally also have $I(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 5.1. □

Remark 5.1. For the corresponding autonomous system of system (2), the conditions (34) and (35) come into

$$\beta \frac{\Lambda}{\mu} \frac{1}{\mu} \le 1 \quad \text{and} \quad \beta \frac{\Lambda}{\mu} \frac{1}{\mu} < 1,$$

which implies $R_0 < 1$. From (Liu et al., 1987), the disease will go to extinct when $R_0 < 1$. It is similar to Remark 4.2 that we put forward the problem: If the conditions (34) and (35) are replaced by

$$R_1^{**} = \limsup_{t \to \infty} \int_t^{t+\lambda} \big[\beta(\theta)N^*(\theta)\varepsilon(\theta) - \big(\mu(\theta) + \varepsilon(\theta)\big)\big(\mu(\theta) + \gamma(\theta)\big)\big]\, d\theta \le 0 \tag{41}$$

and

$$R_2^{**} = \limsup_{t \to \infty} \frac{1}{t} \int_0^t \big[\beta(\theta)N^*(\theta)\varepsilon(\theta) - \big(\mu(\theta) + \varepsilon(\theta)\big)\big(\mu(\theta) + \gamma(\theta)\big)\big]\, d\theta < 0 \tag{42}$$

respectively, will the disease of system (2) go to extinct?

Theorem 5.2. *Suppose that assumptions* (H_1), (H_2) *hold. If there exist two positive constants* r_1 *and* r_2 *such that*

$$\begin{aligned} &\limsup_{t\to\infty}\big[r_1\beta(t)N^*(t) - r_2\big(\mu(t)+\gamma(t)\big)\big] < 0,\\ &\limsup_{t\to\infty}\big[r_2\varepsilon(t) - r_1\big(\mu(t)+\varepsilon(t)\big)\big] < 0 \end{aligned} \tag{43}$$

then infective I *in system* (6) *is extinct i.e.* $\lim_{t\to\infty} I(t) = 0$.

Proof: From (43), choose $\eta > 0$ small enough, then there exists $T > 0$ such that

$$\begin{aligned} &r_1\beta(t)N^*(t) - r_2\big(\mu(t)+\gamma(t)\big) \le -\eta,\\ &r_2\varepsilon(t) - r_1\big(\mu(t)+\varepsilon(t)\big) \le -\eta \end{aligned} \tag{44}$$

for all $t \ge T$. Define $V(t) = r_1E(t) + r_2I(t)$ and obtain

$$\begin{aligned} \dot V|_{(6)}(t) &= \big[r_1\beta(t)\big(N^*(t) - E(t) - I(t) - R(t)\big) - r_2\big(\mu(t)+\gamma(t)\big)\big]I(t)\\ &\quad + \big[r_2\varepsilon(t) - r_1\big(\mu(t)+\varepsilon(t)\big)\big]E(t)\\ &\le \big[r_1\beta(t)N^*(t) - r_2\big(\mu(t)+\gamma(t)\big)\big]I(t)\\ &\quad + \big[r_2\varepsilon(t) - r_1\big(\mu(t)+\varepsilon(t)\big)\big]E(t) \le -\rho V(t). \end{aligned}$$

where $\rho = \min\{\frac{\eta}{r_1}, \frac{\eta}{r_2}\}$. This implies that $V(t) \to 0$ as $t \to \infty$. Hence $\lim_{t\to\infty} I(t) = 0$. □

Remark 5.2. When system (2) transform into the autonomous system, the condition (43) is as follows

$$\begin{aligned} &r_1\beta\frac{\Lambda}{\mu} - r_2(\mu+\gamma) < 0,\\ &r_2\varepsilon - r_1(\mu+\varepsilon) < 0, \end{aligned}$$

which is equivalent to $R_0 < 1$.

Remark 5.3. In Theorem 5.2, if we are replaced the condition (43) by

$$\limsup_{t\to\infty}\frac{\beta(t)N^*(t)}{\mu(t)+\gamma(t)} \cdot \liminf_{t\to\infty}\frac{\varepsilon(t)}{\mu(t)+\varepsilon(t)} < 1, \tag{45}$$

then the infective I is still extinct. In fact, the condition (45) can derive the condition (43).

6. Some corollaries

As consequences of Theorems 4.1 and 5.1, we have the following a series of corollaries.

Corollary 6.1. *Suppose that assumptions* (H_1), (H_2) *hold and* $\mu(t) > 0$ *for all* $t \ge 0$ *and* $0 < \liminf_{t\to\infty}\frac{\Lambda(t)}{\mu(t)} \le \limsup_{t\to\infty}\frac{\Lambda(t)}{\mu(t)} < \infty$. *If there is a constant* $\lambda > 0$ *such that*

$$\liminf_{t\to\infty}\int_t^{t+\lambda} b\left(s, \left(\frac{\Lambda}{\mu}\right)^m\right) ds > 0,$$

then the infective I *of system* (2) *is permanent.*

Corollary 6.2. *Suppose that assumptions* (H_1), (H_2) *hold and* $\mu(t) > 0$ *for all* $t \geq 0$ *and* $0 < \liminf_{t\to\infty} \frac{\Lambda(t)}{\mu(t)} \leq \limsup_{t\to\infty} \frac{\Lambda(t)}{\mu(t)} < \infty$. *If there is a constant* $\lambda > 0$ *such that*

$$\limsup_{t\to\infty} \int_t^{t+\lambda} \beta(\theta)\left(\frac{\Lambda}{\mu}\right)^M - \mu(\theta)\,d\theta \leq 0$$

or

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \beta(\theta)\left(\frac{\Lambda}{\mu}\right)^M - \mu(\theta)\,d\theta < 0$$

then infective I *in system* (2) *is extinct i.e.* $\lim_{t\to\infty} I(t) = 0$.

Corollary 6.3. *Suppose that assumptions* (H_1), (H_2) *hold and there exists a constant* $\lambda > 0$ *such that* $\overline{R}_{01} > 1$, *where*

$$\overline{R}_{01} = \frac{(2\sqrt{\beta\varepsilon z^*})_0}{((\mu+\varepsilon)+(\mu+\gamma))^0}.$$

Here,

$$\left(2\sqrt{\beta\varepsilon z^*}\right)_0 = \liminf_{t\to\infty} \int_t^{t+\lambda} 2\sqrt{\beta(t)\varepsilon(t)z^*(t)}\,dt,$$

$$\left((\mu+\varepsilon)+(\mu+\gamma)\right)^0 = \limsup_{t\to\infty} \int_t^{t+\lambda} \left(\mu(t)+\varepsilon(t)\right)+\left(\mu(t)+\gamma(t)\right)dt.$$

$z^*(t)$ *be some fixed solution of Eq.* (4) *with initial value* $z^*(0) > 0$. *Then the infective* I *is permanent*.

Corollary 6.4. *When system* (2) *is* ω*-periodic and assumptions* (H_1), (H_2) *hold*, *then the infective* I *is permanent provided that*

$$\overline{R}_{02} = \frac{\overline{(2\sqrt{\beta\varepsilon z^*})}}{\overline{((\mu+\varepsilon)+(\mu+\gamma))}} > 1.$$

Here, $z^*(t)$ *is the globally uniformly attractive nonnegative* ω*-periodic solution of Eq.* (4).

Corollary 6.5. *When system* (2) *is almost periodic and assumptions* (H_1), (H_2) *hold*, *then the infective* I *is permanent provided that*

$$\overline{R}_{03} = \frac{m(2\sqrt{\beta\varepsilon z^*})}{m((\mu+\varepsilon)+(\mu+\gamma))} > 1.$$

Here, $z^*(t)$ *is the globally uniformly attractive nonnegative almost periodic solution of Eq.* (4).

Corollary 6.6. *Suppose that assumptions* (H_1), (H_2) *hold and there exists a constant* $\lambda > 0$ *such that* $\tilde{R}_{01} \leq 1$, *where*

$$\tilde{R}_{01} = \frac{(\beta z^*)^0}{(\mu)_0}.$$

Here,

$$(\beta z^*)^0 = \limsup_{t\to\infty} \int_t^{t+\lambda} \beta(t) z^*(t)\, dt,$$

$$(\mu)_0 = \liminf_{t\to\infty} \int_t^{t+\lambda} \mu(t)\, dt.$$

$z^*(t)$ be some fixed solution of Eq. (4) *with initial value $z^*(0) > 0$. Then the infective I is extinct.*

Corollary 6.7. *When system* (2) *is ω-periodic and assumptions* (H_1), (H_2) *hold, then the infective I is extinct provided that*

$$\tilde{R}_{02} = \frac{\overline{(\beta z^*)}}{\overline{\mu}} \le 1.$$

Here, $z^*(t)$ is the globally uniformly attractive nonnegative ω-periodic solution of Eq. (4).

Corollary 6.8. *When system* (2) *is almost periodic and assumptions* (H_1), (H_2) *hold, then the infective I is extinct provided that*

$$\tilde{R}_{03} = \frac{m(\beta z^*)}{m(\mu)} \le 1.$$

Here, $z^*(t)$ is the globally uniformly attractive nonnegative almost periodic solution of Eq. (4).

7. Examples

In this paper, we investigate a class of nonautonomous SEIRS epidemic model. By using analytic method, we give some sufficient conditions for the permanence, extinction of the disease.

In order to testify the validity of our results, we consider the following nonautonomous SEIRS epidemic model.

$$\begin{cases} \dfrac{dS}{dt} = \Lambda(t) - \beta(t)SI - \mu S + \delta R, \\ \dfrac{dE}{dt} = \beta(t)SI - \big(\mu + \varepsilon(t)\big)E, \\ \dfrac{dI}{dt} = \varepsilon(t)E - (\mu + \gamma)I, \\ \dfrac{dR}{dt} = \gamma I - (\mu + \delta)R, \end{cases} \tag{46}$$

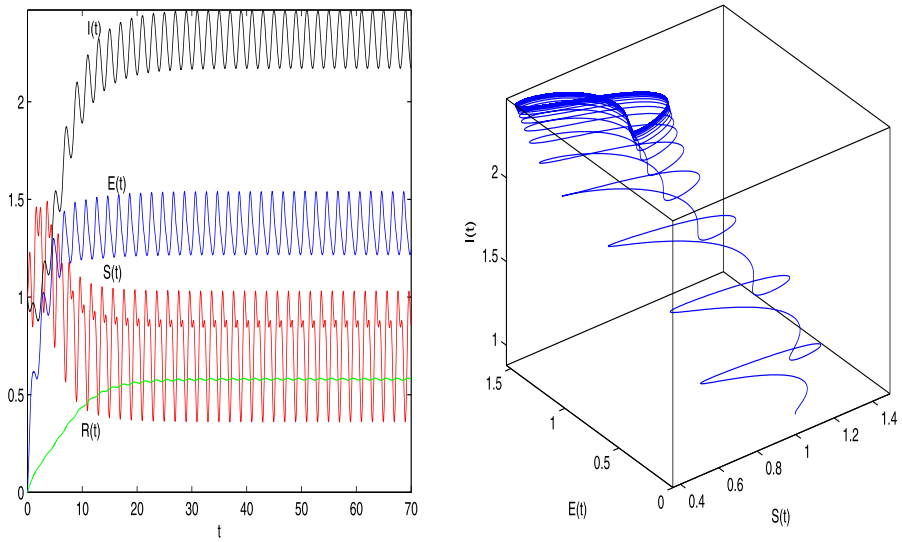

Fig. 1 The left figure shows that movement paths of S, E, I and R as functions of time t. The graph of the trajectory in (S, E, I)-Space is shown in the right figure. $R_0^* = 2.8696 > 0$ and $R_0^{**} = 3.3419 > 0$. The disease is permanent.

Corresponding auxiliary system is

$$\frac{dz}{dt} = \Lambda(t) - \mu z. \tag{47}$$

In system (46), let $\Lambda(t) = 1+\sin(2\pi t)$, $\beta(t) = 0.6+0.5\sin(\pi t)$, $\varepsilon(t) = 0.5+0.3\sin(\pi t)$, $\mu = 0.2$, $\gamma = 0.1$ and $\delta = 0.2$. We easily verify that assumptions (H_1) and (H_2) hold. Therefore, from Lemma 2.1, system (47) has a globally asymptotically stable positive periodic solution $z^*(t)$ with period 1. Here

$$z^*(t) = e^{-\mu t}\left(z_0 + \int_0^t e^{\mu s}\Lambda(s)\,ds\right)$$

where $z_0 = \frac{1}{e^{\mu}-1}\int_0^1 e^{\mu s}\Lambda(s)\,ds$. By (5) and (24), we can solve $R_0^* = 2.8696 > 0$ and $R_0^{**} = 3.3419 > 0$. From Theorem 4.1 and Corollary 6.4, the disease of system (6) is permanent and has a positive periodic solution. Numerical simulation of the above results can seen in Fig. 1.

However, if in system (46), let $\Lambda(t) = 0.5+\sin(2\pi t)$, $\beta(t) = 0.6+0.4\sin(\pi t)$, $\varepsilon(t) = 0.5+0.3\sin(\pi t)$, $\mu = 0.36$, $\gamma = 0.1$ and $\delta = 0.2$. We easily can solve $R_0^* = -0.0612 < 0$ and $R_0^{**} = 0.2183 > 0$, i.e., the condition (5) is invalid but the condition (24) is valid. Computer observations (see Fig. 2) suggest that when (24) holds, system (46) is still permanent.

Furthermore, if in system (46), let $\Lambda(t) = 0.5+\sin(2\pi t)$, $\beta(t) = 0.1+\sin(\pi t)$, $\varepsilon(t) = 0.2+\sin(\pi t)$, $\mu = 0.5$, $\gamma = 0.4$ and $\delta = 0.1$. By (34) and (41), we can easily solve $R_1^* =$

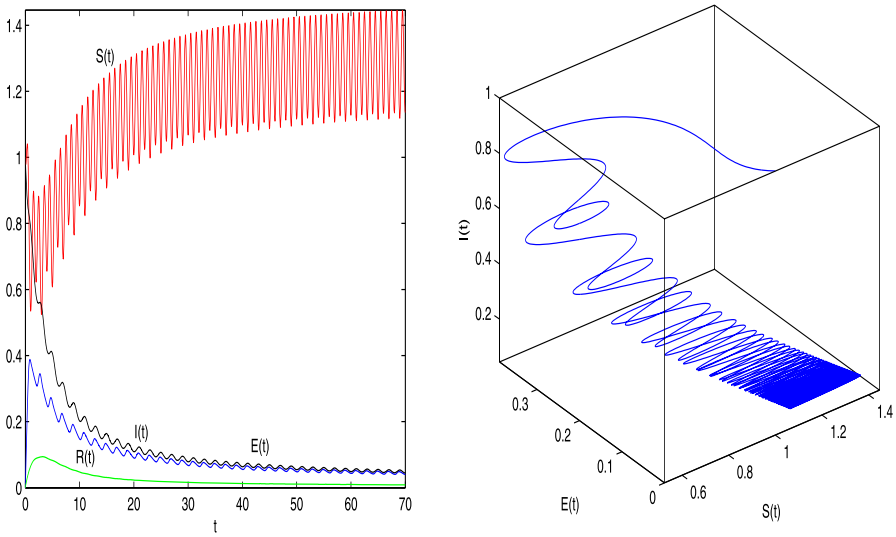

Fig. 2 The left figure shows that movement paths of S, E, I and R as functions of time t. The graph of the trajectory in (S, E, I)-Space is shown in the right figure. $R_0^* = -0.0612 < 0$ and $R_0^{**} = 0.2183 > 0$. The disease is permanent.

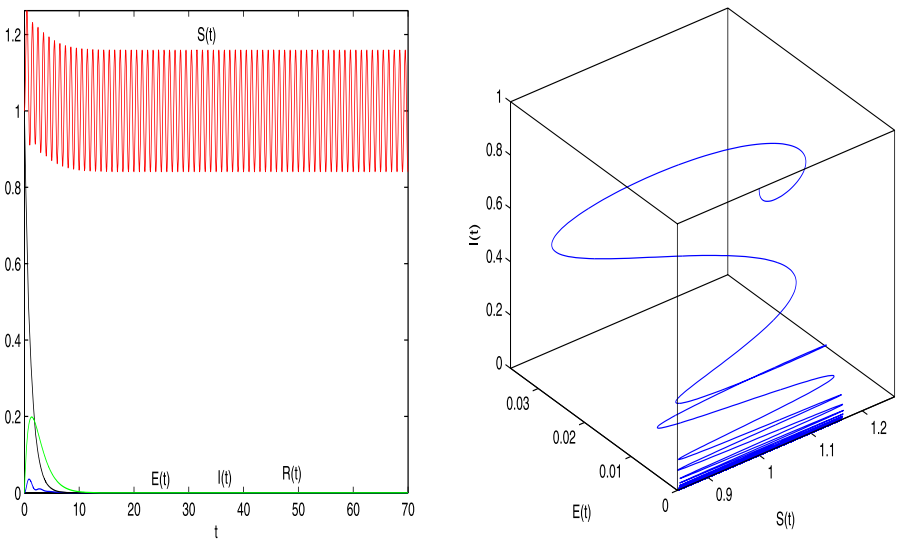

Fig. 3 The left figure shows that movement paths of S, E, I and R as functions of time t. The graph of the trajectory in (S, E, I)-Space is shown in the right figure. $R_1^* = -0.8784 < 0$ and $R_0^{**} = -1.0000 < 0$. The disease is extinct.

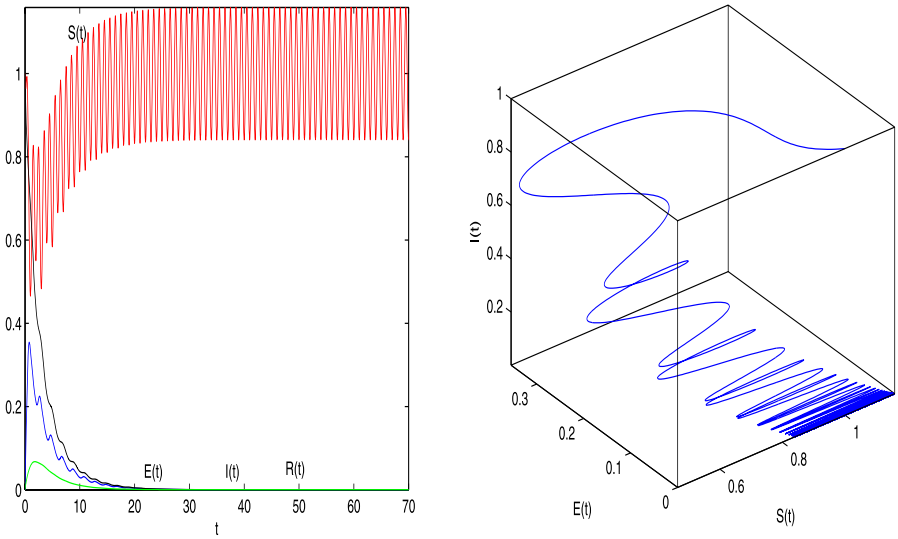

Fig. 4 The left figure shows that movement paths of S, E, I and R as functions of time t. The graph of the trajectory in (S, E, I)-Space is shown in the right figure. $R_1^* = 0.2 > 0$ and $R_0^{**} = -0.4381 < 0$. The disease is extinct.

$-0.8784 < 0$ and $R_1^{**} = -1.0000 < 0$. From Fig. 3, we can obtain that the disease of system (6) is extinct.

Finally, if in system (46), let $\Lambda(t) = 0.5 + \sin(2\pi t)$, $\beta(t) = 0.6 + 0.5\sin(\pi t)$, $\varepsilon(t) = 0.5 + 0.3\sin(\pi t)$, $\mu = 0.5$, $\gamma = 0.1$ and $\delta = 0.2$. By (34) and (40), we can easily solve $R_1^* = 0.2 > 0$ and $R_1^{**} = -0.4381 < 0$. From Fig. 4, we can obtain that the disease of system (6) is extinct.

Obviously, conditions (24), (41) and (42) are the improvement of the condition (5), (34) and (35), respectively. Therefore, as an improvement of Theorems 4.1 and 5.1 we have following interesting open problems.

Question 1. Suppose that assumptions (H_1) and (H_2) hold. Is the disease in system (6) is still permanent when condition (24) holds?

Question 2. Suppose that assumptions (H_1) and (H_2) hold. Is the disease in system (6) is extinct when condition (41) or (42) holds?

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