

Dynamics of Vaccination Strategies via Projected Dynamical Systems

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Abstract Previous game theoretical analyses of vaccinating behaviour have underscored the strategic interaction between individuals attempting to maximise their health states, in situations where an individual's health state depends upon the vaccination decisions of others due to the presence of herd immunity. Here, we extend such analyses by applying the theories of variational inequalities (VI) and projected dynamical systems (PDS) to vaccination games. A PDS provides a dynamics that gives the conditions for existence, uniqueness and stability properties of Nash equilibria. In this paper, it is used to analyse the dynamics of vaccinating behaviour in a population consisting of distinct social groups, where each group has different perceptions of vaccine and disease risks. In particular, we study populations with two groups, where the size of one group is strictly larger than the size of the other group (a majority/minority population). We find that a population with a vaccine-inclined majority group and a vaccine-averse minority group exhibits higher average vaccine coverage than the corresponding homogeneous population, when the vaccine is perceived as being risky relative to the disease. Our model also reproduces a feature of real populations: In certain parameter regimes, it is possible to have a majority group adopting high vaccination rates and simultaneously a vaccine-averse minority group adopting low vaccination rates. Moreover, we find that minority groups will tend to exhibit more extreme changes in vaccinating behaviour for a given change in risk perception, in comparison to majority groups. These results emphasise the important role played by social heterogeneity in vaccination behaviour, while also highlighting the valuable role that can be played by PDS and VI in mathematical epidemiology.

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1. Introduction

Population biology is an inherent part of voluntary vaccination policies. It has been shown that whether or not an individual decides to vaccinate depends partly upon the perceived probability of their becoming infected, which, in turn, depends upon the level of disease prevalence (Goldstein et al., 1996; Chapman and Coups, 1999; Bauch, 2005). Disease prevalence is, in turn, a function of the vaccine coverage in the population (Anderson and May, 1991), which is the collective result of the vaccination decisions of other individuals, if vaccination is voluntary. Hence, the individuals in a given population are effectively engaged in a strategic interaction (a ‘game’) with one another, mediated by transmission dynamics.

Vaccine scares are not uncommon and have occurred for various vaccines, including those for polio, smallpox, pertussis, measles–mumps–rubella, and Hepatitis B (Gangarosa et al., 1998; Durbach, 2000; Albert et al., 2001; Poland and Jacobsen, 2001; Plotkin, 2002; Biroscak et al., 2003; Jansen et al., 2003). At high levels of vaccine coverage, there is a reduced individual incentive to vaccinate, since unvaccinated individuals are already protected through herd immunity. If concerns about the potential health risks of vaccination then develop, high vaccine coverage levels may be prone to destabilise, and vaccine coverage can drop precipitously.

Several mathematical modelling studies have incorporated the effects of human behaviour under a voluntary policy, either explicitly or implicitly, using game theoretical or other techniques (Fine and Clarkson, 1986; Geoffard and Philipson, 1997; Bauch et al., 2003; Bauch and Earn, 2004; Bauch, 2005). However, epidemiological studies, which are usually concerned with individual risk factors or particular focus populations, do not account for this population biological context (Asch et al., 1994; Roberts et al., 1995; Lashuay et al., 2000; Evans et al., 2001; Schmitt, 2002; Smailbegovic et al., 2002; Bellaby, 2003). It is increasingly recognised that incorporation of human behaviour into epidemic models is an important, if challenging, goal (Fischhoff, 2003; McKenzie and Roberts, 2003). The example of strategic interactions under a voluntary vaccination policy is a case in point.

Recent studies explore the application of game theory to vaccinating behaviour under voluntary policies for childhood diseases (Bauch and Earn, 2004; Bauch, 2005), such as measles, mumps, chickenpox, pertussis and rubella (Anderson and May, 1991). In these papers, the authors assume a homogeneous population where all individuals share the same perception of risk. However, in real populations, risk perception can vary significantly across distinct social groups (Durbach, 2000; Lashuay et al., 2000). Many countries with high overall vaccine coverage have minority social or religious groups that vaccinate rarely or never, and which are, therefore, prone to outbreaks. Canada and The Netherlands, for instance, maintain high coverage rates for rubella vaccine but have recently seen rubella outbreaks in minority religious communities with almost no vaccine coverage (Eurosurveillance, 2005). Since the actions of even a small group of nonvaccinators can have a significant impact on disease prevalence when vaccine coverage levels

are high, it is important to understand the influence of social heterogeneity on the dynamics of vaccinating behaviour.

In previous vaccine game studies, individuals implicitly understand the existence of a critical coverage level provided by herd immunity, such that a disease can be eradicated without vaccinating everyone. This has been shown to result in a “Prisoner’s Dilemma,” whereby high coverage levels are unstable due to non-vaccinating behaviour (Bauch et al., 2003; Bauch, 2005). By comparison, in the present study, we assume that individuals do not perceive a critical coverage level. Rather, they perceive a possibility of being infected even when vaccine coverage is very high. In such a situation, we wish to determine whether non-vaccinating behaviour can still occur in some groups while other groups maintain high coverage levels.

Here we study the dynamics of vaccinating behaviour in a population divided into k social groups, each having a different perceived risk of infection and vaccination, and where vaccination is purely voluntary. We assume an infectious disease for which vaccination can take place only shortly after birth, where parents decide on a voluntary basis to vaccinate their children, and in which individuals (children) can be either susceptible, infectious or recovered (immune). These are known as Susceptible–Infectious–Recovered (SIR) models, and have been well-validated and widely applied in infectious disease epidemiology (Anderson and May, 1991).

The mathematical approach used here for deriving the dynamics is that of projected dynamical systems (PDS), via variational inequalities (VI). This approach is widely used in operations research, economic theory, finance and network analysis and mathematical physics (see, for example Dupuis and Nagurney, 1993; Nagurney and Zhang, 1996; Nagurney and Siokos, 1997; Isac and Cojocaru, 2002; Cojocaru, 2005; Cojocaru et al., 2005, the references therein).

A PDS is a dynamical system whose flow is constrained to evolve on a closed and convex subset, generically denoted by \mathbb{K} , of the ambient space. In this paper, we consider the ambient space to be the Euclidean space \mathbb{R}^k and we consider the constraint set \mathbb{K} to be a k -dimensional cube in \mathbb{R}^k . In general, a PDS is produced by projecting the velocity field of an ordinary differential equation onto the boundary of the constraint set \mathbb{K} whenever the velocity is not pointing inside the tangent cone to \mathbb{K} . Hence, a PDS essentially solves a control problem, namely that of keeping all trajectories of the projected velocity within the set \mathbb{K} (in the interior as well as on its boundary). A PDS could be considered a topological dynamical system, since its flow is continuous, but there is little resemblance to the classical dynamical systems theory since a projected system has a nonlinear, discontinuous velocity. The results present in the PDS literature (both on Euclidean space and on more general Hilbert spaces) are based on nonlinear and convex analysis and differential inclusions (see, for example Henry, 1973; Aubin and Cellina, 1984; Isac and Cojocaru, 2002; Cojocaru and Jonker, 2004; Cojocaru, 2006).

The ability to use a projected dynamics means that we are able to handle boundary phenomena (like boundary critical points, their stability, etc.), alongside interior ones, with ease. This is especially useful for modelling vaccinating behaviour in a socially heterogeneous population, since small minority groups adopting pure nonvaccinator behaviour (zero vaccine uptake) correspond to boundary

equilibria. The PDS theory contains existence, uniqueness and local and global stability results for equilibria and periodic solutions of constrained problems, no matter where they occur in \mathbb{K} . In short, a projected dynamics is very useful for studying the time evolution of constrained phenomena.

There exists an intimate relation between PDS and variational inequality (VI) problems. In general, a VI problem is a unifying mathematical tool used to reformulate equilibrium problems from diverse fields in a mathematically consistent and solvable way (see, for example Minty, 1978; Kinderlehrer and Stampacchia, 1980; Baiocchi and Capello, 1984; Isac, 1992, and Isac et al., 2002). Initially, VI problems were introduced (in 1963–1964) in order to help solving boundary value problems; later they were shown to represent mathematical formulations of certain classes of equilibrium problems in applied mathematics (elastic problems in mechanics, differential optimization problems, Nash games), economics (spatial price equilibrium and financial equilibrium problems), operations research and engineering (human migration, transportation and electrical networks problems).

In this paper, we use the finite-dimensional theories of VI and PDS because of the intrinsic relation between certain Nash games and variational inequalities. In fact, the solutions to a VI problem are exactly those of the underlying game and vice versa. Moreover, to each VI problem, one can associate a PDS. The two extremely useful characteristics of such an association are: the critical points of the PDS are the same as the solutions to the associated VI, hence, they coincide with the solutions of the Nash game; the flow of the PDS remains in the constraint set of interest at all times, thus enabling an accurate description of how the system (in our case, the vaccination strategies of various groups within the population) reaches a steady state. All these are exposed in detail in Section 3.

By using the PDS approach, we thus gain greater analytic capabilities, such as the ability to visualise the structure of the game dynamics through theoretical analysis, and to compute the optimal strategy and the respective equilibrium vaccination coverage. We are also able to show robust stability properties of our optimal strategy under perturbations. To our knowledge, this is the first application of VI and PDS theories in mathematical epidemiology.

Our goals in this paper are twofold. First, we wish to use vaccination games to explore the impact of social heterogeneity on vaccinating behaviour. Second, we wish to illustrate the usefulness of VI and PDS techniques for problems in epidemic modelling. In Section 2, we describe the vaccination game. In Section 3, we give a brief introduction to the theories of PDS and VI and their relation to game theory, and we list the mathematical results that are used here. In Section 4, we analyse the dynamics of vaccination strategies, show that equilibria (optimal strategies) exist and are unique and discuss the structure of this dynamics under perturbations. In Section 5, we compute such equilibria in a general setting. In Section 6, we present numerical results, while Section 7 discusses these results.

2. The vaccination game

In this section, we present in brief the vaccination game, using similar notations as in previous publications (Bauch and Earn, 2004). We consider a population consisting of various social groups, where each group may have a different perception

of risks and, therefore, may adopt different strategies. We consider a disease for which there is lifelong natural immunity, and in which individuals are typically infected early in life in the absence of vaccination (this describes the so-called paediatric infectious diseases, such as measles, mumps, rubella, pertussis and chickenpox) (Anderson and May, 1991). Likewise, we consider a vaccine that is administered primarily in the youngest age classes, and in which vaccination coverage is typically low later in life.

As discussed in the Introduction, the decision to vaccinate depends partly upon the perceived risks associated with infection and vaccination. The perceived probability of significant morbidity due to vaccination is denoted by r_v . The perceived probability of becoming infected given that a proportion p of the population is vaccinated, is denoted by π_p and the perceived probability of significant morbidity upon infection is denoted by r_{inf} . The overall perceived probability of experiencing significant morbidity because of not vaccinating is thus $\pi_p r_{\text{inf}}$. We denote by $r := r_v/r_{\text{inf}}$ the relative (perceived) risk of vaccination versus infection. We assume that all individuals within a group share a common assessment of the risks involved with vaccination and infection, but different groups have different relative risk assessments.

Suppose that the strategy set for all individuals in group i is $\{P_i | P_i \in [0, 1_P]\}$, where $1_P < 1$ but large enough and P_i is the probability that a child in group i is vaccinated. We wish to find a Nash equilibrium strategy $\underline{P}^* := (P_1^*, P_2^*, \dots, P_k^*)$, such that when everyone in group i plays P_i^* , no sufficiently small subset of individuals in any group can achieve a higher utility (payoff) by switching to a different strategy $P_i \neq P_i^*$. At P_i^* , there is no incentive to switch strategies, so such strategies should be stable equilibrium solutions of our game. In Sections 3 and 4, we derive a dynamics of the vaccination game that establishes existence and stability properties of such optimal strategies.

The utility function in a group where the perceived relative risk is r , and where the vaccine coverage in the population as a whole is p , is given by

$$u(P_i, p) = -r P_i - \pi_p(1 - P_i) \text{ subject to } P_i \in [0, 1_P], \quad (1)$$

after suitable rescaling. The players in a given round of the game are the parents of a given cohort of children, who play the game only once (they can decide only once whether or not to vaccinate their child). Future rounds of the game are played by the parents of later cohorts.

In order to find a mathematical expression for π_p , one approach is to use equilibrium solutions of a deterministic SIR compartmental model and assume that individuals have perfect knowledge of their probability of eventually becoming infected (Bauch and Earn, 2004). However, individuals do not have perfect knowledge of their probability of being infected. One could, for instance, assume that the perceived probability of eventually becoming infected increases linearly with the current prevalence of disease in the population (Bauch, 2005). Here, we assume for ease of analysis that π_p is a decreasing function of p given by $\pi_p = b/(a + p)$. This expresses the fact that disease prevalence is implicitly a function of how many individuals have been vaccinated, and that greater perceived coverage in the population means a reduced perceived infection risk for susceptible individuals.

Individuals also do not perceive a critical coverage threshold beyond which the disease is eradicated, as discussed in the Introduction. Unfortunately, because appropriate data are generally lacking on perceived risks of vaccination and infection, the validity of this function cannot be tested. However, we use values of a and b that are guided by epidemiologic constraints to ensure plausible results (discussed in Section 6).

Other expressions for π_p can be considered; we address this point in detail in Remark 4.2 of Section 4 later. We also note that time lags may be relevant here, since transmission dynamics can take several years to respond to changes in vaccine coverage. However, for highly transmissible childhood diseases such as measles and pertussis, we assume the effect of this lag to be small, since most vaccination and disease transmission occurs in the youngest age classes.

3. Game theory, variational inequalities and projected dynamical systems

3.1. Games and VI

In this section, we show that a Nash game setting is readily applicable to our study of vaccination behaviour in heterogeneous populations. Historically, the first to study noncooperative behaviour was Cournot in 1838 (Cournot, 1838). Nash formalised and generalised these ideas in Nash (1950, 1951). Here, we reformulate a Nash game as a variational inequality problem, using the setting in Nagurney and Siokos (1997). To this VI, we associate a projected dynamical system whose (mathematical) equilibria are exactly the solutions of the Nash game. There are important consequences for introducing this new way of thinking for the mathematical modelling of the vaccination strategies game, as outlined in the Introduction. The most important is that we are now in possession of a dynamics that gives the conditions for existence, uniqueness and stability properties of equilibria for a game between population groups with heterogeneous risk perceptions.

In this paper, we limit ourselves to the use of VI and PDS theories on the Euclidean space \mathbb{R}^k . By a convex subset $K \subset \mathbb{R}^k$, we understand a set with the property that for any $x, y \in K$ and any $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y \in K$; the set K is closed if any sequence with elements from K has a limit in K . Finally, a mapping $f : K \rightarrow \mathbb{R}^k$ is called convex if, for any $x, y \in K$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

Definition 3.1. Let $K \subset \mathbb{R}^k$ be a closed, convex, nonempty set and $F : K \rightarrow \mathbb{R}^k$ be a mapping. A **variational inequality problem** given by F and K is that of:

$$\text{finding } x \in K \text{ so that } \langle F(x), y - x \rangle \geq 0, \text{ for all } y \in K, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{R}^k , defined by $\langle x, y \rangle = \sum_{i=1}^k x_i y_i$, for any $x, y \in \mathbb{R}^k$.

As discussed in the previous section, it has been shown that Nash equilibria satisfy a VI whenever F is a gradient map (Gabay and Moulin, 1980). In general, we consider a Nash game with m players, each player i having at his/her disposal a strategy vector $x_i = \{x_{i1}, \dots, x_{in}\}$ selected from a closed, convex set $K_i \subset \mathbb{R}^n$, with a utility (or pay-off) function $u_i : K \rightarrow \mathbb{R}$, where $K = K_1 \times K_2 \times \dots \times K_m \subset \mathbb{R}^m$. The rationality postulate is that each player i selects a strategy vector $x_i \in K_i$ that maximises his/her utility level $u_i(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_m)$ given the decisions $(x_j)_{j \neq i}$ of the other players. In this framework one then has:

Definition 3.2 (Nash Equilibrium). A Nash equilibrium is a strategy vector $x^* = (x_1^*, \dots, x_m^*) \in K$ such that

$$u_i(x_i^*, \hat{x}_i^*) \geq u_i(x_i, \hat{x}_i^*), \quad \forall x_i \in K_i, \forall i, \quad \text{where } \hat{x}_i^* = (x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_m^*).$$

This model is used, for example, in market analysis (Nagurney and Siokos, 1997) where m represents the number of investors and n the number of financial instruments. A game of this form can be formulated as a VI as follows (for a proof, see Gabay and Moulin (1980)).

Theorem 3.1. *Provided the utility functions u_i are of class C^1 and concave (meaning $-u_i$ is convex) with respect to the variables x_i , then $x^* \in K$ is a Nash equilibrium if and only if it satisfies the VI*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K, \quad (3)$$

where $F(x) = (-\nabla_{x_1} u_1(x), \dots, -\nabla_{x_m} u_m(x))$ and $\nabla_{x_i} u_i(x) = (\frac{\partial u_i(x)}{\partial x_{i1}}, \dots, \frac{\partial u_i(x)}{\partial x_{in}})$.

3.2. PDS and VI

In general, the theories of PDS and VI are developed on more general spaces (Hilbert, Banach). Hence, the results cited in this subsection are, in fact, formulated on these spaces. However, since we only need to consider the Euclidean space, we present everything in this context.

Let $K \subset \mathbb{R}^k$ be a non-empty, closed, convex subset. We assume the reader is familiar with the concepts of *tangent cone to K at $x \in K$* , $T_K(x)$, and *normal cone to K* , $N_K(x)$ at $x \in K$, defined, respectively, by $T_K(x) = \bigcup_{h>0} (K - x)/h$, and $N_K(x) = \{n \in \mathbb{R}^k \mid \langle n, x - y \rangle \geq 0, \forall y \in K\}$ (otherwise, see, for example, Aubin and Cellina (1984) for an introduction and/or more details). The projection operator of \mathbb{R}^k onto K , denoted by $P_K : \mathbb{R}^k \rightarrow K$ is given by $z \mapsto P_K(z)$, where $P_K(z)$ satisfies

$$\|P_K(z) - z\| = \inf_{x \in K} \|x - z\|.$$

The operator P_K is perhaps better known as *the closest element mapping*, meaning for each $z \in \mathbb{R}^k$, $P_K(z)$ is the vector in K which realises the minimum distance between the vector z and the set K .

The properties of the projection operator are well-known (see, for instance, [Zarantonello \(1971\)](#) or [Kinderlehrer and Stampacchia \(1980\)](#)). However P_K is not differentiable in the usual (Frechet) sense; in turn, we can estimate its one-sided directional derivative, for any $x \in K$ and any direction $v \in \mathbb{R}^k$, as the limit (for a proof, see [Shapiro \(1994\)](#), Lemma 4.6):

$$\Pi_K(x, v) := \lim_{\delta \rightarrow 0^+} \frac{P_K(x + \delta v) - x}{\delta}; \quad \text{moreover} \quad \Pi_K(x, v) = P_{T_K(x)}(v),$$

hence, $\Pi_K(x, v)$ is another projection operator, this time projecting v onto the tangent cone to K at x .

Let $\Pi_K : K \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ be the operator given by $(x, v) \mapsto \Pi_K(x, v)$. Note that Π_K is discontinuous on the boundary of the set K . Detailed characterizations of Π_K are given in [Dupuis and Ishii \(1990\)](#) and [Isac and Cojocaru \(2003\)](#). One characterization we are using here is a consequence of Moreau's Theorem (see [Cojocaru and Jonker, 2004](#), for a proof): there exists $n \in N_K(x)$ such that

$$v = \Pi_K(x, v) + n. \quad (4)$$

The following result gives the existence of PDS.

Theorem 3.2. *Let $K \subset \mathbb{R}^k$ be a non-empty, closed, convex subset. Suppose $x_0 \in K$ and assume one of the following conditions hold:*

- a) $F : K \rightarrow \mathbb{R}^k$ is a Lipschitz continuous vector field;
- b) \mathbb{K} is a convex polyhedral set and F is a vector field with linear growth (i.e. there exists $M > 0$ so that for all $x \in K$, $\|F(x)\| \leq M(1 + \|x\|)$).

Then, the initial value problem

$$\frac{dx(\tau)}{d\tau} = \Pi_K(x(\tau), -F(x(\tau))), \quad x(0) = x_0 \in K, \quad (5)$$

has a unique absolutely continuous solution on the interval $[0, \infty)$.

For a proof, see [Cojocaru and Jonker \(2004\)](#) for a) and [Dupuis and Nagurney \(1993\)](#) for b).

Definition 3.3. A **projected dynamical system** is given by a mapping $\phi : \mathbb{R}_+ \times K \rightarrow K$ given by $(\tau, x) \mapsto \phi_x(\tau)$ which solves the initial value problem (5) with $\phi_x(0) = x \in \mathbb{K}$.

Hence, a PDS is a dynamical system forced to evolve only within the set K (interior as well as boundary). Another important feature of a PDS like the one given earlier is the following (for a proof, see [Cojocaru and Jonker, 2004](#)):

Theorem 3.3. *The critical (or equilibrium) points of (5) coincide with the solutions to the problem represented by (2) and vice versa.*

We note here the alternate sign convention used in the literature for the vector field F in the VI (2) and the PDS (5). In general, the critical points of the PDS (5) are defined as solutions to the equation $\Pi_K(x, -F(x)) = 0$. In the case of PDS, as a consequence of (4), we get the following equivalent definition:

Definition 3.4. All critical points of the PDS (5) are solutions of the inclusion:

$$\text{find } x \in K \text{ so that } -F(x) \in N_K(x).$$

4. Dynamics of vaccination strategies

Using the setup in Section 2 and the theory outlined in Section 3, we model the vaccination game as follows. We consider a population with N individuals divided into k distinct groups. The division is made according to the assumption that all individuals within a group share the same relative risk perception. However, distinct groups have distinct relative risk perceptions. Thus, in the language of Section 3, we consider a game with k players where each player has a 1-dimensional vaccination strategy vector. We denote by $P_i, i \in \{1, 2, \dots, k\}$ the vaccination strategy corresponding to the i -th group and by $\epsilon_i N$ the number of individuals in group i choosing strategy P_i . In this context, we have

$$\epsilon_i \in (0, 1) \text{ and } \sum_{i=1}^k \epsilon_i = 1.$$

Remark 4.1.

- 1) We remark here that we are not interested in $\epsilon_i = 0$. For if this is true for some $i \in \{1, 2, \dots, k\}$, then the problem is reduced to a population with $k - 1$ or less distinct groups.
- 2) We also note that if there exists i with $\epsilon_i = 1$, then the problem is reduced to that of a population where all individuals share the same risk assessment. This reduces to the social homogeneous case considered in previous work ([Bauch and Earn, 2004](#)).
- 3) We denote by r_i the relative risk assessment for the i -th group. We are interested in the case $r_i \neq r_j, \forall i, j \in \{1, 2, \dots, k\}$, otherwise, the problem reduces to the case of a population with $k - 1$ or less distinct groups.

Under these hypotheses, the vaccination coverage level of the entire population is assumed to be $p = \sum_{i=1}^k \epsilon_i P_i$. Following Section 2 and [Bauch and Earn \(2004\)](#), the expected payoff (utility) function for a player is given by

$$u_i(P_i, p) = -r_i P_i - \pi_p(1 - P_i), \quad \forall i \in \{1, 2, \dots, k\}, \quad (6)$$

where π_p is assumed to be of the form $\pi_p = b/(a + p) = b/(a + \sum \epsilon_i P_i)$.

Let $\mathbb{K} := \{\underline{P} := (P_1, \dots, P_k) \mid P_i \in [0, 1_P]\}$ and let the mapping $u : \mathbb{K} \rightarrow \mathbb{R}^k$ be given by $u(\underline{P}) = (u_1(P_1, p), \dots, u_k(P_k, p))$. This game can be formulated, following Theorem 3.1, as the variational inequality problem

$$\text{find } \underline{P}^* \in \mathbb{K} \text{ s.t. } \sum_{i=1}^k \left\langle -\frac{\partial u_i(P_i, p)}{\partial P_i} \mid_{P_i^*}, P_i - P_i^* \right\rangle \geq 0, \quad \forall \underline{P} = (P_1, \dots, P_k) \in \mathbb{K},$$

since each u_i is of class \mathcal{C}^1 and concave with respect to P_i . This VI is further equivalent to

$$\begin{aligned} \text{find } \underline{P}^* \in \mathbb{K} \text{ s.t. } \sum_{i=1}^k \left\langle r_i - \pi_p \mid_{\underline{P}^*} - \frac{b\epsilon_i(1 - P_i)}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \mid_{\underline{P}^*}, P_i - P_i^* \right\rangle \geq 0, \\ \forall \underline{P} \in \mathbb{K}. \end{aligned} \quad (7)$$

Following Section 3, in order to study the proposed vaccination dynamics, we let $F : \mathbb{K} \rightarrow \mathbb{R}^k$ with $F(\underline{P}) = (-\frac{\partial u_1}{\partial P_1}, \dots, -\frac{\partial u_k}{\partial P_k})$ and we associate to the VI problem (7) the projected dynamical system given by $\Pi_{\mathbb{K}}(\underline{P}, -F(\underline{P})) = P_{T_{\mathbb{K}}(\underline{P})}(-F(\underline{P}))$, namely:

$$\begin{aligned} \frac{d\underline{P}(\tau)}{d\tau} = P_{T_{\mathbb{K}}(\underline{P}(\tau))} \left(\pi_{p(\tau)} + \frac{b\epsilon_1(1 - P_1(\tau))}{(a + \sum_{i=1}^k \epsilon_i P_i(\tau))^2} - r_1, \dots, \pi_{p(\tau)} \right. \\ \left. + \frac{b\epsilon_k(1 - P_k(\tau))}{(a + \sum_{i=1}^k \epsilon_i P_i(\tau))^2} - r_k \right), \quad \text{with } \underline{P}(0) \in \mathbb{K}. \end{aligned} \quad (8)$$

According to Theorem 3.3, the stationary points of PDS (8) should coincide with the solutions of the Nash game. Moreover, to study the question of stability of these game solutions under perturbations, we need to introduce

Definition 4.1. A mapping $f : \mathbb{K} \rightarrow \mathbb{R}^k$ is called **monotone** if

$$\langle f(\underline{P}) - f(\underline{Q}), (\underline{P} - \underline{Q}) \rangle \geq 0, \quad \text{for all } \underline{P}, \underline{Q} \in \mathbb{K},$$

and is called **strictly monotone** if

$$\langle f(\underline{P}) - f(\underline{Q}), (\underline{P} - \underline{Q}) \rangle > 0, \quad \text{for all } \underline{P} \neq \underline{Q} \in \mathbb{K}.$$

Monotonicity is a central concept in nonlinear and convex analysis and it has been used and generalised extensively (Henry, 1973; Minty, 1978; Kinderlehrer and Stampacchia, 1980; Aubin and Cellina, 1984; Krasnosleskii and Zabreiko, 1984; Karamardian and Schaible, 1990; Nagurney and Zhang, 1996; Isac and Cojocaru, 2002; Cojocaru and Jonker, 2004; Cojocaru, 2006). As defined and used here

(or in any of the cited references), one should note that monotonicity is a generalization of the usual notion of a monotone real function of one variable. In the theory of PDS, monotonicity and its extensions, like strict monotonicity explained earlier, play a central role in the sense that they give information about the behaviour of perturbed equilibria, as well as about existence of periodic cycles. One of these results states that a PDS with a strictly monotone field F can only have a unique equilibrium and that all solutions are monotonically attracted to this point. The attraction can happen for solutions starting in a neighbourhood of the equilibrium, or can extend to all solutions starting anywhere in the set \mathbb{K} .

Definition 4.2. Let x^* be a critical point of the projected equation (5). Then x^* is a **local strict monotone attractor** if there exists a neighbourhood $\mathcal{N}(x^*) \subset K$ of x^* , so that for any trajectory $x(\tau)$ of (5) starting at $x_0 \in \mathcal{N}(x^*)$, the function $\tau \mapsto \|x(\tau) - x^*\|$ is decreasing.

Moreover, x^* is a **global strict monotone attractor** if the definition above is satisfied for trajectories of (5), starting at any point $x_0 \in K$.

We must note at this point that a (strict) monotone attractor is different than an attractor in the sense of the classical dynamical systems theory. We are now able to prove the most important result of this paper.

Theorem 4.1. *The Nash game introduced earlier has a unique solution. This solution is a global strict monotone attractor for the vaccination strategies dynamics.*

Proof: Step 1. We show first that the field $F : K \rightarrow \mathbb{R}^k$ is strictly monotone on \mathbb{K} . This is relatively easy to see if we keep in mind that for differentiable functions like F , strict monotonicity is equivalent to (see [Nagurny and Zhang, 1996](#))

$$\underline{P}^T (\nabla F) \underline{P} > 0, \quad \text{for all } \underline{P} \neq 0 \in \mathbb{K}. \quad (9)$$

In this case,

$$\begin{aligned} \nabla F = & \frac{b}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \begin{bmatrix} 2\epsilon_1 & \epsilon_2 & \dots & \epsilon_k \\ \dots & \dots & \dots & \dots \\ \epsilon_1 & \epsilon_2 & \dots & 2\epsilon_k \end{bmatrix} \\ & + \frac{2b}{(a + \sum_{i=1}^k \epsilon_i P_i)^3} \begin{bmatrix} \epsilon_1^2(1 - P_1) & \epsilon_1\epsilon_2(1 - P_1) & \dots & \epsilon_1\epsilon_k(1 - P_1) \\ \dots & \dots & \dots & \dots \\ \epsilon_1\epsilon_k(1 - P_k) & \epsilon_2\epsilon_k(1 - P_k) & \dots & \epsilon_k^2(1 - P_k) \end{bmatrix}. \end{aligned}$$

Since $P_i \in [0, 1_P]$, $1_P < 1$, then

$$\begin{aligned} \underline{P}^T(\nabla F)\underline{P} &= \frac{b}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \left[\sum_{i=1}^k \epsilon_i P_i^2 + \left(\sum_{i=1}^k P_i \right) \left(\sum_{i=1}^k \epsilon_i P_i \right) \right] \\ &\quad + \frac{2b}{(a + \sum_{i=1}^k \epsilon_i P_i)^3} \left(\sum_{i=1}^k \epsilon_i P_i \right) \\ &\quad \times (\epsilon_1(1 - P_1)P_1 + \dots + \epsilon_k(1 - P_k)P_k) > 0, \quad \forall \underline{P} \neq 0. \end{aligned}$$

Hence, F is strictly monotone on \mathbb{K} .

Now, by [Kinderlehrer and Stampacchia \(1980\)](#) game (7) has a unique solution.

Step 2. We show next that $-F : K \rightarrow \mathbb{R}^k$ is a vector field with linear growth, i.e. there exists $M > 0$ so that

$$\| -F(\underline{P}) \| \leq M(1 + \|\underline{P}\|), \quad \forall \underline{P} \in \mathbb{K}.$$

Then, by Theorem 3.2, PDS (8), is well-defined and by Theorem 3.3, its critical points are the solutions of our Nash game. By [Nagurney and Zhang \(1996\)](#), [Cojocaru \(2002\)](#), and [Isac and Cojocaru \(2002\)](#), this solution is a strict monotone attractor for the game dynamics.

To see that $-F$ has linear growth, we choose $\underline{P} \in \mathbb{K}$, we set $\epsilon := \max\{\epsilon_1, \dots, \epsilon_k\}$, and we evaluate $\|F(\underline{P})\|$, where $-F = -(f + g_1 + g_2)$, with

$$\begin{aligned} f(\underline{P}) &= \left(\frac{b}{a + \sum_{i=1}^k \epsilon_i P_i} - r_1, \dots, \frac{b}{a + \sum_{i=1}^k \epsilon_i P_i} - r_k \right), \\ g_1(\underline{P}) &= \left(\frac{b\epsilon_1}{(a + \sum_{i=1}^k \epsilon_i P_i)^2}, \dots, \frac{b\epsilon_k}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \right), \end{aligned}$$

and

$$g_2(\underline{P}) = \left(-\frac{b\epsilon_1 P_1}{(a + \sum_{i=1}^k \epsilon_i P_i)^2}, \dots, -\frac{b\epsilon_k P_k}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \right).$$

Then

$$\|f(\underline{P})\| = \left[\sum_{i=1}^k b^2 \left(\frac{1}{a + \sum_{i=1}^k \epsilon_i P_i} \right)^2 \right]^{\frac{1}{2}} + \|(r_1, \dots, r_k)\| \leq \frac{b\sqrt{k}}{a} + \|\underline{r}\|.$$

Similar to our computation earlier, next we have

$$\|g_1(\underline{P})\| = \left[b^2 (\epsilon_1^2 + \dots + \epsilon_k^2) \left(\frac{1}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \right)^2 \right]^{\frac{1}{2}} \leq \frac{b\epsilon\sqrt{k}}{a^2}$$

and

$$\begin{aligned} \|g_2(\underline{P})\|^2 &= b^2 \sum_{i=1}^k \epsilon_i^2 \left(\frac{P_i}{(a + \sum_{i=1}^k \epsilon_i P_i)^2} \right)^2 \leq b^2 \epsilon^2 k \sum_{i=1}^k \frac{P_i^2}{(a + \sum_{i=1}^k \epsilon_i P_i)^4} \\ \implies \|g_2(\underline{P})\| &\leq b\epsilon\sqrt{k} \left[\frac{1}{(a + \sum_{i=1}^k \epsilon_i P_i)^4} \sum_{i=1}^k P_i^2 \right]^{\frac{1}{2}} \leq \frac{b\epsilon\sqrt{k}}{a^2} \|\underline{P}\|. \end{aligned}$$

Finally,

$$\| -F(\underline{P}) \| \leq \frac{b\sqrt{k}}{a} + \|\underline{r}\| + \frac{b\epsilon\sqrt{k}}{a^2} + \frac{b\epsilon\sqrt{k}}{a^2} \|\underline{P}\| \leq M(1 + \|\underline{P}\|),$$

where $M := (b\sqrt{k}/a + \|\underline{r}\| + b\epsilon\sqrt{k}/a^2)$. The proof is complete. \square

Remark 4.2. We remark that other expressions for π_p can be considered, for example, given suitable constants a, b , let $\pi_p := ae^{-b\sum \epsilon_i P_i}$. Then VI problem (7) would be given by a mapping F with components $F_i := r_i - \pi_p(1 + b\epsilon_i(1 - P_i))$, $\forall i \in \{1, \dots, k\}$. If this mapping F satisfies the hypotheses of Theorem 4.1, then we obtain again the existence and uniqueness of a Nash game equilibrium for this new setup. Hence, our methodology up to this point can be applied under other assumptions for the form of π_p .

5. Optimal strategy computation

Given the proposed dynamics, we can theoretically find the structure of the solution of our vaccination game. By Definition 3.4 in Section 3 and Theorem 4.1 earlier, we have that the unique optimal strategy is the point $\underline{P}^* \in \mathbb{K}$ where \underline{P}^* is the solution of the inclusion $-F(\underline{P}) \in N_{\mathbb{K}}(\underline{P})$. To simplify solving this inclusion, we notice that (see [Aubin and Cellina \(1984\)](#)) the projected equation in Eq. (8) is equivalent to the following system of projected equations:

$$\begin{cases} \frac{dP_1(\tau)}{d\tau} = P_{T_{[0,1]P}}(P_1(\tau))(-F_1(\underline{P}(\tau))) \\ \dots \\ \frac{dP_k(\tau)}{d\tau} = P_{T_{[0,1]P}}(P_k(\tau))(-F_k(\underline{P}(\tau))) \end{cases}$$

Hence, the optimal strategy $\underline{P}^* = (P_1^*, \dots, P_k^*)$ for our game is that each P_i^* is a critical point of the respective i th projected equation of the system shown earlier; in other words, P_i^* uniquely solves $-F_i(\underline{P}) \in N_{[0,1_P]}(P_i)$, for all $i \in \{1, \dots, k\}$.

But for each fixed $i \in \{1, \dots, k\}$, the inclusion $-F_i(\underline{P}^*) \in N_{[0,1_P]}(P_i^*)$ is equivalent to the following system

$$\begin{cases} P_i^* \in (0, 1_P) \text{ is an equilibrium} & \text{if } -F_i(\underline{P}^*) = 0 \\ P_i^* = 0 \text{ is an equilibrium} & \text{if } -F_i(\underline{P}^*) < 0 \\ P_i^* = 1_P \text{ is an equilibrium} & \text{if } -F_i(\underline{P}^*) > 0 \end{cases} \quad (10)$$

Since solvability of (10) depends on the sign of the quantity

$$-F_i(\underline{P}) := \frac{b}{a + \sum \epsilon_i P_i} + \frac{b\epsilon_i(1 - P_i)}{(a + \sum \epsilon_i P_i)^2} - r_i,$$

the following scenarios are possible for the unique optimal strategy.

Case 1: The game dynamics could be monotonically attracted to the equilibrium strategy $\underline{P}^* = \underline{0}$; this means that at equilibrium, we have

$$-F_i(\underline{P}^*)|_{\underline{0}} < 0 \Leftrightarrow \frac{b}{a} \left(1 + \frac{\epsilon_i}{a}\right) < r_i, \quad \text{for all } i \in \{1, \dots, k\}.$$

Case 2: The game dynamics could be monotonically attracted to the equilibrium strategy $\underline{P}^* = (1_P, \dots, 1_P)$; this means that at equilibrium

$$F_i(\underline{P}^*)|_{1_P} > 0 \Leftrightarrow \frac{b}{a + 1_P} + \frac{b\epsilon_i(1 - 1_P)}{(a + 1_P)^2} > r_i, \quad \text{for all } i \in \{1, 2, \dots, k\}.$$

Case 3: Without loss of generality, let $i \in \{1, \dots, s\}$, $s < k$. The game dynamics could be monotonically attracted to the equilibrium strategy $\underline{P}^* = (0, \dots, 0, \overset{s \text{ times}}{1_P}, \dots, 1_P)$.

Case 4: The game dynamics could be monotonically attracted to an equilibrium strategy \underline{P}^* , where there exists at least one group i whose strategy $P_i^* \in (0, 1_P)$.

As we see in the next section, in order to find such solutions, depending on the values of the parameters ϵ_i , r_i , a , b , etc., we may need to compute approximate trajectories of Eq. (8), following the method in the constructive proof of Theorem 3.2 (see Cojocaru (2002); Cojocaru and Jonker (2004) where we showed that each trajectory of Eq. (5) is approximated by a linearly piecewise function). For computations and figures, we used MAPLE 8.

6. Results

We consider here examples with 1, 2 and 3 population groups, in order to determine the impact of social heterogeneity in risk perception on vaccine coverage under a voluntary vaccination policy. Before we begin, we remark that according to Theorem 4.1, in all examples there is a unique optimal strategy which is globally monotonically attracting for all time-dependent strategies starting anywhere in the constraint set \mathbb{K} . Moreover, we define a and b in the expression for π_p as follows: the parameter a determines the sensitivity of the perceived probability of infection to the vaccine coverage, i.e. large values of a imply a population where the perceived probability of infection depends weakly upon the vaccine coverage, whereas small values of a imply strong dependence. The parameter combination $\frac{a}{b}$ is the maximum possible perceived probability of infection, achieved at $p = 0$. Clearly, we must have $0 < b < a$ so that $\pi_0 < 1$. For a disease such as measles, the probability that an individual eventually gets infected in the absence of any vaccination programme is close to 90% (Anderson and May, 1991). Hence, when $p = 0$, we set $\pi_0 = b/a = 0.90$. Likewise, when $p \approx p_{\text{crit}}$, where p_{crit} is the critical coverage level required to eradicate a disease, then $\pi_{p_{\text{crit}}} \approx 0$, hence we require that $a \ll p_{\text{crit}}$, and $b \ll p_{\text{crit}}$. For measles, $p_{\text{crit}} \approx 0.9$. With $a = 0.1$, and $b = 0.09$, we are consistent with these restrictions. Significantly smaller values for a and b would yield unrealistic behaviour for intermediate values of p . We use $a = 0.1$, $b = 0.09$ and $1_p := 0.9$ throughout in our numerical results.

6.1. One group model

Let us consider a 1 group model, so that $\epsilon = 1$ and all individuals share the same relative risk assessment r . In this case $\pi_p = b/(a + P)$ and $p = P$. According to the previous section, we search first for points $0 < P^* < 1$ such that

$$\frac{b}{a + P^*} + \frac{b(1 - P^*)}{(a + P^*)^2} = r \implies P^* = -a + \sqrt{\frac{b(a + 1)}{r}}.$$

Clearly, as r decreases, the equilibrium coverage increases. We deduce that for $a = 0.1$, $b = 0.09$, $1_p := 0.9$, and consequently $r \in (0.099, 9.9)$, we have $p^* \in (0, 0.9)$. For all $r \geq 9.9$, the coverage is 0, and for $r \leq 0.099$ the coverage $p^* = 0.9$. This result is intuitive, since it means more people vaccinate as the vaccine is perceived to be increasingly less risky than the disease. Previous work also predicted the existence of a threshold in perceived risk, below which vaccinating behaviour becomes increasingly prevalent and above which no one vaccinates (Bauch and Earn, 2004).

6.2. Two groups model

Here, we study a population consisting of a majority group (of proportion ϵ_1) which is relatively more inclined to vaccinate, and a minority group (of proportion $\epsilon_2 < \epsilon_1$) which is relatively less inclined to vaccinate ($r_2 > r_1$). We determine

the impact of social heterogeneity by comparing the 1 group case to several 2 group cases (for various values of ϵ_1 , ϵ_2 , r_1 , r_2). We wish to determine whether vaccine coverage is higher or lower in the 2 group cases with average relative risk assessment

$$r := \epsilon_1 r_1 + \epsilon_2 r_2, \quad (11)$$

compared to the 1 group case with the same value of r . The following analysis shows that social heterogeneity generally leads to higher vaccine coverage.

Let us take $a = 0.1$, $b = 0.09$ and $1_p := 0.9$ (as justified above). Figures 1 and 2 illustrate the effects of heterogeneity in two cases. Figure 1 shows the average vaccine coverage $p^*(r) = \epsilon_1 P_1^*(r) + \epsilon_2 P_2^*(r)$ versus r with $r_1 = 0.7r$, $r_2 = r(1 - 0.7\epsilon_1)/\epsilon_2$, for various values of ϵ_1 and ϵ_2 . In general, the possible values of the vector of equilibrium strategies are $\underline{P}^* = (P_1^*, P_2^*)$, $P_i^* \in [0, 0.9]$. In the examples of Fig. 1 we obtained only five values, namely $(0.9, 0.9)$, $(0.9, P_2^*(r))$, $(0.9, 0)$, $(P_1^*(r), 0)$, $(0, 0)$. For each of these, we computed and graphed the corresponding equilibrium vaccine coverage $p^*(r)$. The r values at which vaccine coverage is equal or higher in the heterogeneous population than in the homogeneous population for the four cases considered in Fig. 1 are described by the following table:

group sizes	r values for which $p^*(r) \geq p^*(r, 1 \text{ group})$
$\epsilon_1 = 0.9, \epsilon_2 = 0.1$	$r \in (0, 0.027]$ and $r \geq 0.1195$
$\epsilon_1 = 0.8, \epsilon_2 = 0.2$	$r \in (0, 0.049]$ and $r \geq 0.1472$
$\epsilon_1 = 0.7, \epsilon_2 = 0.3$	$r \in (0, 0.068]$ and $r \geq 0.1857$
$\epsilon_1 = 0.6, \epsilon_2 = 0.4$	$r \in (0, 0.086]$ and $r \geq 0.2416$

Figure 2 shows $p^*(r)$ versus r for the case $\epsilon_1 = 0.9$ and $\epsilon_2 = 0.1$, but using various degrees of difference in risk perception between the majority and the minority groups (see table shown later and Fig. 2). Again, we obtained only five equilibrium values, namely $(0.9, 0.9)$, $(0.9, P_2^*(r))$, $(0.9, 0)$, $(P_1^*(r), 0)$, $(0, 0)$. For each of these, we computed and graphed the corresponding equilibrium vaccine coverage $p^*(r)$. The r values at which vaccine coverage is equal or higher in the heterogeneous population than in the homogeneous population for the five cases considered in Fig. 2, are described by the following table:

relative risks perceptions	r values for which $p^*(r) \geq p^*(r, 1 \text{ group})$
$r_1 = 0.9r, r_2 = 1.9r$	$r \in (0, 0.0516]$ and $r \geq 0.1195$
$r_1 = 0.8r, r_2 = 2.8r$	$r \in (0, 0.035]$ and $r \geq 0.1195$
$r_1 = 0.7r, r_2 = 3.7r$	$r \in (0, 0.0265]$ and $r \geq 0.1195$
$r_1 = 0.6r, r_2 = 4.6r$	$r \in (0, 0.0213]$ and $r \geq 0.1195$
$r_1 = 0.5, r_2 = 5.5r$	$r \in (0, 0.0178]$ and $r \geq 0.1195$

Figures 1 and 2 show that, generally speaking, vaccine coverage is higher in populations with social heterogeneity and where there is a majority group and a minority group that is more risk-averse. An exception occurs at lower values of r

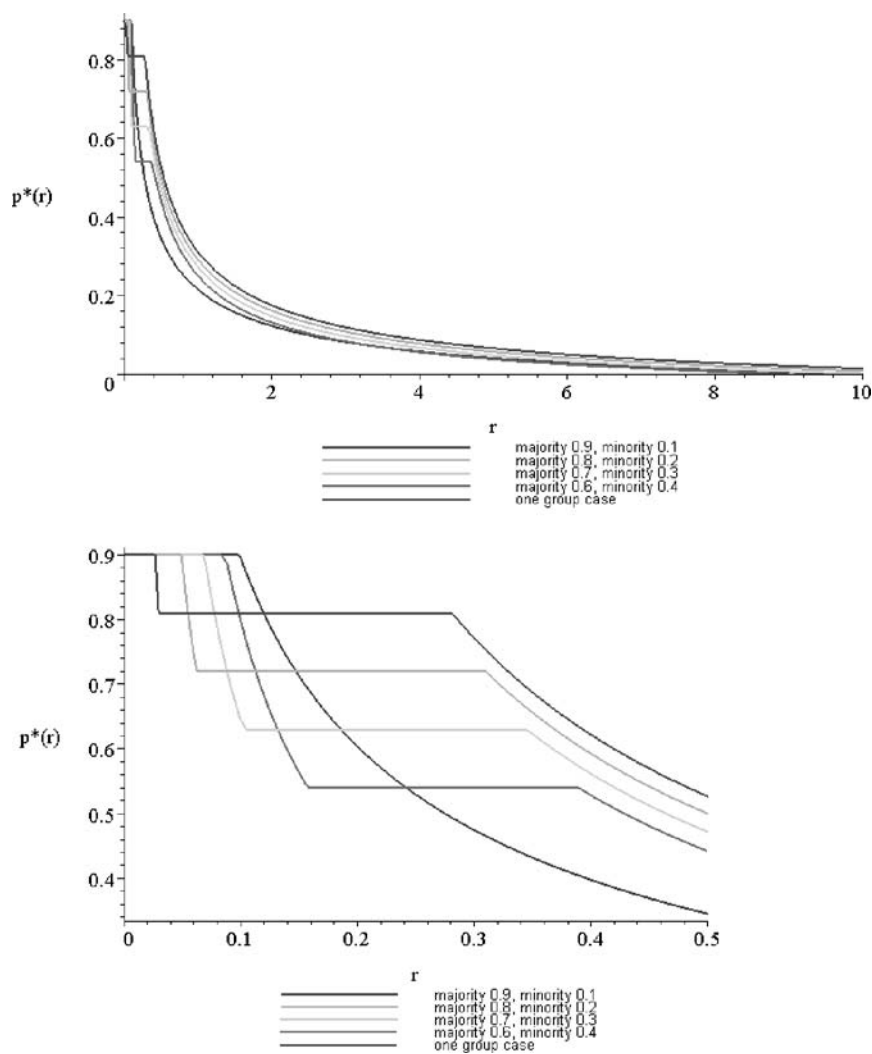


Fig. 1 Equilibrium coverage $p^*(r)$ where $r_1 = 0.7r$ and $r_2 = r - 0.7r\epsilon_1/\epsilon_2$, for one group and several two group cases.

(where the vaccine is perceived to be substantially less risky than the disease): in these cases, the situation is often reversed.

Figure 3 presents surface plots of P_1^* and P_2^* as functions of r_1 and r_2 , for the case $\epsilon_1 = 0.8$, $\epsilon_2 = 0.2$. This figure shows that an increasing perception of vaccine risk in the majority group (r_1) has a proportionate impact on vaccine coverage in that group. By comparison, an increasing perception of vaccine risk in the minority group (r_2) has a more dramatic effect, with a sharp transition to pure nonvaccinating behaviour occuring at a low value of r_2 . This suggests that minority

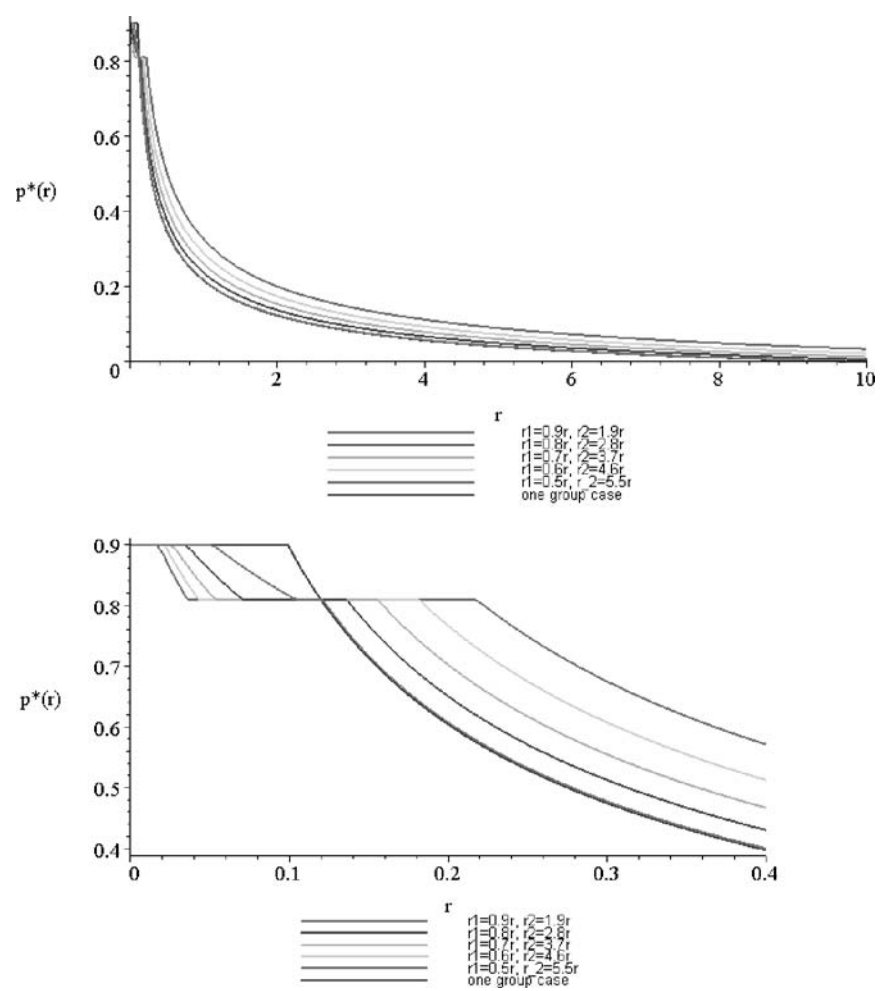


Fig. 2 Equilibrium coverage $p^*(r)$ where $\epsilon_1 = 0.9$ and $\epsilon_2 = 0.1$, for one group and several two group cases.

groups with differing risk perceptions will tend to exhibit more extreme vaccination behaviour than the majority groups.

Figure 4a (resp. Fig. 5a) shows the convergence of a 2-group population with a 0.9 majority and 0.1 minority to the Nash equilibrium $(0.9, 0)$ with average coverage $p^* = 0.81$ (resp. to $(0.9, 0.434)$ with average coverage $p^* = 0.853$) corresponding to low values of r . Figure 4b (resp. Fig. 5b) shows what happens when r is increased by 10%, as might occur during a vaccine scare. After the shift in r , the population converges to a new Nash equilibrium $(0.697, 0)$ with a reduced vaccine coverage of $p^* = 0.6273$ (resp. to $(0.894, 0)$ with a reduced vaccine coverage $p^* = 0.8046$).

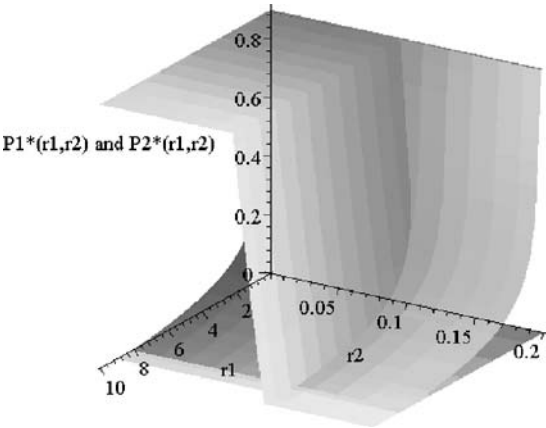


Fig. 3 The equilibrium group coverages $P_1^* = P_1^*(r_1, r_2)$ and $P_2^* = P_2^*(r_1, r_2)$ represented as surfaces in \mathbb{R}^3 in a two group population with $\epsilon_1 = 0.8$ majority and $\epsilon_2 = 0.2$ minority. $P_1^*(r_1, r_2)$ is the green-blue surface and $P_2^*(r_1, r_2)$ is the yellow-red surface. We see here that the minority group exhibits a more extreme vaccination behaviour than the majority group, for a given change in risk perception.

6.3. Three groups model

Here, we study an interesting example of the case with three social groups. We consider a population consisting of three equally-sized groups ($\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{3}$), all of which perceive the vaccine to be less risky than the disease ($r_1 = 3/4, r_2 =$

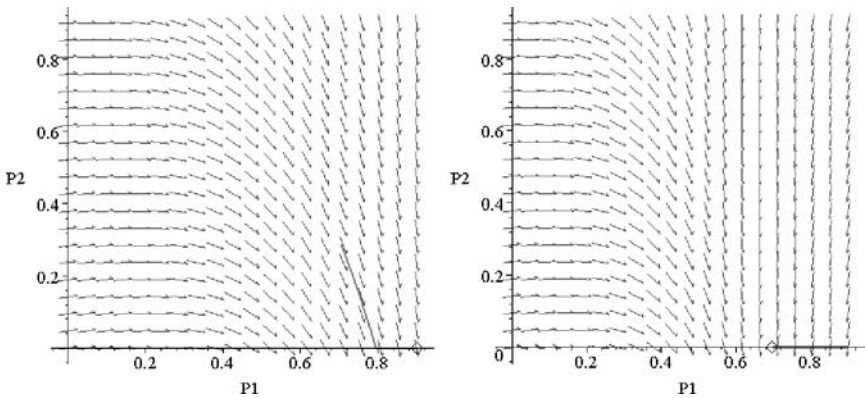


Fig. 4 Effect of a change in average perceived relative risk r on average vaccine coverage level $p^*(r)$, in a population with two social groups ($\epsilon_1 = 0.9, \epsilon_2 = 0.1$). The left panel shows the phase portrait of the time-dependent group strategies $P_1(\tau), P_2(\tau)$ given by the projected system (8), corresponding to $r_1 = 0.07$ and $r_2 = 0.37$. The red curve evolves from the initial strategies $(P_1(0), P_2(0)) = (0.7, 0.3)$ to the Nash equilibrium $(0.9, 0)$, whose overall coverage is $p^* = \epsilon_1 P_1^* + \epsilon_2 P_2^* = 0.81$. The right panel shows how the strategies converge from the former equilibrium state $(0.9, 0)$ to a new Nash equilibrium $(0.697, 0)$ with a reduced coverage $p^* = 0.6273$ after a sudden increase of 10% (to $r_1 = 0.17$ and $r_2 = 0.47$) in relative risk perception, which models the effects of a vaccine scare. The Nash equilibrium points are marked with a diamond symbol.

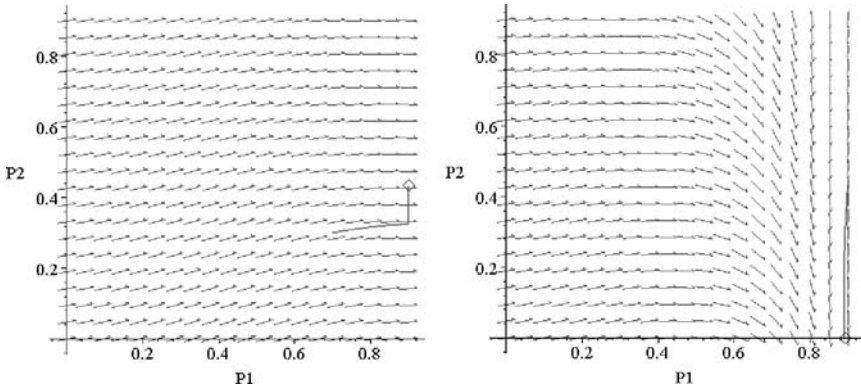


Fig. 5 Effect of a change in average perceived relative risk r on average vaccine coverage levels $p^*(r)$, in a population with two social groups ($\epsilon_1 = 0.9$, $\epsilon_2 = 0.1$). The left panel shows the phase portrait of the time-dependent group strategies $P_1(\tau)$, $P_2(\tau)$ given by the projected system (8), corresponding to $r_1 = 0.01$ and $r_2 = 0.085$. The red curve evolves from the initial strategies $(P_1(0), P_2(0)) = (0.7, 0.3)$ to a Nash equilibrium $P^* = (0.9, 0.434)$ whose overall coverage is $p^* = \epsilon_1 P_1^* + \epsilon_2 P_2^* = 0.853$. The right panel shows how the population converges from the former equilibrium state $(0.9, 0.434)$ to a new Nash equilibrium $P^* = (0.894, 0)$ with coverage $p^* = 0.8046$ after a sudden increase of 10% (to $r_1 = 0.1$ and $r_2 = 0.185$) in relative risk perception, which models the effects of a vaccine scare. The Nash equilibrium points are marked with a diamond symbol.

$1/6, r_3 = 1/5$). With these values,

$$\pi_p = \frac{0.09}{0.1 + \frac{P_1 + P_2 + P_3}{3}} = \frac{0.27}{0.3 + P_1 + P_2 + P_3}.$$

For initial conditions $P_1(0) = 0.7$, $P_2(0) = 0.05$, $P_3(0) = 0.4$, we compute approximate solutions for the PDS (8) and obtain that the equilibrium point is $\underline{P}^* = (0, 0.893, 0.528)$ (Fig. 6). Based on Theorem 4.1, this is the unique equilibrium point and all solutions starting at any initial point $\underline{P}(0) \in \mathbb{K}$ are monotonically attracted to it.

Interestingly, the first group does not vaccinate at all, even though they perceive the vaccine as being less risky than the disease ($r_1 = 3/4$). This group is taking advantage of the herd immunity afforded by the other two groups which have even lower perceived relative risk. Moreover, the difference in perceived risk between groups 2 and 3 is not very large ($r_2 = 1/6, r_3 = 1/5$) and yet group 2 adopts a significantly higher level of vaccination than group 3. This example illustrates how individuals take into account not only their own perception of risk but also what strategies other individuals are adopting, since this influences disease prevalence and hence their own probability of becoming infected.

7. Discussion

This paper applies for the first time the theories and methodologies of PDS and VI to epidemic modelling. There are several reasons why this approach is potentially

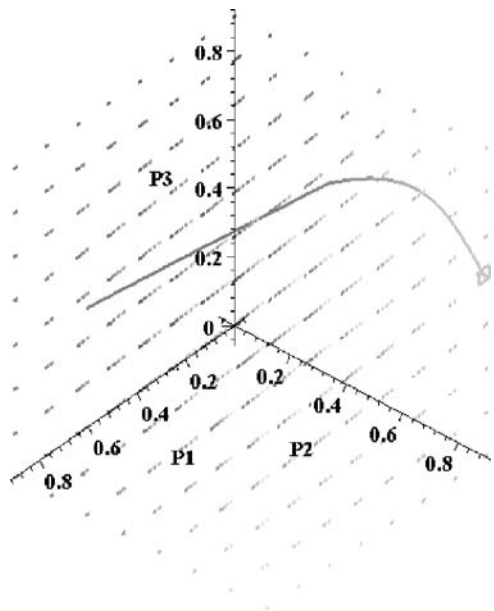


Fig. 6 Game dynamics for a three group model where $\epsilon_i = \frac{1}{3}, \forall i \in \{1, 2, 3\}$ and $r_1 = \frac{3}{4}, r_2 = \frac{1}{6}, r_3 = \frac{1}{5}$. The curve represents the time evolution of each of the three groups' strategies, from the initial state $(P_1(0), P_2(0), P_3(0)) = (0.7, 0.05, 0.4)$ to the Nash equilibrium $P^* = (0, 0.893, 0.526)$. The Nash equilibrium point is marked with a *diamond* symbol.

valuable for biomathematics. First, the solutions of the heterogeneous Nash games have boundary components, and hence a projected dynamics approach must be used instead of a classical dynamical system approach. Second, with these techniques, we have the ability to see the structure of the game solutions through theoretical analysis. We are able to compute the optimal strategy and the respective equilibrium vaccination coverage with relatively little effort. Due to the constraint set being a k -dimensional cube, the expression of the projection of the velocity field F is easily found. The presence of monotonicity is obviously the main factor in analysing the behaviour of the projected flow under perturbations. In this paper, we presented 1, 2 and 3-dimensional examples, so one can readily visualise the results. However, multidimensional ($k > 3$) examples can be computed as well.

It should be noted that these results depend upon our choice of the perceived probability of infection π_p . We expect that qualitatively similar choices for π_p will have qualitatively similar equilibria (Remark 4.2). A potentially interesting case where the results may differ qualitatively is when π_p exhibits an upper threshold beyond which the perceived probability of infection is zero (Section 2) (Bauch and Earn, 2004). This constitutes a topic for future work. When π_p has a critical upper threshold, then a “Prisoner’s Dilemma” may result in coverage levels below that required to eliminate a disease (Introduction). Here we show that a “Prisoner’s Dilemma” is not necessary for nonvaccinating behaviour to develop, at least in certain social groups. Rather, nonvaccinating behaviour can develop due only to

differences in the perception of risk, even if individuals perceive a nonzero risk at arbitrarily high coverage levels.

In this paper, we explored the impact of social heterogeneity in risk perception on vaccine coverage under a voluntary vaccination policy. The PDS approach proved to be particularly valuable here, since minority groups playing a pure non-vaccinator strategy (or majority groups playing a pure vaccinator strategy) correspond to boundary equilibria. In real populations, there are often distinct minority groups with very different perceptions of vaccine risks or vaccine desirability (Eurosurveillance, 2005). We have shown that populations with two groups with distinct risk perceptions tend to exhibit higher average vaccine coverage than the equivalent homogeneous population, except when r is small (i.e. when the vaccine is perceived to be substantially less risky than the disease). Hence, heterogeneous populations should generally exhibit higher vaccine coverage, and perhaps greater stability, in a vaccine scare situation (large r) than suggested by simpler, socially homogeneous models (Bauch and Earn, 2004). Conversely, in situations when r is large, homogenisation of risk perception among different social groups through the influence of mass media may result in lower coverage levels unless there is a systematic bias in favor of vaccines. However, we emphasise that these conclusions may depend upon model assumptions. This model also illustrates how minority and majority groups react differently to changes in risk perception. As illustrated in Fig. 3, a majority group reacts in a relatively gradual way to a change in risk perception, whereas the response of the minority group to a change in risk perception is almost a step function.

Game theoretical models illustrate how vaccine scares and declining vaccine coverage, especially in countries with voluntary vaccination policies such as the United Kingdom, are not isolated historical events, but rather possible instances of inherently unstable dynamics which can apply in any population under a voluntary vaccination policy. While mandatory vaccination would serve the public interest by effectively eradicating diseases, there are also implications for individual rights. Understanding and predicting long-term trends in population vaccination behaviour via game dynamic models is, therefore, valuable for the development of sound, evidence-based public health policy.

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