Strong convergence rate of truncated Euler-Maruyama method for stochastic differential delay equations with Poisson jumps

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Abstract We study a class of super-linear stochastic differential delay equations with Poisson jumps (SDDEwPJs). The convergence and rate of the convergence of the truncated Euler-Maruyama numerical solutions to SDDEwPJs are investigated under the generalized Khasminskii-type condition.

Keywords Truncated Euler-Maruyama method, stochastic differential delay equations, Poisson jumps, rate of the convergence

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1 Introduction

Since the establishment of stochastic differential equations (SDEs) driven by Brownian motions, many scholars have contributed to study properties of SDEs, for example, [1,3,16,20] and references therein. We can observe that the stochastic systems are widely applied in many fields such as biology, chemistry, finance, and economy. When studying the realistic models, it is found that the real state is not only related to the present state, but also related to the past state. Stochastic differential delay equations (SDDEs) are used to describe such systems [4,5,23]. Moreover, if an emergency occurs, its impact on the systems must be taken into account. For example, the sudden outbreak of the new coronavirus has a huge impact on the global economy, leading to a shock in the stock market. Hence, SDEs with jumps which take both the continuous and

discontinuous random effects into consideration are studied to analyze these situations [10,19,28]. In this paper, we will take the delay and jumps into the consideration, i.e., we shall study SDDEs with jumps [15,29,30].

Generally speaking, the true solutions of many equations cannot be calculated, so it is meaningful to investigate the numerical solutions. For instance, the explicit Euler-Maruyama (EM) schemes are very popular to approximate the true solutions [16]. However, when the coefficients grow super-linearly, Hutzenthaler et al. [13] proved that the p th moments of the EM approximations diverge to infinity for all $p \in [1, \infty)$. Thus, many implicit methods have been put forward to approximate the solutions of the equations with nonlinear growing coefficients [2,11,24,27]. In addition, since the explicit schemes require less computation, some modified EM methods have also been established for nonlinear stochastic equations [14,18,25,26]. In particular, the truncated EM method was originally proposed by Mao [21] with drift and diffusion coefficients growing super-linearly. The rate of convergence of the truncated EM method was obtained in [22]. Afterward, there are many papers to study the truncated EM method for stochastic equations whose coefficients grow super-linearly, and we refer to [7–9,12,17] and references therein. Additionally, there are many results on the numerical solutions for SDE with jumps and SDDEs with jumps. For example, the convergence in probability of the EM method for SDDEs with jumps was discussed in [15]. The strong convergence of the EM method for SDDEs with jumps as well as the modified split-step backward Euler method approximation to the true solution was presented in [30]. The semi-implicit Euler method for SDDEs with jumps is convergent with strong order $1/2$ [29]. However, there are few papers studying the numerical solutions of the super-linear SDDEs with Poisson jumps (SDDEwPJs) whose all three coefficients might grow super-linearly. Therefore, in this paper, we will investigate the strong convergence rate of the truncated EM method for super-linear SDDEwPJs in $\mathcal{L}^p$ ($p > 0$) sense.

This paper is organized as follows. We will introduce some necessary notations in Section 2. The rate of convergence in $\mathcal{L}^p$ for $p \geq 2$ will be discussed in Section 3. In Section 4, the rate of convergence in $\mathcal{L}^p$ for $0 < p < 2$ will be presented. Section 5 contains an example to illustrate that our main result could cover a large class of super-linear SDDEwPJs.

2 Mathematical preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. If A is a vector or matrix, its transpose is denoted by A^T . For $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{tr}(A^T A)}$ be its trace norm. By $A \leq 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. If both a, b are real numbers, then

$$a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\}.$$

Let $\lfloor a \rfloor$ denote the largest integer which does not exceed a . Let $\tau > 0$ and $\mathbb{R}_+ = [0, +\infty)$. Denote by $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ the family of continuous functions φ from $[-\tau, 0]$ to $\mathbb{R}^n$ with the norm

$$\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|.$$

If H is a set, denote by $\mathbf{I}_H$ its indicator function; that is, $\mathbf{I}_H(x) = 1$ if $x \in H$ and $\mathbf{I}_H(x) = 0$ if $x \notin H$. Let C stand for a generic positive real constant whose value may change in different appearances.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). Let $\mathbb{E}$ denote the probability expectation with respect to $\mathbb{P}$. For $p > 0$, $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of random variables X with $\mathbb{E}|X|^p < \infty$. Let $\mathcal{C}_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n)$ denote the family of all bounded, $\mathcal{F}_0$ -measurable, $\mathcal{C}([-\tau, 0]; \mathbb{R}^n)$ -valued random variables. Let $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m -dimensional Brownian motion defined on the probability space. Let $N(t)$ be a scalar Poisson process with the compensated Poisson process $\tilde{N}(t) = N(t) - \lambda t$, where the parameter $\lambda > 0$ is the jump intensity. Furthermore, we assume that $B(t)$ and $N(t)$ are independent.

In this paper, we study the truncated EM method for super-linear SDDEwPJs of the form

$$\begin{aligned} dx(t) &= f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t) \\ &\quad + h(x(t^-), x((t-\tau)^-))dN(t), \quad t \geq 0, \end{aligned} \quad (2.1)$$

with the initial value

$$\xi = \{x(\theta) : -\tau \leq \theta \leq 0\} \in \mathcal{C}_{\mathcal{F}_0}^b([-\tau, 0]; \mathbb{R}^n), \quad (2.2)$$

where $x(t^-) = \lim_{s \uparrow t} x(s)$. Here,

$$f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad g: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}, \quad h: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

In order to estimate the convergence rate of the truncated EM method, we assume that the initial value ξ is γ -Hölder continuous, which is a standard constraint.

Assumption 2.1 There exist constants $\bar{K} > 0$ and $\gamma \in (0, 1]$ such that

$$|\xi(t) - \xi(s)| \leq \bar{K}|t - s|^\gamma, \quad s, t \in [-\tau, 0].$$

3 Rate of convergence in $\mathcal{L}^p$ ($p \geq 2$)

Now, in order to obtain the rate of convergence for the truncated EM method for (2.1) in $\mathcal{L}^p$ ($p \geq 2$) sense, we need to impose the following assumptions on coefficients.

Assumption 3.1 There exist constants $K_1 > 0$ and $\beta \geq 0$ such that

$$\begin{aligned} & |f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \\ & \leq K_1(1 + |x|^\beta + |y|^\beta + |\bar{x}|^\beta + |\bar{y}|^\beta)(|x - \bar{x}| + |y - \bar{y}|) \end{aligned}$$

and

$$|h(x, y) - h(\bar{x}, \bar{y})| \leq K_1(|x - \bar{x}| + |y - \bar{y}|)$$

for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

By Assumption 3.1, we obtain

$$|f(x, y)| \vee |g(x, y)| \leq (4K_1 + |f(0, 0)| + |g(0, 0)|)(1 + |x|^{\beta+1} + |y|^{\beta+1})$$

and

$$|h(x, y)| \leq (K_1 + |h(0, 0)|)(1 + |x| + |y|) \quad (3.1)$$

for any $x, y \in \mathbb{R}^n$.

Before stating the next assumption, we need more notations. Let $\mathcal{U}$ denote the family of continuous functions $U: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that for any $b > 0$, there exists a constant $\kappa_b > 0$ satisfying

$$U(x, \bar{x}) \leq \kappa_b |x - \bar{x}|^2 \quad (3.2)$$

for any $x, \bar{x} \in \mathbb{R}^n$ with $|x| \vee |\bar{x}| \leq b$.

Assumption 3.2 There exist constants $K_2 > 0$, $\bar{\eta} > 2$, and $U \in \mathcal{U}$ such that

$$\begin{aligned} & (x - \bar{x})^T(f(x, y) - f(\bar{x}, \bar{y})) + \frac{\bar{\eta} - 1}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ & \leq K_2(|x - \bar{x}|^2 + |y - \bar{y}|^2) - U(x, \bar{x}) + U(y, \bar{y}), \quad \forall x, y, \bar{x}, \bar{y} \in \mathbb{R}^n. \end{aligned}$$

Remark 1 We use an example to illustrate the necessity of setting $U(\cdot, \cdot)$. Let

$$f(x, y) = -5x^3 + \frac{1}{8}|y|^{5/4} + 2x, \quad g(x, y) = \frac{1}{2}|x|^{3/2} + y, \quad x, y \in \mathbb{R}.$$

We could observe that there is no K_2 satisfying

$$\begin{aligned} & (x - \bar{x})^T(f(x, y) - f(\bar{x}, \bar{y})) + \frac{\bar{\eta} - 1}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ & \leq K_2(|x - \bar{x}|^2 + |y - \bar{y}|^2), \end{aligned}$$

but Assumption 3.2 is satisfied. The detailed proof will be provided in Section 5.

Assumption 3.3 There exist constants $K_3 > 0$ and $\bar{p} > \bar{\eta} > 2$ such that

$$x^T f(x, y) + \frac{\bar{p} - 1}{2} |g(x, y)|^2 \leq K_3(1 + |x|^2 + |y|^2), \quad \forall x, y \in \mathbb{R}^n.$$

By using the standard method, we could derive that the moment of the true solution is bounded as follows.

Lemma 3.4 *Let Assumptions 3.1 and 3.3 hold. Then SDDEwPJs (2.1) has a unique global solution $x(t)$. In addition, for any $q \in [2, \bar{p}]$,*

$$\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^q < \infty, \quad \forall T > 0. \quad (3.3)$$

To our best knowledge, there are few results about the strong convergence of the super-linear SDDEwPJs. On the other hand, the truncated EM method developed in [21] is a kind of useful tool to deal with the super-linear terms.

To define the truncated EM scheme, we first choose a strictly increasing continuous function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$ and

$$\sup_{|x| \vee |y| \leq r} (|f(x, y)| \vee |g(x, y)|) \leq \varphi(r), \quad \forall r \geq 1.$$

Let φ^{-1} denote the inverse function of φ . Thus, φ^{-1} is a strictly increasing continuous function from $[\varphi(1), \infty)$ to $[1, \infty)$. Then, we choose $K_0 \geq (1 \vee \varphi(1))$ and a strictly decreasing function $\alpha: (0, 1] \rightarrow (0, \infty)$ such that

$$\lim_{\Delta \rightarrow 0} \alpha(\Delta) = \infty, \quad \Delta^{1/4} \alpha(\Delta) \leq K_0, \quad \forall \Delta \in (0, 1]. \quad (3.4)$$

For example, we could choose

$$\alpha(\Delta) = K_0 \Delta^{-\varepsilon}, \quad \forall \Delta \in (0, 1],$$

for some $\varepsilon \in (0, 1/4]$. For a given step size $\Delta \in (0, 1]$, define the truncated mapping $\pi_\Delta: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\pi_\Delta(x) = (|x| \wedge \varphi^{-1}(\alpha(\Delta))) \frac{x}{|x|}, \quad (3.5)$$

where we let $x/|x| = 0$ when $x = 0$. It is easy to see that the truncated mapping π_Δ maps x to itself when $|x| \leq \varphi^{-1}(\alpha(\Delta))$ and to $\varphi^{-1}(\alpha(\Delta))x/|x|$ when $|x| \geq \varphi^{-1}(\alpha(\Delta))$. We now define the truncated functions

$$f_\Delta(x, y) = f(\pi_\Delta(x), \pi_\Delta(y)), \quad g_\Delta(x, y) = g(\pi_\Delta(x), \pi_\Delta(y)), \quad \forall x, y \in \mathbb{R}^n.$$

By definition, we could easily find that

$$|f_\Delta(x, y)| \vee |g_\Delta(x, y)| \leq \varphi(\varphi^{-1}(\alpha(\Delta))) \leq \alpha(\Delta), \quad \forall x, y \in \mathbb{R}^n. \quad (3.6)$$

Moreover, we could obtain that

$$|\pi_\Delta(x)| \leq |x|, \quad |\pi_\Delta(x) - \pi_\Delta(y)| \leq |x - y|, \quad \forall x, y \in \mathbb{R}^n. \quad (3.7)$$

By Assumption 3.1, we derive that for any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} & |f_\Delta(x, y) - f_\Delta(\bar{x}, \bar{y})| \vee |g_\Delta(x, y) - g_\Delta(\bar{x}, \bar{y})| \\ & \leq K_1(1 + |x|^\beta + |y|^\beta + |\bar{x}|^\beta + |\bar{y}|^\beta)(|x - \bar{x}| + |y - \bar{y}|). \end{aligned} \quad (3.8)$$

If Assumption 3.3 hold, then, for any $\Delta \in (0, 1]$, $x, y \in \mathbb{R}^n$, one has

$$x^T f_{\Delta}(x, y) + \frac{\bar{p} - 1}{2} |g_{\Delta}(x, y)|^2 \leq 3K_3 \left(\frac{1}{\varphi^{-1}(\alpha(1))} \vee 1 \right) (1 + |x|^2 + |y|^2). \quad (3.9)$$

Let us now introduce the discrete-time truncated EM numerical scheme to approximate the true solution of (2.1). Without loss of generality, we assume that τ is a positive number. For some positive integer M , we take step size $\Delta = \tau/M$. Obviously, when we choose M sufficiently large, Δ will become sufficiently small. Define $t_k = k\Delta$ for $k = -M, -M+1, \dots, -1, 0, 1, 2, \dots$. Set $X_{\Delta}(t_k) = \xi(t_k)$ for $k = -M, -M+1, \dots, -1, 0$ and then form

$$\begin{aligned} X_{\Delta}(t_{k+1}) &= X_{\Delta}(t_k) + f_{\Delta}(X_{\Delta}(t_k), X_{\Delta}(t_{k-M}))\Delta \\ &\quad + g_{\Delta}(X_{\Delta}(t_k), X_{\Delta}(t_{k-M}))\Delta B_k \\ &\quad + h(X_{\Delta}(t_k^-), X_{\Delta}(t_{k-M}^-))\Delta N_k \end{aligned} \quad (3.10)$$

for $k = 0, 1, 2, \dots$, where

$$\Delta B_k = B(t_{k+1}) - B(t_k), \quad \Delta N_k = N(t_{k+1}) - N(t_k).$$

As usual, there are two kinds of the continuous-time truncated EM solutions. The first one is

$$\bar{x}_{\Delta}(t) = \sum_{k=-M}^{\infty} X_{\Delta}(t_k) \mathbf{I}_{[t_k, t_{k+1})}(t). \quad (3.11)$$

The second one is defined as follows:

$$\begin{aligned} x_{\Delta}(t) &= \xi(0) + \int_0^t f_{\Delta}(\bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau))ds + \int_0^t g_{\Delta}(\bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau))dB(s) \\ &\quad + \int_0^t h(\bar{x}_{\Delta}(s^-), \bar{x}_{\Delta}((s-\tau)^-))dN(s). \end{aligned} \quad (3.12)$$

It is easy to see that

$$X_{\Delta}(t_k) = \bar{x}_{\Delta}(t_k) = x_{\Delta}(t_k).$$

Additionally, $x_{\Delta}(t)$ is an Itô process on $t \geq 0$ with its Itô differential

$$\begin{aligned} dx_{\Delta}(t) &= f_{\Delta}(\bar{x}_{\Delta}(t), \bar{x}_{\Delta}(t-\tau))dt + g_{\Delta}(\bar{x}_{\Delta}(t), \bar{x}_{\Delta}(t-\tau))dB(t) \\ &\quad + h(\bar{x}_{\Delta}(t^-), \bar{x}_{\Delta}((t-\tau)^-))dN(t). \end{aligned}$$

We now prepare some useful lemmas. Before stating the next lemma, we define

$$\kappa(t) = \left\lfloor \frac{t}{\Delta} \right\rfloor \Delta, \quad \forall t \in [-\tau, T].$$

Lemma 3.5 *Let Assumption 3.1 hold. For any $\Delta \in (0, 1]$ and $t \in [0, T]$, we have*

$$\begin{aligned} &\mathbb{E}(|x_{\Delta}(t) - \bar{x}_{\Delta}(t)|^{\tilde{p}} | \mathcal{F}_{\kappa(t)}) \\ &\leq c_{\tilde{p}}((\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2} + \Delta)(1 + |\bar{x}_{\Delta}(t)|^{\tilde{p}} + |\bar{x}_{\Delta}(t-\tau)|^{\tilde{p}}), \quad \tilde{p} \geq 2, \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} & \mathbb{E}(|x_\Delta(t) - \bar{x}_\Delta(t)|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)}) \\ & \leq c_{\tilde{p}}(\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2} (1 + |\bar{x}_\Delta(t)|^{\tilde{p}} + |\bar{x}_\Delta(t - \tau)|^{\tilde{p}}), \quad 0 < \tilde{p} < 2, \end{aligned} \quad (3.14)$$

where $c_{\tilde{p}}$ is a positive constant which is independent of Δ .

Proof Fix any $\tilde{p} \geq 2$. Then by the Hölder inequality and the Burkholder-Davis-Gundy inequality, we derive from (3.6) that

$$\begin{aligned} & \mathbb{E}(|x_\Delta(t) - \bar{x}_\Delta(t)|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)}) \\ & = \mathbb{E} \left(\left| \int_{\kappa(t)}^t f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) ds + \int_{\kappa(t)}^t g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) dB(s) \right. \right. \\ & \quad \left. \left. + \int_{\kappa(t)}^t h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s - \tau)^-)) dN(s) \right|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)} \right) \\ & \leq c_{\tilde{p}} \left(\mathbb{E} \left(\left| \int_{\kappa(t)}^t f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) ds \right|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)} \right) \right. \\ & \quad \left. + \mathbb{E} \left(\left| \int_{\kappa(t)}^t g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) dB(s) \right|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)} \right) \right. \\ & \quad \left. + \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s - \tau)^-)) dN(s) \right|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)} \right) \right) \\ & \leq c_{\tilde{p}} \left((\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2} + \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) dN(s) \right|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)} \right) \right). \end{aligned}$$

By the characteristic function's argument [6], for $\Delta \in (0, 1]$, we could get

$$\mathbb{E}|N(t) - N(\kappa(t))|^{\tilde{p}} \leq c_0 \Delta,$$

where c_0 is a positive constant independent of Δ . Then, by (3.1), we have

$$\begin{aligned} & \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) dN(s) \right|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)} \right) \\ & = \mathbb{E}(|h(x_\Delta(\kappa(t)), x_\Delta(\kappa(t - \tau)))(N(t) - N(\kappa(t)))|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)}) \\ & \leq |h(x_\Delta(\kappa(t)), x_\Delta(\kappa(t - \tau)))|^{\tilde{p}} \mathbb{E}|N(t) - N(\kappa(t))|^{\tilde{p}} \\ & \leq c_{\tilde{p}}(1 + |\bar{x}_\Delta(t)|^{\tilde{p}} + |\bar{x}_\Delta(t - \tau)|^{\tilde{p}}) \Delta. \end{aligned}$$

Thus, we derive that

$$\begin{aligned} & \mathbb{E}(|x_\Delta(t) - \bar{x}_\Delta(t)|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)}) \\ & \leq c_{\tilde{p}}((\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2} + (1 + |\bar{x}_\Delta(t)|^{\tilde{p}} + |\bar{x}_\Delta(t - \tau)|^{\tilde{p}}) \Delta) \\ & \leq c_{\tilde{p}}((\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2} + \Delta)(1 + |\bar{x}_\Delta(t)|^{\tilde{p}} + |\bar{x}_\Delta(t - \tau)|^{\tilde{p}}). \end{aligned}$$

When $0 < \tilde{p} < 2$, an application of Jensen's inequality yields that

$$\begin{aligned} & \mathbb{E}(|x_\Delta(t) - \bar{x}_\Delta(t)|^{\tilde{p}} \mid \mathcal{F}_{\kappa(t)}) \\ & \leq (\mathbb{E}(|x_\Delta(t) - \bar{x}_\Delta(t)|^2 \mid \mathcal{F}_{\kappa(t)}))^{\tilde{p}/2} \\ & \leq c_{\tilde{p}}((\alpha(\Delta))^2 \Delta + \Delta)^{\tilde{p}/2} (1 + |\bar{x}_\Delta(t)|^2 + |\bar{x}_\Delta(t - \tau)|^2)^{\tilde{p}/2} \\ & \leq c_{\tilde{p}}(\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2} (1 + |\bar{x}_\Delta(t)|^{\tilde{p}} + |\bar{x}_\Delta(t - \tau)|^{\tilde{p}}). \end{aligned}$$

We complete the proof. $\square$

Lemma 3.6 *Let Assumptions 3.1 and 3.3 hold. Then, for any $q \in (2, \bar{p}]$, we have*

$$\sup_{0 < \Delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^q \leq C, \quad \forall T > 0. \quad (3.15)$$

Proof For any $\Delta \in (0, 1]$ and $t > 0$, by Itô's formula and (3.9), we derive that

$$\begin{aligned} & \mathbb{E}|x_\Delta(t)|^q - \|\xi\|^q \\ & \leq \mathbb{E} \int_0^t q|x_\Delta(s)|^{q-2} (x_\Delta^T(s) f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) \\ & \quad + \frac{q-1}{2} |g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))|^2) ds \\ & \quad + \lambda \mathbb{E} \int_0^t (|x_\Delta(s) + h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s - \tau)^-))|^q - |x_\Delta(s)|^q) ds \\ & \leq \mathbb{E} \int_0^t 3qK_3 \left(\frac{1}{\varphi^{-1}(\alpha(1))} \vee 1 \right) |x_\Delta(s)|^{q-2} (1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s - \tau)|^2) ds \\ & \quad + \mathbb{E} \int_0^t q|x_\Delta(s)|^{q-2} |x_\Delta(s) - \bar{x}_\Delta(s)| |f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))| ds \\ & \quad + \lambda \mathbb{E} \int_0^t (|x_\Delta(s) + h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s - \tau)^-))|^q - |x_\Delta(s)|^q) ds \\ & =: A_1 + A_2 + A_3. \end{aligned} \quad (3.16)$$

We now handle A_1 , A_2 , and A_3 , respectively. First, we could see that

$$\begin{aligned} A_1 & \leq C \mathbb{E} \int_0^t (1 + |x_\Delta(s)|^q + |\bar{x}_\Delta(s)|^q + |\bar{x}_\Delta(s - \tau)|^q) ds \\ & \leq C \left(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^q ds \right). \end{aligned} \quad (3.17)$$

Let

$$A_{21} := C \mathbb{E} \int_0^t |\bar{x}_\Delta(s)|^{q-2} |x_\Delta(s) - \bar{x}_\Delta(s)| |f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))| ds \quad (3.18)$$

and

$$A_{22} := C \mathbb{E} \int_0^t |x_\Delta(s) - \bar{x}_\Delta(s)|^{q-1} |f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))| ds. \quad (3.19)$$

Then we get

$$A_2 \leq A_{21} + A_{22}.$$

By Lemma 3.5, (3.4), (3.6), and Young's inequality, we have

$$\begin{aligned} A_{21} &\leq C \int_0^t \mathbb{E}[|\bar{x}_\Delta(s)|^{q-2} |f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))| \mathbb{E}(|x_\Delta(s) - \bar{x}_\Delta(s)| \mid \mathcal{F}_{\kappa(s)})] ds \\ &\leq C \int_0^t (\alpha(\Delta))^2 \Delta^{1/2} \mathbb{E}[|\bar{x}_\Delta(s)|^{q-2} (1 + |\bar{x}_\Delta(s)| + |\bar{x}_\Delta(s-\tau)|)] ds \\ &\leq C \int_0^t (1 + \mathbb{E}|\bar{x}_\Delta(s)|^q + \mathbb{E}|\bar{x}_\Delta(s-\tau)|^q) ds \\ &\leq C \left(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^q ds \right). \end{aligned} \quad (3.20)$$

By (3.6), we obtain that

$$A_{22} \leq C\alpha(\Delta) \int_0^t \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^{q-1} ds. \quad (3.21)$$

This, together with (3.4) and Lemma 3.5, implies

$$A_{22} \leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E}|x_\Delta(l)|^q ds \right). \quad (3.22)$$

Combining (3.20) and (3.22) together, we derive that

$$A_2 \leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E}|x_\Delta(l)|^q ds \right). \quad (3.23)$$

By (3.1), one can see that

$$A_3 \leq C \left(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^q ds \right). \quad (3.24)$$

Substituting (3.17), (3.23), and (3.24) into (3.16), we obtain

$$\sup_{0 \leq u \leq t} \mathbb{E}|x_\Delta(u)|^q \leq C \left(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x_\Delta(u)|^q ds \right).$$

An application of Gronwall's inequality yields that

$$\sup_{0 \leq u \leq T} \mathbb{E}|x_\Delta(u)|^q \leq C,$$

where C is independent of Δ . Since this inequality holds for any $\Delta \in (0, 1]$, the desired result follows. The proof is therefore complete. $\square$

Lemma 3.7 *Let Assumptions 3.1 and 3.3 hold. Then for any $\Delta \in (0, 1]$ and $t \in [0, T]$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^q \leq C((\alpha(\Delta))^q \Delta^{q/2} + \Delta), \quad 2 \leq q \leq \bar{p}, \quad (3.25)$$

and

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^q \leq C(\alpha(\Delta))^q \Delta^{q/2}, \quad 0 < q < 2. \quad (3.26)$$

Hence,

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^q = 0, \quad 0 < q \leq \bar{p}. \quad (3.27)$$

Proof By Lemma 3.6 and (3.13), we get (3.25). Then for any $q \in (0, 2)$, Jensen's inequality gives that

$$\begin{aligned} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^q &\leq (\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^2)^{q/2} \\ &\leq C((\alpha(\Delta))^2 \Delta + \Delta)^{q/2} \\ &\leq C(\alpha(\Delta))^q \Delta^{q/2}. \end{aligned}$$

We complete the proof. $\square$

By using Lemmas 3.4, 3.6, and the Chebyshev inequality, we can immediately have the following lemma.

Lemma 3.8 *Let Assumptions 3.1 and 3.3 hold. For any real number $L > \|\xi\|$, define the stopping time*

$$\tau_L = \inf\{t \geq 0: |x(t)| \geq L\}, \quad \tau_{\Delta,L} = \inf\{t \geq 0: |x_\Delta(t)| \geq L\}. \quad (3.28)$$

Then we have

$$\mathbb{P}(\tau_L \leq T) \leq \frac{C}{L^2}, \quad \mathbb{P}(\tau_{\Delta,L} \leq T) \leq \frac{C}{L^2}. \quad (3.29)$$

Let us now discuss the rate of convergence in $\mathcal{L}^2$ sense for the truncated EM solutions to the true solution.

Theorem 3.9 *Let Assumptions 2.1 and 3.1–3.3 hold. Suppose that there exists a real number $q \in (2, \bar{p})$ such that $q > (1 + \beta)\bar{\eta}$. Then, for any $\Delta \in (0, 1]$, we have*

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \leq C((\varphi^{-1}(\alpha(\Delta)))^{2\beta+2-q} + (\alpha(\Delta))^2 \Delta + \Delta^{(q-2\beta)/q} + \Delta^{2\gamma}) \quad (3.30)$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \leq C((\varphi^{-1}(\alpha(\Delta)))^{2\beta+2-q} + (\alpha(\Delta))^2 \Delta + \Delta^{(q-2\beta)/q} + \Delta^{2\gamma}). \quad (3.31)$$

In particular, let

$$\varphi(r) = c^* r^{1+\beta}, \quad \forall r \geq 1, \quad \alpha(\Delta) = K_0 \Delta^{-\varepsilon}, \quad \varepsilon \in \left(0, \frac{1}{4}\right], \quad (3.32)$$

with

$$c^* = 4K_1 + |f(0, 0)| + |g(0, 0)|.$$

Then it holds for any $\Delta \in (0, 1]$ that

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \leq C\Delta^{[\varepsilon(q-2\beta-2)/(1+\beta)] \wedge [1-2\varepsilon] \wedge [(q-2\beta)/q] \wedge [2\gamma]}, \quad (3.33)$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^2 \leq C\Delta^{[\varepsilon(q-2\beta-2)/(1+\beta)] \wedge [1-2\varepsilon] \wedge [(q-2\beta)/q] \wedge [2\gamma]}. \quad (3.34)$$

Proof Let

$$e_\Delta(t) = x(t) - x_\Delta(t), \quad t \geq 0, \Delta \in (0, 1].$$

Define

$$\rho_{\Delta, L} = \tau_L \wedge \tau_{\Delta, L},$$

that is,

$$\rho_{\Delta, L} = \inf\{t \geq 0: |x(t)| \vee |x_\Delta(t)| \geq L\}.$$

We write $\rho_{\Delta, L} = \rho$ for simplicity. Note that for $\bar{q} \in (2, \bar{\eta})$, we have $q > (1 + \beta)\bar{q}$. By Itô's formula, for any $t \in [0, T]$ and $\Delta \in (0, 1]$, we have

$$\begin{aligned} \mathbb{E}|e_\Delta(t \wedge \rho)|^2 &\leq \mathbb{E} \int_0^{t \wedge \rho} 2 \left(e_\Delta^\top(s) (f(x(s), x(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))) \right. \\ &\quad \left. + \frac{1}{2} |g(x(s), x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right) ds \\ &\quad + \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s) + h(x(s^-), x((s-\tau)^-)) \\ &\quad - h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))|^2 - |e_\Delta(s)|^2) ds \\ &=: I_1 + I_2. \end{aligned} \quad (3.35)$$

First, we estimate I_1 . Note that

$$\begin{aligned} &\frac{1}{2} |g(x(s), x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \\ &\leq \frac{1}{2} \left((\bar{q} - 1) |g(x(s), x(s-\tau)) - g(x_\Delta(s), x_\Delta(s-\tau))|^2 \right. \\ &\quad \left. + \frac{\bar{q} - 1}{\bar{q} - 2} |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right) \\ &= \frac{\bar{q} - 1}{2} |g(x(s), x(s-\tau)) - g(x_\Delta(s), x_\Delta(s-\tau))|^2 \\ &\quad + \frac{\bar{q} - 1}{2(\bar{q} - 2)} |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2. \end{aligned}$$

Then we have

$$I_1 \leq \mathbb{E} \int_0^{t \wedge \rho} 2 \left(e_\Delta^\top(s) (f(x(s), x(s-\tau)) - f(x_\Delta(s), x_\Delta(s-\tau))) \right)$$

$$\begin{aligned}
& + \frac{\bar{q} - 1}{2} |g(x(s), x(s - \tau)) - g(x_\Delta(s), x_\Delta(s - \tau))|^2 \mathrm{d}s \\
& + \mathbb{E} \int_0^{t \wedge \rho} 2 \left(e_\Delta^\mathrm{T}(s) (f(x_\Delta(s), x_\Delta(s - \tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))) \right. \\
& \left. + \frac{\bar{q} - 1}{2(\bar{q} - 2)} |g(x_\Delta(s), x_\Delta(s - \tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))|^2 \right) \mathrm{d}s \\
& =: I_{11} + I_{12}.
\end{aligned} \tag{3.36}$$

By Assumptions 2.1, 3.2, and (3.2), we derive that

$$\begin{aligned}
I_{11} & \leq \mathbb{E} \int_0^{t \wedge \rho} 2 [K_2 (|x(s) - x_\Delta(s)|^2 + |x(s - \tau) - x_\Delta(s - \tau)|^2) \\
& \quad - U(x(s), x_\Delta(s)) + U(x(s - \tau), x_\Delta(s - \tau))] \mathrm{d}s \\
& \leq 4K_2 \mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^2 \mathrm{d}s + 2K_2 \int_{-\tau}^0 |\xi(s) - \xi(\kappa(s))|^2 \mathrm{d}s \\
& \quad + 2\mathbb{E} \int_0^{t \wedge \rho} (-U(x(s), x_\Delta(s)) + U(x(s - \tau), x_\Delta(s - \tau))) \mathrm{d}s \\
& \leq 4K_2 \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + 2\tau K_2 \bar{K}^2 \Delta^{2\gamma} + 2 \int_{-\tau}^0 U(\xi(s), \xi(\kappa(s))) \mathrm{d}s \\
& \leq 4K_2 \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + 2\tau K_2 \bar{K}^2 \Delta^{2\gamma} + 2 \int_{-\tau}^0 \kappa_b |\xi(s) - \xi(\kappa(s))|^2 \mathrm{d}s \\
& \leq 4K_2 \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + c_1 \Delta^{2\gamma},
\end{aligned}$$

where

$$c_1 = 2\tau K_2 \bar{K}^2 + 2\tau \kappa_b \bar{K}^2,$$

and $\kappa(s)$ is defined as before. As for I_{12} , we have

$$\begin{aligned}
I_{12} & \leq \mathbb{E} \int_0^{t \wedge \rho} 2 \left(e_\Delta^\mathrm{T}(s) (f(x_\Delta(s), x_\Delta(s - \tau)) - f_\Delta(x_\Delta(s), x_\Delta(s - \tau))) \right. \\
& \quad + \frac{\bar{q} - 1}{\bar{q} - 2} |g(x_\Delta(s), x_\Delta(s - \tau)) - g_\Delta(x_\Delta(s), x_\Delta(s - \tau))|^2 \Big) \mathrm{d}s \\
& \quad + \mathbb{E} \int_0^{t \wedge \rho} 2 \left(e_\Delta^\mathrm{T}(s) (f_\Delta(x_\Delta(s), x_\Delta(s - \tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))) \right. \\
& \quad \left. + \frac{\bar{q} - 1}{\bar{q} - 2} |g_\Delta(x_\Delta(s), x_\Delta(s - \tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))|^2 \right) \mathrm{d}s \\
& =: I_{121} + I_{122}.
\end{aligned}$$

By Young's inequality, Hölder's inequality, Assumption 3.1, Lemma 3.6, and (3.7), we have

$$I_{121} \leq \mathbb{E} \int_0^{t \wedge \rho} \left(|e_\Delta(s)|^2 + |f(x_\Delta(s), x_\Delta(s - \tau)) - f_\Delta(x_\Delta(s), x_\Delta(s - \tau))|^2 \right) \mathrm{d}s$$

$$\begin{aligned}
& + \frac{2(\bar{q}-1)}{\bar{q}-2} |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2 \mathrm{d}s \\
\leq & \mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^2 \mathrm{d}s \\
& + C \mathbb{E} \int_0^T (|f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2 \\
& + |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2) \mathrm{d}s \\
\leq & \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + C \mathbb{E} \int_0^T [(1 + |x_\Delta(s)|^{2\beta} + |x_\Delta(s-\tau)|^{2\beta} \\
& + |\pi_\Delta(x_\Delta(s))|^{2\beta} + |\pi_\Delta(x_\Delta(s-\tau))|^{2\beta}) \\
& \cdot (|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^2 + |x_\Delta(s-\tau) - \pi_\Delta(x_\Delta(s-\tau))|^2)] \mathrm{d}s \\
\leq & \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + C \int_0^T \mathbb{E} [(1 + |x_\Delta(s)|^{2\beta} + |x_\Delta(s-\tau)|^{2\beta}) \\
& \cdot (|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^2 + |x_\Delta(s-\tau) - \pi_\Delta(x_\Delta(s-\tau))|^2)] \mathrm{d}s \\
\leq & \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + C \int_0^T [\mathbb{E} (1 + |x_\Delta(s)|^q + |x_\Delta(s-\tau)|^q)]^{2\beta/q} \\
& \cdot [\mathbb{E} (|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{2q/(q-2\beta)} \\
& + |x_\Delta(s-\tau) - \pi_\Delta(x_\Delta(s-\tau))|^{2q/(q-2\beta)})]^{(q-2\beta)/q} \mathrm{d}s \\
\leq & \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + C \int_0^T [\mathbb{E} |x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{2q/(q-2\beta)}]^{(q-2\beta)/q} \mathrm{d}s \\
& + C \int_{-\tau}^0 [\mathbb{E} |\xi(\kappa(s)) - \pi_\Delta(\xi(\kappa(s)))|^{2q/(q-2\beta)}]^{(q-2\beta)/q} \mathrm{d}s \\
\leq & \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s \\
& + C \int_0^T [\mathbb{E} (\mathbf{I}_{|x_\Delta(s)| > \varphi^{-1}(\alpha(\Delta))} |x_\Delta(s)|^{2q/(q-2\beta)})]^{(q-2\beta)/q} \mathrm{d}s \\
& + C \int_{-\tau}^0 [\mathbb{E} (\mathbf{I}_{|\xi(\kappa(s))| > \varphi^{-1}(\alpha(\Delta))} |\xi(\kappa(s))|^{2q/(q-2\beta)})]^{(q-2\beta)/q} \mathrm{d}s \\
\leq & \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 \mathrm{d}s + C \int_0^T ([\mathbb{P}(|x_\Delta(s)| > \varphi^{-1}(\alpha(\Delta)))]^{(q-2\beta-2)/(q-2\beta)} \\
& \cdot [\mathbb{E} |x_\Delta(s)|^q]^{2/(q-2\beta)})^{(q-2\beta)/q} \mathrm{d}s \\
& + C \int_{-\tau}^0 ([\mathbb{P}(|\xi(\kappa(s))| > \varphi^{-1}(\alpha(\Delta)))]^{(q-2\beta-2)/(q-2\beta)} \\
& \cdot [\mathbb{E} |\xi(\kappa(s))|^q]^{2/(q-2\beta)})^{(q-2\beta)/q} \mathrm{d}s.
\end{aligned}$$

Then by Chebyshev's inequality, we get

$$\begin{aligned}
I_{121} &\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C \int_0^T \left(\frac{\mathbb{E}|x_\Delta(s)|^q}{(\varphi^{-1}(\alpha(\Delta)))^q} \right)^{(q-2\beta-2)/q} ds \\
&\quad + C \int_{-\tau}^0 \left(\frac{\mathbb{E}|\xi(\kappa(s))|^q}{(\varphi^{-1}(\alpha(\Delta)))^q} \right)^{(q-2\beta-2)/q} ds \\
&\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C(\varphi^{-1}(\alpha(\Delta)))^{2\beta+2-q}. \tag{3.37}
\end{aligned}$$

We can use the same technique to handle I_{122} . By Young's inequality, Hölder's inequality, Lemmas 3.6, 3.7, (3.8), and the inequality $2q/(q-2\beta) \geq 2$, we obtain

$$\begin{aligned}
I_{122} &\leq \mathbb{E} \int_0^{t \wedge \rho} \left(|e_\Delta(s)|^2 + |f_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right. \\
&\quad \left. + \frac{2(\bar{q}-1)}{\bar{q}-2} |g_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right) ds \\
&\leq \mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^2 ds \\
&\quad + C \mathbb{E} \int_0^T (|f_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \\
&\quad + |g_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2) ds \\
&\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C \int_0^T \mathbb{E}[(1 + |x_\Delta(s)|^{2\beta} + |x_\Delta(s-\tau)|^{2\beta} \\
&\quad + |\bar{x}_\Delta(s)|^{2\beta} + |\bar{x}_\Delta(s-\tau)|^{2\beta})(|x_\Delta(s) - \bar{x}_\Delta(s)|^2 \\
&\quad + |x_\Delta(s-\tau) - \bar{x}_\Delta(s-\tau)|^2)] ds \\
&\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C \int_0^T [\mathbb{E}(1 + |x_\Delta(s)|^q + |x_\Delta(s-\tau)|^q \\
&\quad + |\bar{x}_\Delta(s)|^q + |\bar{x}_\Delta(s-\tau)|^q)]^{2\beta/q} [\mathbb{E}(|x_\Delta(s) - \bar{x}_\Delta(s)|^{2q/(q-2\beta)} \\
&\quad + |x_\Delta(s-\tau) - \bar{x}_\Delta(s-\tau)|^{2q/(q-2\beta)})]^{(q-2\beta)/q} ds \\
&\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C \int_0^T (\mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^{2q/(q-2\beta)})^{(q-2\beta)/q} ds \\
&\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C \int_0^T ((\alpha(\Delta))^{2q/(q-2\beta)} \Delta^{q/(q-2\beta)} + \Delta)^{(q-2\beta)/q} ds \\
&\leq \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C((\alpha(\Delta))^2 \Delta + \Delta^{(q-2\beta)/q}). \tag{3.38}
\end{aligned}$$

Let us now estimate I_2 . By Assumptions 2.1, 3.1, and Lemma 3.7, we obtain

$$\begin{aligned}
I_2 &\leq \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^2 + 2|h(x(s^-), x((s-\tau)^-)) \\
&\quad - h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))|^2) ds
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^2 + 4K_1^2(|x(s) - \bar{x}_\Delta(s)|^2 + |x(s - \tau) - \bar{x}_\Delta(s - \tau)|^2)) ds \\
&\leq \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^2 + 8K_1^2(|x(s) - x_\Delta(s)|^2 + |x_\Delta(s) - \bar{x}_\Delta(s)|^2 \\
&\quad + |x(s - \tau) - x_\Delta(s - \tau)|^2 + |x_\Delta(s - \tau) - \bar{x}_\Delta(s - \tau)|^2)) ds \\
&\leq \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^2 + 16K_1^2(|e_\Delta(s)|^2 + |x_\Delta(s) - \bar{x}_\Delta(s)|^2)) ds \\
&\quad + \lambda \int_{-\tau}^0 8K_1^2|\xi(s) - \xi(\kappa(s))|^2 ds \\
&\leq \lambda(16K_1^2 + 1) \mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^2 ds + 16\lambda K_1^2 \int_0^T \mathbb{E} |x_\Delta(s) - \bar{x}_\Delta(s)|^2 ds \\
&\quad + 8\lambda\tau K_1^2 \bar{K}^2 \Delta^{2\gamma} \\
&\leq \lambda(16K_1^2 + 1) \int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 ds \\
&\quad + 16\lambda K_1^2 T(\alpha(\Delta))^2 \Delta + 8\lambda\tau K_1^2 \bar{K}^2 \Delta^{2\gamma}. \tag{3.39}
\end{aligned}$$

Combining (3.35)–(3.39) together, one can see that

$$\begin{aligned}
\mathbb{E} |e_\Delta(t \wedge \rho)|^2 &\leq C \left(\int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^2 ds + (\varphi^{-1}(\alpha(\Delta)))^{2\beta+2-q} \right. \\
&\quad \left. + (\alpha(\Delta))^2 \Delta + \Delta^{(q-2\beta)/q} + \Delta^{2\gamma} \right).
\end{aligned}$$

An application of Gronwall's inequality yields that

$$\mathbb{E} |e_\Delta(T \wedge \rho)|^2 \leq C((\varphi^{-1}(\alpha(\Delta)))^{2\beta+2-q} + (\alpha(\Delta))^2 \Delta + \Delta^{(q-2\beta)/q} + \Delta^{2\gamma}).$$

The desired results (3.30) and (3.31) follow by letting $L \rightarrow \infty$ and using Lemmas 3.7 and 3.8. In particular, by the definition of φ , we can derive (3.33) and (3.34). The proof is therefore complete. $\square$

In the following remark, we get the optimal convergence rate in $\mathcal{L}^2$ when imposing the stronger condition on q .

Remark 2 In Theorem 3.9, if there exists a real number $q \in ((1 + \beta)\bar{\eta}, \bar{p})$ such that $q > (1 + \beta)/\varepsilon$, then, for any $\Delta \in (0, 1]$, we have

$$\mathbb{E} |x(T) - x_\Delta(T)|^2 \leq C \Delta^{[1-2\varepsilon] \wedge [2\gamma]} \tag{3.40}$$

and

$$\mathbb{E} |x(T) - \bar{x}_\Delta(T)|^2 \leq C \Delta^{[1-2\varepsilon] \wedge [2\gamma]}. \tag{3.41}$$

For $q > (1 + \beta)/\varepsilon$, we can derive that

$$\frac{\varepsilon(q - 2\beta - 2)}{1 + \beta} > 1 - 2\varepsilon, \quad \frac{q - 2\beta}{q} > 1 - 2\varepsilon.$$

Then, by (3.33) and (3.34), we get (3.40) and (3.41), respectively.

To obtain the rate of convergence in $\mathcal{L}^p$ sense for $p > 2$, we need to replace Assumption 3.2 with the following assumption.

Assumption 3.10 There exist constants $\widetilde{K}_2 > 0$ and $\widetilde{\eta} \in (2, \bar{p})$ such that

$$\begin{aligned} & (x - \bar{x})^T(f(x, y) - f(\bar{x}, \bar{y})) + \frac{\widetilde{\eta} - 1}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ & \leq \widetilde{K}_2(|x - \bar{x}|^2 + |y - \bar{y}|^2), \quad \forall x, y, \bar{x}, \bar{y} \in \mathbb{R}^n. \end{aligned}$$

Many techniques used in Theorem 3.9 are applied to give the following theorem, so we omit some similar proof procedures.

Theorem 3.11 *Let Assumptions 2.1, 3.1, 3.3, and 3.10 hold. Suppose that there exists a real number $q \in (2, \bar{p})$ such that $q > (1 + \beta)\widetilde{\eta}$. Then, for any $p \in (2, \widetilde{\eta})$ and $\Delta \in (0, 1]$, we have*

$$\begin{aligned} & \mathbb{E}|x(T) - x_\Delta(T)|^p \\ & \leq C((\varphi^{-1}(\alpha(\Delta)))^{p\beta+p-q} + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\beta)/q} + \Delta^{p\gamma}) \end{aligned} \quad (3.42)$$

and

$$\begin{aligned} & \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^p \\ & \leq C((\varphi^{-1}(\alpha(\Delta)))^{p\beta+p-q} + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\beta)/q} + \Delta^{p\gamma}). \end{aligned} \quad (3.43)$$

Proof Let $e_\Delta(t)$ and ρ be the same as before. Note that for $\bar{q} \in (p, \widetilde{\eta})$, we have $q > (1 + \beta)\bar{q}$. By Itô's formula, for any $t \in [0, T]$ and $\Delta \in (0, 1]$, we get

$$\begin{aligned} & \mathbb{E}|e_\Delta(t \wedge \rho)|^p \\ & \leq \mathbb{E} \int_0^{t \wedge \rho} p|e_\Delta(s)|^{p-2} \left(e_\Delta^T(s)(f(x(s), x(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))) \right. \\ & \quad \left. + \frac{p-1}{2} |g(x(s), x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right) ds \\ & \quad + \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s) + h(x(s^-), x((s-\tau)^-)) \\ & \quad - h(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))|^p - |e_\Delta(s)|^p) ds \\ & =: H_1 + H_2. \end{aligned} \quad (3.44)$$

Note that

$$\begin{aligned} & \frac{p-1}{2} |g(x(s), x(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \\ & \leq \frac{\bar{q}-1}{2} |g(x(s), x(s-\tau)) - g(x_\Delta(s), x_\Delta(s-\tau))|^2 \\ & \quad + \frac{(\bar{q}-1)(p-1)}{2(\bar{q}-p)} |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2. \end{aligned}$$

Then we derive that

$$\begin{aligned}
H_1 &\leq \mathbb{E} \int_0^{t \wedge \rho} p |e_\Delta(s)|^{p-2} \left(e_\Delta^\top(s) (f(x(s), x(s-\tau)) - f(x_\Delta(s), x_\Delta(s-\tau))) \right. \\
&\quad \left. + \frac{\bar{q}-1}{2} |g(x(s), x(s-\tau)) - g(x_\Delta(s), x_\Delta(s-\tau))|^2 \right) ds \\
&\quad + \mathbb{E} \int_0^{t \wedge \rho} p |e_\Delta(s)|^{p-2} \left(e_\Delta^\top(s) (f(x_\Delta(s), x_\Delta(s-\tau)) \right. \\
&\quad \left. - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))) \right. \\
&\quad \left. + \frac{(\bar{q}-1)(p-1)}{2(\bar{q}-p)} |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right) ds \\
&=: H_{11} + H_{12}.
\end{aligned} \tag{3.45}$$

By Young's inequality, Assumptions 2.1, 3.10, and (3.2), we obtain that

$$\begin{aligned}
H_{11} &\leq \mathbb{E} \int_0^{t \wedge \rho} p |e_\Delta(s)|^{p-2} [\widetilde{K}_2(|x(s) - x_\Delta(s)|^2 + |x(s-\tau) - x_\Delta(s-\tau)|^2)] ds \\
&\leq C \left(\mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^p ds + \int_{-\tau}^0 |\xi(s) - \xi(\kappa(s))|^p ds \right) \\
&\leq C \left(\int_0^t \mathbb{E} |e_\Delta(s \wedge \rho)|^p ds + \Delta^{p\gamma} \right).
\end{aligned} \tag{3.46}$$

Moreover,

$$\begin{aligned}
H_{12} &\leq \mathbb{E} \int_0^{t \wedge \rho} p |e_\Delta(s)|^{p-2} \left(e_\Delta^\top(s) (f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(x_\Delta(s), x_\Delta(s-\tau))) \right. \\
&\quad \left. + \frac{(\bar{q}-1)(p-1)}{\bar{q}-p} |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2 \right) ds \\
&\quad + \mathbb{E} \int_0^{t \wedge \rho} p |e_\Delta(s)|^{p-2} \left(e_\Delta^\top(s) (f_\Delta(x_\Delta(s), x_\Delta(s-\tau)) \right. \\
&\quad \left. - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))) \right. \\
&\quad \left. + \frac{(\bar{q}-1)(p-1)}{\bar{q}-p} |g_\Delta(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))|^2 \right) ds \\
&=: H_{121} + H_{122}.
\end{aligned}$$

Similar to (3.37) and (3.38), we derive that

$$\begin{aligned}
H_{121} &\leq C \mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^{p-2} (|e_\Delta(s)|^2 \\
&\quad + |f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2 \\
&\quad + |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^2) ds \\
&\leq C \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^p + |f(x_\Delta(s), x_\Delta(s-\tau)) - f_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^p \\
&\quad + |g(x_\Delta(s), x_\Delta(s-\tau)) - g_\Delta(x_\Delta(s), x_\Delta(s-\tau))|^p) ds
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + C \mathbb{E} \int_0^T [(1 + |x_\Delta(s)|^{p\beta} + |x_\Delta(s - \tau)|^{p\beta} \\
&\quad + |\pi_\Delta(x_\Delta(s))|^{p\beta} + |\pi_\Delta(x_\Delta(s - \tau))|^{p\beta}) \\
&\quad \cdot (|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^p + |x_\Delta(s - \tau) - \pi_\Delta(x_\Delta(s - \tau))|^p)] ds \\
&\leq C \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + (\varphi^{-1}(\alpha(\Delta)))^{p\beta+p-q} \right),
\end{aligned}$$

and

$$\begin{aligned}
H_{122} &\leq C \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^p + |f_\Delta(x_\Delta(s), x_\Delta(s - \tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))|^p \\
&\quad + |g_\Delta(x_\Delta(s), x_\Delta(s - \tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))|^p) ds \\
&\leq C \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + C \mathbb{E} \int_0^T [(1 + |x_\Delta(s)|^{p\beta} + |x_\Delta(s - \tau)|^{p\beta} \\
&\quad + |\bar{x}_\Delta(s)|^{p\beta} + |\bar{x}_\Delta(s - \tau)|^{p\beta}) \\
&\quad \cdot (|x_\Delta(s) - \bar{x}_\Delta(s)|^p + |x_\Delta(s - \tau) - \bar{x}_\Delta(s - \tau)|^p)] ds \\
&\leq C \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\beta)/q} \right).
\end{aligned}$$

Thus, we get

$$H_{12} \leq C \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + (\varphi^{-1}(\alpha(\Delta)))^{p\beta+p-q} + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\beta)/q} \right). \quad (3.47)$$

In addition, we derive from Assumptions 2.1, 3.1, and Lemma 3.7 that

$$\begin{aligned}
H_2 &\leq C \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^p + |x(s) - \bar{x}_\Delta(s)|^p + |x(s - \tau) - \bar{x}_\Delta(s - \tau)|^p) ds \\
&\leq C \left(\mathbb{E} \int_0^{t \wedge \rho} |e_\Delta(s)|^p ds + \int_0^T \mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^p ds \right. \\
&\quad \left. + \int_{-\tau}^0 |\xi(s) - \xi(\kappa(s))|^p ds \right) \\
&\leq C \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{p\gamma} + \Delta \right). \quad (3.48)
\end{aligned}$$

Combining (3.44)–(3.48) together yields that

$$\begin{aligned}
\mathbb{E}|e_\Delta(t \wedge \rho)|^p &\leq C \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^p ds + (\varphi^{-1}(\alpha(\Delta)))^{p\beta+p-q} \right. \\
&\quad \left. + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\beta)/q} + \Delta^{p\gamma} \right).
\end{aligned}$$

Applying Gronwall's inequality gives that

$$\mathbb{E}|e_\Delta(T \wedge \rho)|^p \leq C((\varphi^{-1}(\alpha(\Delta)))^{p\beta+p-q} + (\alpha(\Delta))^p \Delta^{p/2} + \Delta^{(q-p\beta)/q} + \Delta^{p\gamma}).$$

We can get (3.42), (3.43) by letting $L \rightarrow \infty$ and using Lemmas 3.7, 3.8. The proof is complete. $\square$

4 Convergence in $\mathcal{L}^p$ for $0 < p < 2$

In this section, we will discuss the convergence and the rate of the convergence of the truncated EM method for (2.1) in $\mathcal{L}^p$ for $0 < p < 2$. To achieve this goal, we need to impose the following assumptions on coefficients.

Assumption 4.1 There exists a positive constant K_L such that

$$\begin{aligned} & |f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \vee |h(x, y) - h(\bar{x}, \bar{y})| \\ & \leq K_L(|x - \bar{x}| + |y - \bar{y}|), \quad \forall x, y, \bar{x}, \bar{y} \in \mathbb{R}^n, \end{aligned}$$

with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq L$.

Assumption 4.2 There exist constants $K_5 > 0$, $K_6 \geq 0$, and $\sigma > 2$ such that

$$\begin{aligned} & 2x^\top f(x, y) + |g(x, y)|^2 + \lambda(2x^\top h(x, y) + |h(x, y)|^2) \\ & \leq K_5(1 + |x|^2 + |y|^2) - K_6|x|^\sigma + K_6|y|^\sigma, \quad \forall x, y \in \mathbb{R}^n. \end{aligned}$$

We could get the following lemma in the similar way as Lemma 3.4 was proved.

Lemma 4.3 *Let Assumptions 4.1 and 4.2 hold. Then SDDEwPJs (2.1) has a unique global solution $x(t)$ which satisfies*

$$\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^2 < \infty, \quad \forall T > 0. \quad (4.1)$$

In the previous section, the jump coefficient h is linear growth, but in Assumptions 4.1 and 4.2, h is allowed to grow super-linearly. Thus, we need to truncate all the three coefficients. In the same way in Section 3, we first choose a strictly increasing continuous function $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\varphi(r) \rightarrow \infty$ as $r \rightarrow \infty$ and

$$\sup_{|x| \vee |y| \leq r} (|f(x, y)| \vee |g(x, y)| \vee |h(x, y)|) \leq \varphi(r), \quad \forall r \geq 1.$$

Choose K_0 and $\alpha: (0, 1] \rightarrow (0, \infty)$ as in (3.4). For a given step size $\Delta \in (0, 1]$, the truncated mapping π_Δ is defined as (3.5), and the truncated functions are define as follows:

$$\begin{aligned} f_\Delta(x, y) &= f(\pi_\Delta(x), \pi_\Delta(y)), \\ g_\Delta(x, y) &= g(\pi_\Delta(x), \pi_\Delta(y)), \quad \forall x, y \in \mathbb{R}^n. \\ h_\Delta(x, y) &= h(\pi_\Delta(x), \pi_\Delta(y)), \end{aligned}$$

It is easy to see that

$$|f_{\Delta}(x, y)| \vee |g_{\Delta}(x, y)| \vee |h_{\Delta}(x, y)| \leq \varphi(\varphi^{-1}(\alpha(\Delta))) \leq \alpha(\Delta), \quad \forall x, y \in \mathbb{R}^n. \quad (4.2)$$

Moreover, if Assumption 4.2 holds, then it holds for any $\Delta \in (0, 1]$, $x, y \in \mathbb{R}^n$ that

$$\begin{aligned} & 2x^T f_{\Delta}(x, y) + |g_{\Delta}(x, y)|^2 + \lambda(2x^T h_{\Delta}(x, y) + |h_{\Delta}(x, y)|^2) \\ & \leq 3K_5 \left(\frac{1}{\varphi^{-1}(\alpha(1))} \vee 1 \right) (1 + |x|^2 + |y|^2) - K_6 |\pi_{\Delta}(x)|^{\sigma} + K_6 |\pi_{\Delta}(y)|^{\sigma}. \end{aligned} \quad (4.3)$$

Since $|h_{\Delta}(x, y)| \leq \alpha(\Delta)$, similar to Lemma 3.5, we have the following lemma.

Lemma 4.4 *Let Assumptions 4.1 and 4.2 hold. For any $\Delta \in (0, 1]$ and $t \in [0, T]$, we have*

$$\mathbb{E}|x_{\Delta}(t) - \bar{x}_{\Delta}(t)|^{\tilde{p}} \leq c_{\tilde{p}}(\alpha(\Delta))^{\tilde{p}} \Delta, \quad \tilde{p} \geq 2, \quad (4.4)$$

and

$$\mathbb{E}|x_{\Delta}(t) - \bar{x}_{\Delta}(t)|^{\tilde{p}} \leq c_{\tilde{p}}(\alpha(\Delta))^{\tilde{p}} \Delta^{\tilde{p}/2}, \quad 0 < \tilde{p} < 2, \quad (4.5)$$

where $c_{\tilde{p}}$ is a positive constant which is independent of Δ . As a result,

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|x_{\Delta}(t) - \bar{x}_{\Delta}(t)|^{\tilde{p}} = 0, \quad \tilde{p} > 0. \quad (4.6)$$

The following lemma states that the numerical solution is bounded in mean square.

Lemma 4.5 *Let Assumptions 4.1 and 4.2 hold. Then we have*

$$\sup_{0 < \Delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}|x_{\Delta}(t)|^2 \leq C, \quad \forall T > 0. \quad (4.7)$$

Proof Since the proof is similar to that of Lemma 3.6, we only highlight how to deal with the jump term. By Itô's formula and (4.3), we derive that, for any $\Delta \in (0, 1]$ and $t \in [0, T]$,

$$\begin{aligned} & \mathbb{E}|x_{\Delta}(t)|^2 \\ & \leq \|\xi\|^2 + \mathbb{E} \int_0^t (2\bar{x}_{\Delta}^T(s) f_{\Delta}(\bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s - \tau)) + |g_{\Delta}(\bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s - \tau))|^2) ds \\ & \quad + \lambda \mathbb{E} \int_0^t (2\bar{x}_{\Delta}^T(s) h_{\Delta}(\bar{x}_{\Delta}(s^-), \bar{x}_{\Delta}((s - \tau)^-)) \\ & \quad + |h_{\Delta}(\bar{x}_{\Delta}(s^-), \bar{x}_{\Delta}((s - \tau)^-))|^2) ds \\ & \quad + \mathbb{E} \int_0^t (2(x_{\Delta}(s) - \bar{x}_{\Delta}(s))^T f_{\Delta}(\bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s - \tau)) \\ & \quad + 2\lambda(x_{\Delta}(s) - \bar{x}_{\Delta}(s))^T h_{\Delta}(\bar{x}_{\Delta}(s^-), \bar{x}_{\Delta}((s - \tau)^-))) ds \end{aligned}$$

$$\begin{aligned}
&\leq \|\xi\|^2 + \mathbb{E} \int_0^t \left(3K_5 \left(\frac{1}{\varphi^{-1}(\alpha(1))} \vee 1 \right) (1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s-\tau)|^2) \right. \\
&\quad \left. - K_6 |\pi_\Delta(\bar{x}_\Delta(s))|^\sigma + K_6 |\pi_\Delta(\bar{x}_\Delta(s-\tau))|^\sigma \right) ds \\
&\quad + 2(\lambda + 1) \mathbb{E} \int_0^t (x_\Delta(s) - \bar{x}_\Delta(s))^T (f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau)) \\
&\quad + h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))) ds.
\end{aligned}$$

Moreover, by (3.4) and Lemma 4.4, we have

$$\begin{aligned}
&\mathbb{E} \int_0^t (x_\Delta(s) - \bar{x}_\Delta(s))^T (f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau)) + h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))) ds \\
&\leq \mathbb{E} \int_0^t |x_\Delta(s) - \bar{x}_\Delta(s)| (|f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))| \\
&\quad + |h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))|) ds \\
&\leq 2\alpha(\Delta) \int_0^t \mathbb{E} |x_\Delta(s) - \bar{x}_\Delta(s)| ds \\
&\leq 2T(\alpha(\Delta))^2 \Delta^{1/2} \\
&\leq 2TK_0^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E} |x_\Delta(t)|^2 &\leq C \left(1 + \int_0^t (\mathbb{E} |\bar{x}_\Delta(s)|^2 + \mathbb{E} |\bar{x}_\Delta(s-\tau)|^2) ds \right) \\
&\leq C \left(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |x_\Delta(u)|^2 ds \right).
\end{aligned}$$

We could observe that the right-hand-side term is nondecreasing in t , and hence,

$$\sup_{0 \leq u \leq t} \mathbb{E} |x_\Delta(u)|^2 \leq C \left(1 + \int_0^t \sup_{0 \leq u \leq s} \mathbb{E} |x_\Delta(u)|^2 ds \right).$$

An application of Gronwall's inequality yields that

$$\sup_{0 \leq u \leq T} \mathbb{E} |x_\Delta(u)|^2 \leq C,$$

where C is independent of Δ . We complete the proof. $\square$

Since the boundedness of the numerical solution, the estimates of stopping times in Lemma 3.8 still hold. Now, we are going to state the convergence of the truncated EM method for SDDEwPJs in $\mathcal{L}^p$ for $0 < p < 2$.

Theorem 4.6 *Let Assumptions 2.1, 4.1, and 4.2 hold. Then, for any $p \in (0, 2)$, we have*

$$\lim_{\Delta \rightarrow 0} \mathbb{E} |x(T) - x_\Delta(T)|^p = 0, \quad (4.8)$$

and

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|x(T) - \bar{x}_\Delta(T)|^p = 0. \quad (4.9)$$

Proof Let $e_\Delta(t) = x(t) - x_\Delta(t)$ for $t \geq 0$ and $\Delta \in (0, 1]$. Define $\rho_{\Delta, L} = \tau_L \wedge \tau_{\Delta, L}$. We write $\rho_{\Delta, L} = \rho$ for simplicity. Obviously,

$$\mathbb{E}|e_\Delta(T)|^p = \mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho > T\}}) + \mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho \leq T\}}). \quad (4.10)$$

Let $\delta > 0$ be arbitrary. By Young's inequality, we have

$$u^p v = (\delta u^2)^{p/2} \left(\frac{v^{2/(2-p)}}{\delta^{p/(2-p)}} \right)^{(2-p)/2} \leq \frac{p\delta}{2} u^2 + \frac{2-p}{2\delta^{p/(2-p)}} v^{2/(2-p)}, \quad \forall u, v > 0.$$

Hence,

$$\mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho \leq T\}}) \leq \frac{p\delta}{2} \mathbb{E}|e_\Delta(T)|^2 + \frac{2-p}{2\delta^{p/(2-p)}} \mathbb{P}\{\rho \leq T\}. \quad (4.11)$$

Applying Lemmas 4.3 and 4.5 yields that

$$\mathbb{E}|e_\Delta(T)|^2 \leq C. \quad (4.12)$$

By Lemma 3.8, we have

$$\mathbb{P}(\rho \leq T) \leq \mathbb{P}(\tau_L \leq T) + \mathbb{P}(\tau_{\Delta, L} \leq T) \leq \frac{C}{L^2}. \quad (4.13)$$

Inserting (4.12) and (4.13) into (4.11) yields that

$$\mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho \leq T\}}) \leq \frac{Cp\delta}{2} + \frac{C(2-p)}{2L^2\delta^{p/(2-p)}}. \quad (4.14)$$

Let ε be arbitrary. We choose δ sufficiently small such that

$$\frac{Cp\delta}{2} \leq \frac{\varepsilon}{3},$$

and choose L sufficiently large such that

$$\frac{C(2-p)}{2L^2\delta^{p/(2-p)}} \leq \frac{\varepsilon}{3}.$$

Thus,

$$\mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho \leq T\}}) \leq \frac{2\varepsilon}{3}. \quad (4.15)$$

Moreover, we could use the similar technique in the proof of [9, Theorem 3.5] to prove that

$$\mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho > T\}}) \leq \frac{\varepsilon}{3}. \quad (4.16)$$

Combining (4.10), (4.15), and (4.16) together, we have

$$\mathbb{E}|e_\Delta(T)|^p \leq \varepsilon.$$

Hence, we get the desired result (4.8). Then combining (4.8) and Lemma 4.4 yields (4.9). We complete the proof. $\square$

Next, in order to estimate the rate of the convergence at time T , we have to impose an extra condition.

Assumption 4.7 There exists a positive constant K_7 such that

$$\begin{aligned} & 2(x - \bar{x})^T(f(x, y) - f(\bar{x}, \bar{y})) + |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ & \quad + 2\lambda(x - \bar{x})^T(h(x, y) - h(\bar{x}, \bar{y})) + \lambda|h(x, y) - h(\bar{x}, \bar{y})|^2 \\ & \leq K_7(|x - \bar{x}|^2 + |y - \bar{y}|^2) - U(x, \bar{x}) + U(y, \bar{y}), \quad \forall x, y, \bar{x}, \bar{y} \in \mathbb{R}^n. \end{aligned}$$

Here, $U(\cdot, \cdot)$ is defined as before.

Lemma 4.8 Let Assumptions 2.1, 4.1, 4.2, and 4.7 hold. Let $\Delta \in (0, 1)$ be sufficiently small such that $\varphi^{-1}(\alpha(\Delta)) \geq L \vee \|\xi\|$. Then we have

$$\mathbb{E}|x(T \wedge \rho_{\Delta, L}) - x_\Delta(T \wedge \rho_{\Delta, L})|^2 \leq C((\alpha(\Delta))^2 \Delta^{1/2} + \Delta^{2\gamma}), \quad (4.17)$$

where $\rho_{\Delta, L} := \tau_L \wedge \tau_{\Delta, L}$ and $\tau_L, \tau_{\Delta, L}$ are defined in Lemma 3.8.

Proof Let $e_\Delta(t) = x(t) - x_\Delta(t)$ for $t \geq 0$ and $\Delta \in (0, 1]$. We write $\rho_{\Delta, L} = \rho$ for simplicity. For $0 \leq s \leq t \wedge \rho$, we observe that

$$|x(s)| \vee |x(s - \tau)| \vee |\bar{x}_\Delta(s)| \vee |\bar{x}_\Delta(s - \tau)| \leq L \leq \varphi^{-1}(\alpha(\Delta)).$$

Recalling the definition of f_Δ, g_Δ , and h_Δ , we obtain for $0 \leq s \leq t \wedge \rho$ that

$$\begin{aligned} f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) &= f(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)), \\ g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) &= g(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)), \\ h_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)) &= h(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau)), \end{aligned}$$

and

$$\begin{aligned} & |f(x(s), x(s - \tau))| \vee |f(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))| \\ & \vee |h(x(s), x(s - \tau))| \vee |h(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))| \leq \alpha(\Delta). \end{aligned}$$

By Itô's formula and Assumption 4.7, for any $t \in [0, T]$, we have

$$\begin{aligned} & \mathbb{E}|e_\Delta(t \wedge \rho)|^2 \\ & \leq \mathbb{E} \int_0^{t \wedge \rho} (2(x(s) - \bar{x}_\Delta(s))^T(f(x(s), x(s - \tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))) \\ & \quad + |g(x(s), x(s - \tau)) - g_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s - \tau))|^2 \\ & \quad + 2\lambda(x(s) - \bar{x}_\Delta(s))^T(h(x(s^-), x((s - \tau)^-)) - h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s - \tau)^-))) \\ & \quad + \lambda|h(x(s^-), x((s - \tau)^-)) - h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s - \tau)^-))|^2) ds \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \int_0^{t \wedge \rho} 2(\bar{x}_\Delta(s) - x_\Delta(s))^T (f(x(s), x(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))) ds \\
& + \mathbb{E} \int_0^{t \wedge \rho} 2\lambda(\bar{x}_\Delta(s) - x_\Delta(s))^T (h(x(s^-), x((s-\tau)^-)) \\
& - h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))) ds \\
& \leq \mathbb{E} \int_0^{t \wedge \rho} K_7(|x(s) - \bar{x}_\Delta(s)|^2 + |x(s-\tau) - \bar{x}_\Delta(s-\tau)|^2) ds \\
& + \mathbb{E} \int_0^{t \wedge \rho} (-U(x(s), \bar{x}_\Delta(s)) + U(x(s-\tau), \bar{x}_\Delta(s-\tau))) ds \\
& + \mathbb{E} \int_0^{t \wedge \rho} 2|\bar{x}_\Delta(s) - x_\Delta(s)| |f(x(s), x(s-\tau)) - f_\Delta(\bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau))| ds \\
& + \mathbb{E} \int_0^{t \wedge \rho} 2\lambda|\bar{x}_\Delta(s) - x_\Delta(s)| |h(x(s^-), x((s-\tau)^-)) \\
& - h_\Delta(\bar{x}_\Delta(s^-), \bar{x}_\Delta((s-\tau)^-))| ds \\
& =: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

By Assumption 2.1 and Lemma 4.4, similar to the proof of Theorem 3.9, we derive that

$$\begin{aligned}
J_1 & \leq 4K_7 \int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + C((\alpha(\Delta))^2 \Delta + \Delta^{2\gamma}), \\
J_2 & \leq \int_{-\tau}^0 U(\xi(s), \xi(\kappa(s))) ds \leq \int_{-\tau}^0 \kappa_b |\xi(s) - \xi(\kappa(s))|^2 ds \leq \tau \kappa_b \bar{K}^2 \Delta^{2\gamma}, \\
J_3 & \leq 4\alpha(\Delta) \mathbb{E} \int_0^{t \wedge \rho} |x_\Delta(s) - \bar{x}_\Delta(s)| ds \leq C(\alpha(\Delta))^2 \Delta^{1/2}, \\
J_4 & \leq C(\alpha(\Delta))^2 \Delta^{1/2}.
\end{aligned}$$

These imply that

$$\mathbb{E}|e_\Delta(t \wedge \rho)|^2 \leq C \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \rho)|^2 ds + (\alpha(\Delta))^2 \Delta^{1/2} + \Delta^{2\gamma} \right).$$

The required assertion follows by the Gronwall inequality. $\square$

Theorem 4.9 *Let Assumptions 2.1, 4.1, 4.2, and 4.7 hold. Let $p \in (0, 2)$, and for any sufficiently small $\Delta \in (0, 1)$, assume that there exists a positive constant c_2 such that*

$$\alpha(\Delta) \geq \varphi(c_2[(\alpha(\Delta))^p \Delta^{p/4}] \vee \Delta^{p\gamma})^{-1/(2-p)}. \quad (4.18)$$

Then, for any $T > 0$, we have

$$\mathbb{E}|x(T) - x_\Delta(T)|^p \leq C([(\alpha(\Delta))^p \Delta^{p/4}] \vee [\Delta^{p\gamma}]), \quad (4.19)$$

and

$$\mathbb{E}|x(T) - \bar{x}_\Delta(T)|^p \leq C([(\alpha(\Delta))^p \Delta^{p/4}] \vee [\Delta^{p\gamma}]). \quad (4.20)$$

Proof We use the notations of $e_\Delta(t)$ and $\rho_{\Delta,L}$ as before. We write $\rho_{\Delta,L} = \rho$ for simplicity. By (4.10) and (4.14), one can see that

$$\begin{aligned}\mathbb{E}|e_\Delta(T)|^p &= \mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho > T\}}) + \mathbb{E}(|e_\Delta(T)|^p \mathbf{I}_{\{\rho \leq T\}}) \\ &\leq \mathbb{E}|e_\Delta(T \wedge \rho)|^p + \frac{Cp\delta}{2} + \frac{C(2-p)}{2L^2\delta^{p/(2-p)}},\end{aligned}$$

for any $\Delta \in (0, 1)$, $L > \|\xi\|$, and $\delta > 0$. Choosing

$$\delta = [(\alpha(\Delta))^p \Delta^{p/4}] \vee [\Delta^{p\gamma}], \quad L = c_2([(\alpha(\Delta))^p \Delta^{p/4}] \vee \Delta^{p\gamma})^{-1/(2-p)},$$

we obtain

$$\mathbb{E}|e_\Delta(T)|^p \leq \mathbb{E}|e_\Delta(T \wedge \rho)|^p + C([(\alpha(\Delta))^p \Delta^{p/4}] \vee [\Delta^{p\gamma}]).$$

By condition (4.18), we derive that

$$\varphi^{-1}(\alpha(\Delta)) \geq c_2([(\alpha(\Delta))^p \Delta^{p/4}] \vee \Delta^{p\gamma})^{-1/(2-p)} = L.$$

Using Lemma 4.8, one has

$$\begin{aligned}\mathbb{E}|x(T) - x_\Delta(T)|^p &\leq (\mathbb{E}|x(T) - x_\Delta(T)|^2)^{p/2} \\ &\leq C([(\alpha(\Delta))^2 \Delta^{1/2}] \vee [\Delta^{2\gamma}])^{p/2} \\ &\leq C([(\alpha(\Delta))^p \Delta^{p/4}] \vee [\Delta^{p\gamma}]).\end{aligned}$$

Combining Lemma 4.4 and (4.19) together, we can derive (4.20). We complete the proof. $\square$

Remark 3 If we impose an additional condition: assume that there exist constants $K_8 > 0$ and $\bar{\beta} \in [0, 1)$ such that

$$\begin{aligned}&|f(x, y) - f(\bar{x}, \bar{y})| \vee |h(x, y) - h(\bar{x}, \bar{y})| \\ &\leq K_8(1 + |x|^{\bar{\beta}} + |y|^{\bar{\beta}} + |\bar{x}|^{\bar{\beta}} + |\bar{y}|^{\bar{\beta}})(|x - \bar{x}| + |y - \bar{y}|)\end{aligned}\quad (4.21)$$

for any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, then we could obtain better convergence rate. However, our main result can cover more equations without this condition (4.21).

5 Example

In this section, we give an example to illustrate our theories. Consider the super-linear scalar SDDEwPJs

$$\begin{aligned}dx(t) &= \left(-5x^3(t) + \frac{1}{8}|x(t-\tau)|^{5/4} + 2x(t)\right)dt \\ &\quad + \left(\frac{1}{2}|x(t)|^{3/2} + x(t-\tau)\right)dB(t) + (x(t^-) + x((t-\tau)^-))dN(t),\end{aligned}\quad (5.1)$$

with the initial value $\xi = \{x(\theta) : -\tau \leq \theta \leq 0\}$ which satisfies Assumption 2.1. Here, $B(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with intensity $\lambda = 0.2$.

Now, we are verifying Assumptions 3.1–3.3.

It is easy to see that

$$\begin{aligned} & |f(x, y) - f(\bar{x}, \bar{y})| \vee |g(x, y) - g(\bar{x}, \bar{y})| \\ &= \left| \left(-5x^3 + \frac{1}{8}|y|^{5/4} + 2x \right) - \left(-5\bar{x}^3 + \frac{1}{8}|\bar{y}|^{5/4} + 2\bar{x} \right) \right| \\ &\quad \vee \left| \left(\frac{1}{2}|x|^{3/2} + y \right) - \left(\frac{1}{2}|\bar{x}|^{3/2} + \bar{y} \right) \right| \\ &\leq 10(1 + |x|^2 + |y|^2 + |\bar{x}|^2 + |\bar{y}|^2)(|x - \bar{x}| + |y - \bar{y}|) \end{aligned} \quad (5.2)$$

and

$$|h(x, y) - h(\bar{x}, \bar{y})| = |(x + y) - (\bar{x} + \bar{y})| \leq 10(|x - \bar{x}| + |y - \bar{y}|). \quad (5.3)$$

Hence, Assumption 3.1 is satisfied with $\beta = 2$. Moreover, we can see that

$$\begin{aligned} & (x - \bar{x})^T (f(x, y) - f(\bar{x}, \bar{y})) \\ &= 5(x - \bar{x})^2(-(x^2 + x\bar{x} + \bar{x}^2)) + 2(x - \bar{x})^2 + \frac{1}{8}(x - \bar{x})(|y|^{5/4} - |\bar{y}|^{5/4}) \\ &\leq 5(x - \bar{x})^2 \left(-\frac{1}{2}(x^2 + \bar{x}^2) \right) + 3(x - \bar{x})^2 + \frac{25}{256}|y - \bar{y}|^2(|y|^{1/4} + |\bar{y}|^{1/4})^2 \\ &\leq -\frac{5}{2}|x - \bar{x}|^2(|x|^2 + |\bar{x}|^2) + 3|x - \bar{x}|^2 \\ &\quad + \frac{25}{64}|y - \bar{y}|^2 + \frac{25}{128}|y - \bar{y}|^2(|y|^2 + |\bar{y}|^2). \end{aligned} \quad (5.4)$$

Let $\bar{\eta} = 3$. In the same way, we can derive that

$$\begin{aligned} \frac{\bar{\eta} - 1}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 &= \left| \left(\frac{1}{2}|x|^{3/2} + y \right) - \left(\frac{1}{2}|\bar{x}|^{3/2} + \bar{y} \right) \right|^2 \\ &\leq \frac{1}{2} ||x|^{3/2} - |\bar{x}|^{3/2}|^2 + 2|y - \bar{y}|^2 \\ &\leq \frac{9}{8}|x - \bar{x}|^2(|x|^{1/2} + |\bar{x}|^{1/2})^2 + 2|y - \bar{y}|^2 \\ &\leq \frac{9}{2}|x - \bar{x}|^2 + \frac{9}{4}|x - \bar{x}|^2(|x|^2 + |\bar{x}|^2) + 2|y - \bar{y}|^2. \end{aligned} \quad (5.5)$$

Combining (5.4) and (5.5) gives that

$$\begin{aligned} & (x - \bar{x})^T (f(x, y) - f(\bar{x}, \bar{y})) + \frac{\bar{\eta} - 1}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ &\leq 8(|x - \bar{x}|^2 + |y - \bar{y}|^2) - \frac{1}{4}|x - \bar{x}|^2(|x|^2 + |\bar{x}|^2) + \frac{1}{4}|y - \bar{y}|^2(|y|^2 + |\bar{y}|^2). \end{aligned}$$

Therefore, Assumption 3.2 is satisfied with

$$U(x, \bar{x}) = \frac{1}{4} |x - \bar{x}|^2 (|x|^2 + |\bar{x}|^2).$$

For $\bar{p} > 3$, we get that

$$\begin{aligned} x^T f(x, y) + \frac{\bar{p} - 1}{2} |g(x, y)|^2 &= x \left(-5x^3 + \frac{1}{8} |y|^{5/4} + 2x \right) + \frac{\bar{p} - 1}{2} \left| \frac{1}{2} |x|^{3/2} + y \right|^2 \\ &\leq C(1 + |x|^2 + |y|^2). \end{aligned}$$

Thus, Assumption 3.3 is satisfied as well.

Additionally, it is easy to see that

$$\sup_{|x| \vee |y| \leq r} (|f(x, y)| \vee |g(x, y)| \vee |h(x, y)|) \leq 5r^3, \quad \forall r \geq 1.$$

Hence, we can choose $\varphi(r) = 5r^3$. This means $\varphi^{-1}(r) = (r/5)^{1/3}$. In order for $q \geq (1 + \beta)/\varepsilon$ to hold, we set $\bar{p} = 26$ such that Assumption 3.3 be satisfied. Then

$$q \in ((1 + \beta)\bar{\eta}, \bar{p}) \implies q \in (9, 26).$$

We choose $q = 25$. Moreover, let $\varepsilon = 1/8$. Then $q \geq (1 + \beta)/\varepsilon$ is satisfied. So $\alpha(\Delta) = K_0 \Delta^{-1/8}$. By Theorem 3.9, we have

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \leq C \begin{cases} \Delta^{2\gamma}, & \forall \gamma \in \left(0, \frac{3}{8}\right), \\ \Delta^{3/4}, & \forall \gamma \in \left[\frac{3}{8}, 1\right], \end{cases}$$

which means that the $\mathcal{L}^2$ -convergence rate of the truncated EM method for SDDEwPJs (5.1) is $(2\gamma) \wedge \frac{3}{4}$.

On the other hand, let us verify Assumptions 4.1, 4.2, and 4.7. By (5.2) and (5.3), we find that Assumption 4.1 is satisfied. In addition, we have

$$\begin{aligned} &2x^T f(x, y) + |g(x, y)|^2 + \lambda(2x^T h(x, y) + |h(x, y)|^2) \\ &= 2x \left(-5x^3 + \frac{1}{8} |y|^{5/4} + 2x \right) + \left| \frac{1}{2} |x|^{3/2} + y \right|^2 + \lambda(2x(x + y) + |x + y|^2) \\ &\leq C(1 + |x|^2 + |y|^2). \end{aligned}$$

Thus, Assumption 4.2 is satisfied. By (5.4) and (5.5), we obtain

$$\begin{aligned} &2(x - \bar{x})^T (f(x, y) - f(\bar{x}, \bar{y})) + |g(x, y) - g(\bar{x}, \bar{y})|^2 \\ &\quad + 2\lambda(x - \bar{x})^T (h(x, y) - h(\bar{x}, \bar{y})) + \lambda|h(x, y) - h(\bar{x}, \bar{y})|^2 \\ &\leq 11(|x - \bar{x}|^2 + |y - \bar{y}|^2) - \frac{11}{4} |x - \bar{x}|^2 (|x|^2 + |\bar{x}|^2) + \frac{11}{4} |y - \bar{y}|^2 (|y|^2 + |\bar{y}|^2) \\ &\quad + 5\lambda|x - \bar{x}|^2 + 3\lambda|y - \bar{y}|^2 \\ &\leq 12(|x - \bar{x}|^2 + |y - \bar{y}|^2) - \frac{11}{4} |x - \bar{x}|^2 (|x|^2 + |\bar{x}|^2) + \frac{11}{4} |y - \bar{y}|^2 (|y|^2 + |\bar{y}|^2). \end{aligned}$$

Hence, Assumption 4.7 is satisfied with

$$U(x, \bar{x}) = \frac{11}{4} |x - \bar{x}|^2 (|x|^2 + |\bar{x}|^2).$$

Choose $\varphi(r) = 5r^3$ and $c_2 = (1/5)^{1/3}$. Let $0 < p < 2/(12\gamma + 1)$ and define

$$\alpha(\Delta) = \Delta^{-\varepsilon}, \quad \forall \varepsilon \in \left[\frac{3p}{8(1+p)} \vee \frac{3p\gamma}{2-p}, \frac{1}{4} \right].$$

Then condition (4.18) is satisfied, that is,

$$\alpha(\Delta) \geq \varphi(c_2((\alpha(\Delta))^p \Delta^{p/4} \vee \Delta^{p\gamma})^{-1/(2-p)}).$$

By Theorem 4.9, we have

$$\mathbb{E}|x(T) - x_\Delta(T)|^p \leq C \Delta^{p((\frac{1}{4} - \varepsilon) \wedge \gamma)},$$

which means that the $\mathcal{L}^p$ -convergence ($p \in (0, 2)$) rate of the truncated EM method for SDDEwPJs (5.1) is $p((\frac{1}{4} - \varepsilon) \wedge \gamma)$.

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