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RESEARCH ARTICLE

Abundant Resonant Behaviors of Soliton Solutions to the (3+1)-dimensional BKP-Boussinesq Equation

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Abstract Resonant phenomena have been observed and investigated in various situations, such as plasma experiments, the maritime security and the microtubule in cell physiology. In this paper, abundant resonant behaviors are studied for the (3+1)-dimensional BKP-Boussinesq equation. We mainly discuss the resonant two- and three-soliton solutions in the (x, y)-plane and (x, z)-plane. The characteristics are given for the kink soliton waves, including expressions, maximums, minimums and velocities. The kink soliton waves in the (x, y)-plane are parallel, and the fusion or fission may occur. The kink soliton waves in the (x, z)-plane are not parallel and the resonant phenomena among them are more complicated.

Keywords Resonant soliton solutions, the (3+1)-dimensional BKP-Boussinesq equation, fusion, fission

MSC2020 35A25, 37K10

1 Introduction

The observation and relevant application of resonant phenomena have been discussed in various situations, such as plasma experiments [40], the maritime security [27], the microtubule in cell physiology [25] and the underdamped Josephson junctions [26]. Resonances of two solitons have been reported theoretically [8, 28]. In one-dimensional space, the two solitons to the Sawada–Kotera equation near the resonant state interact with each other to emit or absorb a third soliton [8]. The resonances of two solitons to the Kadomtsev–Petviashvili

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(KP) equation in superfluid helium films have been investigated based on the asymptotic behavior of solitons [28]. Furthermore, the existence of resonant phenomena among three solitons has been pointed out by using the Hirota bilinear method [34]. The Hirota bilinear method is widely used to construct soliton solutions to nonlinear evolution equations (NLEEs) [2,4,7,15,17,36,39]. The linear superposition principle provides an efficient way to obtain multi-exponential wave solutions to Hirota bilinear equations [22,41].

The B-type KP equation (BKP) [32] is written as

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} = 0, \tag{1.1}$$

which can be used to simulate the evolution process of quasi-one dimensional shallow water waves [11]. As a generalization of Eq. (1.1), the (3+1)-dimensional BKP-Boussinesq equation [32] is given as

$$u_{ty} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xz} + u_{tt} = 0, \tag{1.2}$$

in which the term u_{tt} has significant impact on the phase shift and dispersion relation [32]. Eq. (1.2) can describe the propagation of nonlinear lattice waves or long waves in shallow water [9]. The bilinear form of Eq. (1.2) has been given by virtue of Bell polynomials [30]. Lie group analysis, Bäcklund transformation and conservation laws have been obtained [6,9,35].

Searching for exact solutions to NLEEs has attracted much attention [1,3, 5,10,12,14,16,29,31,33,37,38]. Recently, N-soliton solutions to nonlinear integrable equations have been systematically studied by the Hirota bilinear method for both (1+1)-dimensional integrable equations and (2+1)-dimensional integrable equations [18–21,23]. An algorithm to check the Hirota conditions has been proposed [18, 20, 21]. Different from the soliton solutions, the resonant soliton solutions are soliton solutions without phase shifts. The parameterizations of the constants associated with the wave variables play a vital role in solving the resonant condition. In this paper, we aim to extend the parameterizations and derive the resonant soliton solutions to the (3+1)-dimensional BKP-Boussinesq equation. Compared with the soliton solutions in Ref. [32], the resonant soliton solutions can be used to describe resonant phenomena among soliton waves. Other exact solutions to the (3+1)-dimensional BKP-Boussinesq equation have been deduced, which have their own special properties. The lump solutions in Ref. [13] are a kind of rational function solutions and algebraically decay in all directions. The lump-kink solutions in Ref. [35] can be used to describe the interaction phenomena between lump waves and kink waves. The exact solutions in Ref. [6] were constructed by using the Tanh method on reduced equations. We hope that the resonant phenomena among soliton waves discussed here will be of value for the investigation of nonlinear dynamics.

This paper is organized as follows. In Sect. 2, we will introduce the general method to construct the resonant soliton solutions to NLEEs. In Sect. 3, the resonant soliton solutions to the (3+1)-dimensional BKP-Boussinesq equation will be derived based on the Hirota bilinear form. Finally, some conclusions will be given in Sect. 4.

2 Preliminaries

We consider the following bilinear equation

$$F(D_x, D_y, D_z, D_t)f \cdot f = 0, \tag{2.1}$$

where $D_x,\ D_y,\ D_z$ and D_t are the Hirota bilinear operators [7], and F is an even polynomial with $F(0,0,0,0) = 0$.

The N-wave function $f = \sum_{i=1}^{N} \mu_i e^{k_{i1}x + k_{i2}y + k_{i3}z + k_{i4}t}$ with μ_i and k_{im} $(1 \leq i \leq N,\ 1 \leq m \leq 4)$ as all constants solves Eq. (2.1) if and only if [24]

$$F(k_{i1} - k_{j1}, k_{i2} - k_{j2}, k_{i3} - k_{j3}, k_{i4} - k_{j4}) = 0, \quad 1 \leq i < j \leq N. \tag{2.2}$$

Eq. (2.2) together with

$$F(k_{i1}, k_{i2}, k_{i3}, k_{i4}) = 0, \quad 1 \leq i \leq N, \tag{2.3}$$

generates the resonant soliton solutions to the corresponding NLEEs under the transformations $u = \alpha\big(\ln f\big)_x$ or $u = \alpha\big(\ln f\big)_{xx}$, where α is a constant.

Using the parameter k_i, we firstly introduce the parameterizations of constants k_{i1}, k_{i2}, k_{i3} and k_{i4} $(1 \leq i \leq N)$ as

$$\begin{aligned} &k_{i1} = k_i, \quad k_{i2} = \sum_{n_1=-N_1}^{N_1} b_{n_1} k_i^{n_1}, \\ &k_{i3} = \sum_{n_1=-N_1}^{N_1} c_{n_1} k_i^{n_1}, \quad k_{i4} = \sum_{n_1=-N_1}^{N_1} d_{n_1} k_i^{n_1}, \quad 1 \leq i \leq N, \end{aligned} \tag{2.4}$$

where $b_{n_1},\ c_{n_1}$ and d_{n_1} are constants, and N_1 is a positive integer. Secondly, substituting Eq. (2.4) into Eqs. (2.2) and (2.3), we set the coefficients of each power of the variables k_i and k_j be zero and obtain the relations among $b_{n_1},\ c_{n_1}$ and d_{n_1}. Thus, $f = \sum_{i=1}^{N} \mu_i e^{k_{i1}x + k_{i2}y + k_{i3}z + k_{i4}t}$ solves Eq. (2.1) if the relations among $b_{n_1},\ c_{n_1}$ and d_{n_1} are satisfied, and resonant soliton solutions to the corresponding NLEEs can be derived.

3 Resonant Soliton Solutions to the (3+1)-dimensional BKP-Boussinesq Equation

Under the transformation $u(x,y,z,t) = 2\big[\ln f(x,y,z,t)\big]_x$, Eq. (1.2) is transformed into

$$(D_t D_y - D_x^3 D_y + 3D_x D_z + D_t^2)f \cdot f = 0, \tag{3.1}$$

the polynomial associated with which is

$$F(x,y,z,t)=ty-x^3y+3xz+t^2, \tag{3.2}$$

and the corresponding conditions (2.2) and (2.3) turn out to be

$$\begin{aligned}&F(k_{i1}-k_{j1},k_{i2}-k_{j2},k_{i3}-k_{j3},k_{i4}-k_{j4})\\&\quad=(k_{i4}-k_{j4})(k_{i2}-k_{j2})-(k_{i1}-k_{j1})^3(k_{i2}-k_{j2})\\&\qquad+3(k_{i1}-k_{j1})(k_{i3}-k_{j3})+(k_{i4}-k_{j4})^2=0,\end{aligned} \tag{3.3}$$

and

$$F(k_{i1},k_{i2},k_{i3},k_{i4})=k_{i4}k_{i2}-k_{i1}^3k_{i2}+3k_{i1}k_{i3}+k_{i4}^2=0. \tag{3.4}$$

We substitute Eq. (2.4) into Eq. (3.3) and set all the coefficients of each power of the variables k_i and k_j be zero. We focus on the following coefficients of the variables k_i and k_j,

$$k_i^{N_1+2}k_j:3b_{N_1}, \tag{3.5}$$

$$k_i^{N_1+1}k_j:3b_{N_1-1}, \tag{3.6}$$

$$k_i^nk_j^n\ (4\le|n|\le N_1):-2d_nb_n-2d_n^2, \tag{3.7}$$

$$k_i^nk_j^2\ (4\le n\le N_1):-d_nb_2-d_2b_n-2d_2d_n-3b_{n-1}, \tag{3.8}$$

$$k_i^nk_j\ (4\le n\le N_1,-N_1+2\le n\le -4):-d_nb_1-d_1b_n-2d_1d_n+3b_{n-2}-3c_n, \tag{3.9}$$

$$k_i^4k_j^2:-d_4b_2-d_2b_4-2d_2d_4-3b_3, \tag{3.10}$$

$$k_i^3k_j^3:-2d_3b_3-2d_3^2+b_3, \tag{3.11}$$

$$k_i^2k_j^3:-d_2b_3-d_3b_2-2d_2d_3-2b_2, \tag{3.12}$$

$$k_i^{-N_1}k_j^3:-d_{-N_1}b_3-d_3b_{-N_1}-2d_{-N_1}d_3+b_{-N_1}, \tag{3.13}$$

$$k_i^nk_j^3\ (-N_1\le n\le -4):-d_nb_3-d_3b_n-2d_3d_n+b_n, \tag{3.14}$$

$$k_i^{-N_1}k_j:-d_{-N_1}b_1-d_1b_{-N_1}-2d_{-N_1}d_1-3c_{-N_1}, \tag{3.15}$$

$$k_i^{-N_1+1}k_j:-d_{-N_1+1}b_1-d_1b_{-N_1+1}-2d_{-N_1+1}d_1-3c_{-N_1+1}. \tag{3.16}$$

Eqs. (3.5)–(3.12) indicate that $b_{n_1}=c_{n_2}=d_{n_3}=0$ $(2\le n_1\le N_1, 4\le n_2\le N_1, 3\le n_3\le N_1)$. Eq. (3.7), Eq. (3.9) and Eqs. (3.13)–(3.16) indicate that $b_n=c_n=d_n=0$ $(-N_1\le n\le -4)$. Thus, choosing $N_1=3$ and substituting Eq. (2.4) into Eqs. (3.3) and (3.4), we obtain the following relations among b_{n_1}, c_{n_1} and d_{n_1} $(-3\le n_1\le 3)$,

$$\begin{cases}b_{-3}=0,b_{-2}=0,b_{-1}=0,b_0=0,b_1=-\frac{1}{3}d_2^2,b_2=0,b_3=0,\\ c_{-3}=0,c_{-2}=0,c_{-1}=0,c_0=0,c_1=\frac{1}{9}d_1d_2^2-\frac{1}{3}d_1^2,c_2=\frac{1}{9}d_2^3-\frac{2}{3}d_1d_2,c_3=-\frac{4}{9}d_2^2,\\ d_{-3}=0,d_{-2}=0,d_{-1}=0,d_0=0,d_1=d_1,d_2=d_2,d_3=0.\end{cases} \tag{3.17}$$

Eq. (3.17) yields the relations among the constants associated with wave variables

$$\begin{cases} k_{i1} = k_i, \\ k_{i2} = -\dfrac{1}{3}d_2^2 k_i, \\ k_{i3} = \left(\dfrac{1}{9}d_1 d_2^2 - \dfrac{1}{3}d_1^2\right)k_i + \left(\dfrac{1}{9}d_2^3 - \dfrac{2}{3}d_1 d_2\right)k_i^2 - \dfrac{4}{9}d_2^2 k_i^3, \\ k_{i4} = d_1 k_i + d_2 k_i^2, \quad 1 \le i \le N. \end{cases}$$

The resonant soliton solutions to the (3+1)-dimensional BKP-Boussinesq equation are

$$u = 2\big(\ln f\big)_x = \frac{2\sum_{i=1}^N \mu_i k_i e^{\theta_i}}{\sum_{i=1}^N \mu_i e^{\theta_i}}, \tag{3.18}$$

where $\theta_i = k_i x - \frac{1}{3}d_2^2 k_i y + \big(\left(\frac{1}{9}d_1 d_2^2 - \frac{1}{3}d_1^2\right)k_i + \left(\frac{1}{9}d_2^3 - \frac{2}{3}d_1 d_2\right)k_i^2 - \frac{4}{9}d_2^2 k_i^3\big)z + \left(d_1 k_i + d_2 k_i^2\right)t$ $(1 \le i \le N)$, d_1 and d_2 are arbitrary constants.

Without loss of generality, we assume that $\mu_i = 1$ $(1 \le i \le N)$ and $k_1 > k_2 > \cdots > k_N$. Eq. (3.18) has the following limit expression

$$u = \frac{2\sum_{i=1}^N k_i e^{\theta_i}}{\sum_{i=1}^N e^{\theta_i}} \longrightarrow w_{ij} = k_i + k_j + (k_i - k_j)\tanh\frac{\theta_i - \theta_j}{2} \quad (1 \le i < j \le N),$$

when $\theta_1, \theta_2, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_N \to -\infty$.

We mainly discuss the resonant phenomena among the kink soliton waves in the (x, y)-plane and (x, z)-plane.

3.1 Resonant Soliton Solutions in the (x, y)-plane

The maximum and minimum of the kink soliton wave w_{ij} is $2k_i$ and $2k_j$, respectively. The velocity of the kink soliton wave w_{ij} is

$$\left(v_x = -\frac{9d_1 + 9d_2 p_{ij}}{d_2^4 + 9}, v_y = \frac{3d_1 d_2^2 + 3d_2^3 p_{ij}}{d_2^4 + 9}\right), \tag{3.19}$$

where $p_{ij} = k_i + k_j$, v_x and v_y denote the velocities of the wave along the x-axis and the y-axis, respectively.

In the case of $N = 2$, Eq. (3.18) becomes $u = k_1 + k_2 + (k_1 - k_2)\tanh\frac{\theta_1 - \theta_2}{2}$, and there only exists one kink soliton wave.

When $N \ge 3$, the kink soliton waves are parallel and the fission or fusion may occur. The process of fusion can be described as follows. When $t \to -\infty$, there exist $N - 1$ kink soliton waves $w_{i,i+1}$ $(1 \le i \le N - 1)$. As time goes from $-\infty$ to $+\infty$, two adjacent waves interact and merge into another wave.

To explain this phenomenon in detail, supposing that the two adjacent waves are $w_{i,i+1}$ and $w_{i+1,i+2}$, they merge into another wave $w_{i,i+2}$ at

$$t = \frac{1}{9}(6d_1 - d_2^2 + 4d_2(k_i + k_{i+1} + k_{i+2}))z. \quad (3.20)$$

Eq. (3.20) shows that it is impossible for three adjacent waves to merge into one wave because $k_i \neq k_j$ $(i \neq j)$. When $t \to +\infty$, there only exists one kink soliton wave w_{1N}.

The process of fission can be described as follows. When $t \to -\infty$, there exists one kink soliton wave w_{1N}. As time goes from $-\infty$ to $+\infty$, the one wave splits into two adjacent waves. As an example, the wave $w_{i,i+2}$ splits into the two adjacent waves $w_{i,i+1}$ and $w_{i+1,i+2}$. When $t \to +\infty$, there exist $N-1$ kink soliton waves $w_{i,i+1}$ $(1 \leq i \leq N-1)$.

We illustrate the resonant phenomena among the kink soliton waves in the (x, y)-plane with $N = 2$, 3 and 4 in Eq. (3.18). Figure 1 displays the one-kink soliton wave, which moves with a constant velocity $(v_x = -1.44,\ v_y = 1.92)$. The maximum and minimum of the kink soliton wave are 2 and 1, respectively. Figure 2 shows the fission of kink soliton waves, that is, the wave w_{13} splits into two waves w_{12} and w_{23} at $t = -\frac{58}{27}$. The two waves w_{12} and w_{23} move with the constant velocity and the distance between them increases with time. Figure 3 shows the fusion of kink soliton waves. There exist three kink soliton waves w_{12}, w_{23} and w_{34} at $t = -7$. Then the two adjacent waves w_{23} and w_{34} interact and merge into the wave w_{24} at $t = -\frac{2}{3}$. Two adjacent waves w_{12} and w_{24} interact and merge into the wave w_{14} at $t = \frac{14}{9}$. The maximums, minimums and velocities of the two- and three-kink soliton waves in the (x, y)-plane are shown in Tables 1 and 2, respectively. The sectional plots of the resonant soliton solutions are given in Figure 4.

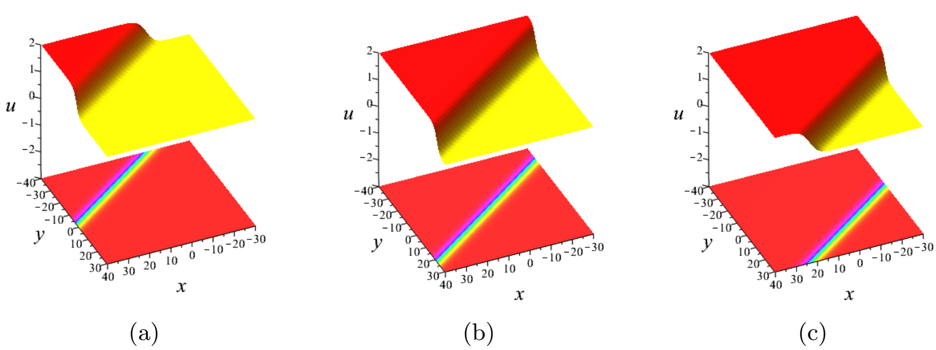
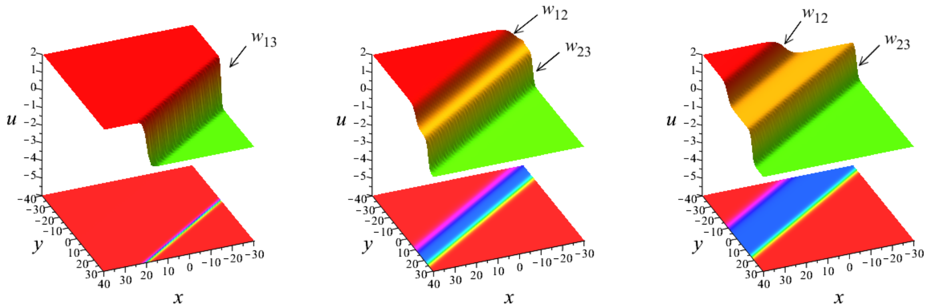
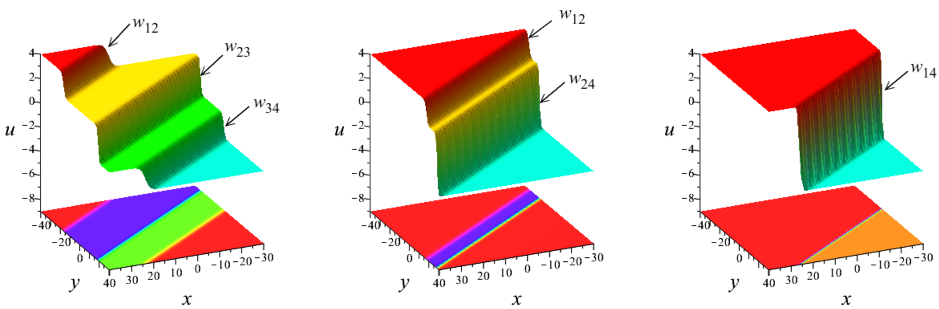


Figure 1. Plots of the one kink soliton wave via Eq. (3.18) in the (x, y)-plane at (a) $t = -10$, (b) $t = -2$ and (c) $t = 5$ with parameters $d_1 = 1$, $d_2 = 2$, $k_1 = 1$, $k_2 = \frac{1}{2}$ and $z = 1$.



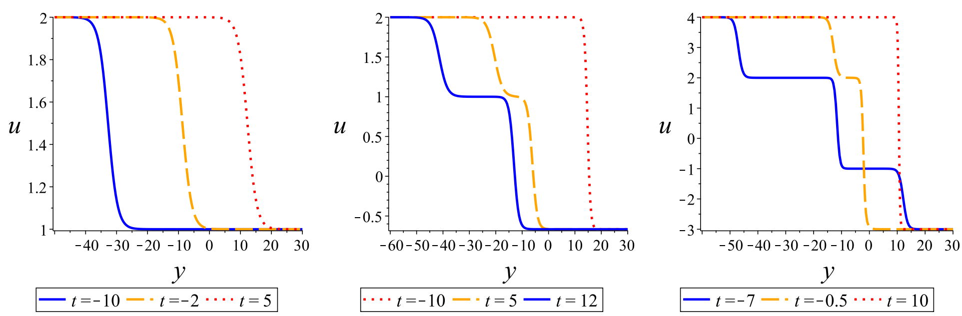
(a) (b) (c)

Figure 2. The resonant phenomena of two kink soliton waves via Eq. (3.18) in the (x, y)-plane at (a) $t = -10$, (b) $t = 5$ and (c) $t = 12$ with parameters $d_1 = -1$, $d_2 = -2$, $k_1 = 1$, $k_2 = \frac{1}{2}$, $k_3 = -\frac{1}{3}$ and $z = 1$.



(a) (b) (c)

Figure 3. The resonant phenomena of three kink soliton waves via Eq. (3.18) in the (x, y)-plane at (a) $t = -7$, (b) $t = -0.5$ and (c) $t = 10$ with parameters $d_1 = 1$, $d_2 = 2$, $k_1 = 2$, $k_2 = 1$, $k_3 = -\frac{1}{2}$, $k_4 = -\frac{3}{2}$ and $z = 1$.



(a) (b) (c)

Figure 4. The sectional plots of resonant soliton solutions in the (x, y)-plane. (a) The sectional plot of Figure 1, (b) The sectional plot of Figure 2, (c) The sectional plot of Figure 3 with $x = 0$.

Table 1 The features of the resonant two-kink soliton waves in the (x, y)-plane

waves	expressions	maximums	minimums	velocities
w_{13}	$k_1 + k_3 + (k_1 - k_3)\tanh\frac{\theta_1-\theta_3}{2}$	$2k_1$	$2k_3$	$\left(v_x = -\frac{9d_1+9d_2p_{13}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{13}}{d_2^4+9}\right)$
w_{12}	$k_1 + k_2 + (k_1 - k_2)\tanh\frac{\theta_1-\theta_2}{2}$	$2k_1$	$2k_2$	$\left(v_x = -\frac{9d_1+9d_2p_{12}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{12}}{d_2^4+9}\right)$
w_{23}	$k_2 + k_3 + (k_2 - k_3)\tanh\frac{\theta_2-\theta_3}{2}$	$2k_2$	$2k_3$	$\left(v_x = -\frac{9d_1+9d_2p_{23}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{23}}{d_2^4+9}\right)$

Table 2 The features of the resonant three-kink soliton waves in the (x, y)-plane

waves	expressions	maximums	minimums	velocities
w_{12}	$k_1 + k_2 + (k_1 - k_2)\tanh\frac{\theta_1-\theta_2}{2}$	$2k_1$	$2k_2$	$\left(v_x = -\frac{9d_1+9d_2p_{12}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{12}}{d_2^4+9}\right)$
w_{23}	$k_2 + k_3 + (k_2 - k_3)\tanh\frac{\theta_2-\theta_3}{2}$	$2k_2$	$2k_3$	$\left(v_x = -\frac{9d_1+9d_2p_{23}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{23}}{d_2^4+9}\right)$
w_{34}	$k_3 + k_4 + (k_3 - k_4)\tanh\frac{\theta_3-\theta_4}{2}$	$2k_3$	$2k_4$	$\left(v_x = -\frac{9d_1+9d_2p_{34}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{34}}{d_2^4+9}\right)$
w_{24}	$k_2 + k_4 + (k_2 - k_4)\tanh\frac{\theta_2-\theta_4}{2}$	$2k_2$	$2k_4$	$\left(v_x = -\frac{9d_1+9d_2p_{24}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{24}}{d_2^4+9}\right)$
w_{14}	$k_1 + k_4 + (k_1 - k_4)\tanh\frac{\theta_1-\theta_4}{2}$	$2k_1$	$2k_4$	$\left(v_x = -\frac{9d_1+9d_2p_{14}}{d_2^4+9}, v_y = \frac{3d_1d_2^2+3d_2^3p_{14}}{d_2^4+9}\right)$

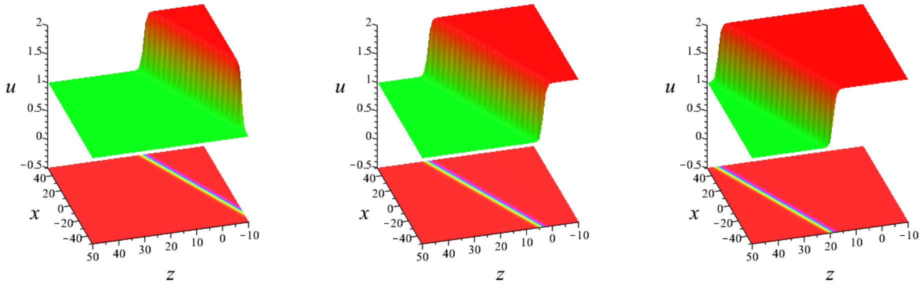
3.2 Resonant Soliton Solutions in the (x, z)-plane

The resonant phenomena among the kink soliton waves in the (x, z)-plane are more complicated than those in the (x, y)-plane, and the kink soliton waves are not parallel. The maximum and minimum of the kink soliton wave w_{ij} is $2k_i$ and $2k_j$, respectively. The velocity of the kink soliton wave w_{ij} is

$$\left(v_x = -\frac{d_1 + d_2p_{ij}}{1 + q_{ij}^2}, v_z = -\frac{(d_1 + d_2p_{ij})q_{ij}}{1 + q_{ij}^2}\right), \tag{3.21}$$

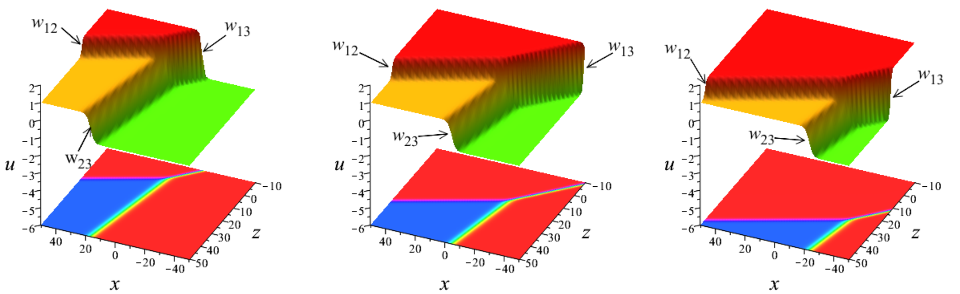
where $q_{ij} = \frac{1}{9}d_1d_2^2 - \frac{1}{3}d_1^2 + (\frac{1}{9}d_2^3 - \frac{2}{3}d_1d_2)(k_i + k_j) - \frac{4}{9}d_2^2(k_i^2 + k_ik_j + k_j^2)$, and v_x and v_z denote the velocities of the wave along the x-axis and the z-axis, respectively.

We illustrate the resonant phenomena among the kink soliton waves in the (x, z)-plane with $N = 2$, 3 and 4 in Eq. (3.18). Figure 5 shows that there only exists one kink soliton wave moving with the velocity ($v_x = -\frac{18}{65}$, $v_z = \frac{66}{65}$). The maximum and minimum of the kink soliton wave are the same as that in Figure 1, because they only depend on k_1 and k_2. Figure 6 shows two kink soliton waves w_{12} and w_{23} interact with each other and generate the third kink soliton wave w_{13}. From Figure 7, we can see that there exist five kink soliton waves $w_{12}, w_{23}, w_{24}, w_{13}$ and w_{34} when $t = -60$. As time goes on, the kink soliton wave w_{23} disappears, while the kink soliton wave w_{14} appears. To study the propagation of the two- and three-kink soliton waves in the (x, z)-plane, the features of the kink soliton waves in the (x, y)-plane are shown in Tables 3 and 4. Figure 8 displays the sectional plots of the resonant soliton solutions.



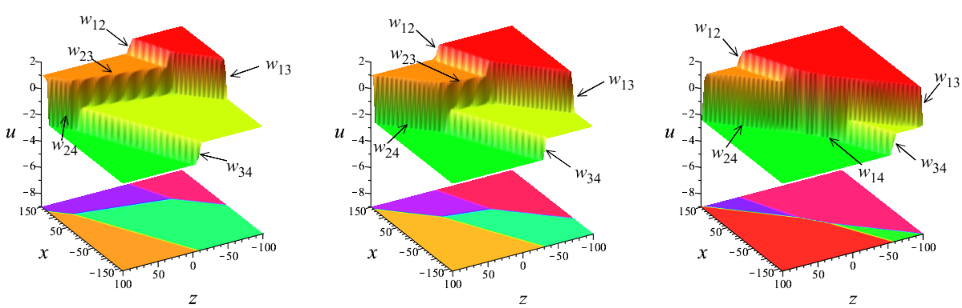
(a) (b) (c)

Figure 5. Plots of the one kink soliton wave in the (x, z)-plane via Eq. (3.18) at (a) $t = 0$, (b) $t = 16$ and (c) $t = 30$ with parameters $d_1 = 1$, $d_2 = 2$, $k_1 = 1$, $k_2 = \frac{1}{2}$ and $y = 1$.



(a) (b) (c)

Figure 6. The resonant phenomena of two kink soliton waves in the (x, z)-plane via Eq. (3.18) at (a) $t = 0$, (b) $t = 16$ and (c) $t = 30$ with parameters $d_1 = 1$, $d_2 = 2$, $k_1 = 1$, $k_2 = \frac{1}{2}$, $k_3 = -\frac{1}{3}$ and $y = 1$.



(a) (b) (c)

Figure 7. The resonant phenomena of three kink soliton waves in the (x, z)-plane via Eq. (3.18) at (a) $t = -60$, (b) $t = -30$ and (c) $t = 5$ with parameters $d_1 = 1$, $d_2 = 2$, $k_1 = 1$, $k_2 = \frac{1}{2}$, $k_3 = -\frac{3}{7}$, $k_4 = -\frac{6}{5}$ and $y = 1$.

Table 3 The features of the resonant two-kink soliton waves in the (x, z)-plane

waves	expressions	maximums	minimums	velocities
w_{12}	$k_1 + k_2 + (k_1 - k_2)\tanh\frac{\theta_1-\theta_2}{2}$	$2k_1$	$2k_2$	$\left(v_x = -\frac{d_1+d_2p_{12}}{1+q_{12}^2}, v_z = -\frac{(d_1+d_2p_{12})q_{12}}{1+q_{12}^2}\right)$
w_{23}	$k_2 + k_3 + (k_2 - k_3)\tanh\frac{\theta_2-\theta_3}{2}$	$2k_2$	$2k_3$	$\left(v_x = -\frac{d_1+d_2p_{23}}{1+q_{23}^2}, v_z = -\frac{(d_1+d_2p_{23})q_{23}}{1+q_{23}^2}\right)$
w_{13}	$k_1 + k_3 + (k_1 - k_3)\tanh\frac{\theta_1-\theta_3}{2}$	$2k_1$	$2k_3$	$\left(v_x = -\frac{d_1+d_2p_{13}}{1+q_{13}^2}, v_z = -\frac{(d_1+d_2p_{13})q_{13}}{1+q_{13}^2}\right)$

Table 4 The features of the resonant three-kink soliton waves in the (x, z)-plane

waves	expressions	maximums	minimums	velocities
w_{12}	$k_1 + k_2 + (k_1 - k_2)\tanh\frac{\theta_1-\theta_2}{2}$	$2k_1$	$2k_2$	$\left(v_x = -\frac{d_1+d_2p_{12}}{1+q_{12}^2}, v_z = -\frac{(d_1+d_2p_{12})q_{12}}{1+q_{12}^2}\right)$
w_{23}	$k_2 + k_3 + (k_2 - k_3)\tanh\frac{\theta_2-\theta_3}{2}$	$2k_2$	$2k_3$	$\left(v_x = -\frac{d_1+d_2p_{23}}{1+q_{23}^2}, v_z = -\frac{(d_1+d_2p_{23})q_{23}}{1+q_{23}^2}\right)$
w_{24}	$k_2 + k_4 + (k_2 - k_4)\tanh\frac{\theta_2-\theta_4}{2}$	$2k_2$	$2k_4$	$\left(v_x = -\frac{d_1+d_2p_{24}}{1+q_{24}^2}, v_z = -\frac{(d_1+d_2p_{24})q_{24}}{1+q_{24}^2}\right)$
w_{13}	$k_1 + k_3 + (k_1 - k_3)\tanh\frac{\theta_1-\theta_3}{2}$	$2k_1$	$2k_3$	$\left(v_x = -\frac{d_1+d_2p_{13}}{1+q_{13}^2}, v_z = -\frac{(d_1+d_2p_{13})q_{13}}{1+q_{13}^2}\right)$
w_{34}	$k_3 + k_4 + (k_3 - k_4)\tanh\frac{\theta_3-\theta_4}{2}$	$2k_3$	$2k_4$	$\left(v_x = -\frac{d_1+d_2p_{34}}{1+q_{34}^2}, v_z = -\frac{(d_1+d_2p_{34})q_{34}}{1+q_{34}^2}\right)$
w_{14}	$k_1 + k_4 + (k_1 - k_4)\tanh\frac{\theta_1-\theta_4}{2}$	$2k_1$	$2k_4$	$\left(v_x = -\frac{d_1+d_2p_{14}}{1+q_{14}^2}, v_z = -\frac{(d_1+d_2p_{14})q_{14}}{1+q_{14}^2}\right)$

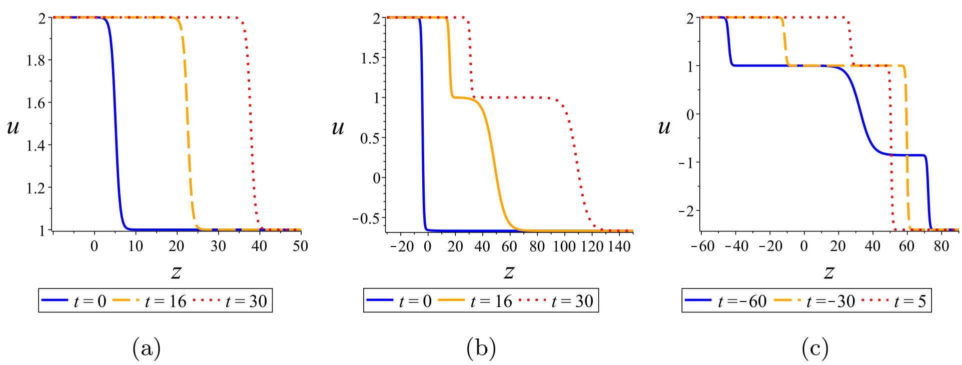


Figure 8. The sectional plots of resonant soliton solutions. (a) The sectional plot of Figure 5 with $x = 20$, (b) The sectional plot of Figure 6 with $x = -5$, (c) The sectional plot of Figure 7 with $x = 80$.

4 Conclusions

We have extended the parameterizations of the constants associated with the wave variables. The resonant soliton solutions to the (3+1)-dimensional BKP-Boussinesq equation have been derived. We have mainly discussed the resonant two- and three-soliton solutions in the (x, y)-plane and (x, z)-plane. The characteristics including expressions, maximums, minimums and velocities of kink soliton waves have been given. The kink soliton waves in the (x, y)-plane are parallel and the fusion or fission may occur. The fusion process can be described as follows. When $t \to -\infty$, there exist $N-1$ kink soliton waves. As time goes from $-\infty$ to $+\infty$, two adjacent waves interact and merge into another wave. When $t \to +\infty$, there only exists one kink soliton wave. The fission process shows that one kink soliton wave splits into $N-1$ kink soliton waves with time. The kink soliton waves in the (x, z)-plane are not parallel and the resonant phenomena are more complicated. In conclusion, we have obtained abundant resonant behaviors of soliton solutions to the (3+1)-dimensional BKP-Boussinesq equation.

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