A closed-form solution for the failure interaction diagrams of pile groups subjected to inclined eccentric load

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Abstract

The work at hand proposes a method for assessing, under reasonable hypotheses from an engineering perspective, the failure envelope of a pile group subjected to generalized loading conditions involving a vertical and a lateral force along with a moment. Following different assumptions of increasing complexity, a simple closed-form expression, which is however capable of considering also the strong dependence of sectional yielding moment on the axial force, is derived. The use of such formula, which allows a practical hand calculation of the interaction diagrams at failure, returns conservative yet very accurate results. As a follow up, with reference to reinforced concrete piles, design considerations involving both structural and geotechnical failure under lateral load are reported. It is found that for most cases, if steel reinforcement is established to resist the design bending moment, the geotechnical Ultimate Limit State checks are automatically satisfied.

Keywords Bearing capacity · Eccentric load · Inclined load · Pile group · Ultimate Limit State

Abbreviations

α_M	Inclination of the applied moment vector	ξ_3	Correlation factor to derive characteristic value
γ	Soil unit weight	ξ_j	Coordinate of pile j -th in the reference system
γ_b, γ_s	Partial resistance factors for pile base and shaft capacity	φ	Soil friction angle
γ_c	Partial resistance factor for concrete	τ_{lim}	Pile unit shaft resistance
γ_{sat}	Saturated unit weight of soil	ω'	Mechanical steel ratio
γ_{st}	Partial resistance factor for steel	A_c	Cross-sectional area of concrete
γ_{su}	Partial factor for undrained shear strength	A_s	Cross-sectional area of the longitudinal reinforcement
γ_T	Partial resistance factor for transverse load	c	Concrete cover of cross-section
ΔM_1	External moment in the Matlab routine	d	Pile diameter
$\Delta N_{i_Nu}, \Delta N_{i_Su}$	Axial forces in the Matlab routine	E, E_{sd}	Effective Young's moduli of the soil
θ	Angle defining the extension of compression zone	E_p	Young's modulus of the concrete
v'	Normalized axial force	f_{ck}	Characteristic strength of concrete
		f_{yk}	Characteristic strength of steel
		f'_{ck}	Effective characteristic strength of concrete
		f'_{yk}	Effective characteristic strength of steel
		f_{cd}	Design strength of concrete
		f_{yd}	Design strength of steel
		H	Horizontal force applied upon pile group

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H_d	Design horizontal load at single pile head
$H_u, H_{ui_Nu}, H_{ui_Su}, H_{u,max}, H_{u,min}, H_{u,1}, H_{u,2}, H_{ug}, H_{us}$	Horizontal capacities of the pile group
$H_{u,i}$	Geotechnical and structural ultimate horizontal loads of the single pile
k_p	Horizontal capacity of the single pile
L	Passive earth pressure coefficient
M	Pile length
M_1	Moment applied upon pile group
$M_u, M_{ui}, M_{uk}, M_{u,1}, M_{u,2}, M_y, M_{y1}, M_{y2}$	Moment in the Matlab routine
$M_{y,c}$	Moment capacities of the pile group
$M_{y,s}$	Yielding moments of pile cross-section
Axial forces	Flexural resistance due to concrete
N_d	Flexural resistance due to steel
N_u	$N, N_i, N_{if}, N_{i_M1}, N_{i_AM1}, N_j$
N_{uj}	Design axial load at single pile head
p	Axial capacity of the single pile in compression
p_u, p_{ud}	Axial capacity in compression of pile j -th
Q	Number of piles
$Q_{u,max}, Q_{u,min}, Q_u, Q_{ui}, Q_{uk}, Q_{u,1}, Q_{u,2}, Q_{u,3}, Q_{u,4}$	Soil resistances to horizontal load
R	Axial load applied upon pile group
$R_{c,d}$	Axial capacities of the pile group
S	Pile radius
S_u	Design resistance of the single pile spacing
S_{uj}	Axial capacity of the single pile in uplift
s_u	Axial capacity in uplift of pile j -th
x_i	Soil undrained shear strength
z	Abscissa of pile i -th
	Depth

1 Introduction

The design of a foundation is aimed at ensuring a satisfactory performance of the foundation itself along with the supported structure. Current practice generally splits the checks in two broad categories: Ultimate Limit State (ULS) checks (i.e. the foundation must guarantee an adequate safety margin against collapse) and, under lower loads' magnitude, the fulfilment of Serviceability Limit State (SLS) requirements (i.e. the foundation displacements must be tolerated by the structure without loss of serviceability). With reference to pile foundations, the assessment of the ultimate load under vertical centered load is often carried out by hand exploiting closed-form formulae available in literature employing either direct results from in-situ tests or estimates of soil mechanical parameters.

In presence of generalized loading conditions, that is when a pile group is subjected to a given inclined eccentric load and the horizontal load component cannot be reasonably neglected (Fig. 1), the evaluation of the group capacity is not trivial. Scientific contributions are limited to semiempirical relationships describing the reduction of bearing capacity as compared to the case of centered vertical load, derived on the basis of some small scale tests on aluminium piles in sand [13, 16, 18] and in clay [15, 17, 21]. Such relationships are of difficult practical implementation, and refer to the case of rigid piles (no plastic hinging at failure) which represents a very rare scenario in practice. So far, no analytical formulation has been reported for the three-dimensional failure interaction diagram for a pile group, contrary to the case of surface foundations where closed-form expressions have been provided in different scientific contributions (e.g. [3, 11, 12, 19, 24, 25]). It follows that the use of numerical methods seems to represent the only way one can follow to assess pile group capacity under an eccentric inclined load (e.g. [4, 10, 20, 22, 23]). Nevertheless, numerical analyses are computationally expensive and requires some expertise along with proper soil characterisation for a reliable constitutive modelling. The work at hand furnishes a contribution towards filling this lack of simple solutions in engineering practice, furnishing a simpler alternative to complex numerical analyses. The methodology proposed, even though under simplifying hypotheses, combines different ingredients, already consolidated in the scientific and technical literature, towards a more rational approach as compared to what is generally done in routine engineering.

Recently, a closed-form, exact solution for vertical eccentric load has been proposed by Di Laora et al. [5] to describe the failure of a group of piles considered as a whole. Along with the straightforward advantage of avoiding the overconservatism inherent in the classical

approach considering the failure of the group coincident with the first single pile failure—which in reality merely represents the onset of yielding—the above work also allows to derive failure envelopes in the vertical force-moment plane through simple hand calculations. From a general standpoint, the advantage of approaching ULS checks by means of failure envelopes is manifold, yet for the scope of this work may be identified in the following two aspects: (1) assess the safety margins of the foundation as a function of the considered load path; (2) in light of a specific design regulation, draw the ‘design failure envelope’ and place in the same plane all the points corresponding to the ‘design action’ associated to any load combination.

For generalized loading condition, this paper extends the above work [5] by providing a simple method to derive failure envelopes in the vertical force-moment-horizontal force space under different simplifying assumptions. Closed-form expressions accounting for the dependence of sectional yielding moment on the axial force are also provided.

Finally, some design considerations are reported to furnish guidance on the practical application in routine design.

2 Failure domains under vertical eccentric load

The method proposed by Di Laora et al. [5], was based on the following hypotheses:

Fig. 1 Problem under investigation: interaction domain for a pile group subjected to a vertical eccentric load

1. Rigid-plastic behaviour of individual piles considered as independent uniaxial elements having two failure loads, i.e. in compression, N_u , and in tension, $-S_u$;
2. Piles' heads connected through a rigid cap of infinite strength;
3. Piles' connections to the cap modelled as hinges (i.e., no bending moments are applied at piles' heads).

The above approach leads to the determination of the moment-axial force interaction diagram of the pile group as that plotted in Fig. 2a for a row of 4 equally spaced, identical piles. Any collapse mechanism for a row of equally spaced identical piles involving a counter-clockwise rotation about a point located between two consecutive piles corresponds to the following pairs:

$$\begin{cases} Q_{ui} = N_u(i-1) - S_u(p-i+1) \\ M_{ui} = \frac{s}{2}(N_u + S_u)(i-1)(p-i+1) \end{cases} \quad i = 1, \dots, p \quad (1)$$

where Q_u and M_u are the axial and the moment capacity of the pile group, respectively; p is the number of piles within the row; i identifies the i -th pile and also refer to the plastic mechanism characterized by a counter-clockwise increment of rotation about a point between piles $(i-1)$ and i ; s is the pile spacing. The whole domain is obtained by repeating the procedure considering the clockwise rotation.

The vertexes of the domain obtained with the above approach can be seen as singularities. To complete the construction of the interaction diagram, such points are connected by straight lines which are proven to correspond to a failure mechanism where the cap rotates about a point located at the head of a pile which is the only element not reaching failure.

For a group of unevenly distributed, dissimilar piles, for which the applied moment vector is inclined by the angle α_M with respect to the y -axis, the vertexes of the failure domain are obtained using the following equations:

$$\begin{cases} Q_{ui} = \sum_{j=1}^{i-1} N_{uj} - \sum_{j=i}^p S_{uj} \\ M_{ui} = -\sum_{j=1}^{i-1} N_{uj}\xi_j + \sum_{j=i}^p S_{uj}\xi_j \end{cases} \quad i = 1, \dots, p \quad (2)$$

$$\begin{cases} Q_{u(k+p)} = \sum_{j=k}^p N_{uj} - \sum_{j=1}^{k-1} S_{uj} \\ M_{u(k+p)} = -\sum_{j=k}^p N_{uj}\xi_j + \sum_{j=1}^{k-1} S_{uj}\xi_j \end{cases} \quad k = 1, \dots, p \quad (3)$$

where k identifies the k -th pile and also refer to the plastic mechanism characterized by a clockwise increment of rotation about a point between piles $(k-1)$ and k ; ξ_j is the

coordinate of each pile in the reference system (ξ, η) , for which the η -axis is coincident with the direction of the resultant moment vector.

The authors also provided a lower-bound approximate solution accounting for the contribution of (axial force dependent) moment capacities of the cross section at pile head, M_y , by modelling the connections to the cap as rigid-plastic internal fixities. Anyhow, this favourable contribution is generally quite small and can be therefore neglected for practical purposes; this conservative hypothesis is also adopted herein.

3 Failure domains under combined vertical eccentric and lateral loads

3.1 Assessment of pile capacity under lateral load

Many methods are available in literature for the assessment of the pile bearing capacity under horizontal load. Among them, the theory proposed by Broms [1, 2], often employed in routine design, is based on the hypothesis of rigid-plastic behaviour of both pile and soil. The solutions are provided for free- and fixed-head piles embedded in purely cohesive or frictional deposits, by writing the rotational and translational equilibrium equations. It is possible to demonstrate that a fixed-head pile, except for extremely low slenderness ratios, always fails according to the ‘long’ mode, where two plastic hinges develop at the head and at some depth down the pile (say, 5 to 10 pile diameters). In this case, the horizontal capacity of the pile, $H_{u,i}$, depends on the pile diameter, d , the soil resistance, p_u , and the yielding moment of the concrete cross-section, M_y :

$$H_{u,i} = H_{u,i}(d, p_u(z), M_y) \quad (4)$$

Without loss of generality, considering for example a soil whose resistance to horizontal load is proportional to depth, the expression of the ultimate horizontal load carried by pile ‘i’ can be manipulated to read:

$$\frac{H_{u,i}}{p_{ud}d^2} = \frac{1}{2} \left(3 \frac{M_{y1} + M_{y2}}{p_{ud}d^3} \right)^{\frac{2}{3}} \quad (5)$$

where p_{ud} is the soil resistance at the reference depth corresponding to the pile diameter; M_{y1} , M_{y2} are the yielding moment of the pile cross-section corresponding to the hinges developed at head and along pile shaft, respectively. While in the original solution such moments are identical, here it is preferred to keep a general formulation distinguishing the moment capacities since, in principle, they are quite different given the reduction of axial force with depth. The choice of employing $M_{y1} \neq M_{y2}$ is a

matter of engineering judgment and is out of the scope of the present work.

Also, note that substituting p_{ud} with $3k_p\gamma d$ (with k_p the passive earth pressure coefficient and γ the soil unit weight) yields the original expression by Broms.

3.2 Construction of interaction domains including lateral load

3.2.1 Cross-section moment capacity independent of axial force

The interaction diagrams reported in Fig. 2a are associated to the absence of a lateral load. It follows that the projections of the vertexes of the domain in the Q_u – H_u and H_u – M_u planes, H_u being the horizontal capacity of the pile group, are located on the x - and y -axis, respectively, as shown in Fig. 2b, c.

When the external load, expressed by a pair (Q, M) , is represented by a point located on the failure envelope, neither the axial load nor the moment may further increase. Instead, piles are able to sustain a horizontal force, which can be reasonably supposed not to alter the axial load distribution on piles, and the horizontal failure load can be obtained as the sum of the piles’ bearing capacities, evaluated using Eq. (4). If one adopts the simplifying hypothesis that the cross-section moment capacity of the pile is independent of the axial force (i.e., $M_{y1} = M_{y2} = \text{constant}$) for which Eq. (5) returns the original equation by Broms:

$$\frac{H_{u,i}}{p_{ud}d^2} = \frac{1}{2} \left(6 \frac{M_y}{p_{ud}d^3} \right)^{\frac{2}{3}} \quad (6)$$

the interaction diagrams in the Q_u – H_u and H_u – M_u planes for selected values of M and Q , respectively, get the shape shown in Fig. 3.

3.2.2 Consideration of N–M relationship for a concrete cross section

The above failure domains may be deemed as oversimplified since, from a general viewpoint, pile horizontal capacity is function of the yielding moment which, in turn, depends on the axial load acting at head. It is therefore possible to consider such a dependence and calculate the yielding moment exploiting the closed-form expression proposed by Di Laora et al. [6] for the design of axial force-moment interaction diagrams relative to circular reinforced concrete cross-section (Fig. 4), properly adapted for the case at hand:

$$M_y = M_{y,c} + M_{y,s} = \frac{2}{3} R^3 \sin^3 \theta f'_{ck} + \frac{2}{\pi} (R - c) A_s \sin \theta f'_{yk} \quad (7)$$

Fig. 2 Interaction domain for vertical eccentric load (a) and projections of the interaction domain in the Q_u – H_u (b) and H_u – M_u (c) planes. In the figure, $S_u = 1/3 N_u$

Fig. 3 Interaction domains for combined loads and cross-section moment capacity independent of the pile axial load

where M_y is the flexural capacity at yielding, equal to the sum of the flexural resistance due to concrete, $M_{y,c}$, and the flexural resistance due to steel, $M_{y,s}$; R is the pile radius; θ is the angle defining the extension of compression zone; $f'_{ck} = 0.9f_{ck}$ and $f'_{yk} = 0.95f_{yk}$, with f_{ck} and f_{yk} the characteristic strengths of concrete and steel; c is the concrete cover of cross-section; A_s is the cross-sectional area of the longitudinal reinforcement.

Fig. 4 N – M interaction domain for reinforced concrete cross-section according to the method by Di Laora et al. [6]

The angle θ is calculated through the following approximate solution, valid for the most common case of $v' \leq 0.5$:

$$\theta = \left(\frac{\pi}{4}\right)^2 \left[-\left(1 + 2\omega' - \frac{4}{\pi}\right) + \sqrt{\left(1 + 2\omega' - \frac{4}{\pi}\right)^2 + \frac{32}{\pi}(\omega' + v')} \right] \quad (8)$$

where

$$\omega' = \frac{A_s f'_{yk}}{A_c f'_{ck}} \quad (9)$$

is the mechanical steel ratio, with $A_c = \pi R^2$ the concrete cross-section, and

$$v' = \frac{N}{A_c f'_{ck}} \quad (10)$$

is the axial force, N , normalized to the effective characteristic sectional strength associated to the sole concrete.

With reference, for sake of simplicity, to the simple case in Fig. 2a (row of 4 equally spaced, identical piles), if the applied (Q, M) pair corresponds to a point located on the failure envelope, the axial load distribution between the piles of the row is known. In particular, since the vertexes of the domain correspond to the failure of all the piles in compression or in tension (Fig. 2a), for a given concrete reinforcement, the cross-section moment capacities are calculated through Eq. (7) for $N = N_u$ and $N = -S_u$, and, therefore, it is possible to derive the horizontal failure loads of each pile using Eq. (4). By summing up the contribution of all the piles, the vertexes of the domains in the Q_u-H_u and H_u-M_u planes are obtained and can be, as a first approximation, connected by a straight line.

Figure 5 reports the interaction diagrams in the Q_u-H_u plane for different values of the moment capacity (i.e. $M_{u,1}$ and $M_{u,2}$). Likewise, in Fig. 6 the domain in the H_u-M_u plane is plotted for four values of the ultimate vertical load (i.e. $Q_{u,1}$, $Q_{u,2}$, $Q_{u,3}$ and $Q_{u,4}$).

Reporting the above failure loci on a single diagram (Fig. 7) it is possible to notice that the Q_u-H_u domains, constructed from the vertexes of the Q_u-M_u interaction diagram, are superimposed. This remarkable result, which can be demonstrated analytically, leads to a very useful simplification for practical purposes, that is the ultimate horizontal load can be derived in a straightforward manner, for any Q and regardless of M , from the formula:

$$H_u = H_{u,\min} + \frac{Q - Q_{u,\min}}{Q_{u,\max} - Q_{u,\min}} (H_{u,\max} - H_{u,\min}) \quad (11)$$

where $Q_{u,\min}$ and $Q_{u,\max}$ correspond to the sum of the piles bearing capacities in tension and in compression, respectively; $H_{u,\min}$ and $H_{u,\max}$ are obtained as follows:

$$\begin{aligned} H_{u,\min} &= p \cdot H_{ui_S_u} (M_y(N = -S_u)) \\ H_{u,\max} &= p \cdot H_{ui_N_u} (M_y(N = N_u)) \end{aligned} \quad (12)$$

Note that for the common case of $v' \leq 0.5$ the increase of the axial load has a beneficial effect on the flexural capacity at failure of the concrete cross-section (Fig. 4). As a consequence, the maximum value of the of the ultimate horizontal load, $H_{u,\max}$, is obtained when all the piles are subjected to compression.

For sake of completeness, it is also possible to derive the Q_u-M_u interaction diagram for a given value of H_u . Since the domains are symmetric with respect to the x -axis, only $H_u \geq 0$ is considered. In Fig. 8 the projections in the Q_u-M_u plane are reported for different values of H_u . It is highlighted that the diagram ABCDE holds the same shape until $H_u = H_{u,\min}$, while for $H_u > H_{u,\min}$ the domains contract. For example, a horizontal line drawn for

Fig. 5 Q_u-H_u interaction domains for combined loads and linear approximation

Fig. 6 H_u - M_u interaction domains for combined loads and linear approximation

Fig. 7 Interaction domains for combined loads and linear approximation

$H_u = H_{u,1}$ intersects the domains AE and BD in the Q_u - H_u plane corresponding to two different moments (identified in red and blue in the Q_u - M_u plane), by the points F and G. The straight line connecting F and G in the Q_u - M_u plane intersects the ABCDE domain at the point I, identifying the

interaction diagram FICDE for $H_u = H_{u,1}$. Likewise, it is possible to derive the domain LME corresponding to $H_u = H_{u,2}$.

The interaction domains are derived at failure. To consider the design conditions, it is only necessary to employ the design axial capacity in compression and in tension and the design horizontal capacity under the above axial loads.

Figure 9 shows a 3D representation of the so-derived failure envelope along with different sections for constant values of Q_u , H_u and M_u .

It may be argued that the above formulation is valid when the (Q, M) pair applied to the foundation is located on the failure envelope, while for a pair lying inside the domain the axial load acting on each pile is not known and should be assessed by equilibrium and compatibility considerations, taking also into account that some piles (but not all of them) might have attained their ultimate load. This means that, while the points in Fig. 7 directly stem from our hypotheses given that they correspond to known distributions of axial force among piles, straight lines in the Q_u - H_u and H_u - M_u diagrams merely represent an assumption, and they could be not exact given the dependence of

Fig. 8 Q_u - M_u and Q_u - H_u interaction domains for combined loads and linear approximation

sectional yielding moment on the specific load acting upon each pile.

To assess the validity of these simplifying hypotheses, a numerical procedure has been used as shown in the ensuing.

4 Comparison with a numerical solution

A mathematical framework has been developed via a Matlab® [14] code. For any (Q, M) pair applied to the foundation, the distribution of axial load acting at the head of each pile is determined assuming an elastic-perfect plastic behaviour of piles and neglecting, for sake of simplicity, pile-to-pile interaction. Pile axial load is computed according to the following steps:

1. the (Q, M) pair is first distributed idealizing piles as independent elastic springs. This leads to the following closed-form expression for the axial load N_i acting upon pile 'i':

$$N_i = \frac{Q(x_i \sum_{i=1}^p x_i - \sum_{i=1}^p x_i^2) + M(p x_i - \sum_{i=1}^p x_i)}{(\sum_{i=1}^p x_i)^2 - p \sum_{i=1}^p x_i^2} \quad \forall i = 1, \dots, p \quad (13)$$

where i is the i -th pile and x_i is x -coordinate of i -th pile;

2. if no pile has attained failure:

$$-S_u \leq N_i \leq N_u \quad \forall i = 1, \dots, p \quad (14)$$

the procedure terminates. If Eq. (14) is not always satisfied, the following quantities are calculated:

$$\begin{aligned} \Delta N_{i_{-S_u}} &= -S_u - N_i \\ \Delta N_{i_{-N_u}} &= N_i - N_u \end{aligned} \quad (15)$$

The largest value obtained by employing Eqs. (15) corresponds to the first pile attaining failure (pile j);

3. the applied moment corresponding to the first pile at failure, M_1 , is determined from Eq. (13), by imposing either $N_j = -S_u$ or $N_j = N_u$;
4. the load distribution is then updated, i.e., $N_{i_{-M_1}}$ associated to the $(Q-M_1)$ couple are evaluated using Eq. (13);
5. the quantity $\Delta M_1 = M - M_1$ is distributed solely between the piles that have not yet achieved the axial capacity by employing Eq. (13) eliminating the x -coordinate of the pile at failure from the x_i vector; this way, the loads $N_{i_{-AM_1}}$ are obtained;
6. the final load distribution is given by $N_{i_{-f}} = N_{i_{-M_1}} + N_{i_{-AM_1}}$;
7. steps from 2 to 6 are repeated until the remaining piles are subjected to a load satisfying Eq. 14.

Once N_i are determined, the horizontal failure loads of each pile are found for a given cross-section reinforcement and, consequently, the points $(Q-H)$ and $(H-M)$ associated to $(Q-M)$ are obtained.

With the aim of facilitating the reader in the implementation of this procedure, a numerical example is reported in the following. The numerical approach is applied to obtain the interaction diagrams for a row of 4 piles 1 m in diameter and equally spaced with $s = 3$ m (Fig. 10). The concrete reinforcement adopted is $A_s = 0.5\% A_c$ while $N_u = 3$ MN and $S_u = 1$ MN. The piles are numbered in ascending order from left to right, and are subjected to the external pair $(Q = 6$ MN, $M = 18$ MNm). The following quantities are derived at each step:

1. the axial load N_i acting upon the piles are: $N_1 = 3.3$ MN, $N_2 = 2.1$ MN, $N_3 = 0.9$ MN, $N_4 = -0.3$ MN;
2. the first pile attaining failure is pile 1 for which $\Delta N_{1_{-N_u}} = 0.3$ MN;
3. $M_1 = 15.00$ MNm is the moment distributed between the piles up to pile 1 failure;

Fig. 9 3D failure envelope for a row of 4 equally spaced identical piles

4. the loads associated to $(Q-M_1)$ pair are: $N_{1_M1} = 3.0$ MN, $N_{2_M1} = 2.0$ MN, $N_{3_M1} = 1.0$ MN, $N_{4_M1} = 0.0$ MN;
5. $\Delta M_1 = 3.00$ MNm is distributed between piles from 2 to 4 and the loads $N_{2_AM1} = 0.5$ MN, $N_{3_AM1} = 0.0$ MN, $N_{4_AM1} = -0.5$ MN are obtained;
6. the final loads acting upon piles are $N_{1,f} = 3.0$ MN, $N_{2,f} = 2.5$ MN, $N_{3,f} = 1$ MN, $N_{4,f} = -0.5$ MN;
7. the axial loads found in the previous step, satisfy Eq. (14).

The yielding moment of the concrete cross section is evaluated using Eq. (7) employing the axial load distribution obtained at step 6 and using $f_{ck} = 25$ MPa, $f_{yk} = 450$ MPa and $c = 0.05$ m. The following values are obtained: $M_{y1} = 1.68$ MNm, $M_{y2} = 1.55$ MNm, $M_{y3} = 1.08$ MNm, $M_{y4} = 0.54$ MNm. Finally, the horizontal capacity is evaluated as the sum of $H_1 = 1.19$ MN, $H_2 = 1.13$ MN, $H_3 = 0.89$ MN, $H_4 = 0.56$ MN, derived employing Eq. (6) with $p_{ud} = 133$ kPa.

From comparison with the domain derived as described in the previous sections, it can be noted that use of the

Fig. 10 Interaction domains for combined loads, numerical approach

linear approximation returns very accurate results which are, however, on the safe side.

It is worth noting that for $M = 0$ each pile of the row is subjected to the following axial force:

$$N_i = \frac{Q_u}{p} \quad \forall i = 1, \dots, p \quad (16)$$

Since the load distribution between the piles is known, the rigorous domain can be also obtained through a simple hand calculation for any applied Q , by employing Eqs. (7) and (4). This can be useful in some circumstances, as will be shown in the ensuing.

For the sake of completeness, in Fig. 11a the domains in the Q_u - M_u plane according to the numerical solution are reported for different values of H_u , while a comparison with the simplified approach is offered in Fig. 11b.

4.1 Piles subjected to high axial loads

For a highly compressed reinforced concrete cross-section, the use of the linear approximation determines a more significant underestimation of the horizontal capacity, especially for low eccentricities. Figure 12 shows a comparison between the numerical approach and the linear assumption for a case in which $N_u = 0.4 f_{ck} A_c$; in addition, the upper bound of lateral capacity coming from Eqs. (7) and (4) employing the average load on the pile (rigorous only for $M = 0$) is also reported. This expression may be therefore used for highly compressed piles and low eccentricities.

5 ‘Structural’ vs ‘geotechnical’ failure under combined loads

The above domains have been constructed taking as lateral capacity the one corresponding to the formation of two plastic hinges, one at pile head and one at some depth, along with the full mobilization of soil resistance in between the hinges. However, one should also verify by means of a pile-soil interaction analysis that this applied horizontal load does not correspond to a moment higher

Fig. 11 **a** Q_u - M_u interaction domains for combined loads, numerical approach; **b** comparison between the linear approximation and the numerical approach

Fig. 12 Upper and lower bound values of horizontal capacity for highly loaded piles

than the sectional capacity. The first failure mechanism is here referred as ‘geotechnical’ failure, while the horizontal load leading to the yielding moment at the top according to an interaction mechanism is indicated in the ensuing as ‘structural’ capacity.

With reference to a reinforced concrete single pile, a parametric study employing the numerical approach has been carried out with reference to three idealized but realistic subsoils, to discuss the role of structural and geotechnical failure in the design of a foundation under combined loads.

The subsoils considered in the analyses are listed below:

- soft clay, for which the undrained shear strength, s_u , linearly increases with depth according to the relation $s_u = 10 + 3z$ kPa, z being the depth in meters, while the effective Young’s modulus to be considered for the interaction analysis, E , is proportional to z , $E = 2.6z$ MPa;
- stiff clay, with a constant $s_u = 100$ kPa and $E = 10.5z$ MPa;
- loose sand, with a constant friction angle $\varphi = 30^\circ$ and $E = 23.1z$ MPa.

Note that soil Young’s modulus is always taken proportional to depth to take into account the larger non-linear effects due to higher displacement close to pile head.

In the analysis, three different diameters (0.5 m, 1.0 m and 1.5 m) are employed and the piles are considered to be subjected to an axial force equal to:

$$N = \alpha f_{ck}' A_c \quad (17)$$

where $\alpha = 0.15; 0.3$.

The concrete reinforcement adopted is in the range $A_s = 0.3 \div 2.5\% A_c$.

The structural ultimate load, H_{us} , is obtained employing the relationship proposed by Di Laora and Rovithis [8] derived by fitting rigorous Finite Element results, in which the active length is calculated by adapting the expressions proposed by Di Laora and Rovithis [7] to consider that the soil stiffness is proportional to depth:

$$H_{us} = \frac{M_y}{0.305 d \left(\frac{E_p}{E_{sd}} \right)^{\frac{1}{3}}} \quad (18)$$

where E_p and E_{sd} are the Young’s modulus of the concrete and of the soil at the depth of one pile diameter, respectively.

The geotechnical ultimate load, H_{ug} , is evaluated with reference to the soil resistance profiles reported in Fig. 13, i.e. by writing the translational and rotational equilibrium equations for $p_u = 9 s_u$, for both soft and stiff clay, and $p_u = 3k_p \gamma z$, for loose sand.

From Fig. 14, it is possible to notice that the structural ultimate load is lower than the geotechnical capacity ($H_{us}/H_{ug} < 1$) for clayey soils and low reinforcement. On the contrary, in the case of loose sand, the geotechnical failure precedes the structural one as evident in Fig. 14. However, even in this case, the ratios H_{us}/N are much larger than the ones commonly encountered in design (say from 0.10 to 0.25) and, therefore, both geotechnical and structural failure are far from the design loads.

It is highlighted that the parametric study refers to failure conditions. If the same analysis were carried out with reference to the design conditions employing the approach proposed, for example, by the Eurocodes, given that the partial factors for the structural checks are lower than those for the geotechnical checks, in many cases H_{us}/H_{ug} would be larger than 1, i.e. the geotechnical failure could rule pile design for high lateral loads (as compared to the axial force). For example, let us consider a bored pile embedded in the soft clay described above and subjected to the design vertical load $N_d = 1.2$ MN and to the design horizontal load $H_d = 0.3$ MN. The use of Eq. (18) employing the design strengths $f_{cd} = f_{ck}'/\gamma_c$ and $f_{yd} = f_{yk}/\gamma_{st}$ (with $\gamma_c = 1.5$ and $\gamma_{st} = 1.15$ the partial resistance factors for concrete and steel, respectively), leads to $d = 1$ m and $A_s = 0.1\% A_c$ as a possible solution which satisfy structural requirements (i.e., the structural design resistance is larger than the design action). Adopting the Design Approach 2 of Eurocode 7 [9] (from the Italian National annex, $\gamma_s = 1.15$, $\gamma_b = 1.35$, $\gamma_T = 1.3$, $\gamma_{su} = 1$ and $\xi_3 = 1.7$, with γ_s and γ_b the partial resistance factors for pile shaft and base capacities, respectively, γ_T the partial resistance factor for transverse load, γ_{su} the partial factor for undrained shear strength and ξ_3 the correlation factor to derive the characteristic value from ground test results), and assuming a saturated unit weight of the soil $\gamma_{sat} = 18$ kN/m³, a unit

Fig. 13 p_u profiles: **a** soft clay; **b** stiff clay; **c** loose sand

Fig. 14 Structural to geotechnical ultimate load and horizontal to axial load ratios vs. reinforcement ratio using $f_{ck} = 25$ MPa and $f_{yk} = 450$ MPa

shaft resistance $\tau_{lim} = 0.5 s_u$ and a pile length $L = 25$ m, the design resistance of the pile obtained from a total stress analysis is $R_{c,d} = 1.36$ MN. The geotechnical check carried out employing Eq. (4) leads to a design horizontal geotechnical resistance equal to 0.26 MN, which is lower than the structural resistance (and therefore than the design load), contrary to the case of failure conditions where the opposite holds regardless of the axial force at which the sectional yielding moment is evaluated. However, even in this circumstance geotechnical checks do not rule the design, given that using the minimum reinforcement prescribed by Codes ($A_s > 0.3\% A_c$), the design structural and geotechnical resistances are equal to 0.39 MN and to 0.31 MN and therefore are both larger than H_d .

6 Discussion and conclusions

The work at hand proposes a simplified methodology to derive the failure envelope for pile groups under the combined action of vertical force, horizontal force and moment. The proposed method has been developed under the assumption, generally adopted in routine engineering also in commercial software employing elastic–plastic springs, that lateral and axial capacity of single piles do not affect each other.

The construction of axial force—moment interaction diagram at failure in absence of horizontal forces has been shown in Di Laora et al. [5]; the present work extends the same concepts to the presence of a lateral load. In addition to the two fundamental ingredients needed for the above diagram (axial capacity in compression and in tension) the proposed method requires only two further ingredients, that is the lateral capacity of a fixed-head pile under its ultimate axial load in compression and in tension, which may be assessed through simplified methods existing in literature such as the Broms [1, 2] formulae. The two capacities are different due to the dependence on the sectional yielding moment, which is in turn function of the axial load and can be determined from a closed-form expression proposed in Di Laora et al. [6]. Once obtained these two quantities, the ultimate lateral load may be established through a simple formula.

The method has been checked against results from an ad-hoc numerical procedure accounting for the actual distribution of axial load upon piles under a (Q, M) pair applied upon pile group. The procedure is always conservative and returns results very close to the numerical ones, except for very large axial loads as compared to the design strength of structural section, like in the case of end-bearing piles, where the procedure is still conservative but deviates more from the rigorous approach. In this case, a

closed-form solution is furnished, which is exact for zero moment and represents an upper bound for eccentric loads.

Note that the interaction domains are fully employable within any design regulation. To overcome specific assumptions on partial safety factors prescribed by different regulations, the above domains have been calculated at failure. To derive the interaction diagrams representing the design resistance, it is only necessary to employ in the procedure the design axial capacities instead of the failure loads.

In addition, a parametric study has been carried out with reference to three realistic soil profiles regarding the geotechnical vs structural failure of a fixed-head axially-loaded reinforced concrete pile under lateral load. More specifically, the horizontal load compatible, in the realm of a Limit Equilibrium approach, with two sectional yielding moment along the pile and ultimate soil pressure in between the hinges (geotechnical failure) is compared to the horizontal load that, by an interaction analysis, leads to the yielding moment at pile top (structural failure). It is found that generally the structural failure anticipates the geotechnical collapse for ordinary reinforcement ratios, except for very soft soils. Nevertheless, even in this case, the horizontal failure load is a very large fraction of the axial load (above 30%) and this represents a rare scenario, at least for seismic design. This leads to the important conclusion that, once the section reinforcement has been established by structural considerations, the geotechnical checks are often redundant. However, this last statement should be interpreted as an indication rather than an operative suggestion possessing general validity since, depending on the specific design regulation, partial factors may be different for structural and geotechnical Ultimate Limit State and the latter in some cases may rule the design, especially for large lateral to axial load ratios.

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