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Identification of Markovian open system dynamics for qubit systems

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We discuss the problem of identification of the dynamical generators for open two-level quantum systems in a Markovian environment. Based on Bloch sphere representation, the identification problem is converted to the estimation of a 3-dimensional real process matrix A and an inhomogeneous term $\boldsymbol{c}$. The parameter identifiability and sufficient conditions for completely identification of A and $\boldsymbol{c}$ are obtained. Further discussion shows that the obtained sufficient conditions are not always necessary. The approach can be generalized to finite-level open quantum systems in an arbitary Markovian environment.

open quantum systems, system identification, Laplace transform, projective measurement

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The characterization of unknown quantum systems and processes is a key problem in quantum information and quantum control [1–3]. In the quantum information domain many techniques have been developed for the characterization of quantum dynamical maps, such as standard quantum process tomography (SQPT) [1, 4, 5], ancilla-assisted process tomography (AAPT) [6–8] and direct characterization of quantum dynamics (DCQD) [9]. All these procedures are often known as quantum process tomography (QPT) [1, 10]. QPT schemes are often used to characterize a global process by estimating the parameters of a superoperator. In the quantum control domain, however, what we are more interested in are generators of the dynamical evolution, such as the Hamiltonian and dissipation generators. The problem of Hamiltonian tomography (HT), in particular direct Hamiltonian estimation from projective measurements, has recently been discussed by several authors [11–15]. Some generalizations to dissipative systems have also been considered [16, 17] but in general these have assumed a particular decoherence model such as dephasing in the Hamiltonian basis [17] or relaxation in a fixed computational basis [16].

Here we consider the generalization of these techniques for two-levels when there are no assumptions on the dissipative processes except that they are Markovian. The evolution of an open quantum system in a Markovian environment can be characterized by the first standard form given in [18] or the Lindblad equation given in [19]. Alternatively, the state of a two-level open system can be characterized by a real vector $\boldsymbol{r}$ and its evolution in a Markovian environment characterized by a system of first order (inhomogeneous) linear differential equations with constant coefficients, which forming a 3-dimensional real matrix A and a 3-dimensional real vector $\boldsymbol{c}$. A and $\boldsymbol{c}$ can be explicitly related to the Hamiltonian and dissipation parameters in the first standard form of the master equation, and therefore the problem of identifying the dynamics of a quantum system in an unknown Markovian environment is equivalent to identifying A and $\boldsymbol{c}$.

When the type of dissipative processes is known then the problem of fully characterizing the dynamics can often be reduced to estimating a small number of parameters such as

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dephasing rates. When there is no a-priori knowledge about the structure of the environment, however, then a large number of parameters have to be simultaneously estimated, and the question arises what resources are required to achieve this, or alternatively, give certain fixed resources, what parameters are identifiable. Here we analyze the problem of identifiability of A and $\boldsymbol{c}$ from time series data obtained from projective measurements. Based on the Bloch sphere representation, the projective measurement is characterized by a 3-dimensional real unit vector $\boldsymbol{m}$. Using the Laplace transform we show that a sufficient condition for complete identification of A and $\boldsymbol{c}$ is that at least three linearly independent projective measurements $\boldsymbol{m}_q (q = 1, 2, 3)$ and three different initial states $\boldsymbol{r}_p(0) (p = 1, 2, 3)$ be permitted, and $\{\boldsymbol{r}_p(0) - \boldsymbol{r}_s\}$ are linearly independent where $\boldsymbol{r}_s$ is a steady state of the system. This condition is also a sufficient condition in the time-domain but it is not always necessary, i.e. experimental requirements can often be relaxed in practice.

The rest of this paper is organized as follows. In Section 1, the system identification problem for open two-level quantum systems in a Markovian environment is characterized. In Section 2 an estimation method based on the Laplace tansform is given. The identifiability of parameters based on this method is analyzed and a sufficient condition for complete identification is obtained. A time-domain method for estimation is also considered, and examples are used to illustrate that the sufficient condition is not always necessary. The conclusions are summarized in Section 3.

1 System identification problem

The evolution of an N-level open quantum system in a Markovian environment is governed by a master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\frac{i}{\hbar}[H,\rho] + \sum_{n,m=1}^{N^2-1} f_{nm} D(F_m, F_n), \tag{1}$$

where H is a Hermitian operator representing the Hamiltonian of the system and $\rho(t)$ is a unit-trace positive operator representing the state of the system at time t. The dissipation generators are given by [18,20]

$$D(F_m, F_n) = F_n\rho F_m^\dagger - \frac{1}{2}\left(\rho F_m^\dagger F_n + F_m^\dagger F_n \rho\right), \tag{2}$$

where the operators $F_n (n = 1, \dots N^2 - 1)$ and $F_{N^2} = \frac{1}{\sqrt{2}}\boldsymbol{I}_2$ are orthonormal

$$(F_n, F_m) \equiv \mathrm{tr}\{F_n^\dagger F_m\} = \delta_{nm} \tag{3}$$

and form a complete basis of the linear operators on N-dimensional Hilbert space. The matrix formed by the coefficients f_{nm} is Hermitian and positive.

For a two-level system we can characterize the system state by a real vector $\boldsymbol{r}$

$$\boldsymbol{r} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\sigma_x\rho) \\ \mathrm{tr}(\sigma_y\rho) \\ \mathrm{tr}(\sigma_z\rho) \end{pmatrix}, \tag{4}$$

where $\sigma_i (i = x, y, z)$ are the Pauli operators. Based on the properties of the Pauli operators, the Hermitian operator H can be written as

$$\frac{H}{\hbar} = h_0\boldsymbol{I}_2 + h_x\sigma_x + h_y\sigma_y + h_z\sigma_z, \tag{5}$$

where the coefficients $h_j (j = 0, x, y, z)$ are real, and F_n can be chosen as $F_1 = \frac{\sqrt{2}}{2}\sigma_x$, $F_2 = \frac{\sqrt{2}}{2}\sigma_y$, $F_3 = \frac{\sqrt{2}}{2}\sigma_z$.

By means of eqs. (1)–(2) and eqs. (4)–(5), the evolution of two-level Markovian open quantum systems can be expressed as

$$\dot{\boldsymbol{r}}(t) = A\boldsymbol{r}(t) + \boldsymbol{c}. \tag{6}$$

Setting $f_{mn} = f_{mn}^R + i f_{mn}^I$ where (f_{mn}^R) is a real symmetric matrix and (f_{mn}^I) are real-anti-symmetric matrix, we have

$$A = \begin{pmatrix} -(f_{22}^R + f_{33}^R) & f_{12}^R - 2h_z & f_{13}^R + 2h_y \\ f_{12}^R + 2h_z & -(f_{11}^R + f_{33}^R) & f_{23}^R - 2h_x \\ f_{13}^R - 2h_y & f_{23}^R + 2h_x & -(f_{11}^R + f_{22}^R) \end{pmatrix},$$
$$\boldsymbol{c} = 2\begin{pmatrix} -f_{23}^I \\ f_{13}^I \\ -f_{12}^I \end{pmatrix}. \tag{7}$$

Eq. (6) implies that the evolution of $\boldsymbol{r}(t)$ is completely characterized by A and $\boldsymbol{c}$ and eq. (7) further indicates that there is a one-to-one correspondence between the Hamiltonian and dissipation parameters and the matrix A and vector $\boldsymbol{c}$ with respect to the Pauli basis. From eq. (7), it is obvious that $f_{nm}(n, m = 1, 2, 3)$ and $h_j (j = x, y, z)$ can be obtained from A and c. Thus, the identifiability problem is equivalent to the estimation of the Bloch matrix A and the vector $\boldsymbol{c}$.

2 Parameter identifiability

2.1 Information from projective measurements

If the system is initialized in the state $\boldsymbol{r}(0)$ and measured by projective measurement $\{|m_0\rangle\langle m_0|, |m_1\rangle\langle m_1|\}$ at time $t_j (j = 1, \cdots N_M)$, then the information obtained is

$$\begin{cases} p_0(t_j) = \mathrm{tr}\{|m_0\rangle\langle m_0|\rho(t_j)\}, \\ p_1(t_j) = \mathrm{tr}\{|m_1\rangle\langle m_1|\rho(t_j)\}, \end{cases} \tag{8}$$

where $p_k(t_j)$ is the probability of obtaining the output k at time t_j and $p_0(t_j) + p_1(t_j) = 1$. Let

$$r_{\boldsymbol{m}}(t_j) = p_0(t_j) - p_1(t_j) = \mathrm{tr}\{(|m_0\rangle\langle m_0| - |m_1\rangle\langle m_1|\rho(t_j))\}, \tag{9}$$

then $r_{\boldsymbol{m}}(t)$ contains all information obtained at time t_j. Expanding this measurement operator with respect to the Pauli matrix

$$|m_0\rangle\langle m_0| - |m_1\rangle\langle m_1| = m_x\sigma_x + m_y\sigma_y + m_z\sigma_z, \tag{10}$$

we obtain

$$r_{\boldsymbol{m}}(t_j) = m_x r_x(t_j) + m_y r_y(t_j) + m_z r_z(t_j) = \boldsymbol{m}^T \boldsymbol{r}(t_j), \tag{11}$$

where $\boldsymbol{m} = (m_x, m_y, m_z)^T$. Note that $\boldsymbol{m}$ is a real vector and $\boldsymbol{m}^T\boldsymbol{m} = 1$. Thereby, the information obtained from projective measurement is $\{r_{\boldsymbol{m}}(t_j) : j = 1, \cdots N\}$.

2.2 Laplace transform estimation of A and $\boldsymbol{c}$

As quantum states form a bounded convex set and the evolution operator must map quantum states to quantum states, it can be shown that quantum systems governed by a Markovian master equation always have at least one steady state. In particular, the completely mixed states $\boldsymbol{r} = 0$ is a steady state of the system only when $\boldsymbol{c} = 0$.

In the long-term limit the state of the system will either converge to a steady state or a limit cycle around a steady state $\boldsymbol{r}_s$. When there are three different projective measurement $\boldsymbol{m}_j (j = 1, 2, 3)$, eq. (11) gives

$$\begin{pmatrix} r^s_{\boldsymbol{m}_1} \\ r^s_{\boldsymbol{m}_2} \\ r^s_{\boldsymbol{m}_3} \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{pmatrix} \boldsymbol{r}_s. \tag{12}$$

Thus $\boldsymbol{r}_s$ can be obtained when $\boldsymbol{m}_j (j = 1, 2, 3)$ are linearly independent and

$$\boldsymbol{c} = -A\boldsymbol{r}_s. \tag{13}$$

Let

$$\boldsymbol{x}(t) = \boldsymbol{r}(t) - \boldsymbol{r}_s, \tag{14}$$

then the eq. (6) can be rewritten as

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t). \tag{15}$$

Taking the Laplace transform over eq. (15) gives

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = A\boldsymbol{X}(s), \tag{16}$$

where $\boldsymbol{X}(s)$ is the Laplace transform of $\boldsymbol{x}(t)$. If there exist three different initial states $\boldsymbol{r}_j(0)$, $j = 1, 2, 3$, with corresponding Laplace transforms $\boldsymbol{X}_j(s)$, then we can use the evolution equations (16) for the three initial states and get

$$(s\boldsymbol{I} - A)[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)] = [\boldsymbol{x}_1(0), \boldsymbol{x}_2(0), \boldsymbol{x}_3(0)]. \tag{17}$$

Thus if $\det[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)] \neq 0$, we formally have

$$A = s\boldsymbol{I} - [\boldsymbol{x}_1(0), \boldsymbol{x}_2(0), \boldsymbol{x}_3(0)]\,[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)]^{-1}. \tag{18}$$

Hence, if we can estimate the Laplace transforms $\boldsymbol{X}_j(s)$ for three initial states $\boldsymbol{r}_j(0)$, $j = 1, 2, 3$, for any value of s, we can reconstruct A, provided the linear independence condition $\det[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)] \neq 0$ holds. $\boldsymbol{c}$ can be reconstructed accordingly by eq. (13).

In principle, the Laplace transform $\boldsymbol{R}(s)$ corresponding to $\boldsymbol{r}(t)$ can be estimated from time series data for $r_{\mathbf{m}}(t_j)$. For example, the Laplace transform of the measurement signals $r_{\mathbf{m}}(t)$ can be estimated via

$$R_{\boldsymbol{m}}(s) = \int_0^\infty r_{\boldsymbol{m}}(t)e^{st}\,\mathrm{d}t \cong \sum_{j=1}^{N} r_{\boldsymbol{m}}(t_j)e^{st_j}\Delta t_j \tag{19}$$

for sufficiently large N_M and sufficiently small $\Delta t_j = t_j - t_{j-1} (j = 1, \cdots N_M)$. From eq. (11) we have $R_{\boldsymbol{m}}(s) = \boldsymbol{m}^T\boldsymbol{R}(s)$. If there are three projective measurements $\boldsymbol{m}_j (j = 1, 2, 3)$,

$$\begin{pmatrix} R_{\boldsymbol{m}_1}(s) \\ R_{\boldsymbol{m}_2}(s) \\ R_{\boldsymbol{m}_3}(s) \end{pmatrix} = \begin{pmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{pmatrix} \boldsymbol{R}(s). \tag{20}$$

If $\det(\boldsymbol{m}_1\ \boldsymbol{m}_2\ \boldsymbol{m}_3) \neq 0$ then

$$\boldsymbol{R}(s) = \begin{pmatrix} \boldsymbol{m}_1^T \\ \boldsymbol{m}_2^T \\ \boldsymbol{m}_3^T \end{pmatrix}^{-1} \begin{pmatrix} R_{\boldsymbol{m}_1}(s) \\ R_{\boldsymbol{m}_2}(s) \\ R_{\boldsymbol{m}_3}(s) \end{pmatrix}. \tag{21}$$

Thus, $\boldsymbol{R}(s)$ can be estimated from the results of projective measurements of the system state at different times when three different projective measurements satisfy $\det[\boldsymbol{m}_1\ \boldsymbol{m}_2\ \boldsymbol{m}_3] \neq 0$. $\boldsymbol{X}(s) = \boldsymbol{R}(s) - \boldsymbol{r}_s/s$ can be obtained accordingly.

Theorem. The Bloch operator A for a two-level system governed by the master equation (7) can be completely identified if the system can be initialized in states $\{\boldsymbol{r}_1(0), \cdots \boldsymbol{r}_p(0)\}$ and measured by the projective measurements $\{\boldsymbol{m}_1, \cdots \boldsymbol{m}_q\}$ satisfying

$$\mathrm{span}\{\boldsymbol{r}_1(0) - \boldsymbol{r}_s, \cdots \boldsymbol{r}_p(0) - \boldsymbol{r}_s\} = \mathrm{span}\{\boldsymbol{m}_1, \cdots \boldsymbol{m}_q\} = \mathbb{R}^3, \tag{22}$$

where $\boldsymbol{r}_s$ is one steady state of the system, $p, q \geqslant 3$ and $\mathbb{R}^3$ is 3-dimensional real space.

Proof. From the analysis given above, we can infer that for $\mathrm{span}\{\boldsymbol{m}_1, \cdots \boldsymbol{m}_q\} = \mathbb{R}^3$, $\boldsymbol{X}(s)$ can be obtained. Thus A is identifiable when $\det[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)] \neq 0$ is satisfied. By computing the determinant of both sides of eq. (17), it is obtained that

$$\det(s\boldsymbol{I} - A)\det[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)] = \det[\boldsymbol{x}_1(s), \boldsymbol{x}_2(0), \boldsymbol{x}_3(0)].$$

Then $\det[\boldsymbol{X}_1(s), \boldsymbol{X}_2(s), \boldsymbol{X}_3(s)] \neq 0$ only when $\det[\boldsymbol{x}_1(0), \boldsymbol{x}_2(0), \boldsymbol{x}_3(0)] \neq 0$ is satisfied. When $\mathrm{span}\{\boldsymbol{r}_1(0) - \boldsymbol{r}_s, \cdots \boldsymbol{r}_p(0) - \boldsymbol{r}_s\} = \mathbb{R}^3$, there exist at least three elements in the set which make $\det[\boldsymbol{x}_1(0), \boldsymbol{x}_2(0), \boldsymbol{x}_3(0)] \neq 0$. Thus A is identifiable. Accordingly, $\boldsymbol{c}$ is identifiable. □

2.3 Time domain estimation of A and c

Alternatively, we can also consider direct time-domain estimation of A. From eq. (15) we know that

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) = C\boldsymbol{e}(t), \tag{23}$$

where C is a 3-dimensional real matrix determined by the eigenvectors of A and the initial state $\boldsymbol{r}(0)$, and $\boldsymbol{e}(t)$ is time-dependent 3-dimensional real vector determined by the eigenvalues and Jordan normal form of A. The possible $\boldsymbol{e}(t)$ are given in Table 1.

Hence, we can estimate A in time domain by following steps:

(1) Determine the type of $\boldsymbol{e}(t)$ of the unknown system and estimate the coefficient matrix C.

(2) Estimate A from $\boldsymbol{e}(t)$ and C via

$$A = [C_1\dot{\boldsymbol{e}}(t)\ C_2\dot{\boldsymbol{e}}(t)\ C_3\dot{\boldsymbol{e}}(t)][C_1\boldsymbol{e}(t)\ C_2\boldsymbol{e}(t)\ C_3\boldsymbol{e}(t)]^{-1} \tag{24}$$

when $\det[C_1\boldsymbol{e}(t), C_2\boldsymbol{e}(t), C_3\boldsymbol{e}(t)] \neq 0$, where C_j is corresponding to different initial state $\boldsymbol{r}_j(0)$.

The full coefficient matrix C in the first step can be obtained if three linearly independent projective measurement $\boldsymbol{m}_p(p = 1,2,3)$ are available. Eq. (23) shows that $\det[C_1\boldsymbol{e}(t), C_2\boldsymbol{e}(t), C_3\boldsymbol{e}(t)] \neq 0$ only when $\boldsymbol{x}_j(0)(j = 1,2,3)$ are linearly independent, i.e. $\boldsymbol{r}_j(0)-\boldsymbol{r}_s(j = 1,2,3)$ are linearly independent. $\boldsymbol{c}$ can be reconstructed accordingly by eq. (13).

Comparing the time domain method with the Laplace transform method, we find that the key point of the time domain method is to estimate $\boldsymbol{e}(t)$ while the Laplace transform approach requires computation of $\boldsymbol{R}(s)$ from the measurement signals $r_{\boldsymbol{m}}(t_j)$. The sufficient condition in the Theorem is also obtained from the time-domain method.

2.4 Necessity of the condition in Theorem

In many cases there are restrictions with regard to the initial states which we can prepare prior to characterization of the dynamics. Suppose that one steady state of the system is 0 and the system can only be initialized in a fixed initial state $\boldsymbol{r}(0)$, then $\boldsymbol{c} = 0$ and $\boldsymbol{x}(t) = \boldsymbol{r}(t)$. Let us assume $\boldsymbol{r}(0) = (0,0,1)^T$. If we still have three linearly independent projective measurements $\boldsymbol{m}_j(j = 1,2,3)$ at our disposal then $\boldsymbol{R}(s)$ can be estimated by eq. (21). In particular, given the same time domain data, we can compute $\boldsymbol{R}(s)$ for different values of s. By eq. (16), if we choose three different s_1, s_2, s_3 then

$$\begin{aligned} &A[\boldsymbol{R}(s_1), \boldsymbol{R}(s_2), \boldsymbol{R}(s_3)] \\ &\quad = [s_1\boldsymbol{R}(s_1) - \boldsymbol{r}(0), s_2\boldsymbol{R}(s_2) - \boldsymbol{r}(0), s_3\boldsymbol{R}(s_3) - \boldsymbol{r}(0)]. \end{aligned} \tag{25}$$

Provided $\det[\boldsymbol{R}(s_1)\ \boldsymbol{R}(s_2)\ \boldsymbol{R}(s_3)] \neq 0$, eq. (25) gives

$$\begin{aligned} A = &[s_1\boldsymbol{R}(s_1) - \boldsymbol{r}(0), s_2\boldsymbol{R}(s_2) - \boldsymbol{r}(0), s_3\boldsymbol{R}(s_3) - \boldsymbol{r}(0)] \\ &\times [\boldsymbol{R}(s_1), \boldsymbol{R}(s_2), \boldsymbol{R}(s_3)]^{-1}. \end{aligned} \tag{26}$$

Assuming $A = (a_{nm})(n,m = 1,2,3)$, by eq. (16) we can get

$$\begin{aligned} &\boldsymbol{R}(s) = (s\boldsymbol{I} - A)^{-1}\boldsymbol{r}(0) \\ &= \frac{\begin{pmatrix} a_{12}a_{23} - a_{13}a_{22} & a_{13} & 0 \\ a_{13}a_{21} - a_{23}a_{11} & a_{23} & 0 \\ a_{11}a_{22} - a_{12}a_{21} & -(a_{11} + a_{22}) & 1 \end{pmatrix}\begin{pmatrix} 1 \\ s \\ s^2 \end{pmatrix}}{\det(s\boldsymbol{I} - A)} \end{aligned} \tag{27}$$

for s which are not eigenvalues of A. Then for $s_j(j = 1,2,3)$ which are not eigenvalues of A, we still have

$$\begin{aligned} &[\det(s_1\boldsymbol{I} - A)\boldsymbol{R}(s_1)\ \det(s_2\boldsymbol{I} - A)\boldsymbol{R}(s_2)\ \det(s_3\boldsymbol{I} - A)\boldsymbol{R}(s_3)] \\ &= \begin{pmatrix} a_{12}a_{23} - a_{13}a_{22} & a_{13} & 0 \\ a_{13}a_{21} - a_{23}a_{11} & a_{23} & 0 \\ a_{11}a_{22} - a_{12}a_{21} & -(a_{11} + a_{22}) & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{pmatrix} \\ &\Rightarrow \det[\boldsymbol{R}(s_1)\ \boldsymbol{R}(s_2)\ \boldsymbol{R}(s_3)]\Pi_{j=1}^3 \det(s_j\boldsymbol{I} - A) \\ &= \begin{vmatrix} a_{12}a_{23} - a_{13}a_{22} & a_{13} \\ a_{13}a_{21} - a_{23}a_{11} & a_{23} \end{vmatrix}(s_2 - s_1)(s_3 - s_1)(s_3 - s_2). \end{aligned} \tag{28}$$

The condition $\det[\boldsymbol{R}(s_1), \boldsymbol{R}(s_2), \boldsymbol{R}(s_3)] \neq 0$ can only be satisfied if

$$\begin{vmatrix} a_{12}a_{23} - a_{22}a_{13} & a_{13} \\ a_{13}a_{21} - a_{11}a_{23} & a_{23} \end{vmatrix} \neq 0. \tag{29}$$

More generally, if we can prepare the initial state $\boldsymbol{r}(0)$, there exists a real orthogonal transformation matrix T_I that maps $\boldsymbol{r}(0)$ to $T_I\boldsymbol{r}(0) = (0,0,r_0)^T$ where $0 < r_0 \leqslant 1$ and eq. (7) can be represented as

$$T_I\boldsymbol{R}(s) = (s\boldsymbol{I} - T_IAT_I^T)^{-1}T_I\boldsymbol{r}(0). \tag{30}$$

In particular we have

$$\det[\boldsymbol{R}(s_1)\ \boldsymbol{R}(s_2)\ \boldsymbol{R}(s_3)] = \det[T_I\boldsymbol{R}(s_1)\ T_I\boldsymbol{R}(s_2)\ T_I\boldsymbol{R}(s_3)], \tag{31}$$

and we can find three different s_1, s_2, s_3 such that $\det[\boldsymbol{R}(s_1), \boldsymbol{R}(s_2), \boldsymbol{R}(s_3)] \neq 0$ only when

$$\begin{vmatrix} a^I_{12}a^I_{23} - a^I_{22}a^I_{13} & a^I_{13} \\ a^I_{13}a^I_{21} - a^I_{11}a^I_{23} & a^I_{23} \end{vmatrix} \neq 0, \tag{32}$$

Table 1 Relation of $\boldsymbol{e}(t)$ and Jordan normal form of A with γ_m, ω real

Jordan form of A	$\boldsymbol{e}^T(t)$
$\begin{pmatrix} \gamma_1 & 1 & 0 \\ 0 & \gamma_1 & 1 \\ 0 & 0 & \gamma_1 \end{pmatrix}$	$(e^{\gamma_1 t}, te^{\gamma_1 t}, t^2e^{\gamma_1 t})$
$\begin{pmatrix} \gamma_1 & 1 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \gamma_2 \end{pmatrix}$	$(e^{\gamma_1 t}, te^{\gamma_1 t}, e^{\gamma_2 t})$
$\mathrm{diag}(\gamma_1, \gamma_2, \gamma_3)$	$(e^{\gamma_1 t}, e^{\gamma_2 t}, e^{\gamma_3 t})$
$\mathrm{diag}(\gamma_1, \gamma_2 + i\omega, \gamma_2 - i\omega)$	$(e^{\gamma_1 t}, e^{\gamma_2 t}\cos(\omega t), e^{\gamma_2 t}\sin(\omega t))$

where $T_I A T_I^T = (a_{ij}^I)$.

This example shows that knowledge of the trajectory of a single initial state is sufficient for complete identification, so the sufficient condition for identifiability is not strictly necessary.

3 Conclusion

The evolution of an open two-level quantum systems subject to a Markovian environment is characterized by a system of first order linear differential equations with constant coefficients, which form a three-dimensional real matrix A, and an inhomogeneous term $\boldsymbol{c}$. There is a one-to-one correspondence between A and $\boldsymbol{c}$ and the Hamiltonian and dissipation parameters. We have considered the problem of identifiability of the dynamic generator A and the vector c from time-series data of a set of observations. Based on the Laplace transform sufficient conditions for identifiability are obtained. The sufficient conditions also apply for the time domain estimation method. The experimental resource requirements can often be relaxed, however, i.e. the sufficient conditions are not always necessary. Furthermore, when there exists some prior information about the dynamics of the system to be identified then the resources for complete identification can be decreased (see e.g. [21]). The minimum requirements for complete identifiability of A under different experimental conditions need be analyzed further. The basic approach can be generalized to N-level systems.

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