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Available control in dynamical decoupled quantum systems

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Dynamical decoupling (DD) eliminates qubit-bath coupling by applying a sequence of instantaneous pulses. While qubit-bath couplings generally lead to qubit relaxation and dephasing, qubit-qubit couplings are often used to manipulate or control quantum states. We investigate the available control operations in two DD schemes, named periodic DD (PDD) and Uhrig DD (UDD), to see whether universal quantum computation can be realized in these decoupled systems. We find that universal control is possible using Heisenberg interaction in both periodically decoupled system and Uhrig decoupled system, and the available control operations under two kinds of DD sequences obey the same commutation relation. In the UDD case, we also derive a rough bound for control errors.

dynamical decoupling, PDD, UDD, universal control

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A quantum bit, commonly represented by a two-level quantum system, is constantly interacting with its surrounding environment. However, the coupling between qubit and environment is detrimental to quantum coherence, which is the crucial resource enabling quantum computation [1]. To protect qubits in quantum registers, dynamical decoupling (DD) methods were introduced to suppress decoherence by isolating the qubit from its environment [2]. During recent years, both theory and experimental progresses have been made to demonstrate the power of various DD sequences in preserving quantum states and generating coherent operations [3–9].

General DD methods employ sequence of discrete pulses to eliminate system-bath coupling. On the other hand, certain couplings need to be switched on for a finite time in order to perform quantum control operations. For example, qubit-qubit interaction is required when implementing two-qubit gates. Besides its ability to keep quantum coherent states in quantum registers, to perform high-fidelity quantum computation on DD-proteced qubits without extra resources, dynamical decoupling process should leave the controlled interaction intact. We will see the ability to accomplish this task is closely related to the selected DD sequence used in different decoupling schemes.

The first type of DD sequence is provided by nuclear magnetic resonance spectroscopy, known as "periodic DD" (PDD) [2], which averages out the non-unitary evolution of a qubit by applying a predetermined sequence periodically. In a single period, the pulse locations are equidistant, and cycle time T_c is required to be as small as possible to ensure the higher-order terms vanish where the propagator during T_c can be expanded into a standard Magnus series. Recently, a kind of optimized DD sequence, Uhrig DD (UDD) [3], makes an enormous progress over PDD by canceling qubit-bath coupling order by order with each additional pulse. In UDD, pulse locations are not equidistant but optimized and restriction on $T_c \rightarrow 0$ is relaxed. It was first discovered for a pure dephasing spin-boson model, and later proved to be applicable to qubit relaxation as well [10].

In this paper, we first investigate the possibility of performing universal control over periodically decoupled quantum systems. Afterwards in Section 2, we analyze the controllability of Uhrig decoupled system. We manage to find out the available qubit-qubit interaction hamiltonian which can be used for quantum control and give a rough estimation of the control errors. Conclusions are put in Section 3.

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1 Control of periodically decoupled system

The general hamiltonian describing the qubit-bath coupling can be written as

$$H = H_{SB} + H_B,$$

where H_{SB} and H_B are interaction and bath hamiltonians respectively. We assume all the DD pulses are applied instantaneously, meaning that the corresponding hamiltonian can be turned on for negligible time with full strength. These control operations are called bang-bang controls [2, 11]. Obviously the realizable bang-bang controls constitute a finite subgroup of all the unitary transformations on the qubit subspace. A decoupling group $\mathcal{G}$ is chosen to be a finite subgroup of the bang-bang controls, with group elements g_j, $j = 0, ..., |\mathcal{G}| - 1$. To implement dynamical decoupling, cycle time T_c is divided into $|\mathcal{G}|$ slices, $T_c = |\mathcal{G}|\Delta t$, and decoupling pulses $D_j = g_j g_{j-1}^\dagger$ are applied exactly at $j\Delta t$. Thus the evolution in a cycle time T_c can be expressed as

$$U(T_c) = \prod_{j=0}^{|\mathcal{G}|-1} g_j^\dagger U_0(\Delta t) g_j = \mathrm{e}^{-iH_{\rm eff}T_c}, \quad (1)$$

$U_0(\Delta t)$ denoting the free evolution between decoupling pulses. $H_{\rm eff}$ is the resulting effective hamiltonian, which can be expanded as Magnus series

$$H_{\rm eff} = \sum_j H^{(j)}, \quad (2)$$

where $H^{(j)}$ is jth order contribution. As indicated in [2], in the limit of fast cycling time $T_c \to 0$, $H_{\rm eff}$ approaches the zeroth term

$$H_{\rm eff} \to H^{(0)} = \frac{1}{|\mathcal{G}|} \sum_{g_j \in \mathcal{G}} g_j^\dagger H g_j = P_{\mathcal{G}}(H_{SB}). \quad (3)$$

This process is similar to the group averaging procedure, projecting the original interaction hamiltonian H_{SB} into the centralizer of $\mathcal{G}$:

$$Z(\mathcal{G}) = \{O | [O, g_j] = 0 \quad \forall g_j \in \mathcal{G}\}.$$

The averaged qubit operators are made invariant under the group action. More specifically, the effective interaction hamiltonian must obey the commutation relation

$$[H_{\rm eff}, H_D] = 0, \quad (4)$$

where H_D is the hamiltonian generating the decoupling group.

If no knowledge is available on the interaction hamiltonian H_{SB}, we have to perform the so-called maximal averaging [2]. Here transformations g_j of the control group $\mathcal{G}$ span the whole space of bounded operators on the qubit subspace and the centralizer $Z(\mathcal{G})$ is consequently trivial, which means $H_{\rm eff} = \lambda I \otimes H_B$. In this case, whatever "slow" controls we impose on the qubits, they will be quenched completely through dynamical decoupling. To achieve universal control, either a universal set of operations can be implemented bang-bang, or a universal set of hamiltonian can be switched very fast. In addition, these controls have to be synchronized with decoupling pulses, posing great challenge to present experimental technique. But if we have certain knowledge of H_{SB}, the situation will be quite different. We can choose the minimum decoupling group $\mathcal{G}$ such that

$$P_{\mathcal{G}}(H_{SB}) = \sum_j^N g_j H_{SB} g_j^\dagger = 0. \quad (5)$$

As long as $\mathcal{G}$ is not full rank, or in other words the qubit component in H_{SB} does not span the whole qubit subspace, the centralizer $Z(\mathcal{G})$ is non-trivial, $Z(\mathcal{G}) \neq \{I\}$, which corresponds to selective averaging. As a matter of fact, any hamiltonians contained in $Z(\mathcal{G})$ can be used for quantum control. By virtue of eq. (1), it is always possible to apply slowly any hamiltonian $H_c \in Z(\mathcal{G})$ in parallel with the decoupling cycle [12].

As an example to illustrate how to perform universal control under PDD, we consider two qubits which are coupled to environment by a linear interaction of the form

$$H_{SB} = \sum_{\alpha=x,y,z} \sum_{j=1,2} \sigma_j^\alpha \otimes \mathcal{B}_j^\alpha.$$

σ_α are Pauli operators. To perform selective decoupling, we choose decoupling group to be $\mathcal{G} = \{I, \otimes_j \sigma_j^x, \otimes_j \sigma_j^y, \otimes_j \sigma_j^z\}$ which contains 3 collective π-rotations. It can be easily verified that $P_{\mathcal{G}}(H_{SB}) = 0$. Noting that $\sum_j \sigma_j^x$, $\sum_j \sigma_j^y$, $\sum_j \sigma_j^z$ generates the group elements $\otimes_j \sigma_j^x, \otimes_j \sigma_j^y, \otimes_j \sigma_j^z$ respectively, two-qubit Heisenberg interaction $H_c = J\vec{\sigma}_1 \cdot \vec{\sigma}_2$ can be turned on simultaneously to implement two-qubit gates for quantum computation since $[H_c, \sum_j \sigma_j^\alpha] = 0$. It is trivial to extend these results to general Heisenberg interaction $H_c = \sum_{i<j} J_{ij} \vec{\sigma}_i \cdot \vec{\sigma}_j$ for large number of qubits. Heisenberg interaction is commonly used to swap any pair of qubits [13], and able to perform universal quantum computation on the decoupled system when combined with fast single-qubit operations. In fact, universal quantum computation is possible using only the Heisenberg interaction without physical level single-qubit gates, provided qubits are encoded into appropriate decoherence-free subspaces [14–16].

Here we have to mention that since $P_{\mathcal{G}}$ is a zeroth-order approximation of effective hamiltonian $H_{\rm eff}$, the control hamiltonian extracted from $Z_{\mathcal{G}}$ can not be performed perfectly. Higher-order corrections will produce non-negligible control errors. A rigorous bound has been derived for these error, which sets a scalability limit on the protection of quantum computation using periodic dynamical decoupling [17].

2 Control of Uhrig decoupled system

A powerful DD scheme was introduced recently by Uhrig to suppress pure dephasing and general decoherence, which optimizes relative pulse locations according to

$$T_j = T\sin^2\frac{j\pi}{2(N+1)}, \quad \text{for} \quad j=1,2,...N, \tag{6}$$

to eliminate qubit-bath coupling up to $O(T^{N+1})$. T is the qubit evolution time. However, the possibility of performing universal control while the qubits are protected by UDD has not been explored yet.

We begin by considering a pure dephasing model

$$H = C + \sigma_z \otimes Z$$

where σ_z is the qubit Pauli matrix along the z-direction and C and Z are arbitrary bath operators. This hamiltonian does not include qubit flip operators σ_x or σ_y so that it leads to pure dephasing. To counter dephasing process, a sequence of π-pulses are applied at T_j obeying eq. (6), which can be expressed as the following hamiltonian:

$$H_{DD} = \sum_{j=1}^{N}\pi\delta(t-T_j)\frac{\sigma_x}{2}. \tag{7}$$

Right after each π-pulse, spin σ_z changes its sign, evolving in the opposite direction. Actually this is the UDD version of selective averaging. The qubit state-dependent propagators under Nth order UDD are

$$U_\pm^N = \mathrm{e}^{-i[C\pm(-1)^N Z](T-T_N)}...\mathrm{e}^{-i[C\mp Z](T_2-T_1)}\mathrm{e}^{-i[C\pm Z]T_1}. \tag{8}$$

Using these expressions, qubit dephasing is characterized by

$$U_-^{N\dagger}U_+^N = I + U_-^{N\dagger}\delta U.$$

To analyze δU, $U_\pm^N$ are transformed in the interaction picture

$$U_\pm^N = \mathrm{e}^{-iCT}\Im\mathrm{e}^{-i\int_0^T \pm F_N(t)Z_I(t)\mathrm{d}t}, \tag{9}$$

where $\Im$ is the time-ordering operator, the modulation function $F_N(t) = (-1)^j$ for $t \in [T_j, T_{j+1}]$ with $T_0 = 0$ and $T_{N+1} = T$, and

$$Z_I(t) = \mathrm{e}^{iCt}Z\mathrm{e}^{-iCt} = \sum_{p=0}^{\infty}\frac{(it)^p}{p!}[C,[C,...[C,Z]...]].$$

Thus the difference δU is given by

$$\delta U = 2\mathrm{e}^{-iCT}\sum_{n=2k+1}^{\infty}(-i)^n\Delta_n, \tag{10}$$

with

$$\Delta_n = \int_0^T F_N(t_N)...\int_0^{t_2}F_N(t_1)[Z_I(t_N)...Z_I(t_1)]\mathrm{d}t_1...\mathrm{d}t_N.$$

Yang and Liu [10] proved the odd-order terms vanish $\Delta_{2k+1} = O(T^{N+1})$, which verifies that qubit pure dephasing is eliminated to Nth order.

While performing high-fidelity universal quantum control, the qubit-qubit interaction has to be turned on in parallel with the UDD sequence to implement two-qubit quantum gates. Here we explore the available qubit-qubit couplings under dynamical decoupling by considering two-qubit hamiltonian of the form

$$H = H_\mathrm{c} + C + (\sigma_z^1 + \sigma_z^2)\otimes Z,$$

where H_c is the qubit-qubit interaction hamiltonian. We have to analyze U_+^N and U_-^N separately to see what kinds of H_c can be applied under UDD. If the available H_c are state independent, then these are the controls we are seeking to implement quantum compuation. First to suppress pure dephasing, the UDD π-pulses are given by

$$H_{DD} = \sum_{j=1}^{N}\frac{\pi}{2}\delta(t-T_j)(\sigma_x^1 + \sigma_x^2). \tag{11}$$

Observing eqs. (8) and (10), we find that C and Z are not necessarily bath operators. In fact for any pair of operators satisfying the following conditions after each UDD pulse

$$C \to C, Z \to -Z,$$

the conclusion $\Delta_{2k+1} = O(T^{N+1})$ also hold true, which means that pure dephasing induced by qubit-bath coupling is canceled to order N. Meanwhile, qubit-qubit couplings included in Z are eliminated as well. We again consider the Heisenberg interaction $H_\mathrm{c} = \sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2$. Making use of the commutation relations $[H_{DD}, \sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2] = 0$, $\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2$ is incorporated into C. The two-qubit hamiltonian with Heisenberg coupling can be simplified as

$$H = C_0 + (\sigma_z^1 + \sigma_z^2)\otimes Z_0,$$

with

$$C_0 = C + \sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2.$$

The propagator $U_\pm^N$ becomes

$$U_\pm^N = \mathrm{e}^{-i[C_0\pm(-1)^N Z_0](T-T_N)}...\mathrm{e}^{-i[C_0\mp Z_0](T_2-T_1)}\mathrm{e}^{-i[C_0\pm Z_0]T_1}. \tag{12}$$

In interaction representation U_+^N is expanded into Taylor series

$$U_+^N = \mathrm{e}^{-iC_0T}\sum_{n=0}^{\infty}(-i)^n\Delta_n. \tag{13}$$

Although odd terms Δ_{2k+1} with $2k+1 \leqslant N$ vanish as usual, now even terms Δ_{2k} play a crucial role in the controlled system evolution. Below we will show that the error rate of the control operation is eliminated to at least second order in T. Defining the relative pulse locations δ_j by $T_j = T\delta_j$, we combine the exponentials in eq. (12) term by term using the formula [18]

$$\mathrm{e}^{TA}\mathrm{e}^{TB} = \mathrm{e}^{TA+TB+\frac{T^2}{2}[A,B]+O(T^3)}. \tag{14}$$

Setting

$$A = -i[C_0 + (-1)^N Z_0](1-\delta_N),$$
$$B = -i[C_0 + (-1)^{N-1} Z_0](\delta_N - \delta_{N-1}),$$

we have

$$\mathrm{e}^{-i[C_0+(-1)^N]T(1-\delta_N)}\mathrm{e}^{-i[C_0+(-1)^{N-1}]T(\delta_N-\delta_{N-1})} = \mathrm{e}^{-i\{C_0T(1-\delta_{N-1})-(-1)^{N-1}Z_0T[(1-\delta_N)-(\delta_N-\delta_{N-1})]\}+\frac{T^2}{2}C_1+O(T^3)},$$

with

$$C_1 = [A,B] = 2(-1)^N(1-\delta_N)(\delta_N-\delta_{N-1})[C_0,Z_0].$$

Because $[C_0, Z_0] = [C, Z_0]$ only contains a combination of bath operators, the second order term $\frac{T^2}{2}C_1$ has no relevance with the qubit evolution. In the next we will prove that after all the exponentials are combined, the second order terms are added to zero, bounding the control error rate up to $O(T^2)$ at least. To do this, we repeat above procedure by setting

$$A_1 = -i\{C_0T(1-\delta_{N-1})-(-1)^{N-1}Z_0T[(1-\delta_N)-(\delta_N-\delta_{N-1})]\} + \frac{T}{2}C_1 + O(T^2),$$
$$B_1 = -i[C_0 + (-1)^{N-2}Z_0](\delta_{N-1}-\delta_{N-2}).$$

Again using eq. (14), and to second order in T, we obtain the expression of three exponentials combined

$$\mathrm{e}^{-i\{C_0T(1-\delta_{N-1}+\delta_{N-1}-\delta_{N-2})+(-1)^{N-2}Z_0T[(1-\delta_N)-(\delta_N-\delta_{N-1})+(\delta_{N-1}-\delta_{N-2})]\}+\frac{T^2}{2}C_1+\frac{T^2}{2}C_2+O(T^3)} \quad (15)$$

with

$$C_2 = 2(-1)^{N-1}(\delta_N-\delta_{N-1})(\delta_{N-1}-\delta_{N-2})[C_0,Z_0].$$

By simply repeating this procedure N times, we find that the approximation form of U_+^N is

$$U_+^N = \mathrm{e}^{-iC_0T\lambda_1-iZ_0T\lambda_2} + T^2[C_0,Z_0]\lambda_3 + O(T^3), \quad (16)$$

where the parameters λ_i can be written in compact form

$$\lambda_1 = \sum_{n=0}^{N}(-1)^n\delta_{N-n},$$
$$\lambda_2 = \sum_{n=0}^{N}(-1)^n(\delta_{N-n+1}-\delta_{N-n}),$$
$$\lambda_3 = \sum_{n=0}^{N}(-1)^{N-n}(\delta_{N-n+1}-\delta_{N-n})\cdot(\delta_{N-n}-\delta_{N-n-1}).$$

It can be easily verified that $\lambda_1 = 1$, $\lambda_2 = 0$ and $\lambda_3 = 0$ for UDD pulse locations δ_j, so the propagator is reduced to

$$U_+^N = \mathrm{e}^{-iC_0T+O(T^3)}. \quad (17)$$

We can also prove the same result for U_-^N in a similar way. Now we can say that at the end of the evolution, Heisenberg interaction $\sigma_x^1\sigma_x^2 + \sigma_y^1\sigma_y^2 + \sigma_z^1\sigma_z^2$ generates the control operation of two-qubit system which is expressed by

$$U(T) = \mathrm{e}^{-i(\sigma_x^1\sigma_x^2+\sigma_y^1\sigma_y^2+\sigma_z^1\sigma_z^2)T+O(T^3)}, \quad (18)$$

indicating the possibility of performing universal control on Uhrig decoupled quantum systems. As in PDD, the available control operations should satisfy $[H_{DD}, H_c] = 0$. At this stage, the control error rate is bounded by $O(T^3)$. Since phase error is eliminated to Nth order by UDD, naturally we would expect to lower the bound of control error to $O(T^{N+1})$ as well. However, the odd order contributions $\sim T^{2k+1}$ can not be eliminated under UDD sequence. To make our controls more accurate, these higher order terms have to be considered. Generally they will introduce a drift term in the qubit evolution hamiltonian in eq. (18). If the drift term can be calculated explicitly, we can make corrections to the control hamiltonian to reduce control errors even further. Currently we are working towards this goal.

The above derivations are based on pure dephasing model. We now briefly discuss the general decoherence problem, where the interaction hamiltonian is modeled as $H = \sigma_x \otimes X + \sigma_y \otimes Y + \sigma_z \otimes Z$. In spite of the great advantages of UDD in dealing with pure dephasing or relaxation, it is difficult to design UDD sequence to cancel both of them. A near-optimal method has been proposed using sequence of the form [19]

$$Z^N X_N(T_N - T_{N-1}) Z X_N(T_{N-1}-T_{N-2}) \ldots Z X_N(T_1).$$

X, Z are π-pulses along axes z and x, and $X_N(T)$ denotes a Nth order UDD using X pulses in a period of time T. Z^N means if N is odd, a Z-pulse is added at the end of the sequence. This so-called quadratic DD (QDD) cancels pure dephasing in inner intervals $T_N - T_{N-1}, T_{N-1} - T_{N-2} \ldots, T_1$ of a common UDD, and relaxation on the secondary level by Z pulses. It is easy to see that operations that commute with X and Z can be performed in parallel with QDD. In paticular, Heisenberg interaction can be implemented.

3 Conclusion

In conclusion, we have proven that the condition for successfully performing control operations over periodically decoupled quantum systems also applies to the UDD case. Universal high-fidelity quantum computation is possible using the Heisenberg interaction while coherence is preserved by PDD or UDD sequence. Other proposed schemes which intend to implement high-fidelity quantum gates against decoherence, such as fault-tolerant quantum computation, quantum error-correction and decoherence-free subspace [20, 21], all require an unrealistic amount of resources. On the contrary, control while performing dynamical decoupling is less resource-consuming, thus made a suitable candidate to implement quantum computation in some circumstances.

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