Geometric algebra method for multidimensionally-unified GIS computation

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Seamless multidimensional handling and coordinate-free characteristics of geometric algebra (GA) provide means to construct multidimensionally-unified GIS computation models. Using the multivector representation for basic geometric objects within GA, we are able to construct adaptable unified geometric-topological structural models of a multidimensional geographical scene. Multidimensional operators found within the geometry, topology and GIS analysis are developed with basic GA operators. A unified computational framework is proposed, it unifies expressions and operation structures, as well as supports the analysis of multidimensional complex scenes. Finally, we illustrate modelling a three-dimensional residential district, which shows that GA-based multidimensionally-unified computation models can effectively represent and analyze complex and multidimensional geographical scenes. The development of the proposed GIS multidimensionally-unified representation, analysis, and modeling enhances current GIS algorithms and geographical models.

geometric algebra, GIS, multidimensionally-unified representation, multidimensionally-unified computation

The representation and arrangement of complex geographical objects and scenes are identified as important in the application and usability of geographic information systems (GIS) [1–4]. Object expression and code construction of operations is an important step in advancing GIS research from static characteristics to a more comprehensive research on temporal and spatially distributed structures, and modeling and forecasting of dynamic processes and mechanisms. It is also an indispensable process in achieving model integration and comprehensive use in GIS and professional areas [5–8]. There still exist many limitations within traditional GIS, based on Euclidean geometry, in maintaining complex representations of geographical objects, multidimensional spatial relations, and topographical analysis [9]. Constructing models that support multidimensionally-unified geographical object representations and operations based on

new mathematical theories is one possible way to reduce the multidimensional complexity of current GIS and to improve architectural efficiencies in its operations and analysis [10,11]. Geometric algebra (GA), known as a unifying descriptive language, can integrate algebra with geometry, mathematics with physics. By defining the spatial operation sets (e.g. geometry, measurement, topology) with GA operators, Euclidean, homogeneous, and conformal spaces can be described and converted among each other, this provide a unification and representation of varied algebraic structures and geometric systems [12,13]. Calculus, geometry, and signal processing (e.g. neural network and wavelet) methods can be reconstructed and expanded under the GA framework [14,15], which then provides a foundation for expansion and reconstruction of topological models. Here, based on GA's multivectors and operators, we construct a geometric framework that enables multidimensionally-unified computation modeling of multiple geometric objects

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and geographical scenes and is validated with the modeling of a 3D residential community.

1   Calculable multidimensionally-unified representation

GA is based on an *n*-dimensional vector space, not necessarily Euclidean, and makes use of inner and outer products to construct geometric objects contained in different dimensional vector-product space. The unified expressions of these geometric objects can directly support calculation. The basic operations and operators in GA can be extended to concise and intuitive spatial analysis operations [16,17], and provide foundation for multidimensionally-unified representation and operation of geometric objects [13,16].

1.1   Complex representation of multidimensional objects based on blades

Conformal GA (CGA) is one of common GA systems [18]. By introducing two additional coordinates e_0 and e_∞, one can easily realize the various conformal transformations, the multidimensional unification of objects, and treatment of coordinate systems given the specific geometric interpretation of the inner and outer products. In the GA space $V(p,q)$, where *p*,*q* corresponding to positive and negative signature pseudo-Euclidean space, the rules of operation between any number of vectors have closure within $V(p,q)$ (it means any operation can be expanded in terms of the geometric product). The geometric product, *ab* of two multi-vectors *a*, *b* can be expanded as

$$ab = a \wedge b + a \cdot b = <ab>_{grade(a)+grade(b)} + <ab>_{grade(a)-grade(b)}.$$
(1)

The definition of the geometric product shows that it is composed of hybrid vectors of various dimensions, i.e. multivectors. Multivectors are basic mathematical structures of GA, which also support as the base of multidimensionally-unified object representation, spatial relation analysis and operation in different dimensional space.

The blade of grade *k*, also termed a *k*-vector, is a special type of multivectors formed by *k* separate vectors generated with only outer products: $A_{<k>} = a_1 \wedge a_2 \wedge \cdots \wedge a_k$. A blade *A* defines a *k* dimensional subspace; the expression $x \wedge A = 0$, i.e. the "meet" of *x* by object *A*, determines the set of all vectors *x* lying in that subspace [19]. In this manner, a blade expresses succinctly such simple geometric objects as points, lines, planes, etc., in the pseudo-Euclidean space forming the GA space. Such subspaces can even be used for expressing intermediate objects of geometric operation, such as point pairs, GA space, etc. (A partial list of outer product representations for basic geometry objects is given in Table 1). In GA, all geometric objects are explicitly constructed by outer products (inner products) relative to the monomorphic representations in the "geometric space" with a Euclidean metric. Therefore, these objects have both explicit geometric interpretation and algebraic expressions, that can be directly applied in geometric operations. For example, project and reflect the line *L* to plane *P* can be expressed as $(L \cdot P)L^{-1}$ and LPL^{-1} respectively. That close connection provides for a tight consistency between structures and operational rules, which provides complete mathematical foundations for object representation.

In CGA, the Grassmann structure hidden in the GA expression is in consistency with the dimensional structure of geometric objects, which ensures that geometric form representation based on outer products can reflect the inter-constructing relationships between geometric forms of different levels. Geometric objects constructed through inner products inherit those geometric features and relationships from the construction of the object, and intrinsic metric properties. Metric parameters such as intrinsic distance or angle are then automatically incorporated in the parametric expression for each geometric object. The coordinate-independent expressions with both inner and outer products make representations of geometric objects dimensional and geometric structure very adaptable, in order that that the calculation process can be simplify and boost in symbolic way. Dorst et al. constructed adaptable fitting algorithms for both spherical and planar objects [13]. The unification of object representations and their geometric meanings leads to

Table 1   Inner and outer product representation of basic geometric objects[a] [18]

Outer products representation		Inner products representation	
Point *A*	*A*	Point *A*	$X \cdot A = 0$
Point pair (*A*, *B*)	$A \wedge B$	Dual mid-plane of point *A*, *B*	$X \cdot (A - B) = 0$
Circle (*A*, *B*, *C*)	$A \wedge B \wedge C$	Dual plane through point *P*, direction A_k	$X \cdot (-P \cdot (A_k e_\infty)) = 0$
Line (*A*, *B*)	$A \wedge B \wedge e_\infty$	Dual plane normal *n*, support δ	$X \cdot (n - \delta \|n\| e_\infty) = 0$
Sphere (*A*, *B*, *C*, *D*)	$A \wedge B \wedge C \wedge D$	Dual sphere center *A*, radius ρ	$X \cdot (A - 0.5\rho e_\infty) = 0$
Plane (*A*, *B*, *C*)	$A \wedge B \wedge C \wedge e_\infty$	Dual plane through *P*, normal *n*	$X \cdot (P \cdot (n \wedge e_\infty)) = 0$
3D Euclidean geometric space	$A \wedge B \wedge C \wedge D \wedge E$	Inner product expression of any geometric objects *S*	$X \cdot S = 0$

a) The bold and italic character means the GA expression of corresponding geometrical object.

adaptable representation of basic geometric objects in a co-ordinate-free manner that contains both structural dimensions and geometrical relationships. Furthermore, it can derive the construction of the geometric representation and modeling based on geometric hierarchy, ubiety, and metric relations. Such characteristics make it possible to unify multidimensional geometric objects and to represent their geometric modeling relationships [20,21].

Similar to simplex-complex structure in GIS, multidimensional geometric objects with different structures can be decomposed into sets of single geometric elements (single geometric feature class, e.g. points, lines, planes, spheres, etc.), of different dimensions. Since blades can be used to express every geometric feature classes, geometric-algebra expressions of complex geometric objects can be achieved with multivectors linking the representation of the above geometric feature class [21] (Figure 1). It can be implemented as: (1) geometric representation: the geometric structure and representation of geographical objects with different levels are decomposed through splitting geographical objects according to objects and element composition. The unified expression and storage of each different dimensional objects is with a multivector, from which the original geographical objects can be reconstructed; (2) organization of objects: the hierarchy of multidimensional geographical objects relating the geometric representation to topological relations are constructed so that a hierarchical decomposition of complex geometric objects can be obtained in terms of single geometric elements of differing dimensions. Aided by GA expressions for the basic geometric elements, the structure of blades in GA space can be attached, thereby reconstructing the geometric objects of different dimensions, and then based on multivectors, a unified integration, representation, and storage of blades can be achieved; and (3) property embedding: based on an object representation in terms of multivectors, semantic descriptions, object links, and property embedding methods are developed to achieve property embedding. From semantic linking of the original data, the representation data based on multivectors can be examined for semantic and property integrity and consistency, thus ensuring the expression for the geographical objects can be interpreted consistently with its properties.
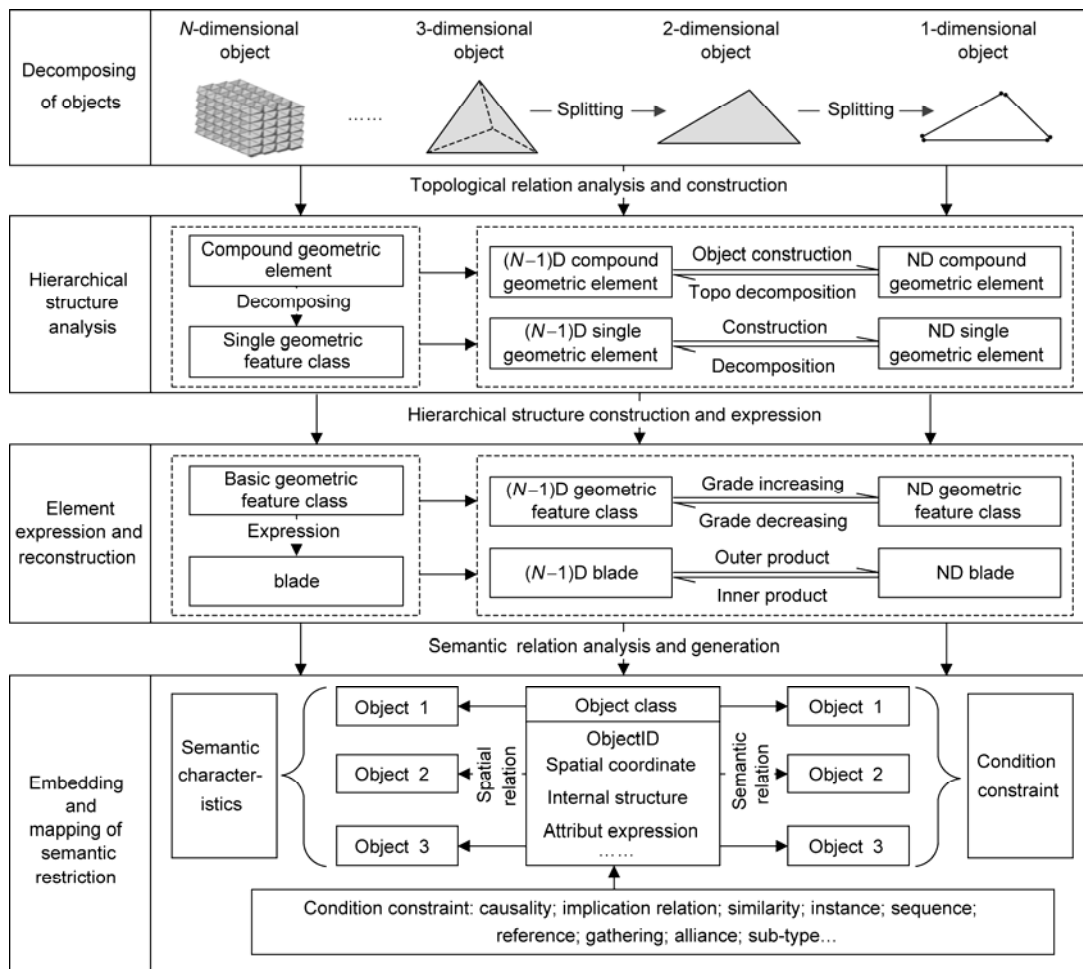

Figure 1   Multidimensionally-unified representation of geometric objects.

1.2   Unified organization of complex multidimensional scene based on multivectors

Based on the blade mathematical structure, we present an integrated representation, storage, and operation of geometric objects. The dimensional operation is intrinsic and directly included in the basic algebraic operations; therefore, we can directly use inner and outer products to realize inter-conversions and constructions of geometric objects, and ensure the consistency of the organizational relations and topological structures of objects composed from elements of different dimensions. Parametric expressions for the different dimensional objects enforce adaptive changes in the structure of geographic objects at the various sub-level geometries. The amount of data storage is simultaneously reduced, as are difficulties in maintaining topological structure. The unified blade expressions and operations provide operational rules and protogenetic mathematical structure for a unified organization of and storage for multidimensional objects, in addition to a multidimensionally-unified geometric operation of geographical scenes. The key to organizing scenes with multivectors is: (1) to form multivector objects with effective data management and operation mechanisms with different blades and blade sets; and (2) to extract the required geometric objects from multivectors according to computational needs. Figure 2 shows the process in constructing a multivector representation of a complex geographical scene from simple geometric blade representations.
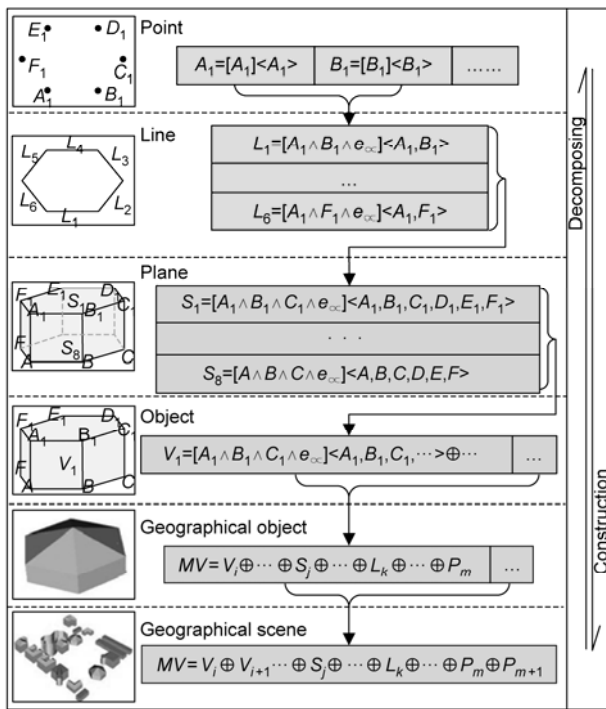

Figure 2   Organization of a geographical scene based on multivectors. The bold and italic characters is the GA representation of points.

Each basic geometric element expression is made of the GA representations bounded by square brackets, while the point sequences are used for boundary restrict of objects. Since the dimension of geometric object sis expressed as blades must be consistent with their Grassmann structure, the basic geometric objects only need point sets of corresponding numbers with Grassmann levels to create an actual representation. For example, ordered segments and polygons only need two and three points to express their respective algebraic representation as lines ($A_1 \wedge B_1 \wedge e_\infty$) and as planes ($A_1 \wedge B_1 \wedge C_1 \wedge e_\infty$). Boundaries can be set by choosing the right sequence of points according to some criterion. For example, the restrictive point sets corresponding to the ordered segment A_1B_1 and polygon $A_1B_1C_1 D_1E_1F_1$ are respectively $<A_1B_1>$ and $<A_1B_1C_1 D_1E_1F_1>$. The representation of basic geometric elements such as spheres of arbitrary dimension can be implemented in a similar way. Composite objects and geographical scenes require connecting basic geometric elements which is represented operationally by the direct sum, symbol "$\oplus$", yielding then a multi-dimensional unified description in terms of multivectors. Although both are designated in the form "[..]", blades represent simple geometric solids whereas multivectors signify complex geometric objects and geographical scenes; GA operators act on both to perform multidimensionally-unified geometric and topological operations. For convenience in managing scene objects and performing spatial analysis, the multivector representation of a scene objects is organized according to type; a formalized expression being written as

$$GeoObjMv = Obj.Points \oplus Obj.Lines \oplus Obj.Planes$$
$$\oplus \cdots \oplus Obj.Sphere, \tag{2}$$

where "$\oplus$" denotes the connector between different blades. *Obj Points*, *Obj Lines*, etc., are the blade lists corresponding to dimensions 0, 1, ... According to the representation of the basic geometric objects, as shown in Figure 2, eq. (2) can be further revised:

$$GeoObjMv = Obj.Points \oplus Obj.Lines < Lines.Pointsindex > \oplus$$
$$Obj.Planes < Planes.Pointsindex > \oplus \cdots \oplus$$
$$Obj.Sphere < Sphere.Pointsindex > . \tag{3}$$

In eq. (3), the blade lists store only the object's representation, i.e., outer products are not real geometric objects but only identifying concordance lists. Boundary information as determined by restriction points is contained in *Lines. Pointsindex*, *Planes.Pointsindex*, etc., but again only the identifying concordance list of points is digitally stored. In general, concordance lists contain incidence and structural relations of objects in the scene. For combinations of various multidimensional scenes, distribution and constructive relations of the different dimensional objects can be effectively presented. Its formal representation enables symbol-

ic-like computations so that rule-based reasoning and judgment can be made. Last, when numeric calculations are required (here the numeric calculation could be of general form, possibly a geometric equation or a representation of geometric objects), these operations can be easily performed on demand that sharply simplify the inherent complexity of the computation [22,23].

Within GA, "$\oplus$" is only used to link objects of different dimensions with little consequence in these numeric calculations. Analysis and subtraction of a blade structure, type and geometric meanings of its multivector can lead to objective-focused geometric representation and operations [22]. The multivector construct provides the fundamental basis for the data storage structures and implementation of design strategies, and structure and work flow optimization. Grade tracking and blade operations can be analyzed, generating objects of different dimensions in the process. Perpendicularity of the blade basis can be fulfilled for objects of different dimensions and can be calculated by appropriate choice of vectors. The multivector representation of geographical objects can yield their geometric components. Moreover, as a complex shape set, the operators can be applied together. Then we can calculate the geometric and measurement relations of its own geometric components and of different geographical objects. In addition, most operators and algorithms associated with spatial analysis can be employed, for example, in Cartesian and spherical coordinates.

2   Multidimensionally-unified computation based on geometric algebra

Distinct from those geometric operations based on "shape" on which Euclidean geometry traditionally was founded, in the GA framework both geometry and geometric relations are based on the algebraic representation of the geometric products. Its representation-intrinsic property, namely the metric property and calculable property, make easier multidimensionally-unified operations that extend to arbitrary object representation as well as relational calculus. By having a universal storage structure, representational and operational structures can be submitted to a more sophisticated but efficient analysis [18,22].

2.1   Unification of representation structure and operation structure

Under the GA framework, the representation and operation of any geometric object can be expressed in terms of geometric products. Clear geometric meaning is retained in the representation of objects. The expression for objects and their computation parameters is directly contained in the representation of operations so that adaptable reconstruction can be performed during a computational procedure. The

integrity of the representation and computational structure provides potentials for direct geometric operation based on GA operators. In addition, the integrity will endow geometric characteristics and meanings that are expressed by the original operators; specifically the results are mostly understandable geometric structures. As an example, we consider the occurrences of intersection here (Figure 3). For an arbitrary polyhedron, the "direct sum" of points, lines, and planes as expressed by the outer products can be arranged according to grade. For this kind of hierarchical expression, the operator "meet" that applies in multidimensional unification decomposes and progressively simplifies the operational procedure according to its topological structure.

Based on the multidimentional-unified characteristic of GA, integrated and formalized expressions of complex geographical objects and spatial relations will be achieved. Thus, a new storage structure that includes both geographical objects and relational sets can be designed. Compared to the storage method of relations between static objects, this implementation is more adaptable and dynamic. The relations can be parameterized and analyzed by using relevant operators addition to the geometric objects expressions. Because this multidimensionally-unified descriptive framework is built based on GA, lots of existing GIS algorithms and models can be effectively inherited, referenced, and carried over to augment the multidimensional representation, analysis, and modeling of geographical objects and phenomena.

2.2   Adaptability of multidimensionally-unified operations

Multidimensionally-unified operations are key advancing geographical model analysis in GIS [24,25]. The universality of GA provides a foundation for multidimensionally-unified expression and operation of geometric objects. GA's coordinate and dimension independency ensures that the geometric characteristics expressed are intrinsic to the space, and the geometric relations formed by various operations are also coordinate-independent relative to all geometric objects. Hence, these operations can be used to establish an adaptable representation, and execute dynamic reconstruction of, and apply relational operations to geometric objects. We illustrate this with the intersection procedure of two geometric objects in Figure 4: the result of the meet operator is automatically adapted to the intersecting objects, and is only related to geometric characteristics and locations of the original two objects but not their dimensions, coordinates, and representation forms. Moreover, it will support the construction of unified operation process in the GA framework.

The representational structure based on multivectors makes the multidimensionally-unified representation of geometric objects and their relations, meanwhile the representation is independent; formalization, parameterization of the expression
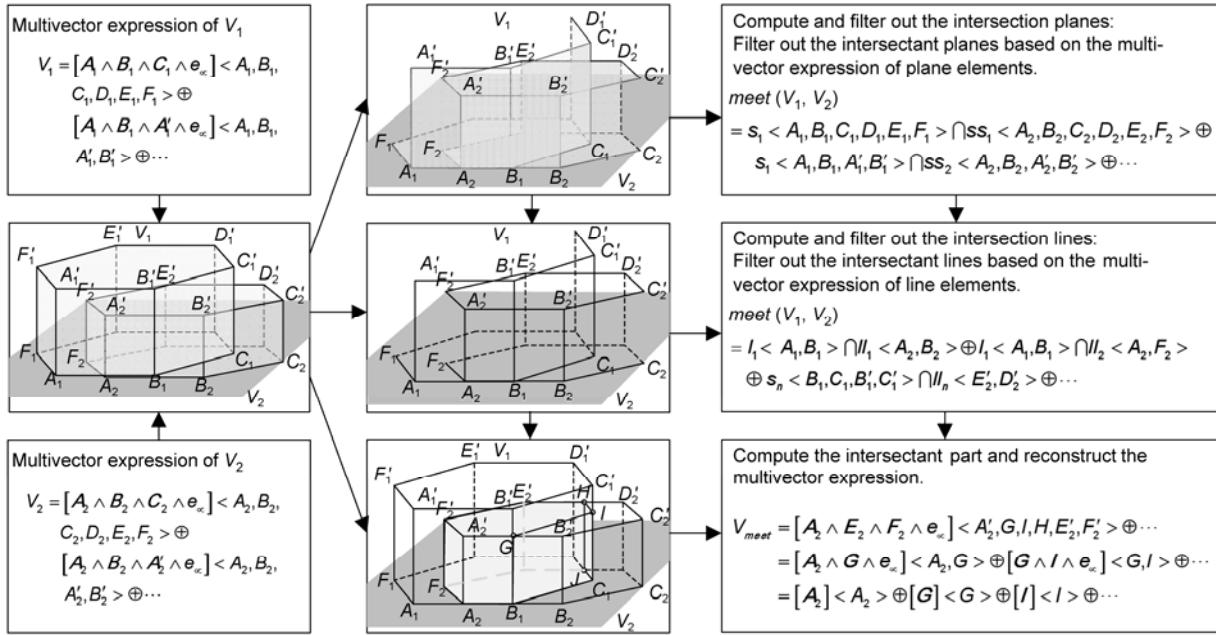
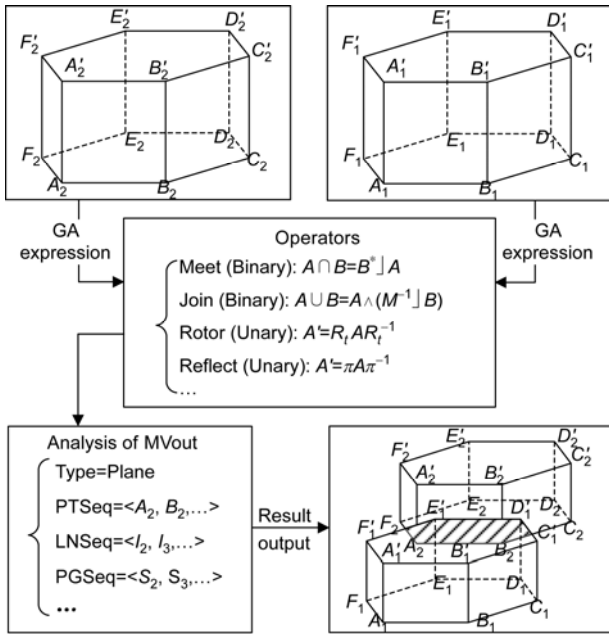**Figure 3**   Unification of representation structure and computation structure.

Figure 4   Self-adaptive computations of multidimensional geometric objects.

and dimensional adaptability ensures simplicity and commonality of algorithms [12,26]. A GIS multidimensional operation implemented by means of GA operators results in a multidimensional unification, object adaptability and operational independency, and results from operations yields multivectors with specific geometric meanings. Therefore, algorithms can be constructed to achieve sophisticated GIS analysis with concordant output.

2.3   GIS analysis-oriented multidimensionally-unified computation process and operators sets

Clear geometric interpretations of CGA representations for GIS operators sustains the multidimensionally-unified GIS computation. The analysis of structural characteristics of transformation operator components, state parameters, such as velocity, angle, rotational axis, etc., can be performed directly [22]. Basic operators and transformation operators are computational stackable, order-independent and structure-preserving. Complex analysis functions can be performed by means of combined sets of operators, and the algorithms can be optimized by an appropriate re-ordered sequence of operators [23,27]. A universal treatment of moving objects would be possible based on CGA. For example, unified representation and interpolating algorithms associated with composite motion, such as translation, rotation, and dilatation, of 3D geometric object, is accommodated by using Versor (a special way to represent orthogonal transformations) constructions [20,28]. Table 2 shows the GA definition of the partial regular operators and their geometric meanings. Geometric measurements, transformations and spatial relation calculation with GA operators mostly directly include formalized representations of geometric objects and have clear geometric meanings. An effective unification of representational and operational structures can be realized to develop various spatial analysis algorithms [18,22].

Object expressions and scene organization based on multivectors can directly facilitate computation, and with the multidimensional GA operators can be combined to form algorithms needed in GIS computations. In accord with a similar treatment of geometric objects, we can then construct

Table 2　Multidimensionally-unified operator sets[a]

Type	Name	GA expression	Functional expression
Basic operators	Project	$P = (A \cdot B)B^{-1}$	Project A on B
	Reflect	$\begin{cases} R = MxM^{-1} \\ R = -MxM^{-1} \end{cases}$	Reflect object x about M
	Dual	$A^* = AI^{-1}$	Resolve the generalized perpendicular subspace and simplify computation of spatial relations
Transformation operators	Scale	$I(x_e) = \dfrac{\rho^2(x_e - c_e)}{(x_e - c_e)^2} + c_e$	Scale objects with cycles or spheres
	Translator	$T_a = 1 + \dfrac{1}{2}ae_\infty = e^{\frac{a}{2}e_\infty}$	Translate objects by a
	Rotor	$R_\theta = \cos\left(\dfrac{\theta}{2}\right) - \sin\left(\dfrac{\theta}{2}\right)L = e^{\frac{\theta}{2}L}$	Generalized rotation with axis L and angle θ
Metric operators	Point-point distance	$d^2(A,B) = -2A \cdot B$	Squared distance between points A and B
	Point-line distance	$d(A,L) = (e_\infty \wedge A \wedge L \wedge B)$	Distance of signed line L to point A
	Point-cycle relations	$\dfrac{A \wedge B \wedge C \wedge D}{e_\infty \wedge A \wedge B \wedge C}$	Relationship between point D and circle ABC: a positive result means the point is in the cycle, whereas a negative result means the point outside the cycle.
	Cycle-cycle relations	$((A \wedge B \wedge C) \cap (D \wedge E \wedge F))^2$	Relationship between cycles ABC and DEF: a positive result means the two are intersecting, zero means these touch at a point, whereas a negative value these are separated.
	Line-cycle relations	$((e_\infty \wedge A \wedge B) \cap (C \wedge D \wedge E))^2$	Relationship of line AB and cycle CDE: a positive result means the two are intersecting, zero means these touch at a point, whereas for negative values these are separated.
Topological operators	Left contraction	$A \rfloor B = A \cdot B$	Extract the orthogonal component of B to A
	Right contraction	$A \lfloor B = B \cdot A$	Extract the orthogonal component of A to B
	Meet	$A \cap B = B^* \cdot A$	Resolve the intersection of A and B
	Join	$A \cup B = A \wedge (M^{-1} \rfloor B)$	Resolve the union of A and B

a) The bold and italic character means the GA expression of corresponding geometrical object.

a unified framework for a multidimensional GIS spatial analysis once data objects and data structure are established. The whole analysis process, comprising data I/O, object analysis, GA operations, parsing of results, and graphical visualization, are all defined within this framework by defining specific functions and operational calculus. The key point of the framework is to construct a series of functions that can extract specific objects from the geographical object class, and then transform these objects into data types and data structures that can be computed in conformal space. Relevant analytic operators and GIS analytic algorithms are applied to solve the problem. The consistency of the geometric representation structure and multidimensional unity of GA operation, coordinates-free property and the intrinsic nature of their properties make most GA operators adaptable to any dimension and type of objects. Moreover, all results are still in a multivector representation form that retains their geometric interpretation.

3　Application models of multidimensionally-unified representation and computation

3.1　Construction of multidimensionally-unified representation and computation model

Robust blade and multivector computational techniques create an environment to handle multidimensionally-unified representation, storage and computation of geometric objects (Figure 5). We propose techniques that transfer that capability over to complex multidimensional geographical scenes with multivectors. In particular, we develop ways to divide and reconstruction scenes of multidimensional objects and to unify expressions, storage, and computation. At the object representation level, multi-source scene data were analyzed and the multidimensionally-unified representation, in terms of inner products and outer products, and multivector expression are used to unify modeling. Computationally, we present spatial transformations and identify corresponding
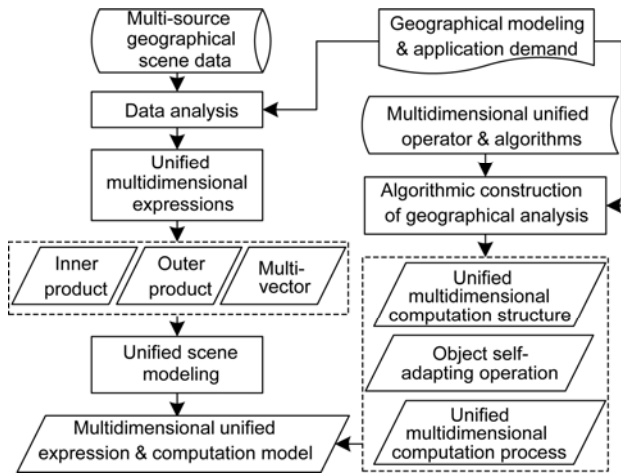
Figure 5   Design of unified dimensional GIS computation models.

relationships between GIS spatial data and mathematical space, and construct corresponding operational rules and calculus to cover the requirements for both the geographical and topological relations as well as more complex spatial analysis. Multidimensionally-unified operation structure, object self-adaptive operation methods and multidimensionally-unified computation process are contained in representation-based multivector framework related to each scene, the framework of which is sufficient support for any computation within the scene.

3.2   A case study

The above multidimensionally-unified representation and computation model is executed within the Clifford algebra-based unified spatial-temporal analytic (CAUSTA [18])
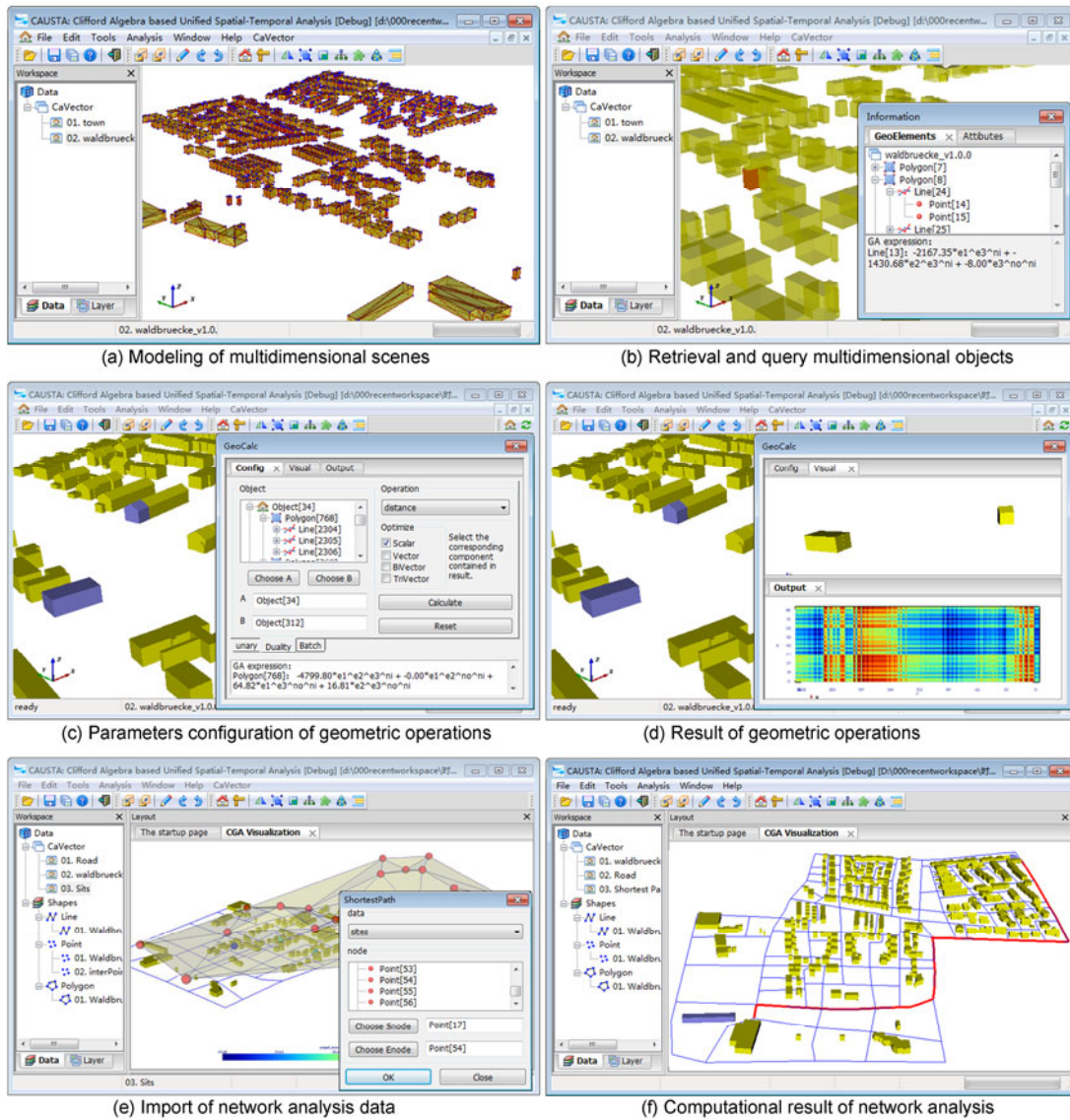

Figure 6   Example of modeling using GA multidimensionally-unified analysis.

environment. As an example of the spatial analysis and application, we analyze the modeling of a 3D residential district of Waldbruecke Village in Germany. We focus on object modeling and scene organization in the complex multidimensional scene and the unified computation of the geometric and topological relations. Space interpolation and network analysis algorithms, which could apply in possible emergency evacuations if contamination occurs, are developed in accord with topological analysis. We express those modeling and computation processes within the GA framework in order to provide directly support to a variety of subsequent spatial analysis.

First, we need to analyze scene data, related to district buildings and road networks in the form of a CityGML file, and then employ the multidimensionally-unified data model and its indexing mechanism based on GA [23,28]. We construct the multivector scene (Figure 6(a)) with space transformations and representation and coding of geometric objects. Inquiry and retrieval methods of scene objects based on GA are constructed with full use of such operators as "Grade" and object index entry. Inquiries and retrievals involve the interactive manipulation of GA expressions for objects and the objects themselves (Figure 6(b)). For all geometric computations, we set parameters associated with each geometric object that gets expression as a multivector, and perform basic geometric and topological operation with the help of GA (Figure 6(c)). We propose simultaneous computation of measurements and relations by exploiting the operational independence of GA operators to compute *en mass* the geographical and topological relationship between objects (Figure 6(d)). The 3D scene, contaminant sampling positions, and road networks are integrated. The discrete spatial distribution of each contaminant is constructed from a combination of the multidimensionally-unified Voronoi algorithm and the shortest-route algorithm [27,29] (Figure 6(e)) Emergency evacuation routes that avoid high contaminant concentrations can then be finally computed (Figure 6(f)).

4   Conclusion

Representation, modeling, and 3D-analysis, extendible to higher dimensions, are prospective issues in GIS [30–35]. In this article, we introduced GA theory and presented the multidimensionally-unified representation of geometric objects. This representation takes advantage of GA's adaptability in handling geometric structures in any dimension. Furthermore, we have integrated its hierarchical relationships of an object's dimension and measurements, and based on GA's multivector elements unified the organization and expression of dynamic multidimensional scenes. We proposed a multidimensionally-unified computation framework based on GA to unify representational, operational, and analysis structures. The framework provides an efficient, unified representation and computation of complex multidimensional scenes. In the analysis of a 3D residential district, we obtained an effective modeling through the construction of a multidimensionally-unified computer representation based on GA, which enabled computation and analysis of complex geographical scenes. GA-based data modeling along with GIS analytic algorithms can provide useful geographical analysis. For the future, we will focus on developing a basic platform that combines this theoretical framework, data models, data representation and data analysis. This platform will provide new techniques for object representation, modeling, and dynamic simulation of geographical scenes.

1   Longley P A, Goodchild M F, Maguire D J, et al. Geographic Information Systems and Science. 2nd ed. New York: Wiley, 2005. 61–83

2   Lan H X, Martin C D, Lim C H. RockFall analyst: A GIS extension for three-dimensional and spatially-distributed rockfall hazard modeling. Comput Geosci, 2007, 33: 262–279

3   Beni L H, Mostafavi M A, Pouliot J, et al. Toward 3D spatial dynamic field simulation within GIS using kinetic Voronoi diagram and Delaunay tetrahedralization. Int J Geogrl Inf Sci, 2011, 25: 25–50

4   Castrillon M, Jorge P A, Lopez I J, et al. Forecasting and visualization of wildfires in a 3D geographical information system. Comput Geosci, 2011, 37: 390–396

5   Xie M W, Zhou G Y, Tetsuro E. Gis component based 3d landslide hazard assessment system: 3dslopegis. Chin Geogr Sci, 2003, 13: 66–72

6   Merwade V, Cook A, Coonrod J. GIS techniques for creating river terrain models for hydrodynamic modeling and flood inundation mapping. Environ Model Software, 2008, 23: 1300–1311

7   Naoum S, Tsanis I K, Fullarton M A. GIS pre-processor for pollutant transport modeling. Environ Model Software, 2005, 20: 55–68

8   Karssenberg D, De Jong K. Dynamic environmental modelling in GIS: 1. Modelling in three spatial dimensions. Int J Geogr Inf Sci, 2005, 19: 559–579

9   McIntosh J, Yuan M. A framework to enhance semantic flexibility for analysis of distributed phenomena. Int J Geogr Inf Sci, 2005, 19: 999–1018

10 Goodchild M F, Yuan M, Cova T J. Towards a general theory of geographic representation in GIS. Int J Geogr Inf Sci, 2007, 21: 239–260

11 Liu Y, Goodchild M F, Guo Q, et al. Towards a general field model and its order in GIS. Int J Geogr Inf Sci, 2008, 22: 623–643

12 Doran C, Lasenby A. Geometric Algebra for Physicists. Cambridge: Cambridge University Press, 2003. 228–264

13 Dorst L, Fongijne D, Mann S. Geometric Algebra for Computer Science: An Object-oriented Approach to Geometry. San Francisco: Morgan Kaufmann, 2007. 355–379

14 Hestenes D, Sobcyk G. Clifford Algebra to Geometric Calculus. Heidelberg: Springer-Verlag, 1984. 1–224

15 Hildenbrand D. Geometric computing in computer graphics using conformal geometric algebra. Comput Graphics-UK, 2005, 29: 795–803

16 Perwass C. Geometric Algebra with Applications in Engineering. Heidelberg: Springer-Verlag, 2009. 255–368

17 Bayro-Corrochano E J, Reyes-Lozano L H, Zamora-Esquivel J. Conformal geometric algebra for robotic vision. J Math Imaging Vision, 2006, 24: 55–81

18 Yuan L W, Yu Z Y, Chen S F, et al. CAUSTA: Clifford algebra based unified spatio-temporal analysis. Trans GIS, 2010, 14(Suppl): 59–83

19 Chao W M, Feng H. Geometric Algebra Ways of Biomimetic Pattern Recognition and Signal Processing (in Chinese). Beijing: Science Press, 2010. 35–36

20 Li H B. Conformal geometric algebra for motion and shape description (in Chinese). J Comput-Aided Design Comput Graph, 2006, 18: 895–901

21 Karimipour F, Delavar M, Frank A U. A simplex-based approach to implement dimension independent spatial analyses. Comput Geosci, 2010, 36: 1123–1134

22 Yuan L W, Yu Z Y, Luo W, et al. A 3D GIS spatial data model based on conformal geometric algebra. Sci China Earth Sci, 2011, 54: 101–112

23 Yu Z Y. Multi-dimensional unified GIS data model based on geometric algebra (in Chinese). Dissertation for the Doctoral Degree. Nanjing: Nanjing Normal University, 2011. 1–117

24 Wang S S, Liu J, Wang X Y, et al. Qualitative spatial reasoning for multi-dimensional topology and size relations (in Chinese). J Comput Res Develop, 2009, 46: 1169–1175

25 Lv G N. Geographic analysis-oriented virtual geographic environment: Framework, structure and functions. Sci China Earth Sci, 2011, 54: 733–743

26 Friedman Y, Gofman Y. Why does the geometric product simplify the equations of physics? Int J Theoretical Phys, 2002, 41: 1841–1855

27 Zhang J Y. GIS network analysis based on Clifford algebra (in Chi-

nese). Dissertation for the Master's Degree. Nanjing: Nanjing Normal University, 2010. 1–53

28 Luo W. Characteristics analysis and motion expression of temporal-spatial field data based on geometric algebra (in Chinese). Dissertation for the Master's Degree. Nanjing: Nanjing Normal University, 2011. 1–59

29 Yi L. Multi-dimensional unified algorithm and application of Voronoi based on conformal geometric algebra (in Chinese). Dissertation for the Master's Degree. Nanjing: Nanjing Normal University, 2011. 1–50

30 Yang B S, Li Q Q, Gong J Y. A robust and rapid algorithm for generating and transmitting multi-resolution three-dimensional models. Chin Sci Bull, 2006, 51: 987–993

31 Li Q Q, Li D R. Research on the conceptual frame of the integration of 3D spatial data model (in Chinese). Acta Geod Cartogr Sin, 1998, 27: 325–330

32 Fang Y, Zhou C H, Jing G F, et al. Research of 4th generation GIS software (in Chinese). J Image Graphics, 2001, 6A: 817–823

33 Chen J, Li Z L, Jiang J, et al. Dynamic and multi-dimensional spatial data modeling: models and methods (in Chinese). Geom Inf Sci Wuhan Univ, 2004, 29: 858–862

34 Wu L X, Chen X X, Chen D F, et al. A GTP-based entity model for underground real 3D integral representation (in Chinese). Geom Inf Sci Wuhan Univ, 2007, 32: 331–335

35 Hu Q W, Ling C F, Yu F, et al. Design and implementation of multi-dimensional GIS-based mineral resource evaluation data management system (in Chinese). Adv Earth Sci, 2010, 25: 990–995