Discrete scale invariance and $\ln(B)$ periodic quantum oscillations in topological semimetals

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Quantum oscillations have played a major role in the history of condensed matter physics—and indeed in physics as a whole. In 1930, Shubnikov and de Haas (SdH) [1] discovered oscillations in the resistance of bismuth as a function of magnetic field, $R(B)$. Because of theoretical work of Landau [2] in the same year, it was soon realized that the oscillations were periodic in $1/B$: if a peak in R is observed at $B = 1$ T and another at $B = 0.5$ T, then we should see one at $B = 0.33$ T, $B = 0.25$ T and so on. This work is famous, but maybe not as much as it should be: *it was the first discovery of a macroscopic effect that depends for its existence on the fact that an electron has a phase.*

The magnetic susceptibility also oscillates. This is the deHaas-van Alphen effect, also a product of the year 1930. The Hall resistance does as well, but in bulk materials the effect is not so big. All of these effects are periodic in $1/B$, just like the SdH oscillations. It must be so, since all of these quantities depend on the density of electron states at the Fermi energy $\rho(E_F)$, and it is this underlying physical quantity that is periodic in $1/B$.

These discoveries were dramatic in that they were a beautiful demonstration of quantum mechanics in bulk materials. The observation of Onsager [3] in 1952 that the period in $(1/B)$ is inversely proportional to the extremal cross section of the Fermi surface meant that SdH and dHvA effects could be used for the measurement of band structure,

even for more complex materials. In the 1950s and early 1960s they were practically the only tool for this, and of course they are still intensively used today.

The SdH effect entered a new era when two-dimensional electron gases (2DEGs) could be made and looked at in detail in the 1970s. SdH oscillations were easy to see from the start. But the big surprise came when 2DEGs became sufficiently pure and the magnetic field big enough. Then the SdH oscillations become huge. Even more dramatically, the Hall resistance oscillates violently and is also quantized, as discovered by von Klitzing, Dorda, and Pepper [4] in 1980. This integer quantum Hall effect has had a huge impact on the entire field of condensed matter physics, starting the whole direction of topological materials.

Macroscopic quantum oscillations were predicted to be observable in interference phenomena of charged particles by Aharonov and Bohm (AB) [5] in 1959. These effects have been observed in many different types of experiments over the years, including in the transport properties of mesoscopic rings. All AB effects are periodic in B , not $1/B$. If there is a peak in a measurable quantity at $B = 1$ T and $B = 2$ T, there should be peaks at $B = 3$ T, $B = 4$ T, and so on.

SdH and all the related $1/B$ phenomena go back to Landau's basic solution of electron motion in a magnetic field. The cyclotron motion perpendicular to the direction of the field is quantized with the Landau level energies forming an arithmetic progression—this leads to $\rho(E_F)$ being periodic in $1/B$. Apart from the periodicity, Landau's theory makes one

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other very important prediction (though he did not note it at the time). Once the electron density is low enough, or the field is high enough, only the lowest Landau level is occupied. *At this point the oscillations should stop!* Increasing the field further should only produce smooth changes in the properties of the system. When this prediction was checked in two dimensional systems it failed spectacularly: this was the discovery of the fractional quantum Hall effect (FQHE) in 1982 by Tsui, Störmer, and Gossard [6]. It is not that Landau was wrong—the original theory does not include interactions, and Laughlin [7] showed that it is interactions that give the fractional effect.

This brings us to the 21st century and the question: what about in three dimensions? The answer is that the theory fails once more—oscillations *still* show up at very strong field strength. By an astounding coincidence, this was observed in Bi in 2007 [8], the system in which quantum oscillations were discovered 87 years before! But the oscillations did not appear to have either the $1/B$ periodicity of SdH or the B periodicity of dHvA. The FQHE has oscillations but no clear periodicity, so early attempts to explain the data made analogies to the FQHE. This hypothesis fits some peaks but others were missing, leading to an unsatisfactory situation in the phenomenology. From the theoretical side, this cannot be the fractional quantum Hall effect, for there is no gap in the excitation spectrum because of the free motion in the direction parallel to the field. It must be something else.

Improvements in sample quality and field strengths have finally settled the periodicity question. The clearest case is

the recent discovery in the layered bulk material ZrTe_5 , where 5 cycles of oscillations have been observed by further increasing the magnetic field when the lowest Landau level is occupied. Amazingly, the oscillations show clear $\ln(B)$ behavior [9], as seen in Figure 1, taken from this reference. The periodicity is entirely new: the peaks in the resistance are periodic in $\ln(B)$, not B or $1/B$. So if there is a peak at $B = 1$ T and a peak at $B = 2$ T, there should be a peak at $B = 4$ T, $B = 8$ T, and so on. The peaks form a *geometric series*.

A theory inspired by this pattern has been suggested by Liu et al. [10]. It also explains puzzling features of old data. When the magnetoresistances of previously investigated systems such as TaAs and Bi were analyzed, they also showed the $\ln(B)$ periodicity. It is not always easy to see, since the effect only starts at high fields and the geometric series increases rapidly, all of which makes it hard to observe very many peaks even if the field range is large.

These systems are all semimetals with Dirac-type dispersion, which was the clue to understanding the oscillations. Coulomb impurities in a Dirac-type systems have a built-in scale invariance: the kinetic energy operator AND the potential energy operator are proportional to $1/r$, where r is the spatial coordinate. This immediately implies that if $\psi(r)$ is a solution to the Schrödinger equation, then so is $\psi(\alpha r)$, for arbitrary α . Boundary conditions constrain α to have discrete values, so we have a case of discrete scale invariance (DSI), a symmetry usually more characteristic of fractals than of quantum systems.

This symmetry becomes visible in transport data because

Figure 1 (Color online) Field-dependent part of the resistance of ZrTe_5 at a temperature of 4.2 K, plotted as a function of (a) $1/B$, (b) B , and (c) $\ln(B)$. Clearly the periodicity is in the variable $\ln(B)$. This is confirmed by plotting the positions of the peaks and dips in (d).

the conduction electrons scatter off the impurities, and the cross-sections are enhanced at the energies of the quasi-bound states. In the strong-coupling regime, these energies show DSI. When a magnetic field B is applied, there is an effective cutoff for the wavefunction at the magnetic length, which is proportional to $B^{-1/2}$. The theory in ref. [10] shows that when all these ingredients are combined, the magnetoresistance will show quantum oscillations and the periodicity is indeed $\ln(B)$, not $1/B$, and not B . This is yet another beautiful macroscopic quantum effect, a complement to the quantum Hall effect and Aharonov-Bohm effects.

1 L. Shubnikov, and W. J. de Haas, *Proc. Royal Nether. Acad. Arts Sci.*

- 33, 130 (1930).
- 2 L. Landau, *Zeitsch. Phys.* **64**, 629 (1930).
- 3 L. Onsager, *Lond. Edinb. Dubl. Phil. Mag.* **43**, 1006 (1952).
- 4 K. V. Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**, 494 (1980).
- 5 Y. Aharonov, and D. Bohm, *Phys. Rev.* **115**, 485 (1959).
- 6 D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).
- 7 R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
- 8 K. Behnia, L. Balicas, and Y. Kopelevich, *Science* **317**, 1729 (2007), arXiv: [0802.1993](#).
- 9 H. C. Wang, H. W. Liu, Y. A. Li, Y. J. Liu, J. F. Wang, J. Liu, J. Y. Dai, Y. Wang, L. Li, J. Q. Yan, D. Mandrus, X. C. Xie, and J. Wang, *Sci. Adv.* **4**, eaau5096 (2018), arXiv: [1704.00995](#).
- 10 H. W. Liu, H. Jiang, Z. Q. Wang, R. Joynt, and X. C. Xie, arXiv: [1807.02459](#).