

Fixed-time stabilization of output-constrained stochastic high-order nonlinear systems

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Received 17 August 2023/Revised 26 October 2023/Accepted 15 December 2023/Published online 27 March 2024

Abstract In this study, the fixed-time stabilization problem of stochastic high-order nonlinear systems with output constraint and high-order and low-order nonlinearities is addressed. A new coordinate transformation is employed to directly convert output-constrained stochastic systems into an equivalent unconstrained form. By fully extracting the characteristics of system nonlinearities and using the stochastic fixed-time stability theorem, a new design and analysis method is constructed to guarantee that the trivial solution of the closed-loop system is stochastically fixed-time stable while fulfilling the output constraint.

Keywords stochastic high-order nonlinear system, output constraint, high-order and low-order nonlinearity, fixed-time stabilization, finite-time stabilization

1 Introduction

To ensure safety and meet performance specifications, the output/state of nonlinear systems needs to be constrained. Violation of the output/state constraints during operation may result in performance reduction, unexpected danger, or system breakdown [1]. In the past three decades, many tools have been proposed to handle output/state constraints, the most representative of which is the barrier Lyapunov function (BLF) first proposed in [2, 3].

Control design and stability analysis of stochastic high-order nonlinear systems are significantly challenging in the nonlinear control community due to the uncontrollability and nonfeedback linearizability of Jacobian linearization. In [4], a key technique called adding power integrators was introduced to solve this problem. Subsequently, many important achievements have been made in this field [5–14].

However, Refs. [5–14] only addressed the asymptotic stabilization problem. In comparison with asymptotic stabilization, finite-time stabilization has some distinct features such as fast response and high accuracy; thus, it is more desirable than asymptotic stabilization in many engineering applications. Due to these benefits, Refs. [15, 16] introduced the concept of stochastic finite-time stability and developed the corresponding Lyapunov criteria. On this basis, Refs. [17, 18] and [19] investigated the finite-time stabilization problem for stochastic strict-feedback and high-order systems, respectively. Ref. [20] proposed a generalized finite-time stability theorem for stochastic nonlinear systems that relaxes the constraint on infinitesimal generators and reveals the important role of white-noise in the finite-time stabilization of stochastic systems. For stochastic nonlinear systems with finite-time stochastic input-to-state stability (FT-SISS) inverse dynamics, Refs. [21, 22], [23], and [24] established finite-time control design strategies for stochastic strict-feedback, high-order systems, and low-order nonlinear systems, respectively. Recently, a weaker concept of stochastic inverse dynamics called finite-time stochastic integral input-to-state stability (FT-SiISS) was introduced in [25]. Despite significant progress in pure finite-time stabilization problem, few results have been obtained in analyzing finite-time stabilization problems for stochastic systems with output constraint. In the only results, Refs. [26, 27] employed the fractional-type BLF to directly handle the output constraint in the finite-time stabilization task. However, these methods are only suitable for a quite limited class of stochastic high-order systems because the nonlinear terms of the system must satisfy the low-order growth condition. By exploring the properties of these nonlinearities, an essential

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breakthrough was achieved by [28], where a finite-time stabilizer is delicately constructed for such systems with high-order and low-order nonlinearities.

Because the settling time of [26–28] depends on the initial condition, the unavailability of the exact initial condition hinders the application of these control schemes in [26–28]. Recently, a new definition of fixed-time stability and its Lyapunov condition were proposed in [29] and further refined in [30]. Subsequently, several studies of stochastic fixed-time control have been conducted. Specifically, by combining with the fuzzy logic technique, Ref. [31] solved the fixed-time stabilization problem of stochastic interconnected systems, Ref. [32] investigated global fixed-time stabilization of stochastic super-twisting systems, Refs. [33] and [34] designed fixed-time event-triggered controllers of stochastic nontriangular and nonstrict-feedback systems, Ref. [35] investigated the fixed-time consensus for stochastic multi-agent systems, and Ref. [36] proposed the stochastic fixed-time control strategy for stochastic switching systems. However, Refs. [31–36] could not guarantee fixed-time convergence of stochastic systems while fulfilling the output constraint.

Based on the aforementioned studies, we can not help but ask a significant question: Can we design a fixed-time state-feedback controller for output-constrained stochastic high-order nonlinear systems with high-order and low-order nonlinearities?

This study provides a satisfactory answer to this problem. Because the system model investigated in this study includes high-order powers, high-order and low-order nonlinearities, and output constraint, the controllers in the aforementioned studies are no longer applicable. In this study, a novel coordinate transformation is employed to directly convert the output-constrained stochastic high-order nonlinear system into an equivalent unconstrained one. For the transformed system, by fully extracting the characteristics of system nonlinearities and using the stochastic fixed-time stability theorem, we construct a state-feedback stabilizer to ensure that the state of the closed-loop system is stochastically fixed-time stable while guaranteeing the achievement of the prespecified output constraint.

This paper is organized as follows. Section 2 states the investigated problem. Section 3 discusses the main results. Section 4 presents a simulation example. Section 5 concludes this paper.

2 Problem statement and preliminaries

2.1 Problem statement

Consider the following class of stochastic high-order nonlinear systems:

$$\begin{aligned} dx_i(t) &= ([x_{i+1}(t)]^{p_i} + f_i(\bar{x}_i(t)))dt + g_i(\bar{x}_i(t))d\omega, \quad i = 1, \dots, n - 1, \\ dx_n(t) &= ([u(t)]^{p_n} + f_n(x(t)))dt + g_n(x(t))d\omega, \\ y(t) &= x_1(t), \end{aligned} \tag{1}$$

with the symmetric output constraint

$$y(t) \in \Omega_y = \{y(t) \in \mathbb{R} : -\epsilon_l < y(t) < \epsilon_l\}, \tag{2}$$

where $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$ are control input and system output, respectively. $\bar{x}_i(t) = (x_1(t), \dots, x_i(t))^T \in \mathbb{R}^i$, $i = 1, \dots, n - 1$, $\bar{x}_n(t) = x(t) \in \mathbb{R}^n$ is the measurable state with the initial value $x(0) = x_0$, and ω is an r -dimensional standard Wiener process defined on a complete probability space (Ω, F, P) with a filtration $\{F_t\}_{t \geq 0}$ satisfying the usual condition. For $i = 1, \dots, n$, $p_i \geq 1$ is called the high order of the system, and $f_i(\bar{x}_i) : \mathbb{R}^i \rightarrow \mathbb{R}$ and $g_i(\bar{x}_i) : \mathbb{R}^i \rightarrow \mathbb{R}^{1 \times r}$ are some of the continuous functions that satisfy $f_i(0) = 0$ and $g_i(0) = 0$, respectively. ϵ_l is a given positive constant. $[\cdot]^\varrho = \text{sgn}(\cdot) \cdot |\cdot|^\varrho$, ϱ is a positive constant, and $\text{sgn}(\cdot)$ represents a sign function.

For system (1) with the given output constraint (2), the control objective is to design a state-feedback controller such that the trivial solution of the closed-loop system is stochastically fixed-time stable while ensuring that Eq. (2) is satisfied. To achieve this aim, we need a key assumption.

Assumption 1. There exist the constants $\varpi_n \leq \varpi_{n-1} \leq \dots \leq \varpi_1 \in (-\frac{1}{\sum_{j=1}^n p_1 \cdots p_{j-1}}, 0)$, $0 \leq \gamma_n \leq \gamma_{n-1} \leq \dots \leq \gamma_1$ and the known smooth functions $f_{ij}(\bar{x}_i)$, $g_{ij}(\bar{x}_i)$, $i = 1, \dots, n$, $j = 1, 2$, such that

$$|f_i(\bar{x}_i)| \leq f_{i1}(\bar{x}_i) \sum_{j=1}^i |x_j|^{\frac{r_i + \varpi_i}{r_j}} + f_{i2}(\bar{x}_i) \sum_{j=1}^i |x_j|^{\frac{h_i + \gamma_i}{h_j}},$$

$$|g_i(\bar{x}_i)| \leq g_{i1}(\bar{x}_i) \sum_{j=1}^i |x_j|^{\frac{2r_j+\varpi_i}{2r_j}} + g_{i2}(\bar{x}_i) \sum_{j=1}^i |x_j|^{\frac{2h_j+\gamma_i}{2h_j}}, \tag{3}$$

where $r_1 = h_1 = 1$, $r_{j+1} = \frac{r_j+\varpi_j}{p_j}$, $h_{j+1} = \frac{h_j+\gamma_j}{p_j}$, $j = 1, \dots, n$. From Assumption 1, we determine that the low-orders $\frac{r_i+\varpi_i}{r_j}$ and $\frac{2r_i+\varpi_i}{2r_j}$ and the high-orders $\frac{h_i+\gamma_i}{h_j}$ and $\frac{2h_i+\gamma_i}{2h_j}$ can take all the values in $(0, \frac{1}{p_j \cdots p_{i-1}}]$ and $[\frac{1}{p_j \cdots p_{i-1}}, \infty)$, respectively.

Remark 1. A significant feature of Assumption 1 is the flexibility to choose the monotonic parameters ϖ_i and γ_i , which enables Assumption 1 to include different assumptions in the existing work on state-feedback stabilization of stochastic systems with output constraint. Specifically, by setting $\gamma_i = 0$, Assumption 1 reduces to Assumption 1 in [26, 27]. If $p_i \in \mathbb{R}_{\text{odd}}^{\geq 1} = \{\frac{p}{q} \in \mathbb{R} | p \geq q \text{ and } p, q \text{ are positive odd integers}\}$, then Assumption 1 is the same as Assumption 1 in [28].

Remark 2. Several practical output-constrained systems can be modeled by (1) with (2), for example, the under-actuated, weakly coupled, unstable mechanical system in [5]:

$$\begin{aligned} dx_1(t) &= x_2(t)dt, \\ dx_2(t) &= \left(x_3(t)^3 + \frac{49}{50} \sin(x_1(t))\right)dt, \\ dx_3(t) &= x_4(t)dt, \\ dx_4(t) &= \left(u(t) + 15x_1(t) + \frac{113}{10}x_3(t) + 15x_3(t)^3\right)dt - \frac{1}{8}(10x_1(t) + x_3(t))d\omega, \\ y(t) &= x_1(t), \end{aligned} \tag{4}$$

with an output constraint $y(t) \in \Omega_y = \{y(t) \in \mathbb{R} : -\frac{\pi}{4} < y(t) < \frac{\pi}{4}\}$. Obviously, Eq. (4) is a special form of (1) with $p_1 = p_3 = p_4 = 1$, $p_2 = 3$, $\varpi_i = -\frac{1}{12} \in (-\frac{1}{8}, 0)$, $\gamma_i = 0$, $i = 1, \dots, 4$, $r_1 = 1$, $r_2 = \frac{r_1+\varpi_1}{p_1} = \frac{11}{12}$, $r_3 = \frac{r_2+\varpi_2}{p_2} = \frac{5}{18}$, $r_4 = \frac{r_3+\varpi_3}{p_3} = \frac{7}{36}$, $r_5 = \frac{r_4+\varpi_4}{p_4} = \frac{1}{9}$, $h_1 = 1$, $h_2 = \frac{h_1+\gamma_1}{p_1} = 1$, $h_3 = \frac{h_2+\gamma_2}{p_2} = \frac{1}{3}$, $h_4 = \frac{h_3+\gamma_3}{p_3} = \frac{1}{3}$, $h_5 = \frac{h_4+\gamma_4}{p_4} = \frac{1}{3}$.

We can easily determine that $f_1(x_1) = 0$, $g_1(x_1) = 0$, $|f_2(\bar{x}_2)| = \frac{49}{50}|\sin x_1| \leq \frac{49}{50}(|x_1|^{\frac{5}{6}} + |x_1|)$, $g_2(\bar{x}_2) = 0$, $f_3(\bar{x}_3) = 0$, $g_3(\bar{x}_3) = 0$, $|f_4(\bar{x}_4)| = |15x_1 + \frac{113}{10}x_3 + 15x_3^3| \leq (\frac{263}{10} + 15x_1^2 + 15x_3^2)(|x_1|^{\frac{1}{9}} + |x_3|)$, $|g_4(\bar{x}_4)| = |\frac{1}{8}(10x_1 + x_3)| \leq (\frac{11}{8} + \frac{5}{4}x_1^2)(|x_1|^{\frac{1}{12}} + |x_3|)$; thus, Assumption 1 is satisfied.

2.2 Preliminaries

To derive the main results, we recall some technical lemmas. Consider the following stochastic nonlinear system:

$$dx(t) = f(x(t))dt + g(x(t))d\omega, \forall t \geq 0, \tag{5}$$

with the initial value $x(0) = x_0$. $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$ are continuous function vectors and matrices, respectively. ω is an r -dimensional standard Wiener process.

Definition 1 ([20]). The trivial solution of system (5) is said to be stochastically finite-time stable for any initial value $x_0 \in \mathbb{R}^n$, if there exists a solution $x(t, x_0)$ in system (5) and the following statements hold.

(i) Stable in probability: For any $\varepsilon \in (0, 1)$ and $\lambda > 0$, if there exists a $\delta(\varepsilon, \lambda) > 0$ such that $P\{|x(t, x_0)| < \lambda(|x_0|), \forall t > 0\} \geq 1 - \varepsilon$ whenever $|x_0| < \delta$.

(ii) Finite-time attractiveness in probability: For any initial value $x_0 \in \mathbb{R}^n \setminus \{0\}$, the first hitting time $\tau_{x_0} = \inf\{t \geq 0 : x(t, x_0) = 0\}$, which is also called the stochastic setting time, is finite almost surely, that is, $P(\tau_{x_0} < \infty) = 1$. Furthermore, $x(t + \tau_{x_0}, x_0) = 0$, a.s., $\forall t \geq 0$.

Definition 2 ([29]). The trivial solution of system (5) is said to be stochastically fixed-time stable if the trivial solution is stochastically finite-time stable and $E(\sigma(x_0)) \leq T_0, \forall x_0 \in \mathbb{R}^n \setminus \{0\}$, where T_0 is a positive constant independent of the initial values.

Lemma 1 ([10]). If $q > 0$, $p \geq 1$ is odd, then $\|c\| - \|d\|^p \leq \|c\|^p - \|d\|^p$, $|c + d|^p \leq 2^{p-1}|c^p + d^p|$, $\lceil c \rceil^{\frac{q}{p}} - \lceil d \rceil^{\frac{q}{p}} \leq 2^{\frac{p-1}{p}} \lceil c \rceil^q - \lceil d \rceil^q$ hold for any $c \in \mathbb{R}, d \in \mathbb{R}$.

Lemma 2 ([17]). Suppose that there exists a nonnegative radially unbounded function $V(x) \in \mathcal{C}^2$. If $\mathcal{L}V \leq 0$, then Eq. (5) has a continuous solution on $[0, \infty)$ for any initial data.

Lemma 3 ([10]). For given $m > 0$ and any $a_i \in \mathbb{R}$, $i = 1, \dots, n$, there holds $(|a_1| + \dots + |a_n|)^m \leq d_m(|a_1|^m + \dots + |a_n|^m)$, where $d_m = n^{m-1}$ if $m \geq 1$ and $d_m = 1$ if $m < 1$.

Lemma 4 ([30]). For system (5), suppose that there exist a function $V \in \mathcal{C}^2$, class \mathcal{K}_∞ functions α_1, α_2 , and constants $c_1 > 0, c_2 > 0, 0 < p < 1, q > 1$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \mathcal{L}V(x) \leq -c_1V(x)^p - c_2V(x)^q.$$

Then, the trivial solution of system (5) is stochastically fixed-time stable.

Lemma 5 ([10]). For a given continuous function $q(x, y)$, there exist smooth functions $q_1(x) \geq 0, q_2(y) \geq 0, q_3(x) \geq 1$, and $q_4(y) \geq 1$ such that $|q(x, y)| \leq q_1(x) + q_2(y), |q(x, y)| \leq q_3(x)q_4(y)$.

Lemma 6 ([10]). The function $y(x) = \lceil x \rceil^a, a \geq 2$ is \mathcal{C}^2 for $x \in \mathbb{R}$, $\dot{y}(x) = a|x|^{a-1}$, and $\ddot{y}(x) = a(a-1)\lceil x \rceil^{a-2}$.

Lemma 7 ([10]). For $m > 0, n > 0, a > 0$, and any $c \in \mathbb{R}, d \in \mathbb{R}$, there holds $|c|^m|d|^n \leq a|c|^{m+n} + (\frac{n}{m+n})(\frac{m+n}{m})^{-\frac{m}{n}}a^{-\frac{m}{n}}|d|^{m+n}$.

Lemma 8 ([10]). For a given function $p : [a, b] \rightarrow \mathbb{R}, b > a$ with $p(a) = 0$, there holds $|\int_a^b p(x)dx| \leq |p(b)| |b - a|$.

3 Main results

3.1 System transformation

In this subsection, we first introduce the following equivalent coordinate transformation:

$$x_1(t) = \lambda_1 \arctan(\xi_1(t)), \quad x_i(t) = \xi_i(t), \quad i = 2, \dots, n, \tag{6}$$

where $\lambda_1 = \frac{2\epsilon_l}{\pi}$. Notably, $x_1(t) = \lambda_1 \arctan(\xi_1(t))$ meets the following properties:

$$\begin{cases} x_1(t) \rightarrow -\epsilon_l, & \text{when } \xi_1(t) \rightarrow -\infty, \\ x_1(t) \rightarrow \epsilon_l, & \text{when } \xi_1(t) \rightarrow \infty. \end{cases} \tag{7}$$

Hence, the constraint (2) is not transgressed almost surely if $\xi_1(t)$ is almost surely bounded. Using (6), we can obtain the following unconstrained systems:

$$\begin{aligned} d\xi_1 &= (D_1(\xi_1)\lceil \xi_2 \rceil^{p_1} + \bar{f}_1(\xi_1))dt + \bar{g}_1(\xi_1)d\omega, \\ d\xi_i &= (\lceil \xi_{i+1} \rceil^{p_i} + \bar{f}_i(\xi_i))dt + \bar{g}_i(\xi_i)d\omega, \quad i = 2, \dots, n-1, \\ d\xi_n &= (\lceil u \rceil^{p_n} + \bar{f}_n(\xi))dt + \bar{g}_n(\xi)d\omega, \end{aligned} \tag{8}$$

where $D_1(\xi_1) = \frac{\partial \xi_1}{\partial x_1} = \frac{1+\xi_1^2}{\lambda_1}, \bar{f}_1(\xi_1) = D_1(\xi_1)f_1(\xi_1) + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2}g_1^T(\xi_1)g_1(\xi_1), \bar{g}_1(\xi_1) = D_1(\xi_1)g_1(\xi_1), \bar{f}_j(\xi_j) = f_j(\xi_j),$ and $\bar{g}_j(\xi_j) = g_j(\xi_j), j = 2, \dots, n$.

3.2 Control design

To facilitate control design, we introduce a proposition, whose proof is given in Appendix A.

Proposition 1. For $k = 1, \dots, n$, we find the known nonnegative smooth functions $h_{k1}(\bar{\xi}_k)$ and $\eta_{k1}(\bar{\xi}_k)$ such that

$$|\bar{f}_k(\bar{\xi}_k)| \leq h_{k1}(\bar{\xi}_k) \sum_{j=1}^k |\xi_j|^{\frac{r_k + \varpi_k}{r_j}}, \tag{9}$$

$$|\bar{g}_k(\bar{\xi}_k)| \leq \eta_{k1}(\bar{\xi}_k) \sum_{j=1}^k |\xi_j|^{\frac{2r_k + \varpi_k}{2r_j}}. \tag{10}$$

Then, we introduce the following key transformations:

$$z_j = \lceil \xi_j \rceil^{\frac{\mu}{r_j}} - \lceil \xi_j^* \rceil^{\frac{\mu}{r_j}}, \quad j = 1, \dots, k,$$

$$\begin{aligned} \xi_1^* &= 0, \\ \xi_k^*(\bar{\xi}_{k-1}) &= -\phi_{k-1}(\bar{\xi}_{k-1})^{\frac{r_k}{\mu}} [z_{k-1}]^{\frac{r_k}{\mu}}, \quad k = 2, \dots, n+1, \end{aligned} \quad (11)$$

where $\mu \geq \max_{1 \leq i \leq n} \{2r_i, r_i + \varpi_i\}$ and $\phi_j, j = 1, \dots, n$, are nonnegative smooth functions to be determined. With the aid of (11), we formulate the following proposition, whose proof is given in Appendix B.

Proposition 2. For system (8), we find the nonnegative smooth functions ϕ_1, \dots, ϕ_n and the following state-feedback controller:

$$u(\xi) = \xi_{n+1}^*(\xi) = -\phi_n(\xi)^{\frac{r_{n+1}}{\mu}} [z_n]^{\frac{r_{n+1}}{\mu}}, \quad (12)$$

such that the Lyapunov function $V_n(\xi) = \sum_{k=1}^n W_k(\bar{\xi}_k)$ satisfies

$$\mathcal{L}V_n \leq -\frac{1}{4} \sum_{j=1}^n |z_j|^4 - \frac{1}{4} \sum_{j=1}^n |z_j|^{\frac{16}{3}}, \quad (13)$$

where

$$W_k(\bar{\xi}_k) = \int_{\xi_k^*}^{\xi_k} \left[[s]^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - r_k - \varpi_k}{\mu}} ds. \quad (14)$$

3.3 Fixed-time stability and constraint analysis

We state the main results.

Theorem 1. For system (1) with the constraint (2), if Assumption 1 holds with $\varpi_1 = \dots = \varpi_n$, then a state-feedback controller (12) guarantees that for any initial value $(x_1(0), \dots, x_n(0)) \in \Omega_y \times \mathbb{R}^{n-1}$,

- (i) The closed-loop system (1), (6), and (12) has a continuous solution on $[0, \infty)$;
- (ii) All of the signals are almost surely bounded and the constraint (2) is achieved almost surely;
- (iii) The trivial solution of the closed-loop system is stochastically fixed-time stable.

Proof. (i) We first prove that $V_n(\xi)$ is \mathcal{C}^2 and radially unbounded by two parts.

Part I. From $V_n(\xi) = \sum_{k=1}^n W_k(\bar{\xi}_k)$ and the definitions of μ and r_i , it follows that $W_k(\bar{\xi}_k)$ in (14) is \mathcal{C}^2 , so is $V_n(\xi)$.

Part II. We next prove that, for $i = 1, \dots, n$,

$$c_{i1} |\xi_i - \xi_i^*|^{\frac{4\mu - \varpi_i}{r_i}} \leq W_i(\bar{\xi}_i) \leq c_{i2} |z_i|^{\frac{4\mu - \varpi_i}{\mu}}, \quad (15)$$

where $c_{i1} = \frac{r_i \cdot 2^{\frac{4\mu - r_i - \varpi_i}{\mu}}}{(4\mu - \varpi_i) \cdot 2^{\frac{4\mu - r_i - \varpi_i}{r_i}}}$ and $c_{i2} = 2^{1 - \frac{r_i}{\mu}}$. Based on Lemma 1, we derive the following expression:

$$W_i(\bar{\xi}_i) \leq |\xi_i - \xi_i^*| |z_i|^{\frac{4\mu - r_i - \varpi_i}{\mu}} \leq 2^{1 - \frac{r_i}{\mu}} |z_i|^{\frac{4\mu - \varpi_i}{\mu}}. \quad (16)$$

Let us prove the left-hand side of (16) through two cases.

Case 1. $\xi_i^* \leq \xi_i$.

(1) If $0 \leq \xi_i^* \leq \xi_i$, then, based on Lemma 1, we derive the following expression:

$$\begin{aligned} W_i(\bar{\xi}_i) &= \int_{\xi_i^*}^{\xi_i} \left[[s]^{\frac{\mu}{r_i}} - [\xi_i^*]^{\frac{\mu}{r_i}} \right]^{\frac{4\mu - r_i - \varpi_i}{\mu}} ds \\ &= \int_{\xi_i^*}^{\xi_i} \left(s^{\frac{\mu}{r_i}} - \xi_i^{*\frac{\mu}{r_i}} \right)^{\frac{4\mu - r_i - \varpi_i}{\mu}} ds \\ &\geq \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu - r_i - \varpi_i}{r_i}} ds. \end{aligned} \quad (17)$$

(2) If $\xi_i^* \leq \xi_i \leq 0$, then, similar to (17), we derive the following expression:

$$W_i(\bar{\xi}_i) = \int_{\xi_i^*}^{\xi_i} \left[[s]^{\frac{\mu}{r_i}} - [\xi_i^*]^{\frac{\mu}{r_i}} \right]^{\frac{4\mu - r_i - \varpi_i}{\mu}} ds$$

$$\begin{aligned}
 &= \int_{\xi_i^*}^{\xi_i} \left(-(-s)^{\frac{\mu}{r_i}} + (-\xi_i^*)^{\frac{\mu}{r_i}} \right)^{\frac{4\mu-r_i-\varpi_i}{\mu}} ds \\
 &\geq \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu-r_i-\varpi_i}{r_i}} ds.
 \end{aligned} \tag{18}$$

(3) If $\xi_i^* \leq 0 \leq \xi_i$, then, based on Lemma 1, we derive the following expression:

$$\begin{aligned}
 W_i(\bar{\xi}_i) &= \int_{\xi_i^*}^0 \left(-(-s)^{\frac{\mu}{r_i}} + (-\xi_i^*)^{\frac{\mu}{r_i}} \right)^{\frac{4\mu-r_i-\varpi_i}{\mu}} ds + \int_0^{\xi_i} \left(s^{\frac{\mu}{r_i}} + (-\xi_i^*)^{\frac{\mu}{r_i}} \right)^{\frac{4\mu-r_i-\varpi_i}{\mu}} ds \\
 &\geq \frac{2^{\frac{4\mu-r_i-\varpi_i}{\mu}}}{2^{\frac{4\mu-r_i-\varpi_i}{r_i}}} \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu-r_i-\varpi_i}{r_i}} ds.
 \end{aligned} \tag{19}$$

By combining (17)–(19), we determine that, when $\xi_i^* \leq \xi_i$,

$$W_i(\bar{\xi}_i) \geq \frac{2^{\frac{4\mu-r_i-\varpi_i}{\mu}}}{2^{\frac{4\mu-r_i-\varpi_i}{r_i}}} \int_{\xi_i^*}^{\xi_i} (s - \xi_i^*)^{\frac{4\mu-r_i-\varpi_i}{r_i}} ds = \frac{r_i \cdot 2^{\frac{4\mu-r_i-\varpi_i}{\mu}}}{(4\mu - \varpi_i) \cdot 2^{\frac{4\mu-r_i-\varpi_i}{r_i}}} (\xi_i - \xi_i^*)^{\frac{4\mu-\varpi_i}{r_i}}. \tag{20}$$

Case 2. When $\xi_i^* \geq \xi_i$, using the same analysis process as (17)–(19), we derive the following expression:

$$W_i(\bar{\xi}_i) \geq \frac{r_i \cdot 2^{\frac{4\mu-r_i-\varpi_i}{\mu}}}{(4\mu - \varpi_i) \cdot 2^{\frac{4\mu-r_i-\varpi_i}{r_i}}} (\xi_i^* - \xi_i)^{\frac{4\mu-\varpi_i}{r_i}}. \tag{21}$$

Based on (20) and (21), the left-hand side of (16) holds. Then, using (16), we derive the following expression:

$$\alpha_1(|\xi|) \leq V_n(\xi) \leq \beta_1(|\xi|), \tag{22}$$

where $\alpha_1(|\xi|) = \sum_{i=1}^n c_{i1} |\xi_i - \xi_i^*|^{\frac{4\mu-\varpi_i}{r_i}}$ and $\beta_1(|\xi|) = \sum_{i=1}^n c_{i2} |z_i|^{\frac{4\mu-\varpi_i}{\mu}}$. Hence, the radial unboundedness of $V_n(\xi)$ can be obtained. From (13), (22), and Lemma 2, we can conclude that the closed-loop system (1), (6), and (12) has a continuous solution on $[0, \infty)$.

(ii) For any $k \in \{2, 3, 4, \dots\}$, let the stopping time $\sigma_k = \inf\{t \geq 0 : |\xi(t)| \geq k\}$. Using (13) and the following Itô's formula, we derive the following expression:

$$\mathbb{E}\{V_n(\xi(\sigma_k \wedge t))\} = V_n(\xi(0)) + \mathbb{E} \int_0^{\sigma_k \wedge t} \mathcal{L}V_n(\xi(s)) ds \leq V_n(\xi(0)). \tag{23}$$

For any $t \geq 0$ and $k > 0$, based on the continuity of $\xi(t)$ and the definition of σ_k , $\{\sup_{0 \leq s \leq t} |\xi(s)| > k\} \in \{\sigma_k \leq t\}$, which, together with (22), means that

$$\begin{aligned}
 \mathbb{E}\{V_n(\xi(\sigma_k \wedge t))\} &\geq \int_{\{\sup_{0 \leq s \leq t} |\xi(s)| > k\}} V_n(\xi(\sigma_k \wedge t)) dP \\
 &= \int_{\{\sup_{0 \leq s \leq t} |\xi(s)| > k\}} V_n(\xi(\sigma_k)) dP \\
 &\geq P \left\{ \sup_{0 \leq s \leq t} |\xi(s)| > k \right\} \inf_{|\xi| \geq k} V_n(\xi) \\
 &\geq P \left\{ \sup_{0 \leq s \leq t} |\xi(s)| > k \right\} \inf_{|\xi| \geq k} \alpha_1(|\xi|), \quad \forall t > 0, \forall k > 0.
 \end{aligned} \tag{24}$$

Substituting (24) into (23) leads to

$$P \left\{ \sup_{0 \leq s \leq t} |\xi(s)| > k \right\} \leq \frac{V_n(\xi(0))}{\inf_{|\xi| \geq k} \alpha_1(|\xi|)}, \quad \forall t > 0. \tag{25}$$

Setting $k \rightarrow \infty$ and $t \rightarrow \infty$, and using the radial unboundedness of $\alpha_1(|\xi|)$, we derive $P\{\sup_{t \geq 0} |\xi(t)| < \infty\} = 1$. Thus, $\xi(t), \xi_1(t), \xi_2(t), \dots, \xi_n(t)$ are almost surely bounded, so are $x_1(t), x_2(t), \dots, x_n(t)$. Keeping this in mind and using the definitions of $\xi_2^*(t), \dots, \xi_n^*(t)$ and $u(t)$, we can recursively prove the almost

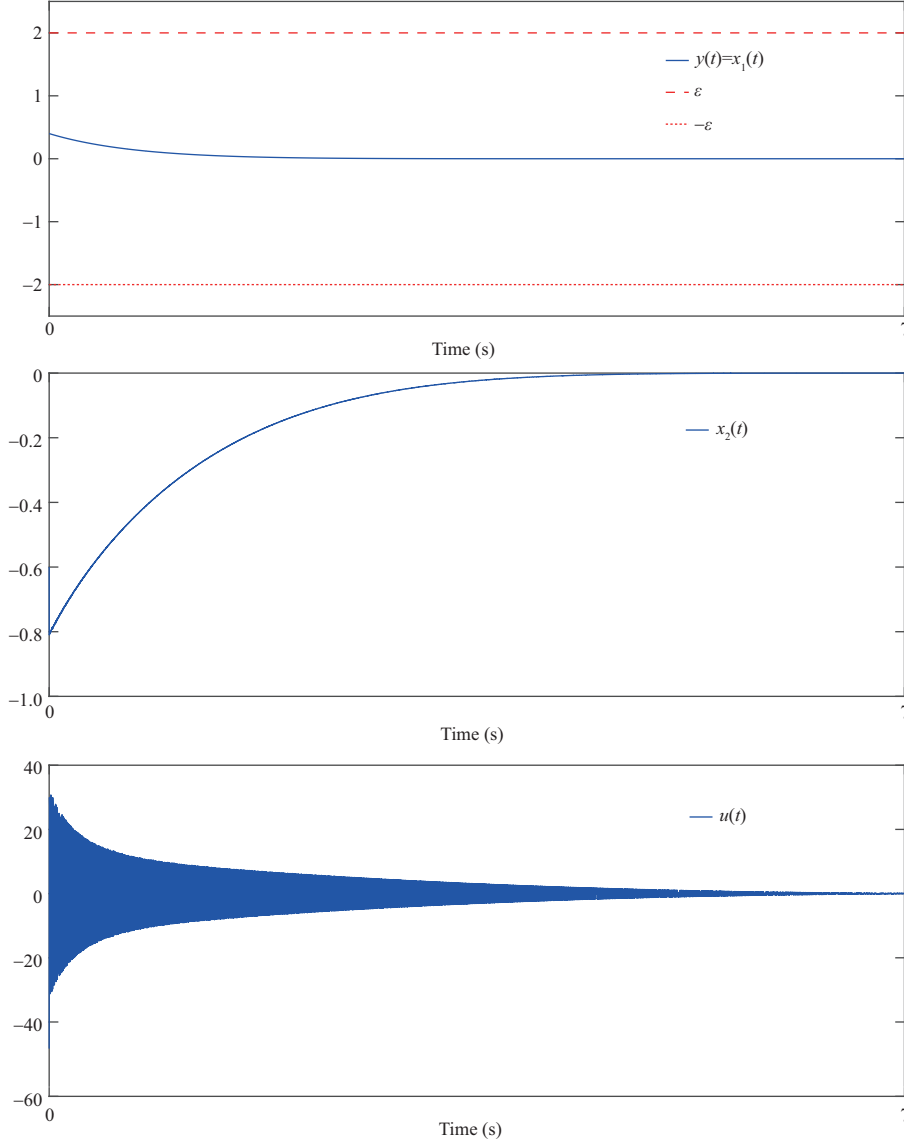


Figure 1 (Color online) Responses of the closed-loop system with $(x_1(0), x_2(0)) = (0.4, -0.6)$.

sure boundedness of $\xi_2^*(t), \dots, \xi_n^*(t), u(t)$. Based on (7) and the almost sure boundedness of $\xi_1(t)$, the constraint (2) is achieved almost surely.

(iii) Based on (16), Lemma 3, and $\varpi_1 = \dots = \varpi_n$, we derive the following expression:

$$\begin{aligned}
 V_n(\xi)^{\frac{4\mu}{4\mu-\varpi_n}} &\leq c_1 \left(\sum_{j=1}^n |z_j|^{\frac{4\mu-\varpi_j}{\mu}} \right)^{\frac{4\mu}{4\mu-\varpi_n}} \leq c_1 \sum_{j=1}^n |z_j|^4, \\
 V_n(\xi)^{\frac{16\mu}{3(4\mu-\varpi_n)}} &\leq c_2 \left(\sum_{j=1}^n |z_j|^{\frac{4\mu-\varpi_j}{\mu}} \right)^{\frac{16\mu}{3(4\mu-\varpi_n)}} \leq c_3 \sum_{j=1}^n |z_j|^{\frac{16}{3}},
 \end{aligned} \tag{26}$$

where $c_1 = (\max_{1 \leq j \leq n} \{c_{j2}\})^{\frac{4\mu}{4\mu-\varpi_n}}$, $c_2 = (\max_{1 \leq j \leq n} \{c_{j2}\})^{\frac{16\mu}{3(4\mu-\varpi_n)}}$, and $c_3 = n^{\frac{16\mu}{3(4\mu-\varpi_n)}-1} c_2$. Substituting (26) into (13) leads to

$$\mathcal{L}V_n(\xi) \leq -c(V_n(\xi))^{\frac{4\mu}{4\mu-\varpi_n}} + V_n(\xi)^{\frac{16\mu}{3(4\mu-\varpi_n)}}, \tag{27}$$

where $c = \min\{\frac{1}{4c_1}, \frac{1}{4c_3}\}$. Because $0 < \frac{4\mu}{4\mu-\varpi_n} < 1$ and $\frac{16\mu}{3(4\mu-\varpi_n)} > 1$, based on (27) and Lemma 4, the trivial solution of the closed-loop system is stochastically fixed-time stable.

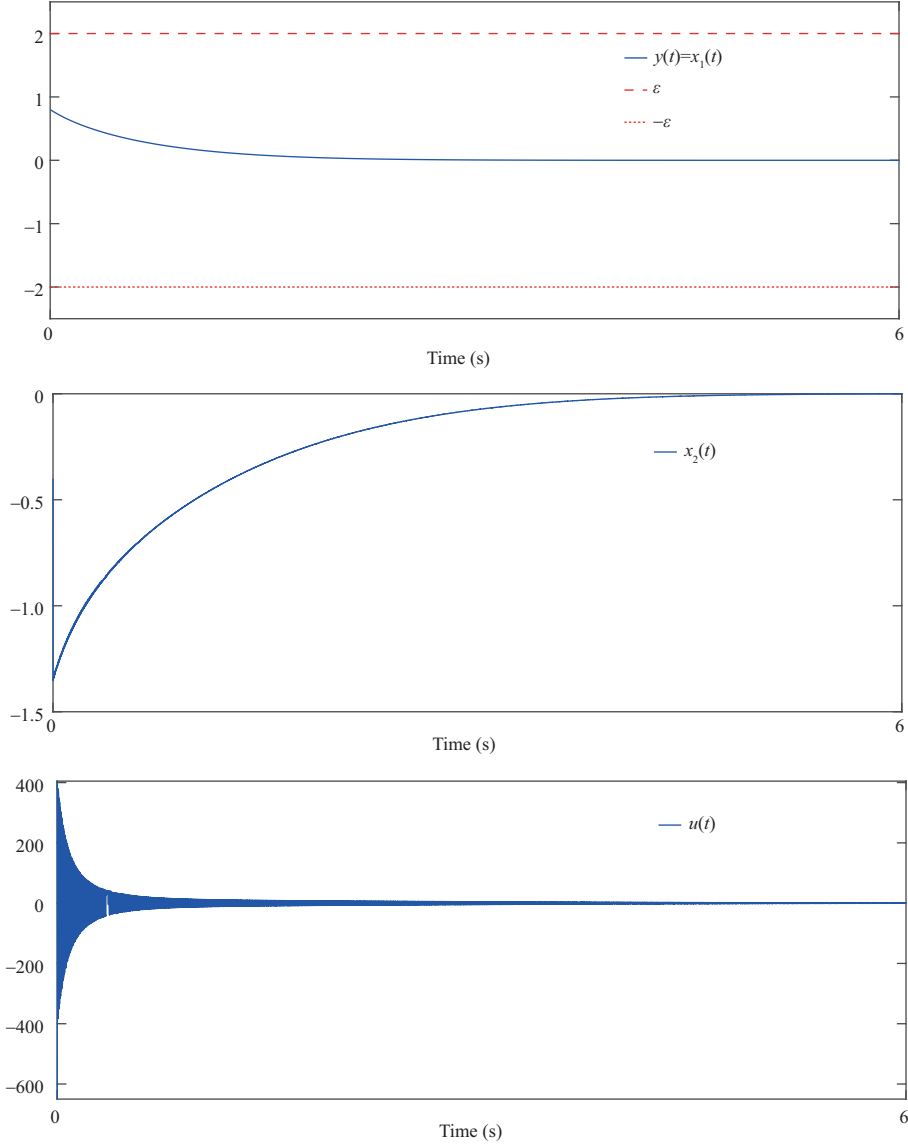


Figure 2 (Color online) Responses of the closed-loop system with $(x_1(0), x_2(0)) = (0.8, -0.5)$.

4 Simulation example

Consider the following system:

$$\begin{aligned} dx_1 &= ([x_2]^{p_1} + f_1(x_1)) dt + g_1(x_1)d\omega, \\ dx_2 &= ([u]^{p_2} + f_2(\bar{x}_2)) dt + g_2(\bar{x}_2)d\omega, \\ y &= x_1, \end{aligned} \tag{28}$$

with the constraint

$$y \in \Omega_y = \{y \in \mathbb{R} : -2 < y < 2\}, \tag{29}$$

where $p_1 = \frac{5}{3}$, $p_2 = 3$, $f_1 = \frac{1}{4}x_1^{\frac{10}{11}}$, $g_1 = 0$, $f_2 = \frac{1}{4}[x_2]^{\frac{5}{6}} + \frac{1}{2}x_2^{\frac{5}{3}}$, and $g_2 = \frac{1}{8}[x_1]^{\frac{1}{2}}$. By choosing $r_1 = 1$, $h_1 = 1$, $\varpi_1 = \varpi_2 = -\frac{1}{11} \in (-\frac{3}{8}, 0)$, and $\gamma_1 = \gamma_2 = \frac{2}{3}$, we obtain $r_2 = \frac{r_1 + \varpi_1}{p_1} = \frac{6}{11}$, $r_3 = \frac{r_2 + \varpi_2}{p_2} = \frac{5}{33}$, $h_2 = \frac{h_1 + \gamma_1}{p_1} = 1$, $h_3 = \frac{h_2 + \gamma_2}{p_2} = \frac{5}{9}$, $f_1 \leq \frac{1}{4}|x_1|^{\frac{10}{11}}$, $f_2 \leq \frac{1}{4}|x_2|^{\frac{5}{6}} + \frac{1}{2}|x_2|^{\frac{5}{3}}$, and $g_2 \leq \frac{1}{8}|x_1|^{\frac{1}{2}}$. Hence, Assumption 1 holds.

Based on

$$\xi_1 = \tan\left(\frac{x_1}{\lambda_1}\right), \quad \xi_2 = x_2, \tag{30}$$

where $\lambda_1 = \frac{4}{\pi}$, Eq. (28) can be rewritten as follows:

$$\begin{aligned} d\xi_1 &= (D_1(\xi_1)[\xi_2]^{p_1} + \bar{f}_1(\xi_1)) dt + \bar{g}_1(\xi_1)d\omega, \\ d\xi_2 &= ([u]^{p_2} + \bar{f}_2(\xi_2)) dt + \bar{g}_2(\xi_2)d\omega, \end{aligned} \tag{31}$$

where $D_1 = \frac{\pi(1+\xi_1^2)}{4}$, $\bar{f}_1 = D_1 f_1 + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2} g_1^\top g_1$, $\bar{g}_1 = D_1 g_1$, $\bar{f}_2 = f_2$, and $\bar{g}_2 = g_2$. Based on $\mu = 2$, $V_1 = \int_{\xi_1^*}^{\xi_1} [[s]^{\frac{\mu}{r_1}} - [\xi_1^*]^{\frac{\mu}{r_1}}]^{\frac{4\mu-r_1-\varpi_1}{\mu}}$ ds with $\xi_1^* = 0$, $z_1 = [\xi_1]^2$, and $\xi_2^* = -(\frac{452(\frac{1787}{1197} + \frac{1}{4}z_1^{\frac{4}{3}} + \frac{304}{1243}z_1)}{355})^{\frac{3}{5}} z_1 \triangleq -\beta_1 z_1$ guarantee that $\mathcal{L}V_1 \leq -\frac{5}{4}z_1^4 - \frac{1}{4}z_1^{\frac{16}{3}} + D_1[z_1]^{\frac{39}{11}}([\xi_2]^{\frac{5}{3}} - [\xi_2^*]^{\frac{5}{3}})$. By setting $z_2 = [\xi_2]^{\frac{4}{3}} - [\xi_2^*]^{\frac{4}{3}}$ and $V_2 = V_1 + \int_{\xi_2^*}^{\xi_2} [[s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}}]^{\frac{4\mu-r_2-\varpi_2}{\mu}}$ ds, the controller

$$u = -\left(\frac{1}{4} + \phi + \frac{1}{4}z_2^{\frac{4}{3}}\right)^{\frac{1}{3}} [z_2]^{\frac{5}{66}} \tag{32}$$

leads to $\mathcal{L}V_2 \leq -\frac{1}{4}\sum_{j=1}^2 z_j^4 - \frac{1}{4}\sum_{j=1}^2 z_j^{\frac{16}{3}}$, where $\beta_2 = \frac{1493}{533}\beta_1^{\frac{6}{5}}(\frac{2}{3}\xi_1^{\frac{5}{3}} + \frac{608}{1243}\xi_1)\xi_1^2 + 2\beta_1^{\frac{11}{5}}\xi_1$ and $\phi = \frac{4131}{385}(D_1\beta_1^{\frac{5}{3}}\beta_2)^{\frac{88}{67}} + \frac{13993}{1938}\beta_2^{\frac{8}{7}} + \frac{751}{324}(D_1\beta_2)^{\frac{88}{67}} + (\frac{3}{4} + \frac{1}{2}\xi_2^2) + \frac{756}{835}(\frac{1}{4}\beta_1^{\frac{5}{6}} + \frac{1}{2}\beta_1^{\frac{5}{6}}\xi_2^{\frac{5}{6}})^{\frac{88}{67}} + \frac{747}{8099} + \frac{115}{1021}\beta_1^{\frac{704}{183}} + \frac{181255}{8}D_1^{\frac{44}{3}}$.

Figures 1 and 2 illustrate the responses of the closed-loop system (28)–(30) and (32) with the mean of 10 sample sizes and two different initial states, that is, $(x_1(0), x_2(0)) = (0.4, -0.6)$ and $(x_1(0), x_2(0)) = (0.8, -0.5)$. Stochastic fixed-time stabilization can be achieved with the expectation of the settling time

being less than $T_{\max} = \frac{2}{4^{\frac{3+p}{4}}(1-\frac{3+p}{4})} + \frac{2(2^{\frac{5+p}{4}}-1)}{4^{\frac{5+p}{4}}(\frac{5+p}{4}-1)} = 168.4$, which is independent of any initial value.

5 Conclusion

This paper studies the fixed-time state feedback control problem of stochastic high-order nonlinear systems with output constraint and high-order and low-order nonlinearities.

Some critical challenges remain unresolved: (i) For the stochastic system with output constraint in [35], the method of designing a fixed-time output feedback controller needs to be clarified. (ii) An important issue corresponding to FT-SISS in [37] is how to define the fixed-time stochastic integral input-to-state stability.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 62221004, 62073166), 333 Project, and Project on the Technological Leading Talent Teams Led by Frontiers Science Center for Complex Equipment System Dynamics (Grant No. FSCCESD220401).

References

- 1 Jin X. Adaptive fault-tolerant control for a class of output-constrained nonlinear systems. *Intl J Robust Nonlinear*, 2015, 25: 3732–3745
- 2 Tee K P, Ge S S, Tay E H. Barrier Lyapunov functions for the control of output-constrained nonlinear systems. *Automatica*, 2009, 45: 918–927
- 3 Tee K P, Ge S S. Control of nonlinear systems with partial state constraints using a barrier Lyapunov function. *Int J Control*, 2011, 84: 2008–2023
- 4 Lin W, Qian C J. Adding one power integrator: a tool for global stabilization of high-order lower-triangular systems. *Syst Control Lett*, 2000, 39: 339–351
- 5 Xie X J, Duan N. Output tracking of high-order stochastic nonlinear systems with application to benchmark mechanical system. *IEEE Trans Automat Contr*, 2010, 55: 1197–1202
- 6 Li W Q, Jing Y W, Zhang S Y. Adaptive state-feedback stabilization for a large class of high-order stochastic nonlinear systems. *Automatica*, 2011, 47: 819–828
- 7 Li W Q, Liu X H, Zhang S Y. Further results on adaptive state-feedback stabilization for stochastic high-order nonlinear systems. *Automatica*, 2012, 48: 1667–1675
- 8 Xie X J, Liu L. A homogeneous domination approach to state feedback of stochastic high-order nonlinear systems with time-varying delay. *IEEE Trans Automat Contr*, 2013, 58: 494–499
- 9 Zhao C R, Xie X J. Global stabilization of stochastic high-order feedforward nonlinear systems with time-varying delay. *Automatica*, 2014, 50: 203–210
- 10 Xie X J, Duan N, Zhao C R. A combined homogeneous domination and sign function approach to output-feedback stabilization of stochastic high-order nonlinear systems. *IEEE Trans Automat Contr*, 2014, 59: 1303–1309
- 11 Xue L R, Zhang T L, Zhang W H, et al. Global adaptive stabilization and tracking control for high-order stochastic nonlinear systems with time-varying delays. *IEEE Trans Automat Contr*, 2018, 63: 2928–2943
- 12 Min H F, Xu S Y, Gu J, et al. Further results on adaptive stabilization of high-order stochastic nonlinear systems subject to uncertainties. *IEEE Trans Neural Netw Learn Syst*, 2020, 31: 225–234
- 13 Cui R H, Xie X J. Adaptive state-feedback stabilization of state-constrained stochastic high-order nonlinear systems. *Sci China Inf Sci*, 2021, 64: 200203

- 14 Cui R H, Xie X J. Exponential stabilization of stochastic feedforward nonlinear systems: a dynamic gain approach. *IEEE Trans Automat Contr*, 2024. doi: 10.1109/TAC.2023.3331682
- 15 Chen W, Jiao L C. Finite-time stability theorem of stochastic nonlinear systems. *Automatica*, 2010, 46: 2105–2108
- 16 Yin J L, Khoo S Y, Man Z H, et al. Finite-time stability and instability of stochastic nonlinear systems. *Automatica*, 2011, 47: 2671–2677
- 17 Khoo S Y, Yin J L, Man Z H, et al. Finite-time stabilization of stochastic nonlinear systems in strict-feedback form. *Automatica*, 2013, 49: 1403–1410
- 18 Min H F, Xu S Y, Zhang Z Q. Adaptive finite-time stabilization of stochastic nonlinear systems subject to full-state constraints and input saturation. *IEEE Trans Automat Contr*, 2021, 66: 1306–1313
- 19 Wang H, Zhu Q X. Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form. *Automatica*, 2015, 54: 284–291
- 20 Yu X, Yin J L, Khoo S Y. Generalized Lyapunov criteria on finite-time stability of stochastic nonlinear systems. *Automatica*, 2019, 107: 183–189
- 21 Zhao P, Feng W, Zhao Y, et al. Finite-time stochastic input-to-state stability of switched stochastic nonlinear systems. *Appl Math Comput*, 2015, 268: 1038–1054
- 22 Zhao G H, Li J C, Liu S J. Finite-time stabilization of weak solutions for a class of non-local Lipschitzian stochastic nonlinear systems with inverse dynamics. *Automatica*, 2018, 98: 285–295
- 23 Jiang M M, Xie X J, Zhang K M. Finite-time stabilization of stochastic high-order nonlinear systems with FT-SISS inverse dynamics. *IEEE Trans Automat Contr*, 2019, 64: 313–320
- 24 Cui R H, Xie X J. Finite-time stabilization of stochastic low-order nonlinear systems with time-varying orders and FT-SISS inverse dynamics. *Automatica*, 2021, 125: 109418
- 25 Cui R H, Xie X J. Finite-time stochastic integral input-to-state stability and its applications. *Automatica*, 2023, 158: 111311
- 26 Fang L D, Ma L, Ding S H, et al. Finite-time stabilization for a class of high-order stochastic nonlinear systems with an output constraint. *Appl Math Comput*, 2019, 358: 63–79
- 27 Fang L D, Ma L, Ding S H, et al. Finite-time stabilization of high-order stochastic nonlinear systems with asymmetric output constraints. *IEEE Trans Syst Man Cybern Syst*, 2021, 51: 7201–7213
- 28 Cui R H, Xie X J. Finite-time stabilization of output-constrained stochastic high-order nonlinear systems with high-order and low-order nonlinearities. *Automatica*, 2022, 136: 110085
- 29 Yu J J, Yu S H, Li J, et al. Fixed-time stability theorem of stochastic nonlinear systems. *Int J Control*, 2019, 92: 2194–2200
- 30 Min H F, Shi S, Xu S Y, et al. Fixed-time Lyapunov criteria of stochastic nonlinear systems and its generalization. *IEEE Trans Automat Contr*, 2023, 68: 5052–5058
- 31 Li K W, Li Y M, Zong G D. Adaptive fuzzy fixed-time decentralized control for stochastic nonlinear systems. *IEEE Trans Fuzzy Syst*, 2021, 29: 3428–3440
- 32 Guerra-Avellaneda F, Basin M. Continuous fixed-time convergent control design for stochastic super-twisting system. *J Franklin Institute*, 2020, 357: 11793–11806
- 33 Yao Y G, Tan J Q, Wu J, et al. Event-triggered fixed-time adaptive neural tracking control for stochastic non-triangular structure nonlinear systems. *Neural Comput Applic*, 2021, 33: 15887–15899
- 34 Qi X J, Xu S Y, Li Y M, et al. Global fixed-time event-triggered control for stochastic nonlinear systems with full state constraints. *Nonlinear Dyn*, 2023, 111: 7403–7415
- 35 Yao D J, Dou C X, Yue D, et al. Event-triggered practical fixed-time fuzzy containment control for stochastic multiagent systems. *IEEE Trans Fuzzy Syst*, 2022, 30: 3052–3062
- 36 Xu K, Wang H, Liu P X. Adaptive fuzzy fixed-time control for non-triangular structural stochastic switching nonlinear systems. *Intl J Robust Nonlinear*, 2023, 33: 4685–4700
- 37 Cui R H, Xie X J. Output feedback stabilization of stochastic planar nonlinear systems with output constraint. *Automatica*, 2022, 143: 110471

Appendix A Proof of Proposition 1

By (6),

$$|x_1|^\eta = |\lambda_1 \arctan(\xi_1)|^\eta \leq \lambda_1^\eta |\xi_1|^\eta, \quad (\text{A1})$$

where $\eta \in \{\frac{r_1+\varpi_1}{r_1}, \frac{h_1+\gamma_1}{r_1}, \frac{2r_1+\varpi_1}{2r_1}, \frac{2h_1+\gamma_1}{2r_1}\}$. By (3), (A1), and Lemmas 3 and 5,

$$\begin{aligned} \bar{f}_1(\xi_1) &\leq |D_1(\xi_1)| |f_1(\xi_1)| + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2} |g_1(\xi_1)|^2 \\ &\leq |D_1(\xi_1)| \left| f_{11}(\xi_1) |x_1|^{\frac{r_1+\varpi_1}{r_1}} + f_{12}(\xi_1) |x_1|^{\frac{h_1+\gamma_1}{h_1}} \right| \\ &\quad + \frac{\xi_1(1+\xi_1^2)}{\lambda_1^2} \left| g_{11}(\xi_1) |x_1|^{\frac{2r_1+\varpi_1}{2r_1}} + g_{12}(\xi_1) |x_1|^{\frac{2h_1+\gamma_1}{2h_1}} \right|^2 \\ &\leq \bar{f}_{11}(\xi_1) |\xi_1|^{\frac{r_1+\varpi_1}{r_1}} + \bar{f}_{12}(\xi_1) |\xi_1|^{\frac{h_1+\gamma_1}{h_1}}, \end{aligned} \quad (\text{A2})$$

where $\bar{f}_{11}(\xi_1)$ and $\bar{f}_{12}(\xi_1)$ are some known nonnegative smooth functions. Similar to (A2), one can find some nonnegative smooth functions $\bar{g}_{11}(\xi_1)$ and $\bar{g}_{12}(\xi_1)$ such that

$$\bar{g}_1(\xi_1) \leq \bar{g}_{11}(\xi_1) |\xi_1|^{\frac{2r_1+\varpi_1}{2r_1}} + \bar{g}_{12}(\xi_1) |\xi_1|^{\frac{2h_1+\gamma_1}{2h_1}}. \quad (\text{A3})$$

For $i = 2, \dots, n$, it follows from (3), (A1), and Lemmas 3 and 5 that

$$|\bar{f}_i(\bar{\xi}_i)| \leq \bar{f}_{i1}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{r_i+\varpi_i}{r_j}} + \bar{f}_{i2}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{h_i+\gamma_i}{h_j}},$$

$$|\bar{g}_i(\bar{\xi}_i)| \leq \bar{g}_{i1}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}} + \bar{g}_{i2}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{2h_i+\gamma_i}{2h_j}}, \tag{A4}$$

where $\bar{f}_{i1}(\bar{\xi}_i)$, $\bar{f}_{i2}(\bar{\xi}_i)$, $\bar{g}_{i1}(\bar{\xi}_i)$, and $\bar{g}_{i2}(\bar{\xi}_i)$ are some known nonnegative smooth functions. By $\varpi_n \leq \varpi_{n-1} \leq \dots \leq \varpi_1 \in (-\frac{1}{\sum_{l=1}^n p_1 \dots p_{l-1}}, 0)$ and $0 \leq \gamma_n \leq \gamma_{n-1} \leq \dots \leq \gamma_1$ in Assumption 1, one can deduce that $\frac{r_i+\varpi_i}{r_j} \leq \frac{1}{p_j \dots p_{i-1}} \leq \frac{h_i+\gamma_i}{h_j}$. Using this fact, (A1), (A4), and Lemma 5, one has

$$\begin{aligned} |\bar{f}_i(\bar{\xi}_i)| &\leq \bar{f}_{i1}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{r_i+\varpi_i}{r_j}} + \bar{f}_{i2}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{h_i+\gamma_i}{h_j}} \\ &= \sum_{j=1}^i \left(\bar{f}_{i1}(\bar{\xi}_i) + \bar{f}_{i2}(\bar{\xi}_i) |\xi_j|^{\frac{h_i+\gamma_i}{h_j} - \frac{r_i+\varpi_i}{r_j}} \right) |\xi_j|^{\frac{r_i+\varpi_i}{r_j}} \\ &\leq h_{i1}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{r_i+\varpi_i}{r_j}}, \end{aligned} \tag{A5}$$

where $h_{i1}(\bar{\xi}_i)$ is a known nonnegative smooth function. Similarly, there exists a known nonnegative smooth function $\eta_{i1}(\bar{\xi}_i)$ such that

$$\begin{aligned} |\bar{g}_i(\bar{\xi}_i)| &\leq \bar{g}_{i1}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}} + \bar{g}_{i2}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{2h_i+\gamma_i}{2h_j}} \\ &= \sum_{j=1}^i \left(\bar{g}_{i1}(\bar{\xi}_i) + \bar{g}_{i2}(\bar{\xi}_i) |\xi_j|^{\frac{2h_i+\gamma_i}{2h_j} - \frac{2r_i+\varpi_i}{2r_j}} \right) |\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}} \\ &\leq \eta_{i1}(\bar{\xi}_i) \sum_{j=1}^i |\xi_j|^{\frac{2r_i+\varpi_i}{2r_j}}. \end{aligned} \tag{A6}$$

Appendix B Proof of Proposition 2

We prove the proposition by induction.

Step 1: Set $z_1 = [\xi_1]^{\frac{\mu}{r_1}}$ and the Lyapunov function $V_1(\xi_1) = W_1(\xi_1) = \frac{r_1}{4\mu-r_1} |\xi_1|^{\frac{4\mu-r_1}{r_1}}$. By (8), Lemma 6, and Itô's formula,

$$\begin{aligned} \mathcal{L}V_1 &\leq D_1(\xi_1) [z_1]^{\frac{4\mu-r_1-\varpi_1}{\mu}} \left(([\xi_2]^{p_1} - [\xi_2^*]^{p_1} + [\xi_2^*]^{p_1}) + [z_1]^{\frac{4\mu-r_1-\varpi_1}{\mu}} \bar{f}_1(\xi_1) \right) \\ &\quad + \frac{4\mu-r_1-\varpi_1}{2r_1} |z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}} \bar{g}_1^T(\xi_1) \bar{g}_1(\xi_1). \end{aligned} \tag{B1}$$

From (9) and Lemmas 5 and 7, it follows that

$$[z_1]^{\frac{4\mu-r_1-\varpi_1}{\mu}} \bar{f}_1(\xi_1) \leq h_{11}(\xi_1) |z_1|^{\frac{4\mu-r_1-\varpi_1}{\mu}} |z_1|^{\frac{r_1+\varpi_1}{\mu}} = \beta_{11}(\xi_1) |z_1|^4, \tag{B2}$$

where $\beta_{11}(\xi_1)$ is a nonnegative smooth function. Using (10) and Lemmas 3, 5, and 7, one can find a nonnegative smooth function $\beta_{12}(\xi_1)$ such that

$$\begin{aligned} \frac{4\mu-r_1-\varpi_1}{2r_1} |z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}} \bar{g}_1^T(\xi_1) \bar{g}_1(\xi_1) &\leq \frac{4\mu-r_1-\varpi_1}{2r_1} |z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}} \left(\eta_{11}(\xi_1) |z_1|^{\frac{2r_1+\varpi_1}{2\mu}} \right)^2 \\ &\leq \frac{4\mu-r_1-\varpi_1}{2r_1} \eta_{11}(\xi_1)^2 |z_1|^{\frac{4\mu-2r_1-\varpi_1}{\mu}} |z_1|^{\frac{2r_1+\varpi_1}{\mu}} \\ &\leq \beta_{12}(\xi_1) |z_1|^4. \end{aligned} \tag{B3}$$

Substituting (B2), (B3), and the virtual controller

$$\xi_2^*(\xi_1) = - \left(\frac{n-1 + \frac{1}{4} + \frac{1}{4} z_1^{\frac{4}{3}} + \beta_{11}(\xi_1)}{D} \right)^{\frac{1}{p_1}} [z_1]^{\frac{r_2}{\mu}} \triangleq -\phi_1(\xi_1) [z_1]^{\frac{r_2}{\mu}} \tag{B4}$$

into (B1) yields

$$\begin{aligned} \mathcal{L}V_1 &\leq D_1(\xi_1) [z_1]^{\frac{4\mu-r_1-\varpi_1}{\mu}} \left(([\xi_2]^{p_1} - [\xi_2^*]^{p_1} + [\xi_2^*]^{p_1}) + \beta_{11}(\xi_1) |z_1|^4 \right) \\ &\leq - \left(n-1 + \frac{1}{4} \right) |z_1|^4 + [z_1]^{\frac{4\mu-r_1-\varpi_1}{\mu}} \left(([\xi_2]^{p_1} - [\xi_2^*]^{p_1}) - \frac{1}{4} |z_1|^{\frac{16}{3}} \right), \end{aligned} \tag{B5}$$

where $D = \frac{1}{\lambda_1}$ and $\beta_1(\xi_1) = \beta_{11}(\xi_1) + \beta_{12}(\xi_1)$.

Step 2: Set $V_2(\bar{\xi}_2) = V_1(\xi_1) + W_2(\bar{\xi}_2) = V_1(\xi_1) + \int_{\xi_2^*}^{\xi_2} [s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}}]^{\frac{4\mu-r_2-\varpi_2}{\mu}} ds$ and $z_2 = [\xi_2]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}}$. Applying (8), (B5), and Itô's formula, one gets

$$\begin{aligned} \mathcal{L}V_2 &= \mathcal{L}V_1 + \frac{\partial W_2}{\partial \xi_1} (D_1(\xi_1) [\xi_2]^{p_1} + \bar{f}_1(\xi_1)) + \frac{\partial W_2}{\partial \xi_2} ([\xi_3]^{p_2} + \bar{f}_2(\bar{\xi}_2)) + \frac{1}{2} \frac{\partial^2 W_2}{\partial \xi_1^2} \bar{g}_1^T(\xi_1) \bar{g}_1(\xi_1) \\ &\quad + \frac{\partial^2 W_2}{\partial \xi_1 \partial \xi_2} \bar{g}_1^T(\xi_1) \bar{g}_2(\bar{\xi}_2) + \frac{1}{2} \frac{\partial^2 W_2}{\partial \xi_2^2} \bar{g}_2^T(\bar{\xi}_2) \bar{g}_2(\bar{\xi}_2) \end{aligned}$$

$$\begin{aligned} &\leq -\left(n-1+\frac{1}{4}\right)|z_1|^4 + [z_1]^{\frac{4\mu-r_1-\varpi_1}{\mu}}([\xi_2]^{p_1} - [\xi_2^*]^{p_1}) - \frac{1}{4}|z_1|^{\frac{16}{3}} + \frac{\partial W_2}{\partial \xi_1}(D_1(\xi_1)[\xi_2]^{p_1} + \bar{f}_1(\xi_1)) \\ &\quad + \frac{\partial W_2}{\partial \xi_2}([\xi_3]^{p_2} + \bar{f}_2(\bar{\xi}_2)) + \frac{1}{2}\frac{\partial^2 W_2}{\partial \xi_1^2}\bar{g}_1^T(\xi_1)\bar{g}_1(\xi_1) + \frac{\partial^2 W_2}{\partial \xi_1 \partial \xi_2}\bar{g}_1^T(\xi_1)\bar{g}_2(\bar{\xi}_2) \\ &\quad + \frac{1}{2}\frac{\partial^2 W_2}{\partial \xi_2^2}\bar{g}_2^T(\bar{\xi}_2)\bar{g}_2(\bar{\xi}_2). \end{aligned} \tag{B6}$$

By the definition of μ and Lemma 6,

$$\frac{\partial W_2}{\partial \xi_1} = -\frac{4\mu - \varpi_2 - r_2}{\mu} \frac{\partial [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1} \int_{\xi_2^*}^{\xi_2} \left| [s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}} \right|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} ds, \tag{B7}$$

$$\frac{\partial W_2}{\partial \xi_2} = [z_2]^{\frac{4\mu - \varpi_2 - r_2}{\mu}}, \tag{B8}$$

$$\begin{aligned} \frac{\partial^2 W_2}{\partial \xi_1^2} &= -\left(\frac{4\mu - \varpi_2 - r_2}{\mu}\right) \left(\frac{\partial^2 [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1^2}\right) \int_{\xi_2^*}^{\xi_2} \left| [s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}} \right|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} ds + \left(\frac{4\mu - \varpi_2 - r_2}{\mu}\right) \\ &\quad \cdot \left(\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}\right) \left(\frac{\partial [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1}\right)^2 \int_{\xi_2^*}^{\xi_2} \left| [s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}} \right|^{\frac{4\mu - \varpi_2 - r_2 - 2\mu}{\mu}} ds, \end{aligned} \tag{B9}$$

$$\frac{\partial^2 W_2}{\partial \xi_2 \partial \xi_1} = \frac{4\mu - \varpi_2 - r_2}{\mu} |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \frac{\partial z_2}{\partial \xi_1}, \tag{B10}$$

$$\frac{\partial^2 W_2}{\partial \xi_2^2} = \frac{4\mu - \varpi_2 - r_2}{\mu} |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \frac{\partial z_2}{\partial \xi_2}. \tag{B11}$$

It follows from (9), (B7), and Lemmas 1, 3, 5, 7, and 8 that

$$\begin{aligned} \frac{\partial W_2}{\partial \xi_1}(D_1(\xi_1)[\xi_2]^{p_1} + \bar{f}_1(\xi_1)) &\leq \left| \frac{4\mu - \varpi_2 - r_2}{\mu} \frac{\partial [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1} \int_{\xi_2^*}^{\xi_2} \left| [s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}} \right|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} ds \right| \\ &\quad \cdot \left(D_1(\xi_1)|\xi_2|^{p_1} + h_1(\xi_1)|\xi_1|^{\frac{r_1 + \varpi_1}{r_1}} \right) \\ &\leq \left| \frac{4\mu - \varpi_2 - r_2}{\mu} \frac{\partial [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1} |\xi_2 - \xi_2^*| |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \right| \\ &\quad \cdot \left(D_1(\xi_1)|z_2 + \phi_1 z_1|^{\frac{r_1 + \varpi_1}{\mu}} + h_{11}(\xi_1)|z_1|^{\frac{r_1 + \varpi_1}{\mu}} \right) \\ &\leq \frac{1}{6}|z_1|^4 + \beta_{21}(\bar{\xi}_2)|z_2|^4, \end{aligned} \tag{B12}$$

where $\beta_{21}(\bar{\xi}_2)$ is a nonnegative smooth function. Similar to (B12), it is clear that

$$\begin{aligned} \frac{\partial W_2}{\partial \xi_2} \bar{f}_2(\bar{\xi}_2) &\leq |z_2|^{\frac{4\mu - \varpi_2 - r_2}{\mu}} \left(h_{21}(\bar{\xi}_2) \sum_{j=1}^2 |\xi_j|^{\frac{r_2 + \varpi_2}{r_j}} \right) \\ &\leq |z_2|^{\frac{4\mu - \varpi_2 - r_2}{\mu}} \left(h_{21}(\bar{\xi}_2) \left(|z_1|^{\frac{r_2 + \varpi_2}{\mu}} + |z_2 + \phi_1 z_1|^{\frac{r_2 + \varpi_2}{\mu}} \right) \right) \\ &\leq \frac{1}{6}|z_1|^4 + \beta_{22}(\bar{\xi}_2)|z_2|^4, \end{aligned} \tag{B13}$$

where $\beta_{22}(\bar{\xi}_2)$ is a nonnegative smooth function. From (10), (B4), (B10), and Lemmas 1, 3, 5, and 7,

$$\begin{aligned} \frac{\partial^2 W_2}{\partial \xi_2 \partial \xi_1} \bar{g}_2^T(\bar{\xi}_2)\bar{g}_1(\xi_1) &\leq \left| \frac{4\mu - \varpi_2 - r_2}{\mu} \frac{\partial z_2}{\partial \xi_1} |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \right| \left(\eta_{21}(\bar{\xi}_2) \sum_{j=1}^2 |\xi_j|^{\frac{2r_2 + \varpi_2}{2r_j}} \right) \left(\eta_{11}(\xi_1) |\xi_1|^{\frac{2r_1 + \varpi_1}{2r_1}} \right) \\ &\leq \left| \frac{4\mu - \varpi_2 - r_2}{\mu} \frac{\partial z_2}{\partial \xi_1} |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \right| \varrho_1(\bar{\xi}_2) \sum_{j=1}^2 |z_j|^{\frac{2r_2 + \varpi_2 + 2r_1 + \varpi_1}{2\mu}} \\ &\leq \frac{1}{6}|z_1|^4 + \beta_{23}(\bar{\xi}_2)|z_2|^4, \end{aligned} \tag{B14}$$

where $\varrho_1(\bar{\xi}_2)$ and $\beta_{23}(\bar{\xi}_2)$ are some nonnegative smooth functions. By (10), (B11), and Lemmas 1, 3, 5, and 7, one has

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 W_2}{\partial \xi_2^2} \bar{g}_2^T(\bar{\xi}_2)\bar{g}_2(\bar{\xi}_2) &\leq \left| \frac{4\mu - \varpi_2 - r_2}{2\mu} |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \frac{\partial z_2}{\partial \xi_2} \right| \left(\eta_{21}(\bar{\xi}_2) \sum_{j=1}^2 |\xi_j|^{\frac{2r_2 + \varpi_2}{2r_j}} \right)^2 \\ &\leq \left| \frac{4\mu - \varpi_2 - r_2}{2r_2} |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} |\xi_2|^{\frac{\mu - r_2}{r_2}} \right| \left(\eta_{21}(\bar{\xi}_2) \left(|z_1|^{\frac{2r_2 + \varpi_2}{2\mu}} \right. \right. \\ &\quad \left. \left. + |z_2 + \phi_1 z_1|^{\frac{2r_2 + \varpi_2}{2\mu}} \right) \right)^2 \\ &\leq \frac{1}{6}|z_1|^4 + \beta_{24}(\bar{\xi}_2)|z_2|^4, \end{aligned} \tag{B15}$$

where $\beta_{24}(\bar{\xi}_2)$ is a nonnegative smooth function. By (10), (B4), (B9), and Lemmas 1, 3, 5, 7, and 8,

$$\frac{1}{2} \frac{\partial^2 W_2}{\partial \xi_1^2} \bar{g}_1^T(\xi_1)\bar{g}_1(\xi_1) \leq \frac{1}{2} \left| \left(\frac{4\mu - \varpi_2 - r_2}{\mu} \right) \left(\frac{\partial^2 [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1^2} \right) \int_{\xi_2^*}^{\xi_2} \left| [s]^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}} \right|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} ds \right|$$

$$\begin{aligned}
 & + \left(\frac{\partial [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1} \right)^2 \left(\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu} \right) \int_{\xi_2^*}^{\xi_2} \left[|s|^{\frac{\mu}{r_2}} - [\xi_2^*]^{\frac{\mu}{r_2}} \right]^{\frac{4\mu - \varpi_2 - r_2 - 2\mu}{\mu}} ds \\
 & \cdot \left(\frac{4\mu - \varpi_2 - r_2}{\mu} \right) \left| \left(\eta_{11}(\xi_1) |\xi_1|^{\frac{2r_1 + \varpi_1}{2r_1}} \right)^2 \right. \\
 \leq & \frac{1}{2} \left| \left(\frac{4\mu - \varpi_2 - r_2}{\mu} \right) \left(\frac{\partial^2 [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1^2} \right) |\xi_2 - \xi_2^*| |z_2|^{\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu}} \right. \\
 & + \left. \left(\frac{\partial [\xi_2^*]^{\frac{\mu}{r_2}}}{\partial \xi_1} \right)^2 \left(\frac{4\mu - \varpi_2 - r_2 - \mu}{\mu} \right) |\xi_2 - \xi_2^*| |z_2|^{\frac{4\mu - \varpi_2 - r_2 - 2\mu}{\mu}} \right. \\
 & \cdot \left. \left(\frac{4\mu - \varpi_2 - r_2}{\mu} \right) \left| \left(\eta_{11}(\xi_1) |z_1|^{\frac{2r_1 + \varpi_1}{2\mu}} \right)^2 \right. \right. \\
 \leq & \frac{1}{6} |z_1|^4 + \beta_{25}(\bar{\xi}_2) |z_2|^4, \tag{B16}
 \end{aligned}$$

where $\beta_{25}(\bar{\xi}_2)$ is a nonnegative smooth function. Since $\frac{r_1 r_2}{\mu} \leq 1$, by Lemmas 1, 5, and 7, there is a nonnegative smooth function $\beta_{26}(\bar{\xi}_2)$ such that

$$\begin{aligned}
 D_1(\xi_1) |z_1|^{\frac{4\mu - \varpi_1 - r_1}{\mu}} (|\xi_2|^{p_1} - [\xi_2^*]^{p_1}) & \leq 2^{1 - \frac{\varpi_1 + r_1}{\mu}} D_1 |z_1|^{\frac{4\mu - \varpi_1 - r_1}{\mu}} |z_2|^{\frac{\varpi_1 + r_1}{\mu}} \\
 & \leq \frac{1}{6} |z_1|^4 + \beta_{26}(\bar{\xi}_2) |z_2|^4. \tag{B17}
 \end{aligned}$$

Substituting (B12)–(B17) into (B6) leads to

$$\mathcal{L}V_2 \leq - \left(n - 2 + \frac{1}{4} \right) |z_1|^4 + \beta_2(\bar{\xi}_2) |z_2|^4 + [z_2]^{\frac{4\mu - \varpi_2 - r_2}{\mu}} (|\xi_3|^{p_2} - [\xi_3^*]^{p_2} + [\xi_3^*]^{p_2}) - \frac{1}{4} |z_1|^{\frac{16}{3}}, \tag{B18}$$

where $\beta_2(\bar{\xi}_2) = \sum_{j=1}^6 \beta_{2j}(\bar{\xi}_2)$. Then, one can design the virtual controller

$$\xi_3^*(\bar{\xi}_2) = - \left(n - 2 + \frac{1}{4} + \frac{1}{4} z_2^{\frac{4}{3}} + \beta_2(\bar{\xi}_2) \right)^{\frac{1}{p_2}} [z_2]^{\frac{r_3}{\mu}} \triangleq -\phi_2^{\frac{r_3}{\mu}}(\bar{\xi}_2) [z_2]^{\frac{r_3}{\mu}}, \tag{B19}$$

such that

$$\mathcal{L}V_2 \leq - \left(n - 2 + \frac{1}{4} \right) \sum_{j=1}^2 |z_j|^4 + [z_2]^{\frac{4\mu - r_2 - \varpi_2}{\mu}} (|\xi_3|^{p_2} - [\xi_3^*]^{p_2}) - \frac{1}{4} \sum_{j=1}^2 |z_j|^{\frac{16}{3}}. \tag{B20}$$

Inductive Step ($3 \leq k \leq n$): Suppose that at Step $k - 1$, there exist a Lyapunov function $V_{k-1}(\bar{\xi}_{k-1})$ and a series of virtual controllers $\xi_2^*(\xi_1), \dots, \xi_k^*(\bar{\xi}_{k-1})$ with the following form:

$$\begin{aligned}
 \xi_2^*(\xi_1) & = -\phi_1^{\frac{r_2}{\mu}}(\xi_1) [z_1]^{\frac{r_2}{\mu}}, \\
 & \vdots \\
 \xi_k^*(\bar{\xi}_{k-1}) & = -\phi_{k-1}^{\frac{r_k}{\mu}}(\bar{\xi}_{k-1}) [z_{k-1}]^{\frac{r_k}{\mu}}, \\
 z_j & = [\xi_j]^{\frac{\mu}{r_j}} - [\xi_j^*]^{\frac{\mu}{r_j}}, \quad j = 1, \dots, k - 1, \tag{B21}
 \end{aligned}$$

such that

$$\mathcal{L}V_{k-1} \leq - \left(n - (k - 1) + \frac{1}{4} \right) \sum_{j=1}^{k-1} |z_j|^4 + [z_{k-1}]^{\frac{4\mu - r_{k-1} - \varpi_{k-1}}{\mu}} (|\xi_k|^{p_{k-1}} - [\xi_k^*]^{p_{k-1}}) - \frac{1}{4} \sum_{j=1}^{k-1} |z_j|^{\frac{16}{3}}. \tag{B22}$$

We next prove that Eq. (B22) still holds at Step k .

Setting $z_k = [\xi_k]^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}}$, we can choose the k -th Lyapunov function $V_k(\bar{\xi}_k) = V_{k-1}(\bar{\xi}_{k-1}) + W_k(\bar{\xi}_k) = V_{k-1}(\bar{\xi}_k) + \int_{\xi_k^*}^{\xi_k} \left[|s|^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - r_k - \varpi_k}{\mu}} ds$. By (8), (B22), and Itô's formula,

$$\begin{aligned}
 \mathcal{L}V_k & \leq \mathcal{L}V_{k-1} + \frac{\partial W_k}{\partial \xi_1} (D_1 |\xi_2|^{p_1} + \bar{f}_1(\xi_1)) + \sum_{j=2}^k \frac{\partial W_k}{\partial \xi_j} (|\xi_{j+1}|^{p_j} + \bar{f}_j(\bar{\xi}_j)) + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 W_k}{\partial \xi_i \partial \xi_j} \bar{g}_i^T(\bar{\xi}_i) \bar{g}_j(\bar{\xi}_j) \\
 & \leq - \left(n - (k - 1) + \frac{1}{4} \right) \sum_{j=1}^{k-1} |z_j|^4 + [z_{k-1}]^{\frac{4\mu - r_{k-1} - \varpi_{k-1}}{\mu}} (|\xi_k|^{p_{k-1}} - [\xi_k^*]^{p_{k-1}}) - \frac{1}{4} \sum_{j=1}^{k-1} |z_j|^{\frac{16}{3}} \\
 & \quad + \frac{\partial W_k}{\partial \xi_1} (D_1 |\xi_2|^{p_1} + \bar{f}_1(\xi_1)) + \sum_{j=2}^k \frac{\partial W_k}{\partial \xi_j} (|\xi_{j+1}|^{p_j} + \bar{f}_j(\bar{\xi}_j)) + \frac{1}{2} \sum_{i,j=1}^k \frac{\partial^2 W_k}{\partial \xi_i \partial \xi_j} \bar{g}_i^T(\bar{\xi}_i) \bar{g}_j(\bar{\xi}_j), \tag{B23}
 \end{aligned}$$

For $i, j = 1, \dots, k - 1$, a simple calculation yields

$$\frac{\partial W_k}{\partial \xi_j} = - \frac{4\mu - \varpi_k - r_k}{\mu} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \int_{\xi_k^*}^{\xi_k} \left[|s|^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} ds, \tag{B24}$$

$$\frac{\partial W_k}{\partial \xi_k} = [z_k]^{\frac{4\mu - \varpi_k - r_k}{\mu}}, \tag{B25}$$

$$\begin{aligned} \frac{\partial^2 W_k}{\partial \xi_i \partial \xi_j} = & -\frac{4\mu - \varpi_k - r_k}{\mu} \left(\frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_i \partial \xi_j} \right) \int_{\xi_k^*}^{\xi_k} \left| [s]^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} ds + \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \\ & \cdot \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_i} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \left(\frac{4\mu - \varpi_k - r_k - \mu}{\mu} \right) \int_{\xi_k^*}^{\xi_k} \left| [s]^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right|^{\frac{4\mu - \varpi_k - r_k - 2\mu}{\mu}} ds, \end{aligned} \tag{B26}$$

$$\frac{\partial^2 W_k}{\partial \xi_k \partial \xi_j} = \frac{4\mu - \varpi_k - r_k}{\mu} |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \frac{\partial z_k}{\partial \xi_j}, \tag{B27}$$

$$\frac{\partial^2 W_k}{\partial \xi_k^2} = \frac{4\mu - \varpi_k - r_k}{\mu} |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \frac{\partial z_k}{\partial \xi_k}. \tag{B28}$$

Since $\xi_k^*(\bar{\xi}_k) = -\phi_{k-1}^{\frac{\mu}{r_k}}(\bar{\xi}_{k-1})[z_{k-1}]^{\frac{r_k}{\mu}}$ and $z_{k-1} = [\xi_{k-1}]^{\frac{\mu}{r_{k-1}}} - [\xi_{k-1}^*]^{\frac{\mu}{r_{k-1}}}$, $k = 2, \dots, n$, one gets

$$\begin{aligned} [\xi_k^*(\bar{\xi}_k)]^{\frac{\mu}{r_k}} &= -\phi_{k-1}[\xi_{k-1}]^{\frac{\mu}{r_{k-1}}} - \phi_{k-1}\phi_{k-2}z_{k-2} \\ &= -\phi_{k-1}[\xi_{k-1}]^{\frac{\mu}{r_{k-1}}} - \phi_{k-1}\phi_{k-2}[\xi_{k-2}]^{\frac{\mu}{r_{k-2}}} - \dots - \phi_{k-1}\phi_{k-2} \dots \phi_1[\xi_1]^{\frac{\mu}{r_1}}. \end{aligned} \tag{B29}$$

Hence, we derive that

$$\begin{aligned} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} = & -\frac{\partial \phi_{k-1}}{\partial \xi_j} [\xi_{k-1}]^{\frac{\mu}{r_{k-1}}} - \frac{\partial(\phi_{k-1}\phi_{k-2})}{\partial \xi_j} [\xi_{k-2}]^{\frac{\mu}{r_{k-2}}} - \frac{\partial(\phi_{k-1}\phi_{k-2} \dots \phi_j)}{\partial \xi_j} [\xi_j]^{\frac{\mu}{r_j}} \\ & - \phi_{k-1}\phi_{k-2} \dots \phi_j \frac{\mu}{r_j} |\xi_j|^{\frac{\mu-r_j}{r_j}} - \dots - \frac{\partial(\phi_{k-1}\phi_{k-2} \dots \phi_1)}{\partial \xi_j} [\xi_1]^{\frac{\mu}{r_1}}, \end{aligned} \tag{B30}$$

$$\begin{aligned} \frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j^2} = & -\frac{\partial^2 \phi_{k-1}}{\partial \xi_j^2} [\xi_{k-1}]^{\frac{\mu}{r_{k-1}}} - \dots - \phi_{k-1}\phi_{k-2} \dots \phi_j \frac{\mu(\mu-r_j)}{r_j^2} [\xi_j]^{\frac{\mu-2r_j}{r_j}} \\ & - \frac{\partial(\phi_{k-1}\phi_{k-2} \dots \phi_j)}{\partial \xi_j} \frac{\mu}{r_j} |\xi_j|^{\frac{\mu-r_j}{r_j}} - \dots - \frac{\partial(\phi_{k-1}\phi_{k-2} \dots \phi_1)}{\partial \xi_j^2} [\xi_1]^{\frac{\mu}{r_1}}. \end{aligned} \tag{B31}$$

By the definition of μ , we know that the zero-division problem of $\frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j^2}$ cannot occur, which indicates that $W_k(\bar{\xi}_k)$ is \mathcal{C}^2 .

For $i, j = 1, \dots, k-1$, by (B29) and Lemmas 5 and 6, there are known nonnegative smooth functions $\hat{\psi}_{k1}(\bar{\xi}_{k-1})$ and $\hat{\psi}_{k2}(\bar{\xi}_{k-1})$ such that

$$\begin{aligned} \left| \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \right| &= \left| -\frac{\partial \phi_{k-1}}{\partial \xi_j} [\xi_{k-1}]^{\frac{\mu}{r_{k-1}}} - \frac{\partial(\phi_{k-1}\phi_{k-2})}{\partial \xi_j} [\xi_{k-2}]^{\frac{\mu}{r_{k-2}}} - \frac{\partial(\phi_{k-1}\phi_{k-2} \dots \phi_j)}{\partial \xi_j} [\xi_j]^{\frac{\mu}{r_j}} \right. \\ &\quad \left. - \phi_{k-1}\phi_{k-2} \dots \phi_j \frac{\mu}{r_j} |\xi_j|^{\frac{\mu-r_j}{r_j}} - \dots - \frac{\partial(\phi_{k-1}\phi_{k-2} \dots \phi_1)}{\partial \xi_j} [\xi_1]^{\frac{\mu}{r_1}} \right| \\ &\leq \hat{\psi}_{k1}(\bar{\xi}_{k-1}) \sum_{l=1}^{k-1} |z_l|^{\frac{\mu-r_l}{\mu}}, \end{aligned} \tag{B32}$$

$$\left| \frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_i \partial \xi_j} \right| \leq \hat{\psi}_{k2}(\bar{\xi}_{k-1}) \sum_{l=1}^{k-1} |z_l|^{\frac{\mu-r_l-r_i}{\mu}}. \tag{B33}$$

By (9), (B21), (B24), (B29), and Lemmas 1, 3, 5, 7, and 8, there is a nonnegative smooth function $\beta_{k1}(\bar{\xi}_k)$ such that

$$\begin{aligned} \frac{\partial W_k}{\partial \xi_1} (D_1[\xi_2]^{p_1} + \bar{f}_1(\xi_1)) + \sum_{j=2}^{k-1} \frac{\partial W_k}{\partial \xi_j} ([\xi_{j+1}]^{p_j} + \bar{f}_j(\bar{\xi}_j)) \\ \leq \left| \frac{4\mu - \varpi_k - r_k}{\mu} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \int_{\xi_k^*}^{\xi_k} \left| [s]^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} ds \right| \cdot \left| D_1[\xi_2]^{p_1} + h_{11}(\xi_1)|\xi_1|^{\frac{r_1 + \varpi_1}{r_1}} \right| \\ + \sum_{j=2}^{k-1} \left| \frac{4\mu - \varpi_k - r_k}{\mu} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \int_{\xi_k^*}^{\xi_k} \left| [s]^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} ds \right| \cdot \left| |\xi_{j+1}]^{p_j} + h_{j1}(\bar{\xi}_j) \sum_{l=1}^j |\xi_l|^{\frac{r_j + \varpi_j}{r_l}} \right| \\ \leq \left| \frac{4\mu - \varpi_k - r_k}{\mu} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} |\xi_k - \xi_k^*| |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \right| \cdot \left| D_1|z_2 + \phi_1 z_1|^{\frac{\varpi_1 + r_1}{\mu}} + h_{11}(\xi_1)|\xi_1|^{\frac{r_1 + \varpi_1}{r_1}} \right| \\ + \sum_{j=2}^{k-1} \left| \frac{4\mu - \varpi_k - r_k}{\mu} \frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} |\xi_k - \xi_k^*| |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \right| \cdot \left| |z_{j+1} + \phi_j z_j|^{\frac{\varpi_j + r_j}{\mu}} + h_{j1}(\bar{\xi}_j) \sum_{l=1}^j |\xi_l|^{\frac{r_j + \varpi_j}{r_l}} \right| \\ \leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k1}(\bar{\xi}_k) |z_k|^4. \end{aligned} \tag{B34}$$

Similarly, it can be deduced that

$$\frac{\partial W_k}{\partial \xi_k} \bar{f}_k(\bar{\xi}_k) \leq |z_k|^{\frac{4\mu - \varpi_k - r_k}{\mu}} \left(h_{k1}(\bar{\xi}_k) \sum_{j=1}^k |\xi_j|^{\frac{r_k + \varpi_k}{r_j}} \right)$$

$$\leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k2}(\bar{\xi}_k) |z_k|^4, \tag{B35}$$

where $\beta_{k2}(\bar{\xi}_k)$ is a nonnegative smooth function. By (10), (B21), (B27), and Lemmas 1, 3, 5, and 7,

$$\begin{aligned} \sum_{j=1}^{k-1} \frac{\partial^2 W_k}{\partial \xi_k \partial \xi_j} \bar{g}_k^T(\bar{\xi}_k) \bar{g}_j(\bar{\xi}_j) &\leq \sum_{j=1}^{k-1} \left| \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \frac{\partial z_k}{\partial \xi_j} \right| \left(\eta_{k1}(\bar{\xi}_k) \sum_{j=1}^k |z_j + \phi_{j-1} z_{j-1}|^{\frac{2r_k + \varpi_k}{2\mu}} \right) \\ &\quad \cdot \left(\eta_{j1}(\bar{\xi}_j) \sum_{l=1}^j |\xi_l|^{\frac{2r_j + \varpi_j}{2r_l}} \right) \\ &\leq \sum_{j=1}^{k-1} \left| \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \frac{\partial z_k}{\partial \xi_j} \right| \bar{\eta}_{k1}(\bar{\xi}_k) \sum_{j=1}^k |z_j|^{\frac{2r_k + \varpi_k + 2r_j + \varpi_j}{2\mu}} \\ &\leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k3}(\bar{\xi}_k) |z_k|^4, \end{aligned} \tag{B36}$$

where $\bar{\eta}_{k1}(\bar{\xi}_k)$ and $\beta_{k3}(\bar{\xi}_k)$ are some nonnegative smooth functions. One can deduce from (10), (B21), (B28), and Lemmas 1, 3, 5, and 7 that

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 W_k}{\partial \xi_k^2} \bar{g}_k^T(\bar{\xi}_k) \bar{g}_k(\bar{\xi}_k) &\leq \left| \left(\frac{4\mu - \varpi_k - r_k}{2\mu} \right) |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \frac{\partial z_k}{\partial \xi_k} \right| \left(\eta_{k1}(\bar{\xi}_k) \sum_{j=1}^k |\xi_j|^{\frac{2r_k + \varpi_k}{2r_j}} \right)^2 \\ &\leq \left| \left(\frac{4\mu - \varpi_k - r_k}{2r_k} \right) |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} |\xi_k|^{\frac{\mu - r_k}{r_k}} \right| \left(\eta_{k1}(\bar{\xi}_k) \sum_{j=1}^k |z_j + \phi_{j-1} z_{j-1}|^{\frac{2r_k + \varpi_k}{2\mu}} \right)^2 \\ &\leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k4}(\bar{\xi}_k) |z_k|^4, \end{aligned} \tag{B37}$$

where $\beta_{k4}(\bar{\xi}_k)$ is a nonnegative smooth function. By (10), (B21), (B26), (B32), (B33), and Lemmas 1, 3, 5, 7, and 8, there is a nonnegative smooth function $\beta_{k5}(\bar{\xi}_k)$ such that

$$\begin{aligned} \frac{1}{2} \sum_{j=1}^{k-1} \frac{\partial^2 W_k}{\partial \xi_j^2} \bar{g}_j^T(\bar{\xi}_j) \bar{g}_j(\bar{\xi}_j) &\leq \frac{1}{2} \sum_{j=1}^{k-1} \left| \int_{\xi_k^*}^{\xi_k} \left[|s|^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} ds \left(\frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j^2} \right) \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \right. \\ &\quad \left. + \left(\frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \right)^2 \left(\frac{4\mu - \varpi_k - r_k - \mu}{\mu} \right) \int_{\xi_k^*}^{\xi_k} \left[|s|^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - \varpi_k - r_k - 2\mu}{\mu}} ds \right. \\ &\quad \cdot \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \left| \left(\eta_{j1}(\bar{\xi}_j) \sum_{l=1}^j |\xi_l|^{\frac{2r_j + \varpi_j}{2r_l}} \right)^2 \right. \\ &\leq \frac{1}{2} \sum_{j=1}^{k-1} \left| |\xi_k - \xi_k^*| |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \left(\frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j^2} \right) \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \right. \\ &\quad \left. + \left(\frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_j} \right)^2 \left(\frac{4\mu - \varpi_k - r_k - \mu}{\mu} \right) |\xi_k - \xi_k^*| |z_k|^{\frac{4\mu - \varpi_k - r_k - 2\mu}{\mu}} \right. \\ &\quad \cdot \left. \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \left| \left(\eta_{j1}(\bar{\xi}_j) \sum_{l=1}^j |z_l + \phi_{l-1} z_{l-1}|^{\frac{2r_j + \varpi_j}{2\mu}} \right)^2 \right. \right. \\ &\leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k5}(\bar{\xi}_k) |z_k|^4. \end{aligned} \tag{B38}$$

Similar to (B38), one can find a nonnegative smooth function $\beta_{k6}(\bar{\xi}_k)$ such that

$$\begin{aligned} &\sum_{l_1, l_2=1, l_1 \neq l_2}^{k-1} \frac{\partial^2 W_k}{\partial \xi_{l_1} \partial \xi_{l_2}} \bar{g}_{l_1}(\bar{\xi}_{l_1})^T \bar{g}_{l_2}(\bar{\xi}_{l_2}) \\ &\leq \sum_{l_1, l_2=1, l_1 \neq l_2}^{k-1} \left| \int_{\xi_k^*}^{\xi_k} \left[|s|^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} ds \left(\frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_{l_1} \partial \xi_{l_2}} \right) \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \right. \\ &\quad \left. + \left(\frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_{l_1}} \right) \left(\frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_{l_2}} \right) \left(\frac{4\mu - \varpi_k - r_k - \mu}{\mu} \right) \int_{\xi_k^*}^{\xi_k} \left[|s|^{\frac{\mu}{r_k}} - [\xi_k^*]^{\frac{\mu}{r_k}} \right]^{\frac{4\mu - \varpi_k - r_k - 2\mu}{\mu}} ds \right. \\ &\quad \cdot \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \left| \left(\eta_{l_1 1}(\bar{\xi}_{l_1}) \sum_{j=1}^{l_1} |\xi_j|^{\frac{2r_{l_1} + \varpi_{l_1}}{2r_j}} \right) \left(\eta_{l_2 1}(\bar{\xi}_{l_2}) \sum_{l=1}^{l_2} |\xi_l|^{\frac{2r_{l_2} + \varpi_{l_2}}{2r_l}} \right) \right. \\ &\leq \sum_{l_1, l_2=1, l_1 \neq l_2}^{k-1} \left| |\xi_k - \xi_k^*| |z_k|^{\frac{4\mu - \varpi_k - r_k - \mu}{\mu}} \left(\frac{\partial^2 [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_{l_1} \partial \xi_{l_2}} \right) \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \right. \\ &\quad \left. + \left(\frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_{l_1}} \right) \left(\frac{\partial [\xi_k^*]^{\frac{\mu}{r_k}}}{\partial \xi_{l_2}} \right) \left(\frac{4\mu - \varpi_k - r_k - \mu}{\mu} \right) |\xi_k - \xi_k^*| |z_k|^{\frac{4\mu - \varpi_k - r_k - 2\mu}{\mu}} \right. \end{aligned}$$

$$\begin{aligned} & \cdot \left(\frac{4\mu - \varpi_k - r_k}{\mu} \right) \left| \left(\eta_{l_1 1}(\bar{\xi}_{l_1}) \sum_{j=1}^{l_1} |z_j + \phi_{j-1} z_{j-1}|^{\frac{2r_{l_1} + \varpi_{l_1}}{2\mu}} \right) \left(\eta_{l_2 1}(\bar{\xi}_{l_2}) \sum_{l=1}^{l_2} |z_l + \phi_{l-1} z_{l-1}|^{\frac{2r_{l_2} + \varpi_{l_2}}{2\mu}} \right) \right| \\ & \leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k6}(\bar{\xi}_k) |z_k|^4. \end{aligned} \tag{B39}$$

By Lemmas 1, 5, and 7, it is not hard to obtain

$$\begin{aligned} [z_{k-1}]^{\frac{4\mu - \varpi_{k-1} - r_{k-1}}{\mu}} ([\xi_k]^{p_{k-1}} - [\xi_k^*]^{p_{k-1}}) & \leq 2^{1 - \frac{\varpi_{k-1} + r_{k-1}}{\mu}} |z_{k-1}|^{\frac{4\mu - \varpi_{k-1} - r_{k-1}}{\mu}} |z_k|^{\frac{\varpi_{k-1} + r_{k-1}}{\mu}} \\ & \leq \frac{1}{7} \sum_{j=1}^{k-1} |z_j|^4 + \beta_{k7}(\bar{\xi}_k) |z_k|^4, \end{aligned} \tag{B40}$$

where $\beta_{k7}(\bar{\xi}_k)$ is a nonnegative smooth function. Substituting (B34)–(B40) into (B23) yields

$$\begin{aligned} \mathcal{L}V_k & \leq - \left(n - k + \frac{1}{4} \right) \sum_{j=1}^{k-1} |z_j|^4 + [z_k]^{\frac{4\mu - r_k - \varpi_k}{\mu}} ([\xi_{k+1}]^{p_k} - [\xi_{k+1}^*]^{p_k} + [\xi_{k+1}^*]^{p_k}) \\ & \quad + \beta_k(\bar{\xi}_k) |z_k|^4 - \frac{1}{4} \sum_{j=1}^{k-1} |z_j|^{\frac{16}{3}}, \end{aligned} \tag{B41}$$

where $\beta_k(\bar{\xi}_k) = \sum_{j=1}^7 \beta_{kj}(\bar{\xi}_k)$. Substituting the virtual controller

$$\xi_{k+1}^* = - \left(n - k + \frac{1}{4} + \frac{1}{4} z_k^{\frac{4}{3}} + \beta_k(\bar{\xi}_k) \right)^{\frac{1}{p_k}} [z_k]^{\frac{r_{k+1}}{\mu}} \triangleq -\phi_k^{\frac{r_{k+1}}{\mu}}(\bar{\xi}_k) [z_k]^{\frac{r_{k+1}}{\mu}} \tag{B42}$$

into (B41) leads to

$$\mathcal{L}V_k \leq - \sum_{j=1}^k \left(n - k + \frac{1}{4} \right) |z_j|^4 - \sum_{j=1}^k \frac{1}{4} |z_j|^{\frac{16}{3}} + [z_k]^{\frac{4\mu - r_k - \varpi_k}{\mu}} ([\xi_{k+1}]^{p_k} - [\xi_{k+1}^*]^{p_k}). \tag{B43}$$

Hence, Eq. (B22) still holds at Step k .

At the last step, setting $V_n(\xi) = V_{n-1}(\bar{\xi}_{n-1}) + W_n(\xi)$, Eq. (12) leads to (13).