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• Preface •

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The contribution of Jing-run Chen to number theory

On the 50th Anniversary of Chen's Theorem

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This year (2023) is the 90th birthday of the great Chinese mathematician Jing-run Chen. It is definitely significant to recall his important contribution to mathematics. Chen's work covers various parts of analytic number theory. Here we only present two major contributions from him.

(1) Representation of sufficiently large even numbers as the sum of a prime and the product of at most two primes $(1+2)$. In 1918, the Norwegian mathematician Brun proved that every sufficiently large even number is the sum of two products, each of which has at most nine prime factors $(9+9)$. This breakthrough opened the door to attack the extremely difficult Goldbach Conjecture via the sieve method. In 1948, the Hungarian mathematician Renyi proved $(1+c)$ via Linnik's large sieve. Here the constant c is sufficiently large and unspecified. In order to obtain such results with the constant c as small as possible, many experts continued to improve the structure of the weighted sieve and the error estimates of the large sieve type. Until 1965, the best result in this direction was $(1+3)$. Due to the limitation of the sieve method itself, it was believed that the stronger result $(1+2)$ could not be achieved by the present method. However, in 1966, it was Chen who announced that he had proved $(1+2)$. One crucial step in his proof is to estimate the number of solutions to the equation $x-p=p_1p_2p_3$, where x is a large even number and p, p_1, p_2 and p_3 are primes running through certain intervals. In such a case the typical estimate (via the Bombieri-Vinogradov theorem directly) is not sharp enough. Chen's approach is first to transfer the equation to $x-p_1p_2p_3=p$ and then to introduce a number of new techniques to obtain a sharper estimate.

(2) Arrangement of error terms in sieve methods. When applying the sieve method, one important problem is to show that the contribution from the error terms is acceptable. Sometimes this can be a difficult task. The traditional treatment of the error terms is to replace them by the corresponding absolute values, and then calculate the upper bounds. In Chen's work on the almost primes in short intervals, he introduced a completely new idea. When dealing with error terms of the type $x-[x]-1/2$, he did not replace them by the trivial bound $O(1)$ directly (as others did); rather, he first applied the Fourier expansion to these terms, and then made a careful arrangement of the resulting expression, that made him able to obtain sharper estimates. Chen's revolutionary idea has been developed and refined by many experts in this field, and has found wide application. For example, the dispersion method, that is one tool used in my work on the bounded gaps between primes, is somewhat based on his idea.

Chen's work and ideas still have important influences in our time.