Erratum to: Every Banach space admits a homogenous measure of non-compactness not equivalent to the Hausdorff measure

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[1, Theorem 4.4] states that every infinite dimensional Banach space admits a homogenous measure of noncompactness not equivalent to the Hausdorff measure. However, there is a gap in the proof. In fact, we found that [1, Lemma 4.3] is not true. In this erratum, we give a corrected proof of [1, Theorem 4.4].

For a Banach space X , let $\mathcal{C}(X)$ (resp. $\mathcal{B}(X)$, $\mathcal{K}(X)$) be the collection of all non-empty bounded closed convex (resp. nonempty bounded, nonempty convex compact) sets of X endowed with the Hausdorff metric. If there is no confusion, we simply denote them by $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{K}$, respectively. We use Ω to denote the closed unit ball B_{X^*} of the dual X^* , and $C_b(\Omega)$, the Banach space of all real-valued bounded norm-continuous functions on Ω endowed with the sup-norm. Let $J : \mathcal{C}(X) \rightarrow C_b(\Omega)$ be defined for $C \in \mathcal{C}(X)$ by

$$J(C)(\omega) = \sigma_C(\omega) \equiv \sup_{c \in C} \langle \omega, c \rangle, \quad \omega \in \Omega. \quad (1)$$

With the symbols as above, we first recall the main results presented in [2], which will be used in the proof.

Theorem 1. (1) (See [2, Theorem 2.3i]) *Given a Banach space X , the collection $\mathcal{C}$ consisting of all nonempty closed bounded convex sets of X endowed with the set addition $A \oplus B = A + B$, the usual scalar multiplication of sets $\lambda C = \{\lambda c : c \in C\}$, and the norm $||| \cdot |||$ defined by $|||C||| = \sup_{c \in C} \|c\|$ is a complete normed convex cone.*

(2) (See [2, Theorem 2.3ii]) *If we endow with the Hausdorff metric d_H on $\mathcal{C}$, i.e.,*

$$d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|b - a\| \right\}$$

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for all $A, B \in \mathcal{C}$, order $\mathcal{C}$ by set inclusion, and order $C_b(\Omega)$ by the usual order of real-valued functions, then the mapping $J : \mathcal{C} \rightarrow C_b(\Omega)$ defined for $C \in \mathcal{C}$ and $\omega \in \Omega$ by

$$J(C)(\omega) = \sigma_C(\omega) = \sup_{c \in C} \langle \omega, c \rangle,$$

is a positively linear order isometry, where $C_b(\Omega)$ denotes the Banach space of all bounded norm continuous functions on Ω endowed with the sup-norm.

(3) (See [2, Theorem 3.2i]) Both $E_{\mathcal{C}} = \overline{J\mathcal{C} - J\mathcal{C}}$ and $E_{\mathcal{K}} = \overline{J\mathcal{K} - J\mathcal{K}}$ are sublattices of $C_b(\Omega)$ and $E_{\mathcal{K}}$ is a lattice ideal of $E_{\mathcal{C}}$, where $\mathcal{K} \subset \mathcal{C}$ is the subcone of $\mathcal{C}$ consisting of all nonempty compact convex subsets of X .

(4) (See [2, Theorem 3.2ii]) The quotient space $QE_{\mathcal{C}} = E_{\mathcal{C}}/E_{\mathcal{K}}$ is an abstract M space; hence, it is order isometric to a sublattice of $C(K)$ for some compact Hausdorff space K .

(5) (See [2, Theorem 3.2iii]) The subcone $TQJ\mathcal{C}$ is contained in the positive cone of $C(K)$, where $T : QE_{\mathcal{C}} \rightarrow C(K)$ is an order isometry.

(6) (See [2, Theorem 5.5]) For every bounded subset $F \subset C(K)^*$ of positive functionals satisfying that for each $u \in TQJ\mathcal{C}$ there exists $\varphi \in F$ so that $\langle \varphi, u \rangle > 0$, the following formula defines a homogenous measure of noncompactness μ on X :

$$\mu(B) = \|T[QJ\overline{\text{co}}(B)]\|_F \quad \text{for all } B \in \mathcal{B}, \quad (2)$$

where $\|u\|_F = \sup_{\varphi \in F \cup -F} \langle \varphi, u \rangle$ for all $u \in C(K)$.

(7) (See [2, Theorem 5.4]) For every bounded subset $F \subset C(K)^*$ of

$$\bigcup_{k \in K} \{\mathbb{R}^+ \delta_k : k \in K\} = \{r\delta_k : r \geq 0, k \in K\}$$

satisfying that for each $u \in TQJ\mathcal{C}$ there exists $\varphi \in F$ so that $\langle \varphi, u \rangle > 0$, the following formula defines a regular measure of noncompactness μ on X :

$$\mu(B) = \|T[QJ\overline{\text{co}}(B)]\|_F \quad \text{for all } B \in \mathcal{B}, \quad (3)$$

where δ_k is the evaluation functional at $k \in K$,

$$\|u\|_F = \sup_{\varphi \in F \cup -F} \langle \varphi, u \rangle$$

for all $u \in C(K)$.

(8) (See [2, Theorem 5.4]) In particular, the Hausdorff measure of non-compactness β can be reformulated as follows:

$$\beta(B) = \|TQJ[\overline{\text{co}}B]\|_{C(K)} \quad \text{for all } B \in \mathcal{B}. \quad (4)$$

Now, we restate and prove the main theorem of [2].

Theorem 2. Every Banach space of infinite dimension admits a homogenous measure of noncompactness not equivalent to the Hausdorff measure.

Proof. Let X be an infinite dimensional Banach space. All symbols will be the same as in Theorem 1.

If it is a Schur space, i.e., the norm sequential convergence coincides with the weakly sequential convergence on X , then we finish the proof by [1, Lemma 3.6].

Suppose that X is not a Schur space. Then it contains a normalized weakly null sequence (y_n) . Thus, there is a basic subsequence $(x_n) (\subset (y_n))$ of X . Without loss of generality, we can assume that $\Delta \equiv (x_n)$ is a monotone basic sequence. (Otherwise, we can renorm X so that Δ is a monotone basic sequence.) Let $X_0 = \overline{\text{span}}(\Delta)$. We denote the sequence of the coefficient functionals corresponding to Δ by $\Delta^* \equiv (\varphi_n) \subset X_0^*$. Then $\|\varphi_n\| \leq 2$ and $\varphi_n \rightarrow 0$ in the w^* -topology of X_0^* . $(\Delta_n)_{n=1}^\infty$ be an infinite σ -partition of Δ , i.e., it satisfies the following conditions: (1) $\bigcup_{n=1}^\infty \Delta_n = \Delta$; (2) $\Delta_n \cap \Delta_m = \emptyset$ whenever $n \neq m$; and (3) each Δ_n is an infinite subset of Δ . Let (Δ_n^*) be the partition of Δ^* corresponding to (Δ_n) . For each $n \in \mathbb{N}$, put $C_n = \overline{\text{co}}(\Delta_n)$. Then it is easy to observe that

- (a) for each $j \in \mathbb{N}$, C_j is weakly compact, $C_m \cap C_n = \{0\}$ whenever $m \neq n$, and
(b) for each selection $z_n \in C_n$ ($n = 1, 2, \dots$) we have $z_n \rightarrow 0$ in the weak topology of X .
We denote by $C(K)^+$ (resp. $C(K)^{*+}$) the positive cone of $C(K)$ (resp. $C(K)^*$). Let

$$f_n = TQJ(C_n) \ (\in C(K)), \quad (5)$$

$$D = \overline{\text{co}(B_{C(K)} \cap C(K)^+ \cup \{nf_n : n \in \mathbb{N}\})}, \quad (6)$$

and let

$$D^\circ = \{\varphi \in C(K)^{*+} : \langle \varphi, d \rangle \leq 1, \text{ for all } d \in D\}. \quad (7)$$

We claim that D and D° have the following properties:

- (i) D contains no nontrivial subcones of $V \equiv TQJ(\mathcal{C})$;
(ii) D° positively separates points of V , i.e., for every $0 \neq x \in V$, there is $\varphi \in D^\circ$ such that $\langle \varphi, x \rangle > 0$.

To show (i), suppose that $g \in V$ with $g = TQJ(B)$ for some closed bounded nonempty convex set B in X satisfies $ng \in D$ for all $n \in \mathbb{N}$. Then by definitions of f_m , D and D° , there exists a sequence of $(g_n) \subset V$ with

$$g_n = b_n + \sum_{j=1}^{m_n} \alpha_{n,j} j f_j \in \text{co}[B_{C(K)} \cap C(K)^+ \cup \{mf_m : m \in \mathbb{N}\}]$$

such that $\|ng - g_n\| \rightarrow 0$, where $b_n \in B_{C(K)} \cap C(K)^+$, $m_n \in \mathbb{N}$, and $\alpha_{n,j} \in \mathbb{R}^+$ with $\sum_{j=1}^{m_n} \alpha_{n,j} \leq 1$. Thus

$$V \ni (1/n)g_n = \frac{b_n}{n} + \left(\frac{1}{n}\right) \sum_{j=1}^{m_n} \alpha_{n,j} j f_j \rightarrow g$$

(as $n \rightarrow \infty$). By Theorems 1(2)–1(5), this is equivalent to that for every $\varepsilon > 0$, there exist a finite set $F \subset X$ and $N \in \mathbb{N}$ such that for all $n \geq N$,

$$B \subset \left(\frac{1}{n}\right) \sum_{j=1}^{m_n} \alpha_{n,j} j C_j + \varepsilon B_X + F \quad \text{and} \quad \left(\frac{1}{n}\right) \sum_{j=1}^{m_n} \alpha_{n,j} j C_j \subset B + \varepsilon B_X + F. \quad (8)$$

Since $(1/n)\alpha_{n,j}j \rightarrow 0$ for $j = 1, 2, \dots$ as $n \rightarrow \infty$, by (b) for every selection

$$w_n \in \left(\frac{1}{n}\right) \sum_{j=1}^{m_n} \alpha_{n,j} j C_j, \quad n = 1, 2, \dots,$$

we have $w_n \rightarrow 0$ in the weak topology of X . Consequently, $B \subset \varepsilon B_X + F$. Since ε is arbitrary, B is compact. Consequently, $g = 0$.

To show (ii), since D does not contain nontrivial cone of V ,

$$D_1 \equiv \overline{\text{co}(D \cup -C(K)^+)}$$

does not contain nontrivial subcone of V . Assume $f \neq 0 \in V$. Then by (i) we have just proven, $mf \notin V$ for some $m \in \mathbb{N}$. By the separation theorem of convex sets, there is a functional $\varphi \in C(K)^*$ such that

$$\infty > M \equiv \langle \varphi, mf \rangle > \sup\{\langle \varphi, d \rangle : d \in D_1\} \geq 0.$$

This entails that $\varphi \in C(K)^{*+}$ and bounded above by M on D . Let $\psi = \frac{1}{M}\varphi$. Then $\psi \in D^\circ$ with $\langle \psi, f \rangle > 0$. Thus, (i) and (ii) have been shown. Therefore, by Theorem 1(6),

$$\mu(B) = \sup_{e \in D^\circ \cup -D^\circ} \langle e, f_B \rangle, \quad f_B = TQJ(\overline{\text{co}(B)}), \quad B \in \mathcal{B}(X)$$

define a homogenous measure of noncompactness on X . Since $\mu(C_n) \leq \frac{1}{n} \rightarrow 0$, and since $\beta(C_n) \geq \frac{1}{2}$ for all $n \in \mathbb{N}$, μ is not equivalent to the Hausdorff measure β . $\square$

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