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Traveling waves for a diffusive SEIR epidemic model with standard incidences

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Abstract This paper is devoted to the existence of the traveling waves of the equations describing a diffusive susceptible-exposed-infected-recovered (SEIR) model. The existence of traveling waves depends on the basic reproduction rate and the minimal wave speed. We obtain a more precise estimation of the minimal wave speed of the epidemic model, which is of great practical value in the control of serious epidemics. The approach in this paper is to use the Schauder fixed point theorem and the Laplace transform. We also give some numerical results on the minimal wave speed.

Keywords traveling waves, SEIR model, Schauder fixed point theorem

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1 Introduction

In 1927, Kermack and McKendrick [8] proposed the Kermack-McKendrick equations

$$\begin{aligned} &\frac{d}{dt}S(t) = -\beta S(t)I(t), \\ &\frac{d}{dt}I(t) = \beta S(t)I(t) - \gamma I(t), \\ &\frac{d}{dt}R(t) = \gamma I(t) \end{aligned}$$

to describe the susceptible-infected-recovered (SIR) model, where S denotes the number of the susceptible population, I and R denote the numbers of the infected and the recovered, respectively, β is the transmission rate between the susceptible and the infected, and γ is the removing rate of the infected. Let $S(0) = S_0$ be the number of the susceptible at the beginning of the epidemic. If the so-called reproductive number $R_0 := \beta/\gamma > 1$, $I(t)$ increases first and then decreases to 0, i.e., an epidemic takes place; whereas $R_0 < 1$, $I(t)$ decreases directly to 0, indicating no epidemic happens. If the effect of spacial diffusion is taken into account, the Kermack-McKendrick equations with standard incidences are

$$\frac{\partial S}{\partial t} = d_1 \frac{\partial^2 S}{\partial x^2} - \frac{\beta SI}{S+I}, \tag{1.1}$$

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$$\frac{\partial I}{\partial t} = d_2 \frac{\partial^2 I}{\partial x^2} + \frac{\beta SI}{S+I} - \gamma I, \tag{1.2}$$

$$\frac{\partial R}{\partial t} = d_3 \frac{\partial^2 R}{\partial x^2} + \gamma I, \tag{1.3}$$

where d_i is the rate of diffusion of each sub-population, $i = 1, 2, 3$ (see [7]). There is no R in the first two equations of the above system according to the assumption that the recovered sub-population is removed from population. Brauer [3] gave the detailed epidemiological consideration of this assumption.

Much work has been done on the existence of traveling waves of System (1.1)–(1.3). By using the Schauder fixed point theorem, Wang et al. [14] proved the existence of traveling wave solution of this system. They proved that for $R_0 := \beta/\gamma > 1$ and $c > c^* := 2\sqrt{d_2(\beta-\gamma)}$, System (1.1)–(1.3) has a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying the boundary conditions $S(\pm\infty) = S_{\pm\infty}$, $I(\pm\infty) = 0$ and $S_{-\infty} > S_{+\infty}$. On the other hand, there is no non-negative non-trivial traveling wave solution if $0 < c < c^*$ or $R_0 \leqslant 1$. More work is done to get the existence of traveling waves in other cases, for example, the diffusion term is non-local, and the reaction term is non-local even with time delay (see [4, 9, 10, 13, 15, 16, 18, 19]). We also notice some recent work on the traveling waves of free boundary problems (see [5, 6]).

In this paper, we consider the corresponding SEIR model with standard incidences and use the assumption that the exposed are of no infectiousness and the recovered are removed from population. We focus on the following diffusive system:

$$\frac{\partial S}{\partial t} = d_1 \frac{\partial^2 S}{\partial x^2} - \frac{\beta SI}{S+I+E}, \tag{1.4}$$

$$\frac{\partial E}{\partial t} = d_2 \frac{\partial^2 E}{\partial x^2} + \frac{\beta SI}{S+I+E} - \alpha E, \tag{1.5}$$

$$\frac{\partial I}{\partial t} = d_3 \frac{\partial^2 I}{\partial x^2} + \alpha E - \gamma I, \tag{1.6}$$

$$\frac{\partial R}{\partial t} = d_4 \frac{\partial^2 R}{\partial x^2} + \gamma I, \tag{1.7}$$

where E is the number of the exposed population and α is the rate of the exposed becoming infected. The number $1/\alpha$ is the average period of the exposed becoming infected. However, it should be pointed out that this system is not the same as the SIR model (1.1)–(1.3) with a time delayed reaction term in that the exposed population has its own spacial diffusion rate.

Many diseases reduce the mobility of the infected individuals, while the exposed individuals are not influenced so much. The classical SIR model (1.1)–(1.3) may underestimate the spread speed of diseases in this case. If a disease is so serious that it disables the infected immediately, it can hardly spread without the participation of the exposed population. On the other hand, several diseases increase the mobility of the infected and Rabies is such an example. The neglect of the exposed population will underestimate or overestimate the spread speed of diseases.

Furthermore, in some scenarios of some serious infectious diseases, such as severe acute respiratory syndrome (SARS) and Ebola, the exposed individuals are traced and their mobility is limited. Thus the diffusion rate d_2 which describes the mobility of the exposed is reduced significantly and the propagation of these diseases is controlled.

The minimal wave speed c^* is the minimum value of c such that System (1.4)–(1.6) has the solution of the form $(\xi_1(x+ct), \xi_2(x+ct), \xi_2(x+ct))$, where ξ_1, ξ_2 and ξ_3 are non-negative and non-trivial. Without ambiguity, we will use S, E and I to denote ξ_1, ξ_2 and ξ_3, respectively hereinafter. The minimal wave speed c^* is important to describe the spread speed of diseases (see [2, 5, 6, 12]). It is interesting to give the relation between these two speeds for our model, which is still an open problem. Our present work shows that the minimal traveling wave speed c^* depends not only on d_3 but also on d_2. Furthermore, we prove that for $R_0 := \beta/\gamma > 1$ and $c > c^*$, System (1.4)–(1.6) has a non-negative and non-trivial traveling wave solution $(S(x+ct), E(x+ct), I(x+ct))$ satisfying $S(\infty) = S_\infty$, $S(-\infty) = S_{-\infty}$ and $E(\pm\infty) = I(\pm\infty) = 0$. In addition, there is no non-negative and non-trivial traveling wave solution if $0 < c < c^*$ or $R_0 \leqslant 1$.

The methods used in this paper are based on Wang et al. [14] and other early studies. First, we use E presenting I and reduce System (1.4)–(1.6) into a two-dimensional problem, which is inspired by Zhao and Wang's work [20] on a two-population epidemic model. We will apply the Schauder fixed point theorem to a non-monotone operator. The most challenging part is to build a suitable invariant convex set for this operator. We use two-side Laplace transform to give the proof of the non-existence of traveling wave solutions.

This paper is organized as follows. In Section 2, we present our main theorem on the existence and non-existence of traveling waves. In Section 3, we outline some properties of the differential and integral operators which will be used in the definition of a non-monotone operator. We also show that the traveling wave solution is the fixed point of this non-monotone operator. To apply the Schauder fixed point theorem, we give the definition of the invariant convex set of this operator. In Sections 4 and 5, we prove some properties of the traveling wave solution and show the existence and non-existence of traveling wave solutions under different values of c and β/γ. In Section 6, we give the discussion.

2 Main results

Since R does not appear in the SEIR model (1.4)–(1.6), it suffices to consider the three-dimensional system for (S, E, I). We look for the non-trivial and non-negative traveling wave solution of the form $(S(x + ct), E(x + ct), I(x + ct))$, which satisfies the following boundary conditions at infinity:

$$S(-\infty) = S_{-\infty}, \quad S(\infty) = S_{\infty} < S_{-\infty}, \quad E(\pm\infty) = I(\pm\infty) = 0. \tag{2.1}$$

Then System (1.4)–(1.6) can be reduced to an ODE system

$$cS' = d_1 S'' - \frac{\beta SI}{S + I + E}, \tag{2.2}$$

$$cE' = d_2 E'' + \frac{\beta SI}{S + I + E} - \alpha E, \tag{2.3}$$

$$cI' = d_3 I'' + \alpha E - \gamma I. \tag{2.4}$$

Our main results are the following.

Theorem 2.1. *There exists a positive constant number c^* such that if $c > c^*$ and $R_0 := \beta/\gamma > 1$, then System* (2.2)–(2.4) *has a non-trivial and non-negative traveling wave solution (S, E, I) satisfying the boundary conditions* (2.1)*. Furthermore, S monotonically decreases, $0 \leqslant E(x) \leqslant S_{-\infty} - S_{\infty}$ and $0 \leqslant I(x) \leqslant S_{-\infty} - S_{\infty}$ for all $x \in \mathbb{R}$, and*

$$\int_{-\infty}^{\infty} \alpha E(x)dx = \int_{-\infty}^{\infty} \gamma I(x)dx = \int_{-\infty}^{\infty} \frac{\beta S(x)I(x)}{S(x) + I(x) + E(x)}dx = c[S_{-\infty} - S_{\infty}].$$

On the other hand, if $R_0 > 1$, $0 < c < c^*$, or $R_0 \leqslant 1$, there exists no non-trivial and non-negative traveling wave solution (S, E, I) satisfying the boundary conditions (2.1)*.*

Remark 2.2. In the SIR model (1.1)–(1.3), the minimal wave speed is given by $c^* := 2\sqrt{d_2(\beta - \gamma)}$, where d_2 is the diffusive coefficient of I. For the SEIR model, as we will see, the minimal wave speed c^* depends not only on d_3, the diffusive rate of I ,but also on d_2, the diffusive rate of E .

3 Preliminaries

Lemma 3.1. *If (S, E, I) is a non-trivial and non-negative solution of System* (2.2)–(2.4)*, satisfying the boundary conditions* (2.1)*, it holds that $\int_{-\infty}^{\infty} E(x)dx < \infty$.*

Proof. Integrating (2.2) from $-\infty$ to x yields

$$d_1 S'(x) = c[S(x) - S_{-\infty}] + \int_{-\infty}^{x} \frac{\beta S(y)I(y)}{S(y) + I(y) + E(y)}dy. \tag{3.1}$$

Since $S(x)$ is uniformly bounded, the integral on the right-hand side of (3.1) should be uniformly bounded. Otherwise, $S'(x)\to\infty$ as $x\to\infty$, thus $S(x)\to\infty$, which leads to a contradiction. Hence,

$$\int_{-\infty}^{x}\frac{\beta S(y)I(y)}{S(y)+I(y)+E(y)}dy \tag{3.2}$$

is integrable on $\mathbb{R}$ and bounded. Integrating (2.3) yields

$$E(x)=\frac{1}{\rho_2}\int_{-\infty}^{x}\mathrm{e}^{\lambda_2^-(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+E(y)}dy+\frac{1}{\rho_2}\int_{x}^{\infty}\mathrm{e}^{\lambda_2^+(x-y)}\frac{\beta S(y)I(y)}{S(y)+I(y)+E(y)}dy$$
$$+C_1\mathrm{e}^{\lambda_2^-x}+C_2\mathrm{e}^{\lambda_2^+x},$$

where C_1 and C_2 are constants and

$$\lambda_2^{\pm}:=\frac{c\pm\sqrt{c^2+4d_2\alpha}}{2d_2},\quad \rho_2:=\sqrt{c^2+4d_2\alpha}=d_2(\lambda_2^+-\lambda_2^-).$$

The integrals in the expression of $E(x)$ are well-defined for the integrability of (3.2). Moreover, by the boundary condition $E(\pm\infty)=0$, C_1 and C_2 must be zeros. By using Fubini's theorem, we have

$$\int_{-\infty}^{\infty}E(x)dx=\frac{1}{\alpha}\int_{-\infty}^{\infty}\frac{\beta S(x)I(x)}{S(x)+I(x)+E(x)}dx<\infty.$$

We finish the proof. □

From (2.4), we get the solution

$$I(x)=C_1\mathrm{e}^{\lambda_1^-x}+C_2\mathrm{e}^{\lambda_1^+x}+\frac{\alpha}{\rho_1}\bigg(\int_{-\infty}^{x}\mathrm{e}^{\lambda_1^-(x-y)}E(y)dy+\int_{x}^{\infty}\mathrm{e}^{\lambda_1^+(x-y)}E(y)dy\bigg),$$

where C_1 and C_2 are constant numbers, and $\lambda_1^-<0<\lambda_1^+$ are the two roots of the equations

$$f_1(\lambda):=-d_3\lambda^2+c\lambda+\gamma=0 \tag{3.3}$$

and

$$\rho_1:=d_3(\lambda_1^+-\lambda_1^-). \tag{3.4}$$

Together with the boundary condition of E in (2.1) and L'Hôpital's rule, the only solution of (2.4) satisfying $\lim_{x\to\pm\infty}I(x)=0$ is of the form

$$[I(E)](x):=\frac{\alpha}{\rho_1}\bigg(\int_{-\infty}^{x}\mathrm{e}^{\lambda_1^-(x-y)}E(y)dy+\int_{x}^{\infty}\mathrm{e}^{\lambda_1^+(x-y)}E(y)dy\bigg), \tag{3.5}$$

where Lemma 3.1 guarantees the integrability of the integrals. Substituting (3.5) into (2.3), we obtain

$$cE'=d_2E''+\frac{\beta SI(E)}{S+I(E)+E}-\alpha E. \tag{3.6}$$

At the equilibrium $(S_{-\infty},0,0)$, (3.6) can be linearized as $cE'=d_2E''+\beta I(E)-\alpha E$. To study the characteristic function we use the form $E(t)=\mathrm{e}^{\lambda t}$, where $\lambda\in(\lambda_1^-,\lambda_1^+)$, then

$$c\lambda=d_2\lambda^2+\frac{\alpha\beta}{-d_3\lambda^2+c\lambda+\gamma}-\alpha.$$

The characteristic function of (3.6) is defined as

$$f(\lambda,c):=-d_2\lambda^2+c\lambda+\alpha-\frac{\alpha\beta}{-d_3\lambda^2+c\lambda+\gamma} \tag{3.7}$$

for $\lambda\in(\lambda_1^-,\lambda_1^+)$.

Lemma 3.2. *Assume $\beta/\gamma > 1$. There exist $\lambda_0 \in (0, \lambda_1^+)$ and $c^* > 0$ such that*

$$f(\lambda_0, c^*) = 0 \quad \textit{and} \quad \frac{\partial f(\lambda_0, c^*)}{\partial \lambda} = 0.$$

Furthermore, if $c > c^*$, $f(\lambda, c) = 0$ has two different real roots λ_1 and λ_2 with $0 < \lambda_1 < \lambda_0 < \lambda_2 < \lambda_1^+$ and $f(\lambda, c) > 0$ if $\lambda \in (\lambda_1, \lambda_2)$; $f(\lambda, c) < 0$ if $\lambda \in (0, \lambda_1) \cup (\lambda_2, \lambda_1^+)$. If $0 < c < c^*$, $f(\lambda, c) < 0$ for $\lambda \in (0, \lambda_1^+)$.

Proof. By calculation, we have

$$\begin{aligned}
&f(0,c) = \alpha - \frac{\alpha\beta}{\gamma} < 0, \quad f(\lambda, \infty) = \infty \quad \text{for} \quad \lambda \in (0, \lambda_1^+),\\
&\frac{\partial f(0,c)}{\partial \lambda} = c(1 + \alpha\beta/\gamma^2) > 0,\\
&\frac{\partial f(\lambda,c)}{\partial c} = \lambda + \frac{\alpha\beta\lambda}{(-d_3\lambda^2 + c\lambda + \gamma)^2} > 0 \quad \text{for} \quad \lambda \in (0, \lambda_1^+),\\
&\frac{\partial^2 f(\lambda,c)}{\partial \lambda^2} = -2d_2 - \frac{2\alpha\beta d_3}{(-d_3\lambda^2 + c\lambda + \gamma)^2} - \frac{2\alpha\beta(-2d_3\lambda + c)^2}{(-d_3\lambda^2 + c\lambda + \gamma)^3} < 0 \quad \text{for} \quad \lambda \in (0, \lambda_1^+).
\end{aligned}$$

By a simple discussion, we can get the existence of such a pair of (λ_0, c^*) from the above inequalities. □

Let $c > c^*$ be fixed. We use the notation

$$f(\lambda) := f(\lambda, c) \tag{3.8}$$

and want to find a $\lambda_* \in (0, \lambda_1^+)$ such that

$$f(\lambda_*) = 0 \quad \text{and} \quad f'(\lambda_*) > 0. \tag{3.9}$$

By Lemma 3.2, we can choose $\lambda_* = \lambda_1$. To get the minimal wave speed c^*, we need to deal with a quartic equation and there is a formula giving its discriminant.

Theorem 3.3. *Given positive numbers d_2, d_3, α, β and γ, the minimal wave speed c^* is the unique positive solution of $\Delta(c) = 0$, where $\Delta(c)$ is defined as follows*:

$$\begin{aligned}
\Delta(c) :=\ & 256A^3E^3 - 192A^2BDE^2 - 128A^2C^2E^2 + 144A^2CD^2E - 27A^2D^4\\
&+ 144AB^2CE^2 - 6AB^2D^2E - 80ABC^2DE + 18ABCD^3 + 16AC^4E\\
&- 4AC^3D^2 - 27B^4E^2 + 18B^3CDE - 4B^3D^3 - 4B^2C^3E + B^2C^2D^2\\
=\ & A_4c^8 + A_3c^6 + A_2c^4 + A_1c^2 + A_0,
\end{aligned}$$

where

$$\begin{aligned}
&A = d_2d_3, \quad B = -c(d_2 + d_3), \quad C = c^2 - d_2\gamma - d_3\alpha,\\
&D = c(\alpha + \gamma), \quad E = \alpha(\gamma - \beta),
\end{aligned}$$

and

$$\begin{aligned}
A_4 =\ & -2d_2d_3\alpha^2 + 4d_2d_3\alpha\gamma - 2d_2d_3\gamma^2 - 2\alpha d_3^2\gamma + 4\alpha d_3^2\beta - 8\alpha d_2d_3\beta - 2\alpha d_2^2\gamma + 4\alpha d_2^2\beta\\
&+ d_3^2\alpha^2 + d_3^2\gamma^2 + d_2^2\alpha^2 + d_2^2\gamma^2,\\
A_3 =\ & -18d_2d_3^2\alpha^2\gamma + 24d_2d_3^2\alpha\gamma^2 + 24d_2^2d_3\alpha^2\gamma - 18d_2^2d_3\alpha\gamma^2 + 14d_2d_3^2\alpha^2\beta + 18\alpha d_3^3\gamma\beta\\
&- 38d_3d_2^2\alpha^2\beta + 6\alpha d_2^3\gamma\beta + 14d_2^2d_3\alpha\gamma\beta - 38\alpha d_2d_3^2\gamma\beta + 2d_2d_3^2\alpha^3 - 8d_2d_3^2\gamma^3 - 8d_2^2d_3\alpha^3\\
&+ 2d_2^2d_3\gamma^3 - 8d_3^3\alpha\gamma^2 - 8d_2^3\alpha^2\gamma + 2d_2^3\alpha\gamma^2 + 6d_3^3\alpha^2\beta + 18d_2^3\alpha^2\beta + 2d_3^3\alpha^3 + 4d_3^3\gamma^3\\
&+ 4d_2^3\alpha^3 + 2d_2^3\gamma^3 + 2d_3^3\alpha^2\gamma,\\
A_2 =\ & -8d_2^2d_3^2\alpha^4 + 8d_2d_3^3\alpha^4 - 6d_3^4\alpha^3\beta - 27\alpha^2d_3^4\beta^2 - 27\alpha^2d_2^4\beta^2 - 8d_2^2d_3^2\gamma^4 + 8d_2^3d_3\gamma^4\\
&+ 8d_3^4\alpha^3\gamma - 8d_3^4\alpha^2\gamma^2 - 8d_2^4\gamma^2\alpha^2 + 8d_2^4\gamma^3\alpha + d_3^4\alpha^4 + d_2^4\gamma^4 - 14d_2^2d_3^2\alpha^3\beta
\end{aligned}$$

$$+ 40d_2d_3^3\alpha^3\beta - 12d_2^3d_3\alpha^3\beta + 36d_2d_3^3\alpha^2\beta^2 - 2d_2^2d_3^2\alpha^2\beta^2 + 36d_2^3d_3\alpha^2\beta^2 - 40d_2^2d_3^2\alpha^3\gamma$$
$$+ 102d_2^2d_3^2\alpha^2\gamma^2 - 40d_2^2d_3^2\alpha\gamma^3 - 4d_2d_3^3\alpha^3\gamma - 40d_2d_3^3\alpha^2\gamma^2 + 32d_2d_3^3\alpha\gamma^3 + 32d_2^3d_3\alpha^3\gamma$$
$$- 40d_2^3d_3\alpha^2\gamma^2 - 4d_2^3d_3\alpha\gamma^3 - 6\alpha d_2^4\gamma^2\beta + 36d_3^4\alpha^2\gamma\beta + 36d_2^4\gamma\alpha^2\beta - 192d_2^2d_3^2\alpha^2\gamma\beta$$
$$- 14d_2^2d_3^2\alpha\gamma^2\beta + 52d_2d_3^3\alpha^2\gamma\beta - 12d_2d_3^3\alpha\gamma^2\beta + 52d_2^3d_3\alpha^2\gamma\beta + 40d_2^3d_3\alpha\gamma^2\beta,$$

$$A_1 = 4d_2d_3^4\alpha^5 - 4d_3^5\alpha^4\beta + 4d_2^4d_3\gamma^5 + 4d_3^5\alpha^4\gamma + 4\alpha d_2^5\gamma^4 + 88d_2^3d_3^2\alpha^3\gamma^2 - 32d_2^3d_3^2\alpha^2\gamma^3$$
$$+ 32d_2^4d_3\alpha\gamma^4 + 32d_2d_3^4\alpha^4\gamma - 48d_2^3d_3^2\alpha\gamma^4 - 48d_2^2d_3^3\alpha^4\gamma - 48d_2d_3^4\alpha^3\gamma^2 - 48d_2^4d_3\alpha^2\gamma^3$$
$$- 4\alpha d_2^5\gamma^3\beta + 48d_2^3d_3^2\alpha^3\beta^2 - 24d_2d_3^4\alpha^4\beta + 60d_2^2d_3^3\alpha^4\beta - 144d_2d_3^4\alpha^3\beta^2 - 32d_2^2d_3^3\alpha^3\gamma^2$$
$$+ 88d_2^2d_3^3\alpha^2\gamma^3 - 104d_2^2d_3^3\alpha^3\gamma\beta - 124d_2^2d_3^3\alpha^2\gamma^2\beta + 48d_2^2d_3^3\alpha^2\gamma\beta^2 - 124d_2^3d_3^2\alpha^3\gamma\beta$$
$$- 104d_2^3d_3^2\alpha^2\gamma^2\beta + 160d_2^3d_3^2\alpha^2\gamma\beta^2 - 24d_2^4d_3\alpha\gamma^3\beta + 60d_2^3d_3^2\alpha\gamma^3\beta + 196d_2d_3^4\alpha^3\gamma\beta$$
$$+ 196d_2^4d_3\alpha^2\gamma^2\beta - 144d_2^4d_3\alpha^2\gamma\beta^2 + 160d_2^2d_3^3\alpha^3\beta^2,$$

$$A_0 = -352d_2^3d_3^3\alpha^3\gamma^2\beta + 512d_2^3d_3^3\alpha^3\gamma\beta^2 + 192d_2^4d_3^2\alpha^2\gamma^3\beta - 128d_2^4d_3^2\alpha^2\gamma^2\beta^2 + 192d_2^2d_3^4\alpha^4\gamma\beta$$
$$- 16d_2^5d_3\alpha\gamma^4\beta + 16d_2^5d_3\alpha\gamma^5 + 16d_2d_3^5\alpha^5\gamma - 16d_2d_3^5\alpha^5\beta - 64d_2^4d_3^2\alpha^2\gamma^4 - 64d_2^2d_3^4\alpha^4\gamma^2$$
$$+ 96d_2^3d_3^3\alpha^3\gamma^3 - 256d_2^3d_3^3\alpha^3\beta^3 - 128d_2^2d_3^4\alpha^4\beta^2.$$

Proof. To study the roots of $f(\lambda)$, we rewrite $f(\lambda) = 0$ as

$$G(\lambda) := (-d_2\lambda^2 + c\lambda + \alpha)(-d_3\lambda^2 + c\lambda + \gamma) - \alpha\beta = 0. \tag{3.10}$$

$f(\lambda) = 0$ and $G(\lambda) = 0$ are of the same roots. Obviously, (3.10) always has two simple real roots $\lambda_{\max} > \max\{\lambda_1^+, \lambda_2^+\}$ and $\lambda_{\min} < \min\{\lambda_1^-, \lambda_2^-\}$; if (3.10) has other real roots, they must be in the interval $(0, \min\{\lambda_1^+, \lambda_2^+\})$, where $\lambda_1^\pm$ and $\lambda_2^\pm$ are the roots of $-d_3\lambda^2 + c\lambda + \gamma = 0$ and $-d_2\lambda^2 + c\lambda + \alpha = 0$, respectively, and $\lambda_i^- < 0 < \lambda_i^+$, $i = 1, 2$. We rewrite (3.10) as

$$d_2d_3\lambda^4 - c(d_2 + d_3)\lambda^3 + (c^2 - d_2\gamma - d_3\alpha)\lambda^2 + c(\alpha + \gamma)\lambda + \alpha(\gamma - \beta) = 0. \tag{3.11}$$

By the theory of the quartic equation, the discriminant of $G(\lambda)$ is Δ defined in Theorem 3.3; $\Delta = 0 \Leftrightarrow$ (3.11) has the multiple roots; $\Delta < 0 \Leftrightarrow$ (3.11) has two simple real roots and two simple complex roots; $\Delta > 0 \Leftrightarrow$ the four simple roots of (3.11) are either all real or all complex (see [11]). Since (3.11) already has two simple real roots $\lambda_{\max}$ and $\lambda_{\min}$, c^* must be the solution of $\Delta(c) = 0$. Furthermore, we show that $\Delta(c) = 0$ has only one real root on $(0, \infty)$. By Lemma 3.2, when $c > c^* > 0$, (3.7) has two simple real roots on $(0, \min\{\lambda_1^+, \lambda_2^+\})$, which means (3.10) has four simple real roots on $\mathbb{R}$ thus $\Delta(c) > 0$; when $0 < c < c^*$, (3.7) has no real roots on $(0, \min\{\lambda_1^+, \lambda_2^+\})$, which means (3.10) has two simple real roots and two simple complex roots on $\mathbb{R}$ thus $\Delta(c) < 0$. Thus for fixed d_2, d_3, α, β and γ, there exists only one $c^* > 0$ satisfying $\Delta(c^*) = 0$. □

Example 3.4. Given $\alpha = 1$, $\beta = 3$, $\gamma = 2$, $d_2 = 4$ and $d_3 = 7$, then $\Delta(c) = 117c^8 + 9804c^6 + 115828c^4 + 7085364c^2 - 50878912$. It can be regarded as a quartic equation of c^2. We use the formula of roots of quartic equation to solve $\Delta = 0$ and get the eight roots ± 2.489728494362, $\pm 9.031162429666\mathrm{i}$ and $\pm 3.543444284346 \pm 4.095346122859\mathrm{i}$. The only positive real root is $c^* = 2.489728494362$. We choose different values of c and the figures of $G(\lambda)$ defined by (3.10) are in Figure 1.

Remark 3.5. The minimal wave speed c^* is important for describing the transmission speed of infectious diseases. We are interested in the relation between the minimal wave speed c^* and the diffusive rates d_2 and d_3. The explicit expression of c^* is too complicated and the surfaces of c^* about d_2 and d_3 are in Figure 2.

The following lemma gives a simple sufficient condition that ensures such a λ_* satisfying (3.9).

Lemma 3.6. *If $d_2 < d_3$, let $c > 2\sqrt{d_3(\beta - \gamma)}$. If $d_2 \geqslant d_3$, let $c > 2\sqrt{d_2(\beta/\gamma - 1)\alpha}$. Then there exists a $\lambda_* \in (0, \lambda_1^+)$ such that*

$$f(\lambda_*) = 0 \quad \textit{and} \quad f'(\lambda_*) > 0.$$

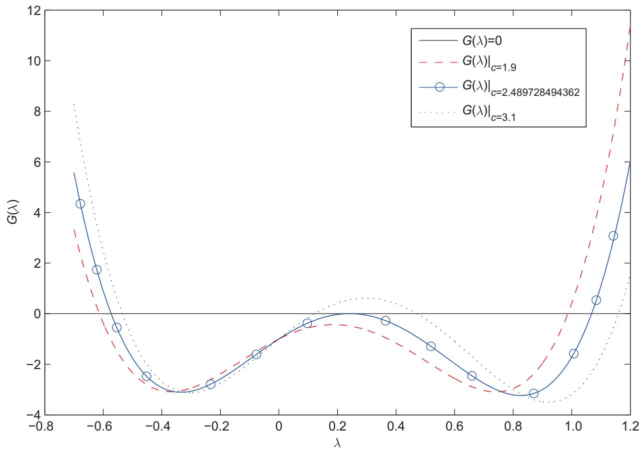

Figure 1 The curves of $G(\lambda)$ when c takes different values. Let $\alpha = 1$, $\beta = 3$, $\gamma = 2$, $d_2 = 4$ and $d_3 = 7$ be fixed. We can see that if $c = c^* = 2.489728494362$, $G(\lambda)$ has two simple real roots and a double real root; if $c = 1.9 < c^*$, $G(\lambda)$ has two real roots; if $c = 3.1 > c^*$, $G(\lambda)$ has four real roots

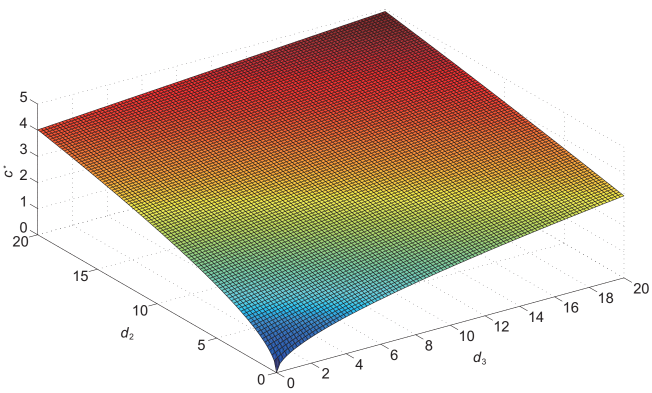

Figure 2 The surface of $c^*(d_2, d_3)$, where α, β and γ take the same values as they are in Example 3.4. As it is expected, c^* increases with respect to d_2 and d_3

Proof. First, we define some new notation

$$g_1(\lambda) := \frac{\alpha\beta}{f_1(\lambda)} = \frac{\alpha\beta}{-d_3\lambda^2 + c\lambda + \gamma}$$

and

$$f_2(\lambda) := -d_2\lambda^2 + c\lambda + \alpha$$

such that

$$f(\lambda) = f_2(\lambda) - g_1(\lambda).$$

We show that if $c > 0$ is small enough, $f(\lambda) = 0$ has no positive solution in $(\lambda_1^-, \lambda_1^+)$, where $\lambda_1^\pm$ are the roots of $f_1 = 0$. If we take $c = 0$, then $g_1(0) = \alpha\beta/\gamma > \alpha$ is its minimum value on $(\lambda_1^-, \lambda_1^+)$, while for f_2, the maximum value is α. Thus $f(\lambda) = 0$ has no solution in $(\lambda_1^-, \lambda_1^+)$. Furthermore, f is continuous with respect to c, thus there is no solution in $(\lambda_1^-, \lambda_1^+)$ for $c > 0$ small enough.

Next, we want to get such a $\lambda_* \in (\lambda_1^-, \lambda_1^+)$ satisfying (3.9) when c is large enough. Notice that $g_1(0) = \alpha\beta/\gamma$ and $f_2(0) = \alpha$. By the assumption $\beta/\gamma > 1$, we obtain $g_1(0) > f_2(0)$. The axis of symmetry of $g_1(\lambda)$ is $\lambda = c/(2d_3) > 0$ and for $f_2(\lambda)$, the axis of symmetry is $\lambda = c/(2d_2) > 0$. Furthermore, $g_1(\lambda)$

takes its minimum value

$$b_1 = \frac{4\alpha\beta d_3}{c^2 + 4d_3\gamma}$$

at $\lambda = c/(2d_3)$, while $f_2(\lambda)$ takes its maximum value

$$b_2 = \frac{c^2}{4d_2} + \alpha$$

at $\lambda = c/(2d_2)$. If c is large enough, b_1 is a sufficiently small positive number and b_2 can be sufficiently large, thus there must be a λ_* satisfying (3.9).

If $c/(2d_2) > c/(2d_3)$, $g_1(\lambda)$ decreases on $[0, c/(2d_3))$. When $c > 2\sqrt{d_3(\beta - \gamma)}$,

$$g_1\left(\frac{c}{2d_3}\right) < \alpha.$$

Since f_2 increases on $[0, \frac{c}{2d_3}]$ and $f_2(0) = \alpha$, we can see that

$$f_2\left(\frac{c}{2d_3}\right) > \alpha.$$

Hence, there exists a solution $\lambda_* \in (0, c/(2d_3))$ such that

$$f(\lambda_*) = 0, \quad f'(\lambda_*) > 0.$$

If $c/(2d_2) \leqslant c/(2d_3)$, $f_2(\lambda)$ increases on $[0, c/(2d_2))$. Let

$$c > 2\sqrt{d_2\left(\frac{\beta}{\gamma} - 1\right)\alpha}.$$

Then

$$f_2\left(\frac{c}{2d_2}\right) > \frac{\alpha\beta}{\gamma},$$

while $g_1(0) = \alpha\beta/\gamma$ and g_1 decreases on $[0, \frac{c}{2d_2})$, which means

$$g_1\left(\frac{c}{2d_2}\right) < \frac{\alpha\beta}{\gamma}.$$

Thus there exists a solution $\lambda_* \in (0, c/(2d_2))$ such that $f(\lambda_*) = 0$, $f'(\lambda_*) > 0$. □

For $i = 1, 2$, we give the second-order linear differential operator Δ_i and its inverse Δ_i^{-1}. Given $a_1 > \beta$ and $a_2 > \alpha$, the roots of the equation

$$-d_i\Lambda^2 + c\Lambda + a_i = 0$$

are

$$\Lambda_i^{\pm} := \frac{c \pm \sqrt{c^2 + 4d_ia_i}}{2d_i}.$$

a_i is chosen so large that

$$-\Lambda_i^- > \lambda_*.$$

We introduce a new symbol

$$R_i := d_i(\Lambda_i^+ - \Lambda_i^-) = \sqrt{c^2 + 4d_ia_i}.$$

The differential operator Δ_i is defined by

$$\Delta_i(h) := -d_ih'' + ch' + a_ih$$

for $h \in C^2(\mathbb{R})$. The corresponding inverse operator Δ_i^{-1} is

$$\Delta_i^{-1}(h(x)) := \frac{1}{R_i}\int_{-\infty}^{x} \mathrm{e}^{\Lambda_i^-(x-y)}h(y)dy + \frac{1}{R_i}\int_{x}^{\infty} \mathrm{e}^{\Lambda_i^+(x-y)}h(y)dy$$

for $h \in C_{\mu^-,\mu^+}(\mathbb{R})$, where

$$\mu^- > \Lambda_i^-, \quad \mu^+ < \Lambda_i^+$$

and

$$C_{\mu^-,\mu^+}(\mathbb{R}) := \Big\{h \in C^2(\mathbb{R}) : \sup_{x\leqslant 0} h(x)\mathrm{e}^{-\mu^- x} + \sup_{x\geqslant 0} h(x)\mathrm{e}^{-\mu^+ x} < \infty\Big\}. \tag{3.12}$$

Furthermore, by a simple calculation,

$$(\Delta_i^{-1}h)'(x) = \frac{\Lambda_i^-}{R_i}\int_{-\infty}^{x} \mathrm{e}^{\Lambda_i^-(x-y)}h(y)dy + \frac{\Lambda_i^+}{R_i}\int_{x}^{\infty} \mathrm{e}^{\Lambda_i^+(x-y)}h(y)dy$$

and

$$(\Delta_i^{-1}h)''(x) = \frac{(\Lambda_i^-)^2}{R_i}\int_{-\infty}^{x} \mathrm{e}^{\Lambda_i^-(x-y)}h(y)dy + \frac{(\Lambda_i^+)^2}{R_i}\int_{x}^{\infty} \mathrm{e}^{\Lambda_i^+(x-y)}h(y)dy + \frac{\Lambda_i^-}{R_i}h(x) - \frac{\Lambda_i^+}{R_i}h(x).$$

The following Lemmas 3.7 and 3.8 on the properties of operators Δ_i and Δ_i^{-1} are from Wang et al. [14]. These two lemmas can be checked by a direct calculation.

Lemma 3.7. *For $i = 1, 2$,*

$$\Delta_i^{-1}(\Delta_i h) = h$$

for any $h \in C^2(\mathbb{R})$ such that $h, h', h'' \in C_{\mu^-,\mu^+}(\mathbb{R})$. Furthermore,

$$\Delta_i(\Delta_i^{-1}h) = h$$

for $h \in C_{\mu^-,\mu^+}(\mathbb{R})$, where $\mu^- > \Lambda_i^-$ and $\mu^+ < \Lambda_i^+$.

Next, define

$$g(x) := \begin{cases} \mathrm{e}^{\lambda x}(1 - M\mathrm{e}^{\varepsilon x}), & x \leqslant x^*, \\ 0, & x > x^*, \end{cases} \tag{3.13}$$

where $x^* = -(\ln M)/\varepsilon$. $g(x)$ can be rewritten as

$$g(x) = \mathrm{e}^{\lambda x}(1 - M\mathrm{e}^{\varepsilon x}) \vee 0$$

by using the new symbol $\vee$ defined as follows:

$$a \vee b := \max\{a, b\}.$$

Lemma 3.8. *For $i = 1, 2$, given any $M > 0$ and $\varepsilon > 0$,*

$$\Delta_i^{-1}(\Delta_i g) \geqslant g$$

holds for $g(x) = \mathrm{e}^{\lambda x}(1 - M\mathrm{e}^{\varepsilon x}) \vee 0$, where

$$\Lambda_i^- < \lambda < \lambda + \varepsilon < \Lambda_i^+.$$

Remark 3.9. Although $g(x)$ is not differentiable at x^*, the integral $\Delta_i^{-1}(\Delta_i g)$ is well-defined in the sense of distribution.

Choosing μ such that

$$\lambda_* < \mu < -\Lambda_i^- < \Lambda_i^+, \quad i = 1, 2, \tag{3.14}$$

we define the functional space

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = \Big\{\phi = (\phi_1, \phi_2) : \phi_i \in C(\mathbb{R}), \sup_{x\in\mathbb{R}} \mathrm{e}^{-\mu|x|}|\phi_i(x)| < \infty, \, i = 1, 2\Big\}$$

with the norm

$$|\phi|_\mu := \max\{|\phi_i|_\mu,\ i = 1, 2\}, \tag{3.15}$$

where

$$|\phi_i|_\mu := \sup_{x\in\mathbb{R}} \mathrm{e}^{-\mu|x|}|\phi_i(x)|.$$

We give the definitions as follows:

$$S_+ := S_{-\infty}, \tag{3.16}$$

$$S_- := S_{-\infty}(1 - M_1\mathrm{e}^{\varepsilon_1 x}) \vee 0, \tag{3.17}$$

$$E_+ := \mathrm{e}^{\lambda_* x}, \tag{3.18}$$

$$E_- := \mathrm{e}^{\lambda_* x}(1 - M_2\mathrm{e}^{\varepsilon_2 x}) \vee 0, \tag{3.19}$$

where $M_i > 0$ is sufficiently large and $\varepsilon_i > 0$ is sufficiently small, $i = 1, 2$. We denote

$$F_1(S, E) := \Delta_1^{-1}\left[a_1 S - \frac{\beta S I(E)}{S + I(E) + E}\right], \tag{3.20}$$

$$F_2(S, E) := \Delta_2^{-1}\left[a_2 E + \frac{\beta S I(E)}{S + I(E) + E} - \alpha E\right]. \tag{3.21}$$

Γ is the convex cone defined by using (3.16)–(3.19), i.e.,

$$\Gamma := \{(S, E) \in B_\mu(\mathbb{R}, \mathbb{R}^2) : S_- \leqslant S \leqslant S_+,\ E_- \leqslant E \leqslant E_+\}.$$

We can see that Γ is uniformly bounded under the norm $|\cdot|_\mu$ defined in (3.15). We show that Γ is invariant under the map $F = (F_1, F_2)$.

Lemma 3.10. *$F = (F_1, F_2)$ maps Γ to Γ, i.e., for $(S, E) \in \Gamma$, $S_- \leqslant S \leqslant S_+$ and $E_- \leqslant E \leqslant E_+$, we have*

$$S_- \leqslant F_1(S, E) \leqslant S_+$$

and

$$E_- \leqslant F_2(S, E) \leqslant E_+.$$

Proof. First, since

$$\Delta_1 F_1(S, E) = a_1 S - \frac{\beta S I(E)}{S + I(E) + E} \leqslant a_1 S_+ = \Delta_1 S_+,$$

by Lemma 3.7, we have

$$F_1(S, E) \leqslant \Delta_1^{-1}(\Delta_1 S_+) = S_+.$$

Next, we show that $F_1(S, E) \geqslant S_-$. For $x > x_1 := -\varepsilon_1^{-1} \ln M_1$, $S_-(x) = 0$, thus $\Delta_1 S_- = 0$. Since $a_1 > \beta$, we have

$$a_1 S - \frac{\beta S I(E)}{S + I(E) + E} \geqslant a_1 S - \beta S \geqslant 0 = \Delta_1 S_-,$$

which implies that $F_1(S, E) \geqslant S_-$ holds for $x \geqslant x_1$. For $x < x_1$, we need to show

$$-\beta I(E_+) \geqslant -d_1 S_-'' + c S_-'.$$

Since $S_-(x) = S_{-\infty}(1 - M_1\mathrm{e}^{\varepsilon_1 x})$, the above inequality is

$$-\beta I(E_+) \geqslant -d_1 S_{-\infty}(1 - M_1\mathrm{e}^{\varepsilon_1 x})'' + c S_{-\infty}(1 - M_1\mathrm{e}^{\varepsilon_1 x})'.$$

By the definition of $I(E)$ and $E_+(x) = \mathrm{e}^{\lambda_* x}$, it implies to prove

$$S_{-\infty} M_1 \varepsilon_1(-d_1\varepsilon_1 + c) \geqslant \frac{\alpha\beta}{-d_3\lambda_*^2 + c\lambda_* + \gamma}\mathrm{e}^{(\lambda_* - \varepsilon_1)x}.$$

Choosing $\varepsilon_1 \in (0, \lambda_*)$ sufficiently small, since the right-hand side of the above inequality monotonically increases with respect to x and $x \leqslant -\varepsilon_1^{-1} \ln M_1$, we only need to show

$$S_{-\infty} M_1 \varepsilon_1 (-d_1 \varepsilon_1 + c) \geqslant \frac{\alpha\beta}{-d_3\lambda_*^2 + c\lambda_* + \gamma} \mathrm{e}^{-\varepsilon_1^{-1}(\lambda_* - \varepsilon_1) \ln M_1}.$$

This inequality holds for $0 < \varepsilon_1 < \min\{\lambda_*, c/d_1\}$ and M_1 large enough.

Next, we verify $E_- \leqslant F_2(S, E) \leqslant E_+$. Since $f(\lambda_*) = 0$ in (3.8), it holds that

$$a_2 E + \frac{\beta S I(E)}{S + I(E) + E} - \alpha E \leqslant a_2 E_+ + \beta I(E_+) - \alpha E_+ = a_2 E_+ - d_2 E_+'' + c E_+' = \Delta_2 E_+. \tag{3.22}$$

By Lemma 3.7, it holds that

$$F_2(S, E) \leqslant \Delta_2^{-1}(\Delta_2 E_+) = E_+.$$

Next, to check

$$a_2 E + \frac{\beta S I(E)}{S + I(E) + E} - \alpha E \geqslant a_2 E_- + \frac{\beta S_- I(E_-)}{S_- + I(E_-) + E_+} - \alpha E_- \geqslant a_2 E_- - d_2 E_-'' + c E_-' = \Delta_2 E_-,$$

it suffices to show

$$\frac{\beta S_- I(E_-)}{S_- + I(E_-) + E_+} - \alpha E_- \geqslant -d_2 E_-'' + c E_-'.$$

For $x \geqslant x_2 := -\varepsilon_2^{-1} \ln M_2$, $E_-(x) = 0$ and the above inequality holds. For $x < x_2$, we subtract both sides by $\beta I(E_-) - \alpha E_-$ and obtain

$$-\frac{\beta I^2(E_-)}{S_- + I(E_-) + E_+} - \frac{\beta I(E_-) E_+}{S_- + I(E_-) + E_+} \geqslant -d_2 E_-'' + c E_-' + \alpha E_- - \beta I(E_-). \tag{3.23}$$

In view of $f(\lambda_*) = 0$ and $E_- = \mathrm{e}^{\lambda_* x}(1 - M_2 \mathrm{e}^{\varepsilon_2 x})$, we obtain

$$-d_2 E_-'' + c E_-' + \alpha E_- - \beta I(E_-) = f(\lambda_*) \mathrm{e}^{\lambda_* x} - M_2 f(\lambda_* + \varepsilon_2) \mathrm{e}^{(\lambda_* + \varepsilon_2) x} = -M_2 f(\lambda_* + \varepsilon_2) \mathrm{e}^{(\lambda_* + \varepsilon_2) x}.$$

To prove (3.23), it suffices to show

$$\frac{\beta I^2(E_-) + \beta I(E_-) E_+}{S_-} \leqslant M_2 f(\lambda_* + \varepsilon_2) \mathrm{e}^{(\lambda_* + \varepsilon_2) x}. \tag{3.24}$$

We take $\varepsilon_2 \in (0, \min\{\varepsilon_1, \lambda_*, \lambda_2 - \lambda_*\})$ small enough, where λ_2 is defined in Lemma 3.2. It holds that

$$I(E_-)(x) = \frac{\alpha \mathrm{e}^{\lambda_* x}}{-d_3 \lambda_*^2 + c\lambda_* + \gamma} - M_2 \frac{\alpha \mathrm{e}^{(\lambda_* + \varepsilon_2) x}}{-d_3 (\lambda_* + \varepsilon_2)^2 + c(\lambda_* + \varepsilon_2) + \gamma}. \tag{3.25}$$

For simplicity, we introduce new notation K_1 and K_2 such that $I(E_-)$ can be rewritten as

$$I(E_-)(x) = K_1 \mathrm{e}^{\lambda_* x} - K_2(\varepsilon_2) M_2 \mathrm{e}^{(\lambda_* + \varepsilon_2) x} > 0$$

for $x < x_2$. (3.24) becomes

$$\frac{\beta \mathrm{e}^{2\lambda_* x}[(K_1 - K_2(\varepsilon_2) M_2 \mathrm{e}^{\varepsilon_2 x})^2 + K_1 - K_2(\varepsilon_2) M_2 \mathrm{e}^{\varepsilon_2 x}]}{S_{-\infty}(1 - M_1 \mathrm{e}^{\varepsilon_1 x})} \leqslant M_2 f(\lambda_* + \varepsilon_2) \mathrm{e}^{(\lambda_* + \varepsilon_2) x}.$$

It suffices to verify

$$M_2 f(\lambda_* + \varepsilon_2) S_{-\infty}(1 - M_1 \mathrm{e}^{\varepsilon_1 x}) \geqslant \beta \mathrm{e}^{(\lambda_* - \varepsilon_2) x}(K_1^2 + K_1)$$

holds for small ε_2. For $x < x_2 = -\varepsilon_2^{-1} \ln M_2$, we need to show

$$M_2 f(\lambda_* + \varepsilon_2) S_{-\infty}(1 - M_1 M_2^{-\varepsilon_1/\varepsilon_2}) \geqslant \beta M_2^{-(\lambda_* - \varepsilon_2)/\varepsilon_2}(K_1^2 + K_1).$$

Let $M_2 = 1/f(\lambda_* + \varepsilon_2)$. As ε_2 goes to zero, the left-hand side of the above inequality goes to $S_{-\infty}$ while the right-hand side goes to zero and thus (3.24) holds. Together with Lemma 3.8, it yields that

$$F_2(S, E) \geqslant \Delta_2^{-1}(\Delta_2 E_-) \geqslant E_-.$$

□

Lemma 3.11. *For $E \in C_{-\mu,\mu}(\mathbb{R})$ whose norm $|E|_\mu$ is bounded, where $|\cdot|_\mu$ is defined in* (3.15), *there exists a constant $K_0 \geqslant 1$ such that $|I(E)|_\mu < K_0|E|_\mu$.*

Proof. Recall the integral form of $I(E)(x)$ is

$$I(E)(x) = \frac{\alpha}{\rho_1}\int_{-\infty}^{x} \mathrm{e}^{\lambda_1^-(x-y)}E(y)dy + \frac{\alpha}{\rho_1}\int_{x}^{\infty} \mathrm{e}^{\lambda_1^+(x-y)}E(y)dy,$$

where ρ_1 and $\lambda_1^\pm$ are defined in (3.4) and (3.3). For any x, we have

$$\begin{aligned}\mathrm{e}^{-\mu|x|}I(E)(x) &= \frac{\alpha}{\rho_1}\bigg(\int_{-\infty}^{x} \mathrm{e}^{\lambda_1^-(x-s)}\mathrm{e}^{-\mu|x|}E(s)ds + \int_{x}^{\infty} \mathrm{e}^{\lambda_1^+(x-s)}\mathrm{e}^{-\mu|x|}E(s)ds\bigg)\\ &\leqslant \frac{\alpha}{\rho_1}\bigg(\int_{-\infty}^{x} \mathrm{e}^{\lambda_1^-(x-s)}ds + \int_{x}^{\infty} \mathrm{e}^{\lambda_1^+(x-s)}ds\bigg)|E|_\mu\\ &= \frac{\alpha}{\rho_1}\bigg(\frac{-1}{\lambda_1^-} + \frac{1}{\lambda_1^+}\bigg)|E|_\mu = (\alpha/\gamma)|E|_\mu.\end{aligned}$$

Take $K_0 = \max\{1, \alpha/\gamma\}$. This completes the proof. □

Lemma 3.12. *The map $F = (F_1, F_2) : \Gamma \to \Gamma$ is continuous and compact with respect to the norm $|\cdot|_\mu$.*

Proof. For $(S_1, E_1) \in \Gamma$ and $(S_2, E_2) \in \Gamma$, since

$$\bigg|\frac{\beta S_1 I(E_1)}{S_1 + I(E_1) + E_1} - \frac{\beta S_2 I(E_2)}{S_2 + I(E_2) + E_2}\bigg| \leqslant \beta(|S_1 - S_2| + |I(E_1) - I(E_2)|),$$

we have

$$\bigg|a_1 S_1 - a_1 S_2 - \frac{\beta S_1 I(E_1)}{S_1 + I(E_1) + E_1} + \frac{\beta S_2 I(E_2)}{S_2 + I(E_2) + E_2}\bigg| \leqslant (a_1 + \beta)(|S_1 - S_2| + |I(E_1) - I(E_2)|).$$

By the definition of the norm $|\cdot|_\mu$ and Lemma 3.11,

$$\begin{aligned}|F_1(S_1, E_1) - F_1(S_2, E_2)|(x)\mathrm{e}^{-\mu|x|} &\leqslant \mathrm{e}^{-\mu|x|}\int_{-\infty}^{x} \mathrm{e}^{\Lambda_1^-(x-y)}(a_1 + \beta)(|S_1 - S_2| + |I(E_1) - I(E_2)|)dy\\ &\quad + \mathrm{e}^{-\mu|x|}\int_{x}^{\infty} \mathrm{e}^{\Lambda_1^+(x-y)}(a_1 + \beta)(|S_1 - S_2| + |I(E_1) - I(E_2)|)dy\\ &\leqslant \frac{a_1 + \beta}{R_1}(|S_1 - S_2|_\mu + K_0|E_1 - E_2|_\mu)C(x)\\ &\leqslant \frac{(a_1 + \beta)K_0}{R_1}(|S_1 - S_2|_\mu + |E_1 - E_2|_\mu)C(x),\end{aligned}$$

where

$$C(x) := \mathrm{e}^{-\mu|x|}\bigg(\int_{-\infty}^{x} \mathrm{e}^{\Lambda_1^-(x-y)+\mu|y|}dy + \int_{x}^{\infty} \mathrm{e}^{\Lambda_1^+(x-y)+\mu|y|}dy\bigg).$$

We have

$$C(-\infty) = \frac{1}{\mu + \Lambda_1^+} - \frac{1}{\mu + \Lambda_1^-}$$

and

$$C(\infty) = \frac{1}{-\mu + \Lambda_1^+} + \frac{1}{\mu - \Lambda_1^-}.$$

Thus $C(x)$ is uniformly bounded on $\mathbb{R}$ and consequently F_1 is continuous with respect to the norm $|\cdot|_\mu$. Similarly, F_2 is also continuous with respect to this norm. To prove the compactness of F, we use the Arzela-Ascoli theorem and the diagonal process. Denote

$$I_k := [-k, k], \quad k \in \mathbb{N}$$

and consider Γ as the bounded subset of $C(I_k,\mathbb{R}^2)$ with the maximum norm. Obviously, F is uniformly bounded on I_k. Next, we show that F is equi-continuous on I_k. For any $(S,E)\in\Gamma$,

$$\begin{aligned}|F_1'(S,E)| &\leqslant \frac{-\Lambda_1^- a_1 S_{-\infty}}{R_1}\int_{-\infty}^{x} \mathrm{e}^{\Lambda_1^-(x-y)}dy + \frac{\Lambda_1^+ a_1 S_{-\infty}}{R_1}\int_{x}^{\infty} \mathrm{e}^{\Lambda_1^+(x-y)}dy\\ &= \frac{2a_1S_{-\infty}}{R_1},\end{aligned}$$

and

$$\begin{aligned}|F_2'(S,E)| &\leqslant \frac{-\Lambda_2^-(a_2+\frac{\alpha\beta}{-d_3\lambda_*^2+c\lambda_*+\gamma}-\alpha)}{R_2}\int_{-\infty}^{x}\mathrm{e}^{\Lambda_2^-(x-y)+\lambda_* y}dy\\ &\quad+\frac{\Lambda_2^+(a_2+\frac{\alpha\beta}{-d_3\lambda_*^2+c\lambda_*+\gamma}-\alpha)}{R_2}\int_{x}^{\infty}\mathrm{e}^{\Lambda_2^+(x-y)+\lambda_* y}dy\\ &= \frac{1}{R_2}\bigg(\frac{-\Lambda_2^-}{\lambda_*-\Lambda_2^-}+\frac{\Lambda_2^+}{\Lambda_2^+-\lambda_*}\bigg)\bigg(a_2+\frac{\alpha\beta}{-d_3\lambda_*^2+c\lambda_*+\gamma}-\alpha\bigg)\mathrm{e}^{\lambda_* x}.\end{aligned}$$

Let $\{u_n\}$ be a subsequence of Γ. $\{u_n\}$ can also be regarded as a bounded subsequence of $C(I_k)$. Since $\{F(u_n)\}$ is uniformly bounded and equi-continuous on I_k, by the Arzela-Ascoli theorem, we can choose a subsequence $\{u_{n_k}\}$ such that

$$v_{n_k} = Fu_{n_k}$$

converges in $C(I_k),\ k\in\mathbb{N}$.

Let v denote the limit of $\{v_{n_k}\}$. We can see that $v\in C(\mathbb{R},\mathbb{R}^2)$. Since $F(\Gamma)\subset\Gamma$ and Γ is closed, thus $v\in\Gamma$. Furthermore, $\mu>\lambda_*>0$, hence $|E_+|_\mu$ is bounded and Γ is also uniformly bounded under the norm $|\cdot|_\mu$. Thus $|v_{n_k}-v|_\mu$ is uniformly bounded for all $n\in\mathbb{N}$. Given any $\varepsilon>0$, we can choose $M\in\mathbb{N}$ independent of u_{n_k} such that

$$\mathrm{e}^{-\mu|x|}|v_{n_k}(x)-v(x)|<\varepsilon,\quad n\in\mathbb{N}$$

holds for any $|x|>M$. On the compact interval $[-M,M]$, $\{v_{n_k}\}$ converges to v with the maximum norm. Thus there exists $K\in\mathbb{N}$ such that

$$\mathrm{e}^{-\mu|x|}|v_{n_k}(x)-v(x)|<\varepsilon$$

for $|x|<M$ and $n>K$. Hence, $\{v_{n_k}\}$ converges to v under the norm $|\cdot|_\mu$. We have finished the proof of the compactness of the map F. □

4 Existence theorem

Since F is continuous and compact on Γ, by the Schauder fixed point theorem, F has a fixed point $(S,E)\in\Gamma$ such that

$$\begin{aligned}S &= F_1(S,E) = \Delta_1^{-1}\bigg(a_1S-\frac{\beta SI(E)}{S+I(E)+E}\bigg),\\ E &= F_2(S,E) = \Delta_2^{-1}\bigg(a_2E+\frac{\beta SI(E)}{S+I(E)+E}-\alpha E\bigg).\end{aligned}$$

Since $S,E\in C_{-\mu,\mu}(\mathbb{R})$ and $\Lambda_i^-<-\mu<\mu<\Lambda_i^+$, $i=1,2$, it holds that

$$\Delta_1 S = a_1S-\frac{\beta SI(E)}{S+I(E)+E}, \tag{4.1}$$

$$\Delta_2 E = a_2E+\frac{\beta SI(E)}{S+I(E)+E}-\alpha E. \tag{4.2}$$

By the definition of Δ_i, (S,E) satisfies

$$cS' = d_1 S'' - \frac{\beta S I(E)}{S+I(E)+E}, \tag{4.3}$$

$$cE' = d_2 E'' + \frac{\beta S I(E)}{S+I(E)+E} - \alpha E. \tag{4.4}$$

Next, we verify that the boundary conditions hold. Since

$$S_- \leqslant S \leqslant S_+ \quad \text{and} \quad E_- \leqslant E \leqslant E_+,$$

it holds that

$$S(x) \to S_{-\infty}, \quad E(x) \sim \mathrm{e}^{\lambda_* x} \quad \text{as} \quad x \to -\infty.$$

In the proof of Lemma 3.1, we have

$$d_1 S'(x) = c[S(x) - S_{-\infty}] + \int_{-\infty}^{x} \frac{\beta S(y) I(y)}{S(y)+I(E)(y)+E(y)} dy.$$

Together with the integrability of (3.2), it holds that S' is uniformly bounded. We have

$$(\mathrm{e}^{-cx/d_1} S'(x))' = \mathrm{e}^{-cx/d_1}(S''(x) - cS'(x)/d_1) = \mathrm{e}^{-cx/d_1} \frac{\beta S(x) I(E)(x)}{d_1[S(x)+I(E)(x)+E(x)]}.$$

Integrating the above equality yields

$$\mathrm{e}^{-cx/d_1} S'(x) = -\int_{x}^{\infty} \mathrm{e}^{-cy/d_1} \frac{\beta S(y) I(E)(y)}{d_1[S(y)+I(E)(y)+E(y)]} dy.$$

Hence S is non-increasing. Since $I(x)$ is not trivial, the integral

$$\int_{x}^{\infty} \mathrm{e}^{-cy/d_1} \frac{\beta S(y) I(E)(y)}{d_1[S(y)+I(E)(y)+E(y)]} dy$$

cannot be identically 0. Thus S' is not trivial and

$$S(\infty) < S(-\infty).$$

By L'Hopital's rule and

$$I(x) = \frac{\alpha}{\rho_1}\left(\int_{-\infty}^{x} \mathrm{e}^{\lambda_1^-(x-y)} E(y) dy + \int_{x}^{\infty} \mathrm{e}^{\lambda_1^+(x-y)} E(y) dy\right),$$

we obtain

$$\lim_{x\to-\infty} I(x) = \lim_{x\to-\infty} \frac{\alpha}{\rho_1}\left(\frac{\mathrm{e}^{-\lambda_1^- x} E(x)}{-\lambda_1^- \mathrm{e}^{-\lambda_1^- x}}\right) + \lim_{x\to-\infty} \frac{\alpha}{\rho_1}\left(\frac{\mathrm{e}^{-\lambda_1^+ x} E(x)}{\lambda_1^+ \mathrm{e}^{-\lambda_1^+ x}}\right) = \lim_{x\to-\infty} \frac{\alpha}{\gamma} E(x) = 0.$$

Recall the integral representation of the first derivative

$$(\Delta_i^{-1} h)'(x) = \frac{\Lambda_i^-}{R_i} \int_{-\infty}^{x} \mathrm{e}^{\Lambda_i^-(x-y)} h(y) dy + \frac{\Lambda_i^+}{R_i} \int_{x}^{\infty} \mathrm{e}^{\Lambda_i^+(x-y)} h(y) dy,$$

where $h \in C_{-\mu,\mu}(\mathbb{R})$. By the definition of F_1 and L'Hopital's rule, we obtain

$$\lim_{x\to-\infty} S'(x) = \lim_{x\to-\infty} \frac{\Lambda_1^-}{R_1}\left(\frac{\mathrm{e}^{-\Lambda_1^- x} a_1 S(x)}{-\Lambda_1^- \mathrm{e}^{-\Lambda_1^- x}}\right) + \lim_{x\to-\infty} \frac{\Lambda_1^+}{R_1}\left(\frac{\mathrm{e}^{-\Lambda_1^+ x} a_1 S(x)}{\Lambda_1^+ \mathrm{e}^{-\Lambda_1^+ x}}\right) = 0.$$

Similarly, by the definition of F_2 and L'Hopital's rule,

$$E'(x) \to 0, \quad I'(x) \to 0, \quad \text{as} \quad x \to -\infty.$$

By (2.2)–(2.4), we obtain the limits of the second derivatives

$$S''(x) \to 0, \quad E''(x) \to 0, \quad I''(x) \to 0, \quad \text{as} \quad x \to -\infty.$$

Next, we give the asymptotic behaviors of $S(x)$, $E(x)$ and $I(x)$ as $x \to \infty$.

Similarly, by (3.5), we have

$$\int_{-\infty}^{\infty} I(x)dx = \frac{\alpha}{\gamma}\int_{-\infty}^{\infty} E(x)dx = \frac{1}{\gamma}\int_{-\infty}^{\infty} \frac{\beta S(x)I(E)(x)}{S(x)+I(E)(x)+E(x)}dx. \tag{4.5}$$

Since

$$E'(x) = \frac{\lambda_2^-}{\rho_2}\int_{-\infty}^{x} \mathrm{e}^{\lambda_2^-(x-y)}\frac{\beta S(y)I(E)(y)}{S(y)+I(E)(y)+E(y)}dy + \frac{\lambda_2^+}{\rho_2}\int_{x}^{\infty} \mathrm{e}^{\lambda_2^+(x-y)}\frac{\beta S(y)I(E)(y)}{S(y)+I(E)(y)+E(y)}dy,$$

we obtain

$$|E'(x)| \leqslant \frac{\beta}{d_2}\int_{-\infty}^{\infty} I(E)(x)dx.$$

We can see that

$$E(x) \to 0, \quad \text{as} \quad x \to \infty.$$

Otherwise, since $|E'|$ is bounded, we can choose a sequence $x_i \to \infty$, $\varepsilon > 0$ and $\delta > 0$, such that $E(x) > \varepsilon$ on $(x_i - \delta, x_i + \delta)$, which contradicts the integrability of E on $\mathbb{R}$. Similarly,

$$I(x) \to 0, \quad \text{as} \quad x \to \infty.$$

Furthermore, we use the same methods dealing with $I'(x)$ and $E'(x)$ and obtain

$$E'(x) \to 0, \quad I'(x) \to 0, \quad \text{as} \quad x \to \infty.$$

From (2.2)–(2.4), we also have

$$S''(x) \to 0, \quad E''(x) \to 0, \quad I''(x) \to 0, \quad \text{as} \quad x \to \infty.$$

Integrating (4.3) from $-\infty$ to ∞ yields

$$\int_{-\infty}^{\infty} \frac{\beta S(x)I(E)(x)}{S(x)+I(E)(x)+E(x)}dx = c[S(-\infty) - S(\infty)].$$

We want to prove that

$$E(x) \leqslant S(-\infty) - S(\infty)$$

for all $x \in \mathbb{R}$. Define

$$J(x) := E(x) + \frac{\alpha}{c}\int_{-\infty}^{x} E(y)dy + \frac{\alpha}{c}\int_{x}^{\infty} \mathrm{e}^{\frac{c}{d_2}(x-y)}E(y)dy. \tag{4.6}$$

By the property at infinity of $E(x)$ and L'Hopital's rule,

$$\lim_{x\to-\infty} J(x) = 0, \quad \lim_{x\to\infty} J(x) = \frac{\alpha}{c}\int_{-\infty}^{\infty} E(y)dy = S(-\infty) - S(\infty).$$

Similarly, by differentiating (4.6), together with $E(x) \to 0$ as $x \to \pm\infty$, we obtain

$$J'(x) = E'(x) + \frac{\alpha}{d_2}\int_{x}^{\infty} \mathrm{e}^{\frac{c}{d_2}(x-y)}E(y)dy.$$

Hence,

$$\lim_{x\to-\infty} J'(x) = 0, \quad \lim_{x\to\infty} J'(x) = 0$$

from the above equality. We differentiate (4.6) twice, i.e.,

$$J''(x) = E''(x) - \frac{\alpha}{d_2}E(x) + \frac{c\alpha}{d_2^2}\int_x^{\infty} \mathrm{e}^{\frac{c}{d_2}(x-y)}E(y)dy$$

and obtain

$$-d_2J'' + cJ' = -d_2E'' + cE' + \alpha E = \beta SI(E)/(S + I(E) + E).$$

We integrate the above equality from x to ∞ and obtain

$$J'(x) = \frac{1}{d_2}\int_x^{\infty} \mathrm{e}^{\frac{c}{d_2}(x-y)}\frac{\beta S(y)I(E)(y)}{S(y) + I(E)(y) + E(y)}dy \geqslant 0. \tag{4.7}$$

We have

$$J(\infty) = S(-\infty) - S(\infty)$$

and from (4.7) we can see that $J(x)$ is non-decreasing and thus

$$J(x) \leqslant S(-\infty) - S(\infty)$$

for all $x \in \mathbb{R}$. Recall that $E(x) \leqslant J(x)$ by (4.6), so

$$E(x) \leqslant S(-\infty) - S(\infty)$$

for all $x \in \mathbb{R}$.

To study I, define

$$H(x) := I(x) + \frac{\gamma}{c}\int_{-\infty}^{x} I(y)dy + \frac{\gamma}{c}\int_x^{\infty} \mathrm{e}^{\frac{c}{d_3}(x-y)}I(y)dy.$$

Using the similar methods as in studying (4.6), we obtain

$$\lim_{x\to-\infty} H(x) = 0, \quad \lim_{x\to\infty} H(x) = \frac{\gamma}{c}\int_{-\infty}^{\infty} I(y)dy = S(-\infty) - S(\infty).$$

Differentiating $H(x)$ we obtain

$$H'(x) = I'(x) + \frac{\gamma}{d_3}\int_x^{\infty} \mathrm{e}^{\frac{c}{d_3}(x-y)}I(y)dy,$$

and furthermore,

$$\lim_{x\to-\infty} H'(x) = 0, \quad \lim_{x\to\infty} H'(x) = 0.$$

Differentiate $H(x)$ twice, i.e.,

$$H''(x) = I''(x) - \frac{\gamma}{d_3}I(x) + \frac{c\gamma}{d_3^2}\int_x^{\infty} \mathrm{e}^{\frac{c}{d_3}(x-y)}I(y)dy,$$

and thus

$$-d_3H'' + cH' = -d_3I'' + cI' + \gamma I = \alpha E.$$

From the above equality, we have

$$H'(x) = \frac{1}{d_3}\int_x^{\infty} \mathrm{e}^{\frac{c}{d_3}(x-y)}E(y)dy.$$

Since $H(\infty) = S(-\infty) - S(\infty)$ and $H'(x) \geqslant 0$,

$$I(x) \leqslant H(x) \leqslant S(-\infty) - S(\infty)$$

holds for all $x \in \mathbb{R}$. This completes the proof.

5 Non-existence

Theorem 5.1. *If $\beta/\gamma \leqslant 1$, for any $c > 0$, there is no non-trivial non-negative traveling wave solution satisfying the boundary conditions* (2.1).

Proof. Recall (4.5), i.e.,

$$\int_{-\infty}^{\infty} I(x)dx = \frac{1}{\gamma}\int_{-\infty}^{\infty} \frac{\beta S(x)I(E)(x)}{S(x)+I(E)(x)+E(x)}dx.$$

If $\beta \leqslant \gamma$ and $I(x)$ is a non-trivial and non-negative function, then

$$\int_{-\infty}^{\infty} I(x)dx = \frac{\beta}{\gamma}\int_{-\infty}^{\infty} \frac{S(x)I(E)(x)}{S(x)+I(E)(x)+E(x)}dx < \frac{\beta}{\gamma}\int_{-\infty}^{\infty} I(x)dx \leqslant \int_{-\infty}^{\infty} I(x)dx.$$

We get a contradiction. □

Theorem 5.2. *If $\beta/\gamma > 1$ and $0 < c < c^*$, there is no non-negative and non-trivial traveling wave solution satisfying the boundary conditions* (2.1).

Proof. Let (S, E) be the solution of (2.2)–(2.3) satisfying the boundary conditions (2.1). Since

$$\frac{\beta S}{S+I+E} \to \beta, \quad \alpha E(x) \to \gamma I(x)$$

as $x \to -\infty$, there exists an $\bar{x} < 0$ such that for $x < \bar{x}$,

$$\beta SI/(S+I+E) - \alpha E > \delta E \geqslant 0,$$

where $\delta := \alpha(\beta-\gamma)/(2\gamma)$. Applying the above inequality to (2.3), we obtain

$$cE' - d_2E'' > \delta E \geqslant 0 \tag{5.1}$$

for $x < \bar{x}$. Since

$$E(\pm\infty) = 0, \quad E'(\pm\infty) = 0, \quad E''(\pm\infty) = 0, \tag{5.2}$$

we can see that $cE' - d_2E''$ is integrable at $-\infty$. Together with the Lebesgue dominated convergence theorem and (5.1), E is also integrable at $-\infty$. We denote

$$K(x) := \int_{-\infty}^{x} E(y)dy.$$

Integrating (5.1) yields

$$\delta K(x) \leqslant cE(x) - d_2E'(x)$$

for $x < \bar{x}$. Integrate the above inequality again and we obtain

$$\int_{-\infty}^{x} K(y)dy \leqslant \frac{c}{\delta}K(x)$$

for all $x < \bar{x}$. We also notice that $K(x)$ is non-decreasing, hence

$$\eta K(x-\eta) \leqslant \int_{x-\eta}^{x} K(y)dy \leqslant \frac{c}{\delta}K(x)$$

for all $\eta > 0$ and $x < \bar{x}$. By choosing $\eta > 2c/\delta$,

$$K(x-\eta) < K(x)/2$$

for all $x < \bar{x}$. Denote $\mu_0 := \min\{(\ln 2)/\eta,\ \lambda_1^+/2\}$ and

$$L(x) := K(x)\mathrm{e}^{-\mu_0 x}.$$

It holds that

$$L(x-\eta) < L(x)$$

for $x < \bar{x}$, which implies that $L(x)$ is bounded as $x \to -\infty$. (5.1) and (5.2) yield

$$cE' > d_2 E'', \quad cE > d_2 E', \quad cK > d_2 E.$$

Hence, we conclude that

$$E''(x)\mathrm{e}^{-\mu_0 x}, \quad E'(x)\mathrm{e}^{-\mu_0 x} \quad \text{and} \quad E(x)\mathrm{e}^{-\mu_0 x}$$

are all bounded as $x \to -\infty$. Together with (5.2), they are uniformly bounded on $\mathbb{R}$. By (3.5),

$$\lim_{x\to\pm\infty} E(x)/I(x) = \gamma/\alpha$$

and we can see that $I(x)\mathrm{e}^{-\mu_0 x}$ is also bounded on $\mathbb{R}$. Since $I/(S+I+E) \leqslant 1$ and $S(x)+I(x)+E(x) \to S_{-\infty}$ as $x \to -\infty$, both

$$\frac{I(x)\mathrm{e}^{-\mu x}}{S(x)+I(x)+E(x)}$$

and

$$\frac{[I^2(x)/E(x)]\mathrm{e}^{-\mu x}}{S(x)+I(x)+E(x)}$$

are uniformly bounded on $\mathbb{R}$ for $\mu \in (0, \mu_0]$. Since

$$\frac{\beta SI}{S+I+E} = \beta I - \frac{\beta I^2}{S+I+E} - \frac{\beta IE}{S+I+E},$$

together with (2.3) we obtain

$$-d_2 E'' + cE' + \alpha E - \beta I = -\frac{\beta I^2}{S+I+E} - \frac{\beta IE}{S+I+E}. \tag{5.3}$$

Moreover, it holds that

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{e}^{-\mu x} I(x)dx &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mu x}[I(E)](x)dx \\
&= \frac{\alpha}{\rho_1}\int_{-\infty}^{\infty} \mathrm{e}^{-\mu x}\bigg(\int_{-\infty}^{x} \mathrm{e}^{\lambda_1^-(x-y)}E(y)dy + \int_{x}^{\infty} \mathrm{e}^{\lambda_1^+(x-y)}E(y)dy\bigg)dx \\
&= \frac{\alpha}{\rho_1}\int_{-\infty}^{\infty} \mathrm{e}^{-\mu x}\bigg(\int_{0}^{\infty} \mathrm{e}^{\lambda_1^- y}E(x-y)dy - \int_{0}^{\infty} \mathrm{e}^{\lambda_1^+ y}E(x-y)dy\bigg)dx \\
&= \frac{\alpha}{\rho_1}\int_{-\infty}^{\infty} \mathrm{e}^{-\mu(x-y)}E(x-y)\bigg(\int_{0}^{\infty} \mathrm{e}^{(\lambda_1^- -\mu)y}dy - \int_{0}^{\infty} \mathrm{e}^{(\lambda_1^+ -\mu)y}dy\bigg)dx \\
&= \frac{\alpha}{-d_3\mu^2 + c\mu - \gamma}\int_{-\infty}^{\infty} \mathrm{e}^{-\mu x}E(x)dx.
\end{aligned}$$

We use the two-side Laplace transform on both sides of (5.3) and then obtain

$$f(\mu)\int_{-\infty}^{\infty} E(x)\mathrm{e}^{-\mu x}dx = -\beta\int_{-\infty}^{\infty}\bigg(\frac{I^2(x)/E(x)}{S(x)+I(x)+E(x)} + \frac{I(x)}{S(x)+I(x)+E(x)}\bigg)E(x)\mathrm{e}^{-\mu x}dx, \tag{5.4}$$

where

$$f(\mu) = -d_2\mu^2 + c\mu + \alpha - \frac{\alpha\beta}{-d_3\mu^2 + c\mu + \gamma}.$$

The integrals on both sides of the above equality are well-defined for any $\mu \in (0, \mu_0)$. By the assumption that $c < c^*$, $f(\mu)$ is always negative for all $\mu \in [0, \lambda_1^+)$. All the three integrals in (5.4) can be analytically continued to the interval $[0, \lambda_1^+)$. Otherwise, by the theory of convergence region of two-side Laplace transform (see [14, 16, 17]), the integral

$$\int_{-\infty}^{\infty} E(x)\mathrm{e}^{-\mu x}dx$$

has a singularity at $\mu = \mu^* \in (0, \lambda_1^+)$ and is analytic for all $\mu \in (0, \mu^*)$. At the same time, we check the integral

$$\int_{-\infty}^{\infty} \left(\frac{I^2(x)/E(x)}{S(x)+I(x)+E(x)} + \frac{I(x)}{S(x)+I(x)+E(x)} \right) E(x) e^{-\mu x} dx. \tag{5.5}$$

Notice

$$\frac{I(x)}{S(x)+I(x)+E(x)} e^{-\mu_1 x}$$

and

$$\frac{I^2(x)/E(x)}{S(x)+I(x)+E(x)} e^{-\mu_1 x}$$

are uniformly bounded for $\mu_1 = \min\{(\lambda_1^+ - \mu^*)/2, \mu_0\}$, so the integral (5.5) is analytic for all $\mu < \mu^* + \mu_1$. We get a contradiction.

We rewrite (5.4) as

$$\int_{-\infty}^{\infty} e^{-\mu x} E(x) \left[f(\mu) + \frac{\beta I^2(x)/E(x)}{S(x)+I(x)+E(x)} + \frac{\beta I(x)}{S(x)+I(x)+E(x)} \right] dx = 0.$$

This leads to a contradiction again in that

$$f(\mu) + \frac{\beta I^2(x)/E(x)}{S(x)+I(x)+E(x)} + \frac{\beta I(x)}{S(x)+I(x)+E(x)} \to -\infty$$

as $\mu \to \lambda_1^+ - 0$, while $e^{-\mu x} E(x) > 0$ for all $\mu \in (0, \lambda_1^+)$. Thus we conclude that if $R_0 > 1$ and $c < c^*$ there is no non-negative and non-trivial traveling wave solution satisfying the boundary conditions (2.1). □

6 Discussion

Due to the significant epidemic meaning of our diffusive SEIR model, the speed of spatial spread of epidemics is an important problem in mathematical epidemiology. Aronson and Weinberger [2] have proved the coincidence of the minimal wave speed and the asymptotic speed of propagation for the Fisher's equation, where $c_0 > 0$ is the asymptotic speed if for any $c > c_0$ the solution tends to zero uniformly in the spatial-time region $\{(x,t) : |x| \geqslant ct\}$, while for any $0 < c < c_0$ the solution is bounded and away from zero uniformly in the region $\{(x,t) : |x| \leqslant ct\}$ as $t \to \infty$. Aronson [1] also gave an analogous result for an SIR epidemic model with non-local reaction, which is called the Kendall model. This result implies that if you travel toward $+\infty$, then you escape from the epidemic region if your speed is larger than the minimal speed c^*, but if your speed is less than c^*, the infection eventually overtakes you.

Recently, the spreading speed of reaction-diffusion has been intensively studied by many researchers. Thieme and Zhao [12] generalized the concept of spreading speeds and monotone traveling waves to the non-linear integral equations and their results can be used to a large number of non-local reaction-diffusion population models. They showed the spreading speed coincides with the minimal wave speed. Despite these existing results, the relation between these two speeds of our diffusive SEIR model is still an open problem and it will be our successive work. Furthermore, as it is mentioned in [16], the non-locality of the reaction and time delay may increase or decrease the speed of traveling waves of a diffusive SIR model. We also get some analogous results for the corresponding diffusive SEIR model.

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