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Factorization of proper holomorphic maps on irreducible bounded symmetric domains of rank $\geqslant 2$

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract We obtain rigidity results on arbitrary proper holomorphic maps F from an irreducible bounded symmetric domain Ω of rank $\geqslant 2$ into any complex space Z. After lifting to the normalization of the subvariety $F(\Omega) \subset Z$, we prove that F must be the canonical projection map to the quotient space of Ω by a finite group of automorphisms. The approach is along the line of the works of Mok and Tsai by considering radial limits of bounded holomorphic functions derived from F and proving that proper holomorphic maps between bounded symmetric domains preserve certain totally geodesic subdomains. In contrast to the previous works, in general we have to deal with multivalent holomorphic maps for which Fatou's theorem cannot be applied directly. We bypass the difficulty by devising a limiting process for taking radial limits of correspondences arising from proper holomorphic maps and by elementary estimates allowing us to define distinct univalent branches of the underlying multivalent map on certain subsets. As a consequence of our rigidity result, with the exception of Type-IV domains, any proper holomorphic map $f : \Omega \to D$ of Ω onto a bounded convex domain D is necessarily a biholomorphism. In the exceptional case where Ω is a Type-IV domain, either f is a biholomorphism or it is a double cover branched over a totally geodesic submanifold which can be explicitly described.

Keywords bounded symmetric domain, proper holomorphic map, Fatou's theorem, correspondence, discriminant, G-structure

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1 Introduction

In Several Complex Variables the subject of proper holomorphic maps on bounded domains is well studied in the case of strictly pseudoconvex domains, especially in the case of complex unit balls B^n, which are precisely the bounded symmetric domains of rank 1. By contrast proper holomorphic maps on irreducible bounded symmetric domains Ω of rank $\geqslant 2$ are not well understood. Earlier works include the work of Henkin-Novikov [4] on proper holomorphic maps on classical domains. One of the authors introduced in a joint work with Tsai (Mok-Tsai [11]) a method for the study of proper holomorphic maps on irreducible bounded symmetric domains of rank $\geqslant 2$ by considering boundary values of the holomorphic map and associated bounded holomorphic functions defined on totally geodesic complex submanifolds which are biholomorphic to reducible bounded symmetric domains. By the Polydisk Theorem (Wolf [17]) the

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existence of such subspaces is a main feature which distinguishes the case of rank $\geqslant 2$ from the rank-1 case. Inspired by results on strong rigidity concerning harmonic maps and on Hermitian metric rigidity, Mok [5] formulated a conjecture on proper holomorphic maps from an irreducible bounded symmetric domain Ω of rank $r \geqslant 2$ into a bounded symmetric domain Ω' of rank $r' \leqslant r$, according to which r' must agree with r and such maps must necessarily be totally geodesic. This conjecture was resolved in the affirmative by Tsai [14] which exploited further the method of considering radial limits of bounded holomorphic functions in conjunction with methods from Kähler geometry. A new proof illustrated by the case of Type-I classical symmetric domains was recently devised by Mok [8] in which methods of Kähler geometry were replaced by methods concerning geometric structures for non-equidimensional holomorphic maps between uniruled projective manifolds.

Other results on proper holomorphic maps were obtained by Tu [15] in the equidimensional case and Tu [16] in the non-equidimensional case. In [15] it was proven that any surjective proper holomorphic map $F : \Omega \to \Omega'$ between bounded symmetric domains must necessarily be a biholomorphism provided that Ω is irreducible and of rank $\geqslant 2$. In [16] rigidity results were obtained for proper holomorphic maps $F : \Omega \to \Omega'$ for certain special pairs of irreducible bounded symmetric domain (Ω, Ω') in which $\mathrm{rank}(\Omega) \geqslant 2$ and $\mathrm{rank}(\Omega') - \mathrm{rank}(\Omega) = 1$, and nonexistence results of proper holomorphic maps were also established for certain pairs of (Ω, Ω') with the same gap of 1 between their ranks. In Mok [9] nonexistence results were further established for certain pairs (Ω, Ω') of irreducible bounded symmetric domains of rank $\geqslant 2$ with an arbitrarily large gap between their ranks. The general structure theory of proper holomorphic maps in the case of unequal rank and of rank $\geqslant 2$ remains unexplored.

In this article we consider rigidity results on proper holomorphic maps $F : \Omega \to Z$ for an irreducible bounded symmetric domain of rank $\geqslant 2$ into any complex space Z. Replacing Z by the normalization of the image $Y \subset Z$ of Ω under the proper map F we may assume that Z is a normal complex space and that F is surjective. In this case we prove that F is the canonical holomorphic map onto the quotient space $Z = \Omega/H$ of Ω by the action of a finite group of automorphisms of Ω. Thus, our general result is a factorization theorem for proper holomorphic maps of Ω onto any complex space, and we obtain rigidity results when additional conditions are imposed on the target space. When the target space is assumed to be smooth, for instance itself biholomorphic to a bounded domain, then excepting for a series of explicit examples we show that F is necessarily a biholomorphism. The series of exceptions are branched double covers of an irreducible bounded symmetric domain D_n^{IV} of Type-IV and of dimension $n \geqslant 3$, and the double covers are restrictions to open subsets of the standard branched double cover of the hyperquadric $\mathbf{Q}^n$ onto the projective space $\mathbf{P}^n$ ramified along a smooth hyperplane section of the hyperquadric. This result strengthens the equidimensional result of Tu [15] in which the target space is assumed to be also a bounded symmetric domain. Combined with the rigidity results of Mok-Tsai [11] on convex realizations of irreducible bounded symmetric domains of rank $\geqslant 2$, our factorization result also says that with the same exceptions of D_n^{IV}, $n \geqslant 3$, any proper holomorphic map from Ω onto a bounded convex domain is necessarily the Harish-Chandra realization up to an affine linear transformation on the target space.

As no conditions are imposed on the target complex space Z, the latter also plays no role in the proof, and our approach is to translate the study of the proper holomorphic map $F : \Omega \to Z$ to the problem of characterizing the proper holomorphic correspondence $S \subset \Omega \times \Omega$ defined by declaring $(x, y) \in S$ if and only if $F(x) = F(y)$. Whereas the approach of Mok-Tsai [11] and Tsai [14] is to consider boundary values of F and associated bounded holomorphic maps when restricted to totally geodesic complex submanifolds which are product domains, working with proper holomorphic correspondences on Ω our task is to devise a method for defining boundary values of holomorphic families of proper holomorphic correspondences obtained by restricting to fibers of product domains. Interpreting the correspondences as multivalent maps the principal difficulty of taking radial limits arises from the discriminant locus. In the case of *bona fide* holomorphic maps as in Tsai [14], the starting point is to prove that, taking radial boundary values, boundary components lying on certain product domains are transformed into boundary components. In the Harish-Chandra realization boundary components are domains on affine-linear subspaces, and to show that they are transformed under boundary maps to boundary components one is led to prove that the image lies on affine-linear subspaces, a property that can be checked by testing the linear dependence of sets of vectors obtained by taking derivatives in the fiber directions. This produces by Cauchy estimates bounded holomorphic functions, but in the presence of the discriminant locus the corresponding functions, even if well-defined, need not be bounded. By elementary estimates on the domain of univalence of holomorphic maps restricted to fibers of product domains we show that it is still possible to derive bounded holomorphic functions from the multivalent maps making it still possible to apply Fatou's Theorem on radial limits, and this allows us to show that boundary components are still transformed to boundary components under the "multivalent" boundary maps. From the latter statement we conclude a proof of the Main Theorem by applying results on analytic continuation concerning G-structures modeled on irreducible Hermitian symmetric manifolds of the compact type and of rank $\geqslant 2$ due to Ochiai [12], which allows us to recover a finite group of automorphisms from the intertwining maps of the proper holomorphic map $F:\Omega\to Z$ defined by switching connected components of inverse images of small open sets.

It is hoped that the general structural result obtained in the current article on proper holomorphic maps on irreducible bounded symmetric domains of rank $\geqslant 2$ can serve as a motivation for the further study of proper holomorphic maps between bounded symmetric domains of unequal rank.

2 Statement of results

In this article we prove a general result on the structure of proper holomorphic maps defined on irreducible bounded symmetric domains Ω of rank $\geqslant 2$, where the target space is an arbitrary complex space. Our principal result is given by

Main Theorem. *Let Ω be an irreducible bounded symmetric domain of rank $\geqslant 2$, Z be a complex space and $F:\Omega\to Z$ be a proper holomorphic map. Then, there exists a finite group H of automorphisms of Ω such that, denoting by $X=\Omega/H$ the quotient space equipped with the unique structure as a normal complex space with respect to which the canonical projection $\pi:\Omega\to X$ is holomorphic, we have the factorization $F=\nu\circ\pi$ of the proper holomorphic map $F:\Omega\to Z$, where $\nu:X\to Z$ is the normalization of its image $Y:=F(\Omega)$ (which is a complex-analytic subvariety of Z).*

When the target space is a complex manifold and the proper holomorphic mapping $F:\Omega\to Z$ is surjective, we deduce from the Main Theorem the following result.

Theorem 1. *Let Ω be an irreducible bounded symmetric domain of rank $\geqslant 2$, Z be a complex manifold, and $F:\Omega\to Z$ be a surjective proper holomorphic map. Then,*

(a) *If Ω is not isomorphic to a Type-IV classical symmetric domain D_n^{IV} of dimension $n\geqslant 3$, then $F:\Omega\to Z$ is necessarily a biholomorphic map.*

(b) *If $\Omega=D_n^{IV}$, $n\geqslant 3$, then either $F:D_n^{IV}\to Z$ is a biholomorphism, or F is a two-fold branched covering of Z ramified along a totally geodesic smooth complex hypersurface $J\subset D_n^{IV}$ (which is necessarily biholomorphic to D_{n-1}^{IV} and embedded in D_n^{IV} in the standard way).*

Combining Theorem 1 with the rigidity results of Mok-Tsai [11] on convex realizations of irreducible bounded symmetric domains of rank $\geqslant 2$ we have

Corollary 1. *Let Ω be an irreducible bounded symmetric domain of rank $\geqslant 2$ which is not isomorphic to a Type-IV classical symmetric domain D_n^{IV} of dimension $n\geqslant 3$. Let $F:\Omega\to D$ be a proper holomorphic map onto a bounded convex domain D. Then, F is a biholomorphism and D is up to an affine-linear transformation the Harish-Chandra realization of Ω.*

3 Background on characteristic subdomains of irreducible bounded symmetric domains

Let $\Omega\Subset\mathbb{C}^n\subset S$ be an irreducible bounded symmetric domain of rank $\geqslant 2$ in its Harish-Chandra realization, where $\Omega\subset S$ is the Borel embedding. Let $\alpha\in T_0(\Omega)$ be a non-zero highest weight vector of the isotropy representation of $K := \mathrm{Aut}_0(\Omega; 0)$ at 0, equivalently a minimal rational tangent of S as a uniruled projective manifold of Picard number 1. Thus, there exists a minimal rational curve C_α on S passing through 0 such that $T_0(C_\alpha) = \mathbb{C}\alpha$. (We refer the reader to Hwang-Mok [3] for the notion of minimal rational tangents, and to Mok [6] for minimal rational tangents on irreducible Hermitian symmetric manifolds of the compact type.) Denote by R the curvature tensor of (Ω, g) with respect to the Bergman metric on Ω. Denote by $\mathcal{N}_\alpha \subset T_0(\Omega)$ the null-space associated to α, i.e., the vector subspace consisting of all (1,0)-tangent vectors ζ at 0 such that $R_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0$. Then $\mathcal{N}_\alpha$ is orthogonal to α. Write $q = \dim(\mathcal{N}_\alpha)$. There is a q-dimensional totally geodesic complex submanifold $\Theta_\alpha \subset \Omega$ such that $T_0(\Theta_\alpha) = \mathcal{N}_\alpha$. $\Theta_\alpha \subset \Omega$ will be called a maximal characteristic subdomain on Ω passing through 0. A maximal characteristic subdomain on Ω is by definition given by $\gamma(\Theta_\alpha)$ for some Θ_α as in the above and for some $\gamma \in \mathrm{Aut}(\Omega)$. There is a $(q+1)$-dimensional totally geodesic complex submanifold $\Pi_\alpha \subset \Omega$ passing through $0 \in \Omega$ such that $T_0(\Pi_\alpha) = \mathbb{C}\alpha \oplus \mathcal{N}_\alpha$. Π_α is a domain on a complex vector subspace $V_\alpha \subset \mathbb{C}^n$. Moreover, there exists a complex linear isometry $\eta : V_\alpha \cong \mathbb{C}^{q+1}$ such that $\eta(\Pi_\alpha) = \Delta \times \Omega'$, and $\eta(\Theta_\alpha) = \{0\} \times \Omega'$ where $\Omega' \Subset \mathbb{C}^q$ is a bounded domain in its Harish-Chandra realization.

Replacing Ω by any Θ_α we obtain analogously maximal characteristic subdomains on Θ_α. Proceeding inductively we have obtained the characteristic subdomains $\Theta \subset \Omega$ passing through 0. Their images under automorphisms of Ω will be referred to as characteristic subdomains on Ω. Characteristic subdomains on Ω enjoy the remarkable property that given any $\gamma \in \mathrm{Aut}(\Omega)$, $\gamma(\Theta)$ is the intersection of Ω with an affine-linear subspace $A \subset \mathbb{C}^n$. The latter property defines the more general notion of invariantly affine-linear subdomains, which were introduced and completely classified in Tsai [14].

A minimal rational tangent on Ω will be referred to as a maximal characteristic vector. For a general reference on the role of such vectors in Hermitian metric rigidity cf. Mok [5, Chapter 6] (called characteristic vectors there) and [7, (2.1)]. For the notion of (maximal) characteristic subdomains of a bounded symmetric domain we refer the reader to Mok-Tsai [11], where the notion is introduced.

On an irreducible Hermitian symmetric manifold S of rank $\geqslant 2$ the varieties of minimal rational tangents define a geometric structure called a flat (an integrable) holomorphic S-structure. For such geometric structures we have the following result of Ochiai [12] which serves as a prototype of results on analytic continuation of local holomorphic maps preserving varieties of minimal rational tangents.

Theorem 2 (Ochiai [12]). *Let S be an irreducible compact Hermitian symmetric manifold of the compact type and of rank $\geqslant 2$; $U, V \subset S$ be connected open subsets, and $g : U \to V$ be a biholomorphism. Suppose for every $x \in U$, $dg(x)$ preserves characteristic vectors. Then, there exists an automorphism $\gamma \in \mathrm{Aut}(S)$ such that $\gamma|_U \equiv g$.*

The following lemma will be needed when we apply Theorem 2 to the study of proper holomorphic correspondences.

Lemma 1 (Mok-Ng [10]). *Let $\Omega \Subset \mathbb{C}^n \subset S$ be an irreducible bounded symmetric domain of rank $\geqslant 2$ in its Harish-Chandra realization, where furthermore $\Omega \subset S$ denotes the Borel embedding. Suppose b is a smooth point on $\partial\Omega$. Let $U_b \subset \mathbb{C}^n$ be an open neighborhood of b in $\mathbb{C}^n$ and $\gamma \in \mathrm{Aut}(S)$ such that $\gamma(U_b \cap \Omega) \subset \Omega$ and $\gamma(U_b \cap \partial\Omega) \subset \partial\Omega$. Then, $\gamma(\Omega) = \Omega$, i.e., $\gamma|_\Omega \in \mathrm{Aut}(\Omega)$.*

A much stronger statement is true. In fact, if $\gamma \in \mathrm{Aut}(S)$ is replaced by a holomorphic map $f : U_b \to \mathbb{C}^n$ such that $f(U_b \cap \Omega) \subset \Omega$ and $f(U_b \cap \partial\Omega) \subset \partial\Omega$, then we have proved in Mok-Ng [10, (3.1), Theorem 2] that f is the restriction of some $\gamma \in \mathrm{Aut}(\Omega)$ to U_b. The latter result is an Alexander-type extension result which is based on the method of taking radial limits of bounded holomorphic functions on totally geodesic complex submanifolds of Ω of the form $\Delta \times \Omega' \subset \Omega$, where $\Omega' \subset \Omega$ is a maximal characteristic subdomain on Ω, together with an application of Ochiai's Theorem on S-structures in the above. The special case of Lemma 1 is used in the final step of the proof of [10, (3.1), Theorem 2]. For a proof of Lemma 1 we refer the reader to [10, (3.3), last paragraph].

Regarding characteristic subdomains $\Theta \subset \Omega$ the following discussion in relation to invariantly affine-linear subdomains is helpful to streamline our discussion on boundary values of proper holomorphic correspondences on Ω. We briefly recall the basic notions that can be read from Mok-Tsai [11] and Tsai [14]. Let $\Omega \Subset \mathbb{C}^n \subset S$ denote simultaneously the Harish-Chandra realization $\Omega \Subset \mathbb{C}^n$ and the

Borel embedding $\Omega \subset S$ of an irreducible bounded symmetric domain Ω of arbitrary rank. Let g be the canonical Kähler-Einstein metric on Ω and g_c be the dual Kähler-Einstein metric on S. A complex submanifold $Q \subset S$ is said to be invariantly geodesic if and only if $(\gamma(Q), g_c|_{\gamma(Q)}) \hookrightarrow (S, g_c)$ is a totally geodesic submanifold for any $\gamma \in \mathrm{Aut}(S)$. Invariantly geodesic complex submanifolds on S can be determined in Lie-theoretic terms. For any such complex submanifold $Q \subset S$ either $Q \cap \mathbb{C}^n$ is empty or it is an affine-linear subspace of $\mathbb{C}^n$. In the latter case we call $W := Q \cap \mathbb{C}^n$ an invariantly geodesic affine-linear subspace. Furthermore, $Q \cap \Omega$ is either empty or it is a bounded domain on the affine-linear subspace W. In the latter case we call $\Gamma := Q \cap \Omega$ an invariantly affine-linear subdomain. In this case $(\Gamma, g|_\Gamma) \hookrightarrow (\Omega, g)$ is a totally geodesic submanifold. By [11] the characteristic subdomains $\Gamma \subset \Omega$ are examples of invariantly affine-linear subdomains. The set of invariantly geodesic complex submanifolds $Q \subset S$ are completely classified. They form a finite number of connected components in the Chow space $\mathrm{Chow}(S)$ of the projective manifold S. Each of these connected component is a projective manifold which is an $\mathrm{Aut}(S)$-orbit in $\mathrm{Chow}(S)$.

As an illustration consider $S = G(p, q)$, the Grassmannian of p-planes in $\mathbb{C}^{p+q}$. Then, up to automorphisms of $G(p, q)$, an invariantly geodesic complex submanifold $Q \subset G(p, q)$ is precisely of the form $G(r, s) \hookrightarrow G(p, q)$, where $1 \leqslant r \leqslant p$. $1 \leqslant s \leqslant q$, embedded by the standard embedding. The maximal characteristic subspaces in $G(p, q)$ are up to biholomorphisms of $G(p, q)$ given by $G(p-1, q-1) \hookrightarrow G(p, q)$. If $q = p + 1$, then there are precisely two connected components of invariantly geodesic complex submanifolds of dimension $p(p-1)$, given up to an automorphism of $G(p, q)$ by $G(p-1, p) \hookrightarrow G(p, p+1)$ and $G(p, p-1) \hookrightarrow G(p, p+1)$ embedded by standard embeddings. The first connected component corresponds to the maximal characteristic subspaces.

4 On boundary values of holomorphic correspondences

In this section we lay out the scheme for the proof of the Main Theorem. Recall that Ω is an irreducible bounded symmetric domain of rank $\geqslant 2$, and $F : \Omega \to Z$ is a proper holomorphic map into a complex space Z. For each point $z \in Z$, by properness the inverse image $F^{-1}(z) \subset \Omega$ is a compact complex-analytic subvariety, which is necessarily finite since Ω is a domain. By the Proper Mapping Theorem, $Y := F(\Omega) \subset Z$ is a complex-analytic subvariety. For the proof of the Main Theorem, replacing Z by $Y \subset Z$ if necessarily without loss of generality we may assume that the finite proper holomorphic map $F : \Omega \to Z$ is also surjective. Let $B \subset Z$ be union of the branching locus of F and the singular part of Z and let $R = F^{-1}(B)$. Then $F : \Omega - R \to Z - B$ is a finite unbranched holomorphic covering map. Suppose F is not generically one-to-one. Choose a point $a \in Z - B$, and $x, y \in \Omega - R$, $x \neq y$, such that $F(x) = F(y) = a$. By our choice, there exist neighborhoods U_x and U_y of x and y respectively such that $F_x : U_x \to F(U_x)$ and $F_y : U_y \to F(U_y)$ are biholomorphisms, where $F_x = F|_{U_x}$ and $F_y = F|_{U_y}$. We can define the intertwining map $\varphi_{x,y} : U_x \to U_y$ by $\varphi_{x,y}(z) = F_y^{-1}(F_x(z))$.

Our main goal is to show that the intertwining map $\varphi_{x,y}$ preserves the space of maximal characteristic vectors. We will adopt a similar approach as in the work of Mok-Tsai [11] and Tsai [14]. One of the main steps in these works is to consider radial limits of certain bounded holomorphic functions, which are the restrictions of a proper holomorphic map of Ω into its subdomains of the form $\Delta \times \Omega'$, where Ω' is a maximal characteristic subdomain of Ω. In our situation, the intertwining map $\varphi_{x,y}$ is only locally defined and its extension is a "multivalent" map in general and hence we have difficulty in taking the radial limit of it. To bypass this difficulty, we consider the underlying correspondences, and devise a method for taking radial limits of such correspondences. In so doing, there is the difficulty caused by the discriminant loci associated to the correspondences, and the holomorphic functions arising from considering boundary behavior of such correspondences may no longer be bounded. We solve the problem nonetheless by an estimate of such functions which allows us to derive from them bounded holomorphic functions, making it possible still to apply Fatou's Theorem.

First of all we consider the correspondence $S \subset \Omega \times \Omega$ associated to the map $F : \Omega \to Z$ defined by $(x, y) \in S \Longleftrightarrow F(x) = F(y)$. S is an analytic subvariety of $\Omega \times \Omega$ and it is symmetric in the sense that it is invariant by the transformation of $\Omega \times \Omega$ given by $(x, y) \mapsto (y, x)$. For $i = 1, 2$ the restriction on S of the projection map onto the ith direct factor $\rho_i : \Omega \times \Omega \to \Omega$ is finite as f is finite. We will still use ρ_i to denote the restriction, and write ρ for ρ_1. Let p be the number of elements in $\rho^{-1}(x)$ for a general point $x \in \Omega$. Note that S can be considered as the graph of the multivalent extension of $\varphi_{x,y}$ over Ω. From now on we will forget about our initial choice of (x, y) and view S as a graph of a multivalent map f from Ω into Ω. There is a subset $\mathcal{R} \subset S \subset \Omega \times \Omega$ consisting of points where either ρ_1 or ρ_2 fails to be a local biholomorphism. We call $\mathcal{R} \subset S$ the ramification locus of the correspondence S, and its image $\mathcal{B} = \rho(\mathcal{R}) \subset \Omega$ will be called the branching locus of S. (In Section 3, following standard conventions the symbol S was used as a generic symbol for an irreducible Hermitian symmetric space of the compact type and of rank $\geqslant 2$, e.g., in the term 'S-structure'. From now on the symbol S will stand for the holomorphic correspondence as defined. For the irreducible bounded symmetric domain Ω of rank $\geqslant 2$ we will use $\Omega \subset M$ to denote the Borel embedding of Ω into its dual compact manifold M.)

Taking the correspondence $S \subset \Omega \times \Omega$ as the graph of a multivalent holomorphic map we will define boundary values of the correspondence S and formulate a reduction of the proof of the Main Theorem. The proof of the Main Theorem will be given in Section 5.

Let Ω' be a maximal characteristic symmetric subspace of Ω (c.f. [11, 14]). We have $\mathrm{rank}(\Omega') = \mathrm{rank}(\Omega) - 1 = r - 1$. There are totally geodesic subdomains of Ω biholomorphic to $\Delta \times \Omega'$ and Ω is a union of such subdomains. Fix a subdomain $\Delta \times \Omega'$ containing $0 \in \Omega$ and for each $\{x\} \times \Omega' \subset \Delta \times \Omega'$, we define $S_x = S \cap ((\{x\} \times \Omega') \times \Omega)$. In other words, S_x is the restriction of the correspondence to the maximal symmetric subdomain $\{x\} \times \Omega'$. Following the line of thought in [11, 14], we would like to consider the radial limit of S_x towards the boundary of Δ. That is, to obtain a limiting correspondence S_b defined on $(\{b\} \times \Omega') \times \mathbb{C}^n$ when x tends to $b \in \partial\Delta$ radially.

For a generic point $z \in \Omega$, we can find p branches of f over a neighborhood U_z of z. We may take $z = 0$ and denote the branches by $f^{(\alpha)} : U_0 \to \Omega$, $1 \leqslant \alpha \leqslant p$, where U_0 is an open neighborhood of $0 \in \Omega$. Denote also the component functions of each $f^{(\alpha)}$ by $f_j^{(\alpha)}$, $1 \leqslant j \leqslant n$. For the purpose of proving the Main Theorem we want to study boundary values of the proper holomorphic correspondence when restricted to fibers $\{x\} \times \Omega'$ of $\Delta \times \Omega'$ as x converges radially to a boundary point $b \in \partial\Delta$. For this purpose we are free to perform linear transformations on the target domain Ω. We will now choose Euclidean coordinates $(\zeta_1, \ldots, \zeta_n)$ of the target domain Ω such that, writing $f = (f_1, \ldots, f_\ell)$, more precisely $f^{(\alpha)} = (f_1^{(\alpha)}, \ldots, f_n^{(\alpha)})$, for $1 \leqslant \ell \leqslant n$ and for a general point $z \in \Omega - \mathcal{B}$, the complex numbers $f_\ell^{(1)}(z), \ldots, f_\ell^{(p)}(z)$ are distinct. Thus, we are in general using different coordinates for the two factors of $\Omega \times \Omega$. This choice of coordinates is for convenience only and not absolutely necessary in the argument. It is chosen so that the discriminant function $\mathfrak{D}_\ell$ for each of the multivalent holomorphic function f_ℓ, $1 \leqslant \ell \leqslant n$, defined by

$$\mathfrak{D}_\ell = \prod_{\alpha \neq \beta} (f_\ell^{(\alpha)} - f_\ell^{(\beta)})$$

does not vanish identically on Ω. Define $\mathfrak{D} := \mathfrak{D}_1 \cdots \mathfrak{D}_n$. We may assume that $\mathfrak{D} \neq 0$ on the neighborhood U_0 of $0 \in \Omega$.

To study the radial limits of the correspondence $S \subset \Omega \times \Omega$ we first "enlarge" S to a bigger correspondence so as to facilitate the limiting process. Fix a component index l, we can form the elementary symmetric polynomials σ_ℓ^k, $1 \leqslant k \leqslant p$, so that $\{f_\ell^{(\alpha)} : 1 \leqslant \alpha \leqslant n\}$ are the solutions of the equation

$$\xi^p - \sigma_\ell^1 \xi^{p-1} + \sigma_\ell^2 \xi^{p-2} - \cdots + (-1)^p \sigma_\ell^p = 0.$$

Note that the elementary symmetric polynomials are globally defined holomorphic functions on Ω. Now we define a correspondence $\widetilde{S} \subset \Omega \times \mathbb{C}^n$ by

$$(z, \xi) \in \widetilde{S} \Longleftrightarrow \xi_\ell^p - \sigma_\ell^1(z) \xi_\ell^{p-1} + \sigma_\ell^2(z) \xi_\ell^{p-2} - \cdots + (-1)^p \sigma_\ell^p(z) = 0,$$

where $\xi = (\xi_1, \ldots, \xi_n)$. By our construction, we obviously have $S \subset \widetilde{S}$. Fixed a maximal symmetric subdomain $\{x\} \times \Omega' \subset \Delta \times \Omega'$, we similarly define $\widetilde{S}_x = \widetilde{S} \cap ((\{x\} \times \Omega') \times \mathbb{C}^n)$. If we write the coordinates in $(\{x\} \times \Omega') \times \mathbb{C}^n$ as $((x, w), \xi)$, then $\widetilde{S}_x$ is the correspondence in $(\{x\} \times \Omega') \times \mathbb{C}^n$ defined by the equations

$$\xi_\ell^p - \sigma_\ell^1(x, w) \xi_\ell^{p-1} + \sigma_\ell^2(x, w) \xi_\ell^{p-2} - \cdots + (-1)^p \sigma_\ell^p(x, w) = 0,$$

where $\xi = (\xi_1, \ldots, \xi_n)$.

Regarding bounded holomorphic functions on the unit disk we will use the following version of Fatou's Theorem (cf. Rudin [13, Theorem 11.32, Theorem 17.18]).

Lemma 2. *To every bounded holomorphic function h defined on the unit disk Δ, there corresponds a function $h^\star \in L^\infty(\partial\Delta)$, defined almost everywhere by $h^\star(e^{i\theta}) = \lim_{r\to 1} h(re^{i\theta})$. The Cauchy Integral Formula is valid for h in terms of the radial limits $h^\star$. In other words, for $z \in \Delta$, we have*

$$h(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{h^\star(\zeta)d\zeta}{\zeta - z}\,.$$

Moreover, if h is not identically zero on Δ, we have $h^\star(\zeta) \neq 0$ for almost all $\zeta \in \partial\Delta$.

An application of Fatou's Theorem on the disk gives the following result for a product domain $\Delta \times W$ (cf. Mok-Tsai [11, Proposition 2.2]).

Lemma 3. *Suppose that f is a bounded holomorphic function defined on the space $\Delta \times W$, and W is a bounded domain in $\mathbb{C}^k$. Write (z, w), $w = (w_1, \ldots, w_k)$, for the holomorphic coordinates of $\Delta \times W$. Define $f_r(e^{i\theta}, w) = f(re^{i\theta}, w)$ for $r < 1$. Then there exists a measurable function $f^\star(e^{i\theta}, w)$ which is defined on $\partial\Delta \times W$ and satisfies*

$$\lim_{r\to 1} f_r(e^{i\theta}, w) = f^\star(e^{i\theta}, w)$$

for almost everywhere on $\partial\Delta \times W$. As a consequence, $\lim_{r\to 1} f_r(e^{i\theta}, \cdot) = f^\star(e^{i\theta}, \cdot)$ in $L^1(W)$ for almost all θ. Moreover, for almost all θ, the function $f^\star(e^{i\theta}, w)$ is holomorphic in w.

We can now define the radial limit of $\widetilde{S}_x$ by applying the above proposition on the elementary symmetric functions $\sigma^k_\ell(x, w)$, where $(x, w) \in \Delta \times \Omega'$. Write $\sigma^{k\star}_\ell(b, w)$ as the limit functions given by Lemma 3, where $(b, w) \in \partial\Delta \times \Omega'$. For almost all b, $\sigma^{k\star}_\ell(b, w)$ are holomorphic in w. We define the correspondence $\widetilde{S}_b \subset (\{b\} \times \Omega') \times \mathbb{C}^n$ by the equations

$$\xi^p_\ell - \sigma^{1\star}_\ell(b, w)\xi^{p-1}_\ell + \sigma^{2\star}_\ell(b, w)\xi^{p-2}_\ell - \cdots + (-1)^p\sigma^{p\star}_\ell(b, w) = 0,$$

where $((b, w), \xi) \in (\{b\} \times \Omega') \times \mathbb{C}^n$ and $\xi = (\xi_1, \ldots, \xi_n)$. Obviously $\widetilde{S}_b$ lies in the closure of $\widetilde{S}$.

Since $\mathfrak{D} = \mathfrak{D}_1 \cdots \mathfrak{D}_n$ is non-zero on the neighborhood $U_0 \subset \Omega$ of 0, restricting from Ω to $\Delta \times \Omega'$ and applying Lemmas 2 and 3 from Fatou's Theorem, the radial limits $\mathfrak{D}^\star_{\ell,b}(w) := \mathfrak{D}^\star_\ell(b, w)$ as bounded holomorphic functions on Ω' are defined and non-zero for almost every $b \in \partial\Delta$. We now recall the following two well-known facts for solving polynomial equations:

Lemma A. *Let $\mathcal{U} \subset \mathbb{C}^m$ be a domain and $c_1(z), \ldots, c_p(z)$ be holomorphic functions on $\mathcal{U}$. Let X be an indeterminate and consider the equation $X^p + X^{p-1}c_1(z) + X^{p-2}c_2(z) + \cdots + c_p(z) = 0$. Suppose that the discriminant $D(z)$ of this equation is non-vanishing on $\mathcal{U}$. Then there exist p distinct holomorphic functions $X_\alpha(z)$, $1 \leqslant \alpha \leqslant p$ defined on $\mathcal{V} \subset \mathcal{U}$, $\mathcal{V} \simeq \Delta^m$, such that $X^p_\alpha(z) + X^{p-1}_\alpha(z)c_1(z) + X^{p-2}_\alpha(z)c_2(z) + \cdots + c_p(z) \equiv 0$ on $\mathcal{V}$.*

Lemma B. *Let $X^p + X^{p-1}c_1 + X^{p-2}c_2 + \cdots + c_p = 0$ be a polynomial equation in X. Let $(c_1(t), \ldots, c_p(t))$, $0 \leqslant t \leqslant 1$, be a continuous curve in $\mathbb{C}^p$. Suppose there exist p continuous functions $X_1(t), \ldots, X_p(t)$ defined on $[0, 1) \subset \mathbb{R}$ such that $\{X_1(t), \ldots, X_p(t)\}$ are the p roots of the equation $X^p + X^{p-1}c_1(t) + X^{p-2}c_2(t) + \cdots + c_p(t) = 0$ for every $t \in [0, 1)$ and $\{x_1, \ldots, x_p\}$ are the p roots of $X^p + X^{p-1}c_1(1) + X^{p-2}c_2(1) + \cdots + c_p(1) = 0$. Then after a permutation of indices if necessary, $(X_1(t), \ldots, X_p(t))$ converges to $(x_1, \ldots, x_p)$ in the Euclidean topology of $\mathbb{C}^p$ as $t \to 1$.*

At almost every $b \in \partial\Delta$, we can find a small connected neighborhood V_0 of $0 \in \Omega'$ such that the radial limit of every elementary symmetric function $\sigma^k_\ell(x, w)$ and the radial limit of every $\mathfrak{D}_\ell(x, w)$ exist for all $w \in V_0$ and $\mathfrak{D}^\star_\ell(b, w) \neq 0$. By Lemma A, by replacing V_0 by a smaller polydisk containing the origin, we have that $\widetilde{S}_b \cap ((\{b\} \times V_0) \times \mathbb{C}^n) = \bigcup_{i=1}^s W_i$, $s = p^n$, where W_i are disjoint and they are graphs of holomorphic maps from $\{b\} \times V_0$ into $\mathbb{C}^n$. Let $0 < \epsilon < 1$, define $I_\epsilon = \{rb : r \in (1 - \epsilon, 1)\}$. By Lemma A again, for a sufficiently small $\epsilon > 0$, we have similarly that $\widetilde{S} \cap ((I_\epsilon \times V_0) \times \mathbb{C}^n) = \bigcup_{i=1}^s W_{\epsilon;i}$, where $W_{\epsilon;i}$ are disjoint and for every $i \in \{1, \ldots, s\}$ and every $r \in (1 - \epsilon, 1)$, $W_{\epsilon;i} \cap ((\{rb\} \times V_0) \times \mathbb{C}^n)$ is the graph of some holomorphic map from $\{rb\} \times V_0$ into $\mathbb{C}^n$. Furthermore, by Lemma B, W_i lies in the closure of $W_{\epsilon;i}$ for every $i \in \{1, \ldots, s\}$ after a permutation of indices. Among the branches $W_{\epsilon;i}$, $1 \leqslant i \leqslant s$, we may assume without loss of generality that $W_{\epsilon;i} \subset S$, for $1 \leqslant i \leqslant p \leqslant s$. Now we define $S_b \subset \widetilde{S}_b$ to be the union of the irreducible components of $\widetilde{S}_b$ which contains at least one of the W_i, where $1 \leqslant i \leqslant p$. We have the following description for S_b:

Proposition 1. *$S_b \subset (\{b\} \times \Omega') \times \partial\Omega$, i.e., S_b is a correspondence from $\{b\} \times \Omega'$ to $\partial\Omega$.*

Here and in what follows, given complex manifolds X and Y, by abuse of language, by a correspondence from X and Y we will mean a subvariety $Q \subset X \times Y$ of dimension equal to $\dim(X)$ such that the canonical projection from Q into X is proper and finite. (By the Proper Mapping Theorem the canonical projection is also surjective.) If $E \subset Y$ is a subset such that $Q \subset X \times E$, we also speak of Q as a correspondence from X to E even though E may not carry any natural complex structure. (For our extended use of the term 'correspondence', the transpose $Q^T \subset Y \times X$ of a correspondence (i.e., $(y, x) \in Q^T$ if and only if $(x, y) \in Q$) is not necessarily a correspondence, and hence we generally speak of a correspondence from X to Y rather than a correspondence between X and Y.)

Proof of Proposition 1. Take an irreducible component V of S_b. We know that V contains some W_i, where $1 \leqslant i \leqslant p$. By our definition, W_i lies in the closure of S. Take a point $(x, y) \in W_i \subset \partial\Omega \times \mathbb{C}^n$ and let $\{(x_n, y_n) \in S \subset \Omega \times \Omega,\ n \in \mathbb{N}\}$ be a sequence of points converging to (x, y). We have $F(x_n) = F(y_n)$ by the definition of S. Since $F : \Omega \to Z$ is proper, $x_n \to x \in \partial\Omega$ implies that $F(x_n) = F(y_n)$ escapes to infinity (this means every compact subset of Z contains only a finite number of $F(x_n)$) when $n \to \infty$. Therefore, being the limit point of y_n, y must lie on $\partial\Omega$, hence $(x, y) \in \partial\Omega \times \partial\Omega$ and $W_i \subset \partial\Omega \times \partial\Omega$. Note that $\partial\Omega$ is defined by $\varphi = 0$, where φ is a real-analytic function on $\mathbb{C}^n$. As $W_i \subset \partial\Omega \times \partial\Omega$, we have $\varphi(x) = \varphi(y) = 0$ if $(x, y) \in W_i$. V contains W_i as an open subset and hence we also have $\varphi(x) = \varphi(y) = 0$ if $(x, y) \in V$ because φ is real-analytic. We finally conclude that $\varphi(x) = \varphi(y) = 0$ if $(x, y) \in S_b$ as V is arbitrary and the proof of the proposition is complete. $\square$

By Proposition 1, in the subdomain $\Delta \times \Omega'$, for almost every $b = e^{i\theta} \in \partial\Delta$, there exists an open subset $U \subset \Omega'$ and $\epsilon > 0$ such that any branch of $f = f^{(\alpha)}$ when restricted to $\{re^{i\theta}\} \times U$, $1 - \epsilon < r < 1$, converges radially to $f^{(\alpha)\star}(b, w)$. Dropping the index (α) for specifying the branch and writing also $f_b^\star(w) = f^\star(b, w)$, we have $f_b^\star(U) = f^\star(\{b\} \times U) \subset \partial\Omega$. For almost every $b \in \partial\Delta$, $f_b^\star(U)$ is an open set contained in some maximal face Γ_b of Ω. Thus, $S_b \subset (\{b\} \times \Omega') \times \Gamma_b$. Let $\dim(\Omega') = q$ which is also the dimension of any maximal face of Ω.

We are going to prove

Proposition 2. *For every $x \in \Delta$, S_x is a correspondence between $\{x\} \times \Omega'$ and $\mathcal{L}_x$, where $\mathcal{L}_x$ is a maximal characteristic subdomain of Ω, i.e. $S_x \subset (\{x\} \times \Omega') \times \mathcal{L}_x$.*

To pinpoint the difficulty in proving Proposition 2, we consider an analogous situation in which

(a) $S_x \subset (\{x\} \times \Omega') \times \Omega$ is in fact the graph of $f|_{\{x\}\times\Omega'}$, where $f : \Delta \times \Omega' \to \Omega$ is a (univalent) holomorphic map,

(b) for almost every $b \in \partial\Delta$ the boundary maps $f^\star\big|_{\{b\}\times\Omega'} : (\{b\} \times \Omega') \to \mathbb{C}^n$ obtained by taking radial limits is defined, $f^\star(b, w) := \lim_{t\to 1^-} f(rb, w)$, and $f^\star(\{b\} \times \Omega') \subset \partial\Omega$.

(c) for almost every $b \in \partial\Delta$, $f_b^\star(w) := f^\star(b, w)$ is of maximal rank at $w = 0$.

The situation here is a special case of what has been dealt with in Mok-Tsai [11]. Let $\mathcal{G}$ be the Grassmannian of all q-dimensional affine-linear subspace of $\mathbb{C}^n$. For the situation described (a)–(c) in the above the conclusion is that f induces a meromorphic map $f^\sharp : \Delta \to \mathcal{G}$ (cf. Mok-Tsai [11, Section 2, Proposition 2.3]). Equivalently, this means that for all but at most a discrete set of base points $x \in \Delta$, $f\big|_{\{x\}\times\Omega'}$ maps $\{x\} \times \Omega'$ into the affine-linear subspace $f^\sharp(x) \in \mathcal{G}$, of dimension $q = \dim(\Omega')$, such that $f_x(w) := f(x, w)$ is of rank equal to q at some point. Let $\gamma \in \mathrm{Aut}(\Omega)$. Then, exactly the same argument can be applied to the holomorphic map $\gamma \circ f : \Omega \to \Omega$. We conclude that for each $x \in \Delta$, $f^\sharp(x) \in \mathcal{G}$ is a q-dimensional affine-linear space V_x enjoying the special property that $\gamma(V_x)$ is affine-linear for every $\gamma \in \mathrm{Aut}(\Omega) \hookrightarrow \mathrm{Aut}(M)$. Let $\mathcal{H}_0 \subset \mathcal{G}$ denote the complex submanifold consisting of affine-linear subspaces W such that $W \cap \Omega$ is a maximal characteristic subdomain. Let $\Theta \subset \Omega$ be a maximal characteristic subdomain passing through $0 \in \Omega$ and write $\Theta \subset Q$ for its Borel embedding, so that $Q \subset M$ is a maximal characteristic subspace. Denote by $\mathcal{H}$ the orbit of $[Q]$ in $\mathrm{Chow}(M)$ under the action of $\mathrm{Aut}(M)$. Then $\mathcal{H}$ is a projective manifold, and it is a connected component of $\mathrm{Chow}(M)$. By associating $[\Theta] \in \mathcal{H}_0$ to the point $[Q] \in \mathrm{Chow}(M)$ we may identify $\mathcal{H}_0$ as an open subset of $\mathcal{H} \subset \mathrm{Chow}(M)$.

We have seen that $(\gamma \circ f)^\sharp(x) = \gamma(f^\sharp(x)) \in \mathcal{G}$ for any $\gamma \in \mathrm{Aut}(\Omega)$, so that $f^\sharp(x)$ is an invariantly affine-linear subdomain. Equivalently, we may identify $f^\sharp(x)$ as a point in $\mathrm{Chow}(M)$ corresponding to an invariantly geodesic complex submanifold of M. For almost every point $b \in \partial\Delta$, by Lemma 3 the radial limit $f_b^\star : \{b\} \times \Omega' \to \partial\Omega$ is defined, of maximal rank (equal to q) at some point, and $f_b^\star(\{b\} \times \Omega')$ is contained in a boundary component $(f^\sharp)^\star(b) := \Theta_b \in \mathcal{H}$. Thus, for almost every point $b \in \partial\Delta$, the points $f^\sharp(rb)$ converges in the natural metric topology of $\mathcal{G}$ to $(f^\sharp)^\star(b)$ as r increases to 1. Since $\mathcal{H}$ is one of the (finitely many) connected components of $\mathrm{Chow}(M)$ consisting of invariantly geodesic complex submanifolds, we conclude that $f^\sharp(x) \in \mathcal{H}$ for $x = rb$ and $r < 1$ sufficiently close to 1. It follows therefore that $f^\sharp(x) \in \mathcal{H}$ for *every* point $x \in \Delta$.

As we have seen, in order to prove that $f(\{x\} \times \Omega') \subset \mathcal{L}_x$ for some characteristic subdomain $\mathcal{L}_x$, it is sufficient to show that $f(\{x\} \times \Omega')$ is contained in some q-dimensional affine-linear subspace. This reduction remains applicable when we deal with the general situation of Proposition 2 in which we have a proper holomorphic correspondence $S \subset \Omega \times \Omega$ which gives a multivalent holomorphic map f, say with branches $f^{(1)}, \ldots, f^{(p)}$ in place of a (univalent) holomorphic map f as described in the above. At almost every boundary point $b \in \partial\Delta$ and a general point $w \in \Omega'$ we still have for each branch the convergence $f^{(\alpha)}(x, w) \to f^{(\alpha)\star}(b, w)$ as x converges radially to b. In order to prove Proposition 2, our task is reduced to proving that, for $x = rb$ with $r < 1$ sufficiently close to 1 and for an open neighborhood V_w of w on Ω' such that the branch $f^{(\alpha)}$ is defined on $\{x\} \times V_w$, the image $f^{(\alpha)}(\{x\} \times V_w)$ is contained in a q-dimensional affine-linear subspace.

Finally, for the purpose of making the arguments adaptable to give a proof of Proposition 2, we recall and slightly reformulate, in the case of a univalent holomorphic map $f : \Delta \times \Omega' \to \Omega$ as in the above, how one proves that $f^\sharp(x) \in \mathcal{G}$ except for possibly a discrete set of points x_i. From the assumption (b), for almost every $b \in \partial\Delta$, we have $f_b^\star(\Omega') \subset \Gamma_b$ where Γ_b is a maximal face on $\partial\Omega$. Γ_b is of dimension $q := \dim(\Omega')$, and for almost every b, by assumption (c) the set $f_b^\star(\Omega')$ contains a non-empty open subset of Γ_b. Given $x \in \Delta$ the statement that $f^\sharp(x) \in \mathcal{G}$ can be translated as follows.

(†) Let $I = (i_1, \ldots, i_q)$ be a non-zero q-tuple of nonnegative integers such that $I \neq 0$. Writing $|I| := i_1 + \cdots + i_q$, consider the partial derivative

$$\frac{\partial^{|I|} f}{\partial w_1^{i_1} \cdots \partial w_q^{i_q}}(x; 0) := \eta_I(x) \in \mathbb{C}^n.$$

Let V_x be the linear span of all $\eta_I(x)$ as I varies over the set of non-zero q-tuples I of nonnegative integers. Then, V_x is a q-dimensional vector space. Moreover $\frac{\partial f}{\partial w_1}(x; 0), \ldots, \frac{\partial f}{\partial w_q}(x; 0)$ are linearly independent and they span V_x.

Verification of condition (†) can further be implemented as follows. For a positive integer $s \leqslant n$, let $\mathbf{I} = (I_1, \ldots, I_s)$ be an s-tuple of distinct non-zero multi-indexes I_k of nonnegative integers, $1 \leqslant k \leqslant s$, $I_k = (i_1(k), \ldots, i_q(k))$. Suppose $J = (j_1, \ldots, j_s)$ is an s-tuple of distinct integers j_ℓ, where $1 \leqslant \ell \leqslant s$, $1 \leqslant j_\ell \leqslant n$. Write $\mathbf{A} = (\mathbf{I}, J)$. We say that $\mathbf{A}$ is an index of order s. To each index $\mathbf{A}$ of order s we can associate an s-tuple of column s-vectors, such that the k-th column vector is given by the transpose of $(\eta_{I_k}^{j_1}(x), \ldots, \eta_{I_k}^{j_s}(x))$. (Here we write a vector ξ in the target Euclidean space $\mathbb{C}^s$ as $\xi = (\xi^1, \ldots, \xi^s)$.) The s-by-s matrix thus obtained will be denoted by $\mathfrak{M}_{\mathbf{A}}(x)$ and its determinant will be written as $h_{\mathbf{A}}(x)$. By Cauchy estimates on derivatives with respect to the variables $w_1, \ldots, w_q$, we see that $h_{\mathbf{A}}(x)$ is a bounded holomorphic function in x. Here in the application of Cauchy estimates on derivatives, it suffices to note the trivial fact that there exists some $\rho > 0$ such that for $x \in \Delta$, $f_x(w) := f(x, w)$ is holomorphic on a fixed polydisk of polyradii $(\rho, \ldots, \rho)$ centred at $0 \in \Omega'$ and each component f_x^ℓ of $f_x := (f_x^1, \ldots, f_x^n)$ is uniformly bounded by a constant C independent of $x \in \Delta$. For $E_k = (0, \ldots, 1, 0, \ldots, 0)$ with 1 in the k-th position, by assumption (c), for almost every $b \in \partial\Delta$, for $\mathbf{I} = (E_1, \ldots, E_q)$ and for some $J = (j_1, \ldots, j_q)$ where $j_1, \ldots, j_q$ are distinct positive integers, $1 \leqslant j_1, \ldots, j_s \leqslant n$, we have $h_{\mathbf{A}}^\star(b) \neq 0$ for $\mathbf{A} = (\mathbf{I}, J)$, where $h^{\star}_{\mathbf{A}}(b)$ denotes the radial limit at b. Since $h_{\mathbf{A}}(rb)$ converges to $h^{\star}_{\mathbf{A}}(b)$ as $r \to 1^-$, we must have $h_{\mathbf{A}}(rb) \neq 0$ for $r < 1$ sufficiently close to 1. Thus, by using just one choice of J it is enough to show that $f_x(w) = f(x, w)$ is of maximal rank at $w = 0$ for all $x \in \Delta$ except possibly for a discrete set of points x_i. To verify (†) it remains therefore to show that $h_{\mathbf{A}}(x) = \det(\mathfrak{M}_{\mathbf{A}}(x))$ vanishes for any $\mathbf{A} = (\mathbf{I}, J)$ of order $q+1$. Let now $\mathbf{A}$ be of order $q+1$. By assumption $h^{\star}_{\mathbf{A}}(b) = 0$ for almost every $b \in \partial\Delta$. By Lemma 2, the bounded holomorphic function $h_{\mathbf{A}}(x)$ can be recovered from its boundary values $h^{\star}_{\mathbf{A}}(b)$ by the Cauchy Integral formula, from which it follows that the (bounded) holomorphic function $h_{\mathbf{A}}$ is identically zero on Δ, and the verification of (†), and hence a proof of the analogue of Proposition 2 in the case of univalent holomorphic maps is complete.

5 Proof of the Main Theorem

To adapt the arguments of Mok-Tsai [11] and Tsai [14] to the current situation, the difficulty arises from the fact that in place of holomorphic maps f we have now multivalent 'holomorphic maps'. At a general point $z \in \Omega$, the multivalent map f consists of p distinct branches $f^{(\alpha)}$, $1 \leqslant \alpha \leqslant p$. Here each $f^{(\alpha)}$ is more precisely a germ of holomorphic map, and may be taken as being defined on the same open neighborhood U_z of z. Because of the presence of the branching locus $\mathcal{B}$ of the correspondence S, owing to monodromy around $\mathcal{B}$ it is in general not possible to fix the indexing of the branches. For the purpose of proving Proposition 2 we need to show that, outside of the branching locus $\mathcal{B}$ of the correspondence $S \subset \Omega \times \Omega$, an open subset of a maximal characteristic subdomain $\Theta \subset \Omega$, of dimension q, is mapped under each branch of the multivalent holomorphic map f into a q-dimensional affine-linear subspace of $\mathbb{C}^n$ (and hence into a maximal characteristic subdomain). Recall that $\mathfrak{D} = \mathfrak{D}_1 \cdots \mathfrak{D}_n$ is the product of the discriminants $\mathfrak{D}_\ell$, of the multivalent holomorphic function f_ℓ. Here the branches are $f^{(\alpha)} = (f_1^{(\alpha)}, \ldots, f_n^{(\alpha)})$, and Euclidean coordinates have been chosen for the target bounded symmetric domain Ω to guarantee that each discriminant $\mathfrak{D}_\ell$ is not identically 0 on Ω. The discriminant functions $\mathfrak{D}_\ell$ are defined on Ω but we will only be considering their restriction to a generic product domain $\Pi \subset \Omega$, which is isomorphic to $\Delta \times \Omega' \subset \Omega$ under an automorphism of Ω. Also, we will identify $x \in \Delta$ with $(x; 0) \in \Delta \times \Omega'$. Hence, we will write $\mathfrak{D}_\ell(x) = \mathfrak{D}_\ell(x; 0)$ and $\mathfrak{D}(x) = \mathfrak{D}(x; 0)$. We have

Lemma 4. *Let $U \subset \Delta$ be a non-empty open set such that $\mathfrak{D}$ is non-zero on $U \times \{0\}$. Let $1 \leqslant s \leqslant n$ and $\mathbf{I} = (I_1, \ldots, I_s)$ be an s-tuple of distinct multi-indexes of nonnegative integers given by $I_k = (i_1(k), \ldots, i_q(k)) \neq 0$ for $1 \leqslant k \leqslant s$. Suppose $J = (j_1, \ldots, j_s)$ is an s-tuple of distinct integers j_ℓ, where $1 \leqslant \ell \leqslant s$, $1 \leqslant j_\ell \leqslant n$. Write $\mathbf{A} = (\mathbf{I}, J)$. Then, there exists a positive integer $N_{\mathbf{A}}$ and a constant $C_{\mathbf{A}} > 0$ independent of x, α such that the holomorphic function $h^{(\alpha)}_{\mathbf{A}}(x)$ on $U \subset \Delta$ satisfies*

$$|h^{(\alpha)}_{\mathbf{A}}(x)| \leqslant \frac{C_{\mathbf{A}}}{|\mathfrak{D}(x)|^{N_{\mathbf{A}}}}\,.$$

Proof. Let $\delta(x) > 0$ be the largest real number such that the branch $f^{(\alpha)}$ as a holomorphic map can be defined on the q-dimensional polydisk $\{x\} \times P^q(0; \delta(x))$, where $P^q(0; r) := \Delta(0, r) \times \ldots \times \Delta(0, r)$ is the q-dimensional polydisk of polyradii $(r, \ldots, r)$ centred at 0. Since each component $f^{(\alpha)}_\ell$ is uniformly bounded in absolute values by a positive real number, say C, by Cauchy estimates of derivatives in the variables $(w_1, \ldots, w_q)$, it follows that there exists a constant $C'_{\mathbf{A}} > 0$ such that

$$|h^{(\alpha)}_{\mathbf{A}}(x)| \leqslant \frac{C'_{\mathbf{A}}}{\delta(x)^{|I_1| + \cdots + |I_s|}}\,.$$

To prove Lemma 4 it remains to verify that there exists some constant $c > 0$ such that $\delta(x) \geqslant c\,|\mathfrak{D}(x)|$. Now $f^{(\alpha)}$ can be well-defined on $\{x\} \times P^q(0; \delta(x))$ whenever each $f^{(\alpha)}_\ell$ is well-defined there for $1 \leqslant \ell \leqslant n$. For the latter to hold true it suffices to verify that the discriminant $\mathfrak{D}_\ell(x, w) \neq 0$ whenever $|w| < \delta(x)$. Now

$$\mathfrak{D}_\ell(x, w) = \mathfrak{D}_\ell(x, 0) + \int_{\Lambda_x(0,w)} d\mathfrak{D}_\ell\,.$$

where $\Lambda_x(0,w)$ denotes the directed Euclidean line segment joining $(x,0)$ to (x,w). Estimating $d\mathfrak{D}_\ell$ by Cauchy estimates for first derivatives, noting that $\mathfrak{D}_\ell$ is bounded, it follows readily that there exists some $c_1>0$ such that $\mathfrak{D}_\ell(x,w)\neq 0$ whenever $\Lambda_x(0,w)$ is of length dominated by $c_1|\mathfrak{D}_\ell(x,0)|$. Thus we have

$$\delta(x)\geqslant \frac{c_1}{\sqrt{q}}\cdot \min_{1\leqslant \ell\leqslant n}\big|\mathfrak{D}_\ell(x,0)\big|\geqslant c|\mathfrak{D}(x)|$$

for some $c>0$. The proof of Lemma 4 is complete. □

By means of the estimates in Lemma 4 we have the following lemma with which the proof of Proposition 2 in Section 4 can be completed.

Lemma 5. *Let* $\mathbf{A}$ *is an index* $(\mathbf{I},J)$ *of order* $q+1$, *and* $x_0\in\Delta$ *be a point where* $\mathfrak{D}(x_0)\neq 0$. *Let* $f^{(\alpha)},1\leqslant\alpha\leqslant p$, *be the* p *branches of the multivalent holomorphic map* f *defined on a product neighborhood* $U=U_0\times U'\subset\Delta\times\Omega'$ *of* $(x_0,0)$. *For* $1\leqslant\alpha\leqslant p$, *define* $h^{(\alpha)}_{\mathbf{A}}(x)=\det\big(\mathfrak{M}^{(\alpha)}_{\mathbf{A}}(x)\big)$, *where* $\mathfrak{M}^{(\alpha)}_{\mathbf{A}}(x)$ *is the* $(q+1)$*-by-*$(q+1)$ *matrix at* x *associated to the index* A *of order* $q+1$ *as in the above and defined using the the branch* $f^{(\alpha)}$ *of the multivalent holomorphic map* f. *Then,* $h^{(\alpha)}_{\mathbf{A}}(x)=0$ *for every* $x\in U_0$.

Proof. For a set of p indeterminates $X_1,\dots X_p$, for $1\leqslant e\leqslant p$ denote by $\sigma_e(X_1,\dots,X_p)$ the e-th elementary symmetric polynomial in p variables. As before we will identify Δ with $\Delta\times\{0\}$. At a point $x\in\Delta$ where $\mathfrak{D}(x)\neq 0$, for $1\leqslant e\leqslant p$ we define $g^e_{\mathbf{A}}(x)=\sigma_e\big(h^{(1)}_{\mathbf{A}}(x),\dots,h^{(p)}_{\mathbf{A}}(x)\big)$. Write $\mathcal{Z}:=\big\{z=(x,w)\in\Delta\times\Omega':\mathfrak{D}(z)=0\big\}$, and $\mathcal{Z}_0:=\mathcal{Z}\cap(\Delta\times\{0\})$. Each $g^e_{\mathbf{A}}$ is a holomorphic function on $\Delta-\mathcal{Z}_0$. By Lemma 4 we have

$$|g^e_{\mathbf{A}}(x)|\leqslant\frac{C_{\mathbf{A}}}{|\mathfrak{D}(x)|^{eN_{\mathbf{A}}}}.$$

As a consequence, for $x\in\Delta-\mathcal{Z}_0$ we have

$$|g^e_{\mathbf{A}}\mathfrak{D}^{eN_{\mathbf{A}}}(x)|\leqslant C_{\mathbf{A}}.$$

For $x\in\Delta-\mathcal{Z}_0$ write $\mu^e_{\mathbf{A}}(x):=g^e_{\mathbf{A}}\mathfrak{D}^{eN_{\mathbf{A}}}$. From the estimates in the above $\mu^e_{\mathbf{A}}:\Delta-\mathcal{Z}_0\to\mathbb{C}$ is a bounded holomorphic function, and hence it extends by Riemann extension to a bounded holomorphic function on Δ, to be denoted by the same symbol. By Fatou's Theorem, the bounded holomorphic function $\mu^e_{\mathbf{A}}$ admits a radial limit $\mu^{e\star}_{\mathbf{A}}(b)$ for almost every $b\in\partial\Delta$. By Lemma 2, for almost every $b\in\partial\Delta$, each radial limit $\mu^{e\star}_{\mathbf{A}}(b),1\leqslant e\leqslant p$, exists and the radial limit $\mathfrak{D}^\star(b)$ of the bounded holomorphic function $\mathfrak{D}(x)$ on Δ also exists and furthermore $\mathfrak{D}^\star(b)\neq 0$ for almost every b. For such b we have $\lim_{r\to 1^-}\mathfrak{D}(rb)=\mathfrak{D}^\star(b)\neq 0$. By our construction of the boundary correspondence (cf. the paragraph before Proposition 1), there exists then a fixed open neighborhood $V_0\Subset\Omega'$ of $0\in\Omega'$, and $\epsilon>0$ such that for $I_\epsilon=(1-\epsilon,1)$, we have well-defined branches $f^{(1)},\dots,f^{(p)}$ of f such that each $f^{(\alpha)}$, $1\leqslant\alpha\leqslant p$, is continuous on $I_\epsilon\times V_0$ and restricts to $\{rb\}\times V_0$, $1-\epsilon<r<1$ to give a holomorphic map $f^{(\alpha)}_{rb}:V_0\to\mathbb{C}^n$, and such that moreover $f^{(\alpha)}_{rb}$ converges uniformly to $f^{(\alpha)\star}_b:V_0\to\partial\Omega$ (c.f. the paragraph before Proposition 1). Thus, for an index $\mathbf{A}=(\mathbf{I},J)$ we have $\lim_{r\to 1^-}h^{(\alpha)}_{\mathbf{A}}(rb)=h^{(\alpha)\star}_{\mathbf{A}}(b)$. Now if the index $\mathbf{A}$ is of order $q+1$ from the fact that $f^{(\alpha)}(V_0)\subset\Gamma_b$ for some maximal face Γ_b on $\partial\Omega$, of dimension q, it follows readily that $h^{(\alpha)\star}_{\mathbf{A}}(b)=0$. As a consequence, for $1\leqslant e\leqslant p$,

$$\lim_{r\to 1^-}g^e_{\mathbf{A}}\mathfrak{D}^{eN_{\mathbf{A}}}(rb)=0\,.$$

Thus, the boundary values of the bounded holomorphic functions $\mu^{e\star}_{\mathbf{A}}(b),1\leqslant e\leqslant p$ are all equal to 0 for almost all b. By the Cauchy Integral Formula in Lemma 2 we conclude that $\mu^e_{\mathbf{A}}\equiv 0$ on Δ, hence $g^e_{\mathbf{A}}\equiv 0$ for $1\leqslant e\leqslant p$, implying that the branches $h^{(\alpha)}_{\mathbf{A}}\equiv 0$ wherever defined on Δ. □

From Lemma 4 we deduce Proposition 2 in Section 4.

Proof of Proposition 2. By Lemma 4, the proposition follows from the scheme of proof given in Section 4 in the case where f is univalent, in which case the proof was reduced to verifying of the condition (†) as given there. □

We are now ready to prove the Main Theorem.

Proof of the Main Theorem. We first recall the setting and notations. Without loss of generality, we may assume that F is a surjective proper holomorphic map from an irreducible bounded symmetric domain Ω onto a complex space Z. Associated to F, the correspondence $S \subset \Omega \times \Omega$ is defined by $(x, y) \in S \iff F(x) = F(y)$. S can be regarded as the graph of a multivalent map f from Ω into Ω. We can assume that f is unbranched in a neighborhood U_0 of $0 \in \Omega$. From Section 3 for a maximal characteristic subdomain $\Theta \subset \Omega$, there is a totally geodesic complex submanifold $\Pi \subset \Omega$ isomorphic to $\Delta \times \Omega' \hookrightarrow \Omega$ such that Π contains Θ as a fiber with respect to the projection map $\Pi \cong \Delta \times \Omega' \to \Delta$. Ω is a union of maximal characteristic subdomains. We fix a subdomain of such type which contains the origin, and define the restrictions $S_x = S \cap ((\{x\} \times \Omega') \times \Omega)$ of the correspondence S.

By Proposition 2 in Section 4, we see that each S_x is in fact a correspondence for $\{x\} \times \Omega'$ to $\mathcal{L}_x$, where $\mathcal{L}_x$ is also a maximal characteristic symmetric subdomain of Ω. As all the maximal characteristic symmetric subdomains are isomorphic, we can do induction on the rank (down to rank one) by considering the induced correspondence between $\{x\} \times \Omega'$ and $\mathcal{L}_x$. Namely, for any $z \in U_0$ and any minimal disk Δ_z passing through z, if we define the restriction $S_{\Delta_z} := S \cap (\Delta_z \times \Omega)$, we actually have $S_{\Delta_z} \subset \Delta_z \times D_z$, where D_z is also a minimal disk in Ω. Hence each branch of f on U_0 preserves characteristic vectors for every point $z \in U_0$. By Theorem 2 in Section 3 of Ochiai [12], the branches of f extend to automorphisms of M, the compact dual of Ω. By Lemma 1 in Section 3 such automorphisms of M preserve Ω and they restrict to automorphisms of Ω.

Coming back to our original surjective proper holomorphic map $F : \Omega \to Z$, we define $H \subset \mathrm{Aut}(\Omega)$ to be the subset consisting of all automorphisms γ of Ω satisfying $F \circ \gamma \equiv F$ on Ω. Clearly $H \subset \mathrm{Aut}(\Omega)$ is a finite subgroup. Let $a \in Z$ at which F is unbranched and $F^{-1}(a) = \{y_1, \ldots, y_p\}$. By the above, we have proven that for each intertwining map $\varphi_{i,j} : U_i \to U_j$ between open neighborhoods U_i of y_i and U_j of y_j, $\varphi_{i,j}$ extends to an automorphism of Ω. It follows that H acts transitively on $F^{-1}(a)$. In fact, H consists of precisely p elements, say $H = \{\gamma_{1,1} = id_\Omega, \gamma_{1,2}, \ldots, \gamma_{1,p}\}$. Denote now by $X = \Omega/H$ the quotient space, and by $\pi : \Omega \to X$ the canonical projection map. By a result of Cartan [1], X has a unique structure as a normal complex space. For any $z, z' \in \Omega$, if $\pi(z) = \pi(z')$, then there exists $\gamma \in H$ such that $\gamma(z) = z'$. By our construction of H, γ is the extension of an intertwining map induced from $F : \Omega \to Z$ and since intertwining maps are fibre-preserving (with respect to F), we have $F(z) = F(z')$. Therefore we can define the map $\nu : X \to Z$ by $\nu(x) = F(\pi^{-1}(x))$. From the definition of the complex structure of the quotient space X, ν is holomorphic because $\nu \circ \pi = F$ is holomorphic. Since any compact subvariety of Ω consists of finitely many points, $\nu : X \to Z$ must be finite. Let $\widetilde{Z}$ be the normalization of Z. We can then lift ν to $\widetilde{\nu} : X \to \widetilde{Z}$, which remains finite, proper, surjective and generically one to one, and we have an inverse map $\nu : \widetilde{Z} \to X$ which is weakly holomorphic. Since $\widetilde{Z}$ is normal, every weakly holomorphic function extends holomorphically and therefore we see that $\widetilde{\nu}$ is actually biholomorphic. Thus, X is the normalization of Z.

6 Proper holomorphic maps onto manifolds

We are ready to prove Theorem 1 in Section 2 regarding proper holomorphic maps $F : \Omega \to Z$ onto a complex manifold Z for an irreducible bounded symmetric domain Ω of rank $\geqslant 2$.

Proof of Theorem 1. Recall that $\Omega \subset M$ is the Borel embedding of Ω into its compact dual M. If F is not a biholomorphism, then the ramification locus $R \subset \Omega$ is the fixed point set of the group $H \subset \mathrm{Aut}(\Omega)$, and as such R must be a smooth totally geodesic hypersurface. Thus, R is itself a Hermitian symmetric manifold of the semisimple and noncompact type. Let $R \subset Q$ be the Borel embedding of R into its compact dual Q. Then, Q can be identified naturally as a complex submanifold of M. Furthermore, the finite group $H \subset \mathrm{Aut}(\Omega)$ extends as a finite group of automorphisms fixing the hypersurface Q. The irreducible Hermitian symmetric manifold M belongs to the class of rational homogeneous manifolds of Picard number 1. For the latter class of manifolds we have the following general result in Hwang-Mok [2, Prop.15] concerning finite groups of automorphisms proven in relation to a solution of Lazarsfeld's Problem.

Proposition 3. *Let Y be a rational homogeneous manifold of Picard number* 1 *of dimension $n \geqslant 3$ different from the projective space $\mathbf{P}^n$. Suppose there exists a nontrivial finite cyclic group $H \subset \mathrm{Aut}(Y)$*

which fixes a (smooth) hypersurface $E \subset Y$ pointwise. Then, Y is the hyperquadric $\mathbf{Q}^n$, E is equal to an $\mathcal{O}(1)-$hypersurface, which is itself a hyperquadric, H is a group of order 2, and the quotient of Y by H, endowed with the standard structure as a normal complex space, is the projective space $\mathbf{P}^n$.

Proof of Theorem 1 *continued.* If Ω is not biholomorphic to D_n^{IV} for some $n \geqslant 3$, then by Proposition 3 (and the discussion on duality preceding it) any proper holomorphic map $F : \Omega \to Z$ onto a complex manifold Z must be a biholomorphism. This gives the alternative (a) in Theorem 1. For alternative (b), i.e., the case where $\Omega = D_n^{IV}$, it remains to identify the cases of $F : D_n^{IV} \to D_n^{IV}/H \cong Z$ where H is a non-trivial finite group of automorphisms. We will describe the non-trivial automorphism $\sigma \in H$ of order 2 explicitly to check that it gives rise to an automorphism of $\Omega = D_n^{IV}$ by restriction. (For the description of the Harish-Chandra and Borel embeddings $D_n^{IV} \Subset \mathbb{C}^n \subset \mathbf{Q}^n$ we refer the reader to Mok [5, Chapter 4, (3.1), pp. 82–83]). Represent the hyperquadric $\mathbf{Q}^n \subset \mathbf{P}^{n+1}$ by (†) $z_1^2 + \cdots + z_n^2 - 2z_{n+1}z_{n+2} = 0$ in the homogeneous coordinates $[z_1, \ldots, z_{n+2}]$. Then, the Euclidean space $\mathbb{C}^n$ is embedded in $\mathbf{Q}^n$ as an open subset by the mapping $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n, 1, \frac{1}{2}(z_1^2 + \cdots + z_n^2))$. The classical domain D_n^{IV} of Type-IV is identified via the Harish-Chandra realization as $D_n^{IV} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 < 2 \text{ and } \|z\|^2 < 1 + |\frac{1}{2}(z_1^2 + \cdots + z_n^2)|^2\}$. For $[z_1, \ldots, z_{n+2}] \in \mathbf{Q}^n$ define now $F([z_1, \ldots, z_{n+2}]) = [z_1, \ldots, z_{n-1}, z_{n+1}, z_{n+2}]$. By the defining equation (†) of the hyperquadric we see that $z_1 = \cdots = z_{n-1} = z_{n+1} = z_{n+2} = 0$ implies $z_n = 0$, so that $F : \mathbf{Q}^n \to \mathbf{P}^n$ is a holomorphic map. Clearly F is a double cover ramified precisely along the smooth hyperplane section $E \subset \mathbf{Q}^n$ defined by $z_n = 0$, and we have $F(x) = F(\sigma(x))$ for $\sigma \in \mathrm{Aut}(\mathbf{Q}^n)$ given by $\sigma([z_1, z_2, \ldots, z_{n+2}]) = [z_1, \ldots, z_{n-1}, -z_n, z_{n+1}, z_{n+2}]$. The involution σ generates a finite group H of order 2, and the holomorphic mapping $F : \mathbf{Q}^n \to \mathbf{P}^n$ agrees with the canonical quotient map $\mathbf{Q}^n \to \mathbf{Q}^n/H \cong \mathbf{P}^n$. Now, $z = (z_1, \ldots, z_n) \in D_n^{IV}$ if and only if $\|z\|^2 < 2$ and $\|z\|^2 < 1 + |\frac{1}{2}(z_1^2 + \cdots + z_n^2)|^2$. Obviously, for $z \in D_n^{IV}$, the two conditions are satisfied when z is replaced by $\sigma(z)$, so that $\sigma|_{D_n^{IV}} \in \mathrm{Aut}(D_n^{IV})$. It follows that $F|_{D_n^{IV}} : D_n^{IV} \to \mathbb{C}^n \subset \mathbf{P}^n$ is a proper holomorphic map onto some open subset $G := F(D_n^{IV}) \Subset \mathbb{C}^n$. F is ramified precisely along the totally geodesic hypersurface $J = E \cap D_n^{IV}$, and the proof of Theorem 1 is complete. □

Remarks. Alternatively, one can show that $F : \Omega \to Z$ is an isomorphism unless Ω is biholomorphic to $D_n^{IV}, n \geqslant 3$ by a case-by-case checking on dimensions of totally geodesic complex submanifolds. The proof of Proposition 3 given in [2] is however more conceptual.

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