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American Barrier Option Pricing Formulas for Currency Model in Uncertain Environment*

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Abstract Option pricing problem is one of the central issue in the theory of modern finance. Uncertain currency model has been put forward under the foundation of uncertainty theory as a tool to portray the foreign exchange rate in uncertain finance market. This paper uses uncertain differential equation involved by Liu process to dispose of the foreign exchange rate. Then an American barrier option of currency model in uncertain environment is investigated. Most important of all, the authors deduce the formulas to price four types of American barrier options for this currency model in uncertain environment by rigorous derivation.

Keywords Barrier option, currency model, option pricing, uncertain process.

1 Introduction

An option is a contract which confers the right for holders to buy or sell their assets at a set price at any time before or on the maturity time. Option trading is buying and selling of this right. In the transaction of option, the rights and obligations of the buyer and the seller are not symmetrical. Then the option is not free and the buyer must pay a certain amount of option fee, which creates the option pricing problem. The problem of pricing option acts as a significant part in quantitative finance[1]. In 1900, Bachelier[2], the father of option pricing theory, first raised that the stock prices were subject to Brownian motion. This is

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the first attempt in the history of mathematics to apply advanced mathematics to financial problems. Black and Scholes[3], and Merton[4] independently established and developed the Black-Scholes formulas for pricing option in 1973, which laid the foundation for the reasonable pricing of various derivative financial instruments in the emerging derivative finance market, including stocks, bonds, currencies and commodities. Subsequently, many scholars extended the Black-Scholes option pricing formula. For example, in 1979, Cox, et al.[5] proposed the Binomial model, which established the groundwork for the numerical method of option pricing, and priced the American option. In 1983, Garman and Kohlhagen[6] proposed a modified model for foreign exchange option based on Black-Scholes option pricing formula, G-K model, which is now widely used in the pricing of foreign exchange option.

Barrier option refers to the fact that the profit of the option depends on whether or not the price of the underlying asset has reached a set level (threshold value) during a certain period of time. Before the Chicago Board of Options Exchange (CBOE) appeared, barrier options were sold sporadically in the United States. Merton[4] first presented a formula for a down-and-out call option in 1973. From then on, the academic research on barrier option pricing began to develop rapidly. In 1991, pricing formulas for several European barrier options were derived by Rich[7], Rubinstein and Reiner[8]. Heynen and Kat[9], and Carr[10] introduced rainbow barrier options and partial barrier options. Boyle and Lau[11], and Kat and Verdonk[12] respectively discussed that how to apply the standard binomial model to solving the pricing problem of barrier options. The pricing problem of capped options under European scenario and American scenario were investigated by Broadie and Detemple[13]. The "adaptive mesh" technique to barrier options was applied by Figlewski and Gao[14]. The case of producing static hedges for barrier options was also considered by Carr, et al.[15]. Under the constant volatility framework, a series of one-asset barrier option pricing formulas and multi-asset barrier option pricing formulas were presented by Rich[7], and Wong and Kwok[16], respectively. Option pricing problems with both American early exercise features and knock-out barrier can be found in [17].

As is known to us all, probability theory is used as a helpful tool to depict indeterminate phenomena. However, the function of cumulative probability distribution should close enough to the real frequency (which is determined by the law of large numbers) if you need to utilize probability theory. Obtaining the cumulative probability distribution needs sufficient sample data, which requires a lot of independent and repeatable experiments. However, due to economic and technical reasons, these independent repetitive tests cannot be carried out. For example, the bridge in use cannot obtain the data of its bearing weight. In other words, some data are not obtainable in real life or are unavailable under special circumstances, such as the demand for masks in pandemic (e.g., COVID-19 in 2020). Without enough data, the cumulative probability distribution cannot be obtained, so it needs experts in related fields to estimate the belief degree of events occurring according to their existing experience and knowledge. Since the conservatism of human estimation, if we make use of probability theory to describe degree of belief, there may trigger a counterintuitive result which was demonstrated by some credible examples in [18]. At this point, in order to deal with this problem that there are small samples or even no any sample, we have to find new tools.

Liu[19, 20] proposed uncertainty theory with introducing normality axiom, duality axiom, subadditivity axiom and product axiom. Within the uncertainty theory context, uncertain variable is raised to depict the uncertain quantity and uncertain measure is proposed for modeling the belief degree, respectively. In addition, uncertainty distribution was served as a tool to directly describe uncertain variables, and expected value operator was defined in [19] for the sake of ranking uncertain variables. So far, uncertainty theory has been widely applied in lots of fields such as uncertain risk and reliability analysis (Gao and Yao[21, 22]), uncertain programming (Liu[23]), as well as uncertain finance (Liu[24]).

The concept of uncertain process was put forward in [25] to describe the uncertain variables varying with time. For the uncertain process, its uncertainty distribution, independence and operational law are studied respectively. Further, on this basis, independent incremental process is proposed, in which the increment is independent[26]. Subsequently, Liu[20] proposed a stationary and independent increment process where its increments are independent and identically distributed (i.i.d.) uncertain variables. Furthermore, if these iid uncertain variables obey the normal uncertainty distribution, then Liu process is produced. Driven by Liu process, both uncertain calculus and uncertain differential equation which have many properties developed by Liu[20]. Chen and Liu[27] verified some properties of solutions for uncertain differential equations including existence, uniqueness and stability. Yao and Chen[28] firstly raised Yao-Chen formula to turn uncertain differential equations into ordinary differential equations.

With its maturity and perfection, the application of uncertain differential equation can be found in sophisticated finance market far and wide. Liu[20] proposed an uncertain stock model motivated by the uncertain differential equation firstly. Under the guidance of the uncertain stock model, Chen[29], Liu[20], and Sun and Chen[30] priced American option, European option and Asian option relative to finance markets determined by the uncertain stock model in succession, as well as American barrier option were priced by Gao, et al.[31]. Furthermore, Chen and Gao[32] constructed an uncertain interest rate model on the premise that the interest rate follows an uncertain differential equation. Jiao and Yao[33] studied the zero-coupon bond model, and Zhang, et al.[34] proposed the interest rate ceiling model as well as the interest rate floor model.

Foreign exchange plays a significant part in the capital market, since it can promote international economic and trade development, adjust international capital surplus. Liu, et al.[35] supposed that the exchange rate follows an uncertain differential equation, then put forward the uncertain currency model. In addition, many researches are based on this assumption and this model. Shen and Yao[36] proposed a mean-reverting uncertain currency model to describe the foreign exchange rate in the long term. Wang and Ning[37] made some improvements referring to Liu-Chen-Ralescu model, and presented an uncertain currency model with floating interest rates. Different from Wang-Ning's currency model, Wang and Chen[38] considered the long-term fluctuations of the exchange rate and the changing of the interest rates from time to time, and put forward a mean-reverting uncertain currency model with floating interest rates to simulate the foreign exchange market. In this paper, we will discuss four types of American barrier options and introduce the corresponding pricing formulas for uncertain currency model. The rest of the paper is arranged as follows: Some basic knowledge related to uncertain variable, uncertain process and uncertain differential equation is given one by one in Section 2. The contents of the Section 3 and Section 4 are about the formulas for pricing the American barrier option in two scenarios: the knock-in scenario and the knock-out scenario. At last, in Section 5 a brief conclusion is given.

2 Preliminaries

Several basic definitions and related properties about uncertain variable, uncertain process, and uncertain differential equation will be introduced in this section.

2.1 Uncertain Variable

Definition 2.1 (see [19]) Suppose that $(\Gamma, \mathcal{L})$ is a measurable space. If $\mathcal{M}$ is a set function and satisfies three axioms as follows:

Axiom 1 (Normality Axiom) For the universal set Γ, $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2 (Duality Axiom) For arbitrary event Λ, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$;

Axiom 3 (Subadditivity Axiom) For each countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Then we say the $\mathcal{M}$ an uncertain measure and the triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ an uncertain space.

Theorem 2.2 (see [19]) *Suppose that $\mathcal{M}$ is an uncertain measure. Then it is a set function which is monotonically increasing. Then, the following inequality*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$$

holds for arbitrary events Λ_1 and Λ_2 with $\Lambda_1 \subset \Lambda_2$.

Definition 2.3 (see [39]) Assuming that there is an uncertain variable ξ whose uncertainty distribution $\Phi(x)$ is regular. Then ξ possesses an inverse uncertainty distribution which is the inverse function $\Phi^{-1}(\alpha)$.

Definition 2.4 (see [20]) The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are regarded as independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for arbitrary Borel sets $B_1, B_2, \cdots, B_n$ of real numbers.

Liu[19] presented the concept of expected value to ranking uncertain variables as follows:

Theorem 2.5 (see [39]) *Suppose that ξ is an uncertain variable which possesses regular uncertainty distribution Φ and suppose that there exists its expected value. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha) d\alpha.$$

2.2 Uncertain Process

Definition 2.6 (see [25]) Assuming that there is a totally ordered set T and an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If the function X_t is measurable from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, that is, for arbitrary given $\{t \in T\}$, the set

$$\{X_t \in B\} = \{\gamma | X_t(\gamma) \in B\}$$

is an event for arbitary Borel set B. Then X_t is called uncertain process.

Definition 2.7 (see [26]) The uncertain process X_t whose uncertainty distribution is $\Phi_t(x)$ which is defined by

$$\Phi_t(x) = \mathcal{M}\{X_t \leq x\}$$

for arbitrary number x and arbitrary time t.

Definition 2.8 (see [20]) The uncertain process C_t is regarded as a Liu process, suppose that

(i) its initial value is zero, i.e., $C_0 = 0$, and almost all of its sample paths are Lipschitz continuous,

(ii) C_t possesses independent and stationary increments,

(iii) each increment $C_{s+t} - C_s$ is an uncertain variable with normal uncertainty distribution whose expected value is 0 and variance is t^2, that is, the uncertainty distribution of $C_{s+t} - C_s$ is

$$\Phi(x) = \left(1 + \exp\left(\frac{-\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

2.3 Uncertain Differential Equation

Definition 2.9 (see [28]) Let

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t \tag{1}$$

be an uncertain differential equation whose initial value is X_0. If for arbitrary α $(0 < \alpha < 1)$, X_t^α relative to t is the solution of the corresponding equation

$$dX_t^\alpha = f(t, X_t^\alpha)dt + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)dt,$$

in which

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}, \quad 0 < \alpha < 1,$$

the deterministic function X_t^α is the α-path of Equation (1).

Theorem 2.10 (see [28]) *Considering an uncertain differential equation*

$$dX_t = f(t, X_t)dt + g(t, X_t)dC_t, \quad t \in [0, s],$$

where its α-path is X_t^α. If the functions $f(t, x)$ and $g(t, x)$ are continuous, then we derive

$$\mathcal{M}\{X_t \leq X_t^\alpha, \ \forall t\} = \alpha, \quad \mathcal{M}\{X_t > X_t^\alpha, \ \forall t\} = 1 - \alpha.$$

Chen and Yao[28] firstly found that X_t possesses the inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = X_t^\alpha, \quad 0 < \alpha < 1.$$

3 Knock-in Options

In this section, we investigate one kind of barrier option, the knock-in option which is divided into up-and-in scenario and down-and-in scenario. Knock-in option refers to the option begins to function as a normal option only in which the market price of the underlying asset hits a certain price level within a specified period. In addition, we derive and give their concerning pricing formulas.

In the following derivation, we need to use an indictor function which was defined by

$$I_L = \begin{cases} 1, & \text{if } L \leq x, \\ 0, & \text{otherwise,} \end{cases}$$

where L is a given number which acts for the trigger point.

The uncertain currency model raised by Liu, et al.[35] as follows:

$$\begin{cases} dX_t = aX_t dt, \\ dY_t = \beta Y_t dt, \\ dZ_t = \mu Z_t dt + \upsilon Z_t dC_t, \end{cases} \tag{2}$$

where X_t denotes the domestic currency whose domestic interest rate is a, Y_t denotes the foreign currency whose foreign interest rate is β, and Z_t denotes the exchange rate means that the domestic currency price of one unit of foreign currency at time t, C_t represents a Liu process, μ and υ represent the drift and diffusion, respectively.

For one thing, we study the up-and-in scenario that goes into effect in which the exchange rate is beneath the trigger point L and goes up within its maturity time. In the uncertain finance market, the currency model for the American up-and-in call option whose exercise price is K, maturity time is T and trigger point is L was defined by Model (2).

Assume that the contract price is f_{ui} in domestic currency. At time 0, the buyer pays f_{ui} to purchase the contract. And the buyer possesses a present value of the revenue in domestic currency at time t which is

$$\sup_{0\leq t\leq T} I_L\left(\sup_{0\leq s\leq t} Z_s\right)(Z_t - K)^+.$$

So at time 0, the net revenue of the buyer is

$$-f_{ui} + \sup_{0\leq t\leq T} \exp(-at) I_L\left(\sup_{0\leq s\leq t} Z_s\right)(Z_t - K)^+.$$

In contrast, at time 0, the seller obtains f_{ui}. And the seller obtains $(1 - K/Z_t)^+$ in foreign currency at time t, then the seller need to pay

$$\sup_{0\leq t\leq T} I_L\left(\sup_{0\leq s\leq t} Z_s\right) Z_0(1 - K/Z_t)^+.$$

So at time 0, the net revenue of the seller is

$$f_{ui} - \sup_{0\leq t\leq T} \exp(-\beta t) I_L\left(\sup_{0\leq s\leq t} Z_s\right) Z_0(1 - K/Z_t)^+.$$

The derived contract price should make the expected revenues of the buyer and the seller equal, which means that,

$$\begin{aligned}&-f_{ui}+E\left[\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+\right]\\=&f_{ui}-E\left[\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)Z_0(1-K/Z_t)^+\right].\end{aligned}$$

Through the above analysis, we give the pricing formula as follows.

Definition 3.1 Assuming that there exists an American up-and-in call option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American up-and-in call option for Model (2) is

$$\begin{aligned}f_{ui}=&\frac{1}{2}E\left[\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+\right]\\&+\frac{1}{2}E\left[\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)Z_0(1-K/Z_t)^+\right].\end{aligned}$$

Theorem 3.2 *Assuming that there exists an American up-and-in call option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American up-and-in call option is controlled by the following equality*

$$\begin{aligned}f_{ui}=&\frac{1}{2}\int_b^1\sup_{0\le t\le T}\exp(-at)\left(Z_0\exp\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+d\alpha\\&+\frac{1}{2}\int_b^1\sup_{0\le t\le T}\exp(-\beta t)\left(Z_0-K/\exp\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+d\alpha,\end{aligned}$$

in which

$$b=\left(1+\exp\left(\frac{\pi(\mu t+\ln Z_0-\ln L)}{\sqrt{3}\upsilon t}\right)\right)^{-1}.$$

Proof In the first place, we verify that the following uncertain variable

$$\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+,$$

in which

$$Z_t^\alpha=Z_0\exp\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\begin{aligned}&\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+\le\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\ \supseteq&\left\{\sup_{0\le s\le t}Z_s\le\sup_{0\le s\le t}Z_s^\alpha,Z_t\le Z_t^\alpha,t\in[0,T]\right\}\\ \supseteq&\left\{Z_t\le Z_t^\alpha,t\in[0,T]\right\}\end{aligned}$$

and

$$\begin{aligned}&\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+>\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\ \supseteq&\left\{\sup_{0\le s\le t}Z_s>\sup_{0\le s\le t}Z_s^\alpha,Z_t>Z_t^\alpha,t\in[0,T]\right\}\\ \supseteq&\left\{Z_t>Z_t^\alpha,t\in[0,T]\right\},\end{aligned}$$

we obtain

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+\le\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\ \ge&M\left\{Z_t\le Z_t^\alpha,t\in[0,T]\right\}\\ =&\alpha\end{aligned}\tag{3}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+>\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\ \ge&M\left\{Z_t>Z_t^\alpha,t\in[0,T]\right\}\\ =&1-\alpha\end{aligned}\tag{4}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+\le\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\ &+\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+>\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\ =&1.\end{aligned}\tag{5}$$

It follows Equations (3)–(5) that

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+\le\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}=\alpha.$$

Then the uncertain variable

$$\sup_{0\le t\le T}\exp(-at)I_L\left(\sup_{0\le s\le t}Z_s\right)(Z_t-K)^+$$

possesses the inverse uncertain distribution which is

$$\phi^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-at) I_L\left(\sup_{0\le s\le t} Z_s^\alpha\right)(Z_t^\alpha - K)^+.$$

So the following equality

$$f_{ui} = \int_0^1 \sup_{0\le t\le T} \exp(-at) I_L\left(\sup_{0\le s\le t} Z_s^\alpha\right)(Z_t^\alpha - K)^+ d\alpha$$

holds due to Theorem 2.5.

In the second place, we verify that the following uncertain variable

$$\sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s\right)(1 - K/Z_t)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s^\alpha\right)(1 - K/Z_t^\alpha)^+,$$

in which

$$Z_t^\alpha = Z_0 \exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi} \ln \frac{\alpha}{1-\alpha}\right).$$

Since

$$\left\{\sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s\right)(1 - K/Z_t)^+ \le \sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s^\alpha\right)(1 - K/Z_t^\alpha)^+\right\}$$
$$\supseteq \left\{\sup_{0\le s\le t} Z_s \le \sup_{0\le s\le t} Z_s^\alpha, Z_t \le Z_t^\alpha, t\in[0,T]\right\}$$
$$\supseteq \{Z_t \le Z_t^\alpha, t\in[0,T]\}$$

and

$$\left\{\sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s\right)(1 - K/Z_t)^+ > \sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s^\alpha\right)(1 - K/Z_t^\alpha)^+\right\}$$
$$\supseteq \left\{\sup_{0\le s\le t} Z_s > \sup_{0\le s\le t} Z_s^\alpha, Z_t > Z_t^\alpha, t\in[0,T]\right\}$$
$$\supseteq \{Z_t > Z_t^\alpha, t\in[0,T]\},$$

we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s\right)(1-K/Z_t)^+ \le \sup_{0\le t\le T} \exp(-\beta t) I_L\left(\sup_{0\le s\le t} Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}$$
$$\ge M\{Z_t \le Z_t^\alpha, st\in[0,T]\}$$
$$=\alpha \tag{6}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)(1-K/Z_t)^+ > \sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ \ge& M\left\{Z_t > Z_t^\alpha, t\in[0,T]\right\}\\ =&1-\alpha \qquad (7)\end{aligned}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)(1-K/Z_t)^+ \le \sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ +&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)Z_0(1-K/Z_t)^+ > \sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ =&1. \qquad (8)\end{aligned}$$

It follows from Equations (6)–(8) that

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)(1-K/Z_t)^+ \le \sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}=\alpha.$$

Then the uncertain variable

$$\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s\right)Z_0(1-K/Z_t)^+$$

possesses the inverse uncertain distribution which is

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+.$$

So the following equality

$$f_{ui}=\int_0^1\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+d\alpha$$

holds due to Theorem 2.5.

Moreover, the equation

$$I_L\left(\sup_{0\le s\le t}Z_s^\alpha\right)=1$$

holds if and only if

$$\sup_{0\le t\le T}Z_t^\alpha=\sup_{0\le t\le T}Z_0\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\ge L \qquad (9)$$

holds for arbitrary given $t\in[0,T]$.

Because of $Z_0<L$, (9) can be reduced to

$$Z_0\left(\mu T+\frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\ge L,$$

which shows that

$$\alpha \geq \left(1+\exp\left(\frac{\pi(\mu T+\ln Z_0-\ln L)}{\sqrt{3}\upsilon T}\right)\right)^{-1} \triangleq b.$$

Consequently, the pricing formula of American up-and-in call option is

$$\begin{aligned}
f_{ui} &= \frac{1}{2}\int_0^1 \sup_{0\leq t\leq T} \exp(-at) I_L\left(\sup_{0\leq s\leq t} Z_s^\alpha\right)(Z_t^\alpha-K)^+ d\alpha \\
&\quad + \frac{1}{2}\int_0^1 \sup_{0\leq t\leq T} \exp(-\beta t) I_L\left(\sup_{0\leq s\leq t} Z_s^\alpha\right) Z_0(1-K/Z_t^\alpha)^+ d\alpha \\
&= \frac{1}{2}\int_b^1 \sup_{0\leq t\leq T} \exp(-at)(Z_t^\alpha-K)^+ d\alpha + \frac{1}{2}Z_0\int_b^1 \sup_{0\leq t\leq T} \exp(-\beta t)(1-K/Z_t^\alpha)^+ d\alpha \\
&= \frac{1}{2}\int_b^1 \sup_{0\leq t\leq T} \exp(-at)\left(Z_0\exp\left(\mu T+\frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+ d\alpha \\
&\quad + \frac{1}{2}\int_b^1 \sup_{0\leq t\leq T} \exp(-\beta t)\left(Z_0-K/\exp\left(\mu T+\frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha.
\end{aligned}$$

Thus the proof is finished. ∎

Corollary 3.3 *Let f_{ui} be the price of American up-and-in call option under uncertain currency Model* (2). *Then,*

1) *f_{ui} is a decreasing function of a.* 2) *f_{ui} is a decreasing function of β.*

3) *f_{ui} is an increasing function of μ.* 4) *f_{ui} is an increasing function of υ.*

5) *f_{ui} is an increasing function of Z_0.* 6) *f_{ui} is a decreasing function of K.*

Example 3.4 Considering Model (2) where we set $a=0.03,\ \beta=0.025,\ \mu=0.04,\ \upsilon=0.05$. And assuming that $Z_0=3,\ K=5,\ T=10,\ L=8$. Then $f_{ui}=0.786$ according to Theorem 3.2.

In Figure 1, we directly obtain that f_{ui} is monotonously decreasing with respect to L providing the other parameters are unchanged.

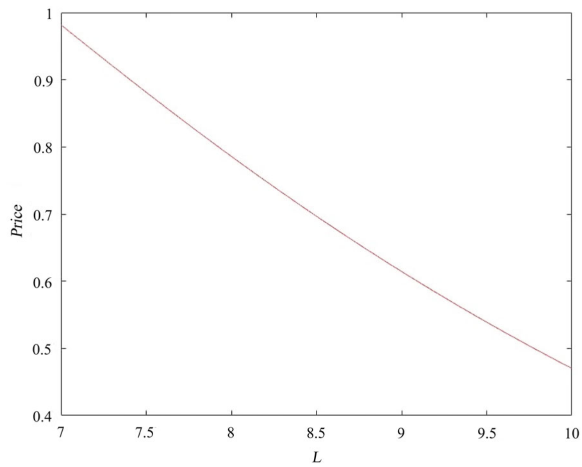

Figure 1 The price f_{ui} in regard to barrier level L in Example 3.4

For another thing, we study the down-and-in scenario that goes into effect in which the exchange rate is above the trigger point L and goes down within its maturity time. In the uncertain finance market, the currency model for the American down-and-in put option whose exercise price is K, maturity time is T and trigger point is L was defined by Model (2).

Assume that the contract price is f_{di} in domestic currency. At time 0, the buyer pays f_{di} to purchase the contract. And the buyer possesses a present value of the revenue in domestic currency at time t which is

$$\sup_{0\leq t\leq T}\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)(K-Z_t)^+.$$

So at time 0, the net revenue of the buyer is

$$-f_{di}+\sup_{0\leq t\leq T}\exp(-at)\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)(K-Z_t)^+.$$

In contrast, at time 0, the seller obtains f_{di}. And the seller obtains $(K/Z_t-1)^+$ in foreign currency at time t, then the seller need to pay

$$\sup_{0\leq t\leq T}\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)Z_0(K/Z_t-1)^+.$$

So at time 0, the net revenue of the seller is

$$f_{di}-\sup_{0\leq t\leq T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)Z_0(K/Z_t-1)^+.$$

The derived contract price should make the expected revenues of the buyer and the seller equal, which means that,

$$\begin{aligned}&-f_{di}+E\left[\sup_{0\leq t\leq T}\exp(-at)\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)(K-Z_t)^+\right]\\ =&f_{di}-E\left[\sup_{0\leq t\leq T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)Z_0(K/Z_t-1)^+\right].\end{aligned}$$

Through the above analysis, we give the pricing formula as follows.

Definition 3.5 Assuming that there exists an American down-and-in put option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American down-and-in put option for Model (2) is

$$\begin{aligned}f_{di}=&\frac{1}{2}E\left[\sup_{0\leq t\leq T}\exp(-at)\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)(K-Z_t)^+\right]\\ &+\frac{1}{2}E\left[\sup_{0\leq t\leq T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\leq s\leq t} Z_s\right)\right)Z_0(K/Z_t-1)^+\right].\end{aligned}$$

Theorem 3.6 *Assuming that there exists an American down-and-in put option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American* *down-and-in put option is controlled by the following equality*

$$f_{di} = \frac{1}{2}\int_0^b \sup_{0\le t\le T} \exp(-at)\left(K - Z_0\exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha$$
$$+\frac{1}{2}\int_0^b \sup_{0\le t\le T} \exp(-\beta t)\left(K\Big/\exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ d\alpha,$$

in which

$$b = \left(1+\exp\left(\frac{\pi(\mu t + \ln Z_0 - \ln L)}{\sqrt{3}\upsilon t}\right)\right)^{-1}.$$

Proof In the first place, we verify that the following uncertain variable

$$\sup_{0\le t\le T} \exp(-at)\left(1 - I_L\left(\inf_{0\le s\le t} Z_s\right)\right)(K - Z_t)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-at)\left(1 - I_L\left(\inf_{0\le s\le t} Z_s^{1-\alpha}\right)\right)(K - Z_t^{1-\alpha})^+,$$

in which

$$Z_t^{1-\alpha} = Z_0\exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}$$
$$\supseteq\left\{\inf_{0\le s\le t}Z_s \ge \inf_{0\le s\le t}Z_s^{1-\alpha}, Z_t \ge Z_t^{1-\alpha}, t\in[0,T]\right\}$$
$$\supseteq\left\{Z_t \ge Z_t^{1-\alpha}, t\in[0,T]\right\}$$

and

$$\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ > \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}$$
$$\supseteq\left\{\inf_{0\le s\le t}Z_s < \inf_{0\le s\le t}Z_s^{1-\alpha}, Z_t < Z_t^{1-\alpha}, t\in[0,T]\right\}$$
$$\supseteq\left\{Z_t < Z_t^{1-\alpha}, t\in[0,T]\right\},$$

we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}$$
$$\ge\mathcal{M}\left\{Z_t \ge Z_t^{1-\alpha}, t\in[0,T]\right\}$$
$$=\alpha \tag{10}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ > \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\ \ge&\mathcal{M}\left\{Z_t<Z_t^{1-\alpha},t\in[0,T]\right\}\\ =&1-\alpha \end{aligned}\tag{11}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\ &+\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ > \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\ &=1. \end{aligned}\tag{12}$$

It follows from Equations (10)–(12) that

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}=\alpha.$$

Then the uncertain variable

$$\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+$$

possesses the inverse uncertain distribution which is

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+.$$

So the following equality

$$f_{di}=\int_0^1\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+d\alpha$$

holds due to Theorem 2.5.

In the second place, we verify that the following uncertain variable

$$\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+,$$

in which

$$Z_t^{1-\alpha}=Z_0\exp\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$\supseteq\left\{\inf_{0\le s\le t}Z_s\ge\inf_{0\le s\le t}Z_s^{1-\alpha},Z_t\ge Z_t^{1-\alpha},t\in[0,T]\right\}$$
$$\supseteq\left\{Z_t\ge Z_t^{1-\alpha},t\in[0,T]\right\}$$

and

$$\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+>\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$\supseteq\left\{\inf_{0\le s\le t}Z_s<\inf_{0\le s\le t}Z_s^{1-\alpha},Z_t<Z_t^{1-\alpha},t\in[0,T]\right\}$$
$$\supseteq\left\{Z_t<Z_t^{1-\alpha},t\in[0,T]\right\},$$

we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$\ge\mathcal{M}\left\{Z_t\ge Z_t^{\alpha},st\in[0,T]\right\}$$
$$=\alpha \tag{13}$$

and

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+>\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$\ge\mathcal{M}\left\{Z_t<Z_t^{\alpha},t\in[0,T]\right\}$$
$$=1-\alpha \tag{14}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$+\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+>\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$=1. \tag{15}$$

It follows from Equations (13)–(15) that

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}=\alpha.$$

Then the uncertain variable

$$\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\inf_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+$$

possesses the inverse uncertain distribution which is

$$\phi^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-\beta t)\left(1 - I_L\left(\inf_{0\le s\le t} Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha} - 1)^+.$$

So the following equality

$$f_{di} = \int_0^1 \sup_{0\le t\le T} \exp(-\beta t)\left(1 - I_L\left(\inf_{0\le s\le t} Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha} - 1)^+ d\alpha$$

holds due to Theorem 2.5.

Moreover, the equation

$$I_L\left(\inf_{0\le s\le t} Z_s^{1-\alpha}\right) = 0$$

holds if and only if

$$\inf_{0\le s\le t} Z_s^{1-\alpha} = \inf_{0\le t\le T} Z_0\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right) < L \tag{16}$$

holds for arbitrary given $t \in [0, T]$.

Because of $Z_0 \ge L$, (16) can be reduced to

$$Z_0\left(\mu T + \frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) < L,$$

which shows that

$$\alpha < \left(1 + \exp\left(\frac{\pi(\mu T + \ln Z_0 - \ln L)}{\sqrt{3}\upsilon T}\right)\right)^{-1} \triangleq b.$$

Consequently, the pricing formula of American down-and-in put option is

$$\begin{aligned}
f_{di} &= \frac{1}{2}\int_0^1 \sup_{0\le t\le T} \exp(-at)\left(1 - I_L\left(\inf_{0\le s\le t} Z_s^{1-\alpha}\right)\right)(K - Z_t^{1-\alpha})^+ d\alpha \\
&\quad + \frac{1}{2}\int_0^1 \sup_{0\le t\le T} \exp(-\beta t)\left(1 - I_L\left(\inf_{0\le s\le t} Z_s^{1-\alpha}\right)\right) Z_0(K/Z_t^{1-\alpha} - 1)^+ d\alpha \\
&= \frac{1}{2}\int_0^b \sup_{0\le t\le T} \exp(-at)(K - Z_t^{1-\alpha})^+ d\alpha + \frac{1}{2} Z_0 \int_0^b \sup_{0\le t\le T} \exp(-\beta t)(K/Z_t^{1-\alpha} - 1)^+ d\alpha \\
&= \frac{1}{2}\int_0^b \sup_{0\le t\le T} \exp(-at)\left(K - Z_0 \exp\left(\mu T + \frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha \\
&\quad + \frac{1}{2}\int_0^b \sup_{0\le t\le T} \exp(-\beta t)\left(K/\exp\left(\mu T + \frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ d\alpha.
\end{aligned}$$

Thus, the proof is finished. ▮

Corollary 3.7 *Let f_{di} be the price of American down-and-in put option under uncertain currency Model* (2). *Then,*

1) *f_{di} is a decreasing function of a.* 2) *f_{di} is a decreasing function of β.*

3) *f_{di} is a decreasing function of μ.* 4) *f_{di} is a decreasing function of υ.*

5) *f_{di} is a decreasing function of Z_0.* 6) *f_{di} is an increasing function of K.*

Example 3.8 Considering Model (2) where we set $a = 0.03,\ \beta = 0.025,\ \mu = -0.02,\ \upsilon = 0.05$. And assuming that $Z_0 = 10,\ K = 7,\ T = 10,\ L = 8$. Then $f_{di} = 2.6496$ according to Theorem 3.6.

In Figure 2, we directly obtain that f_{di} is monotonously increasing with respect to L providing the other parameters are unchanged.

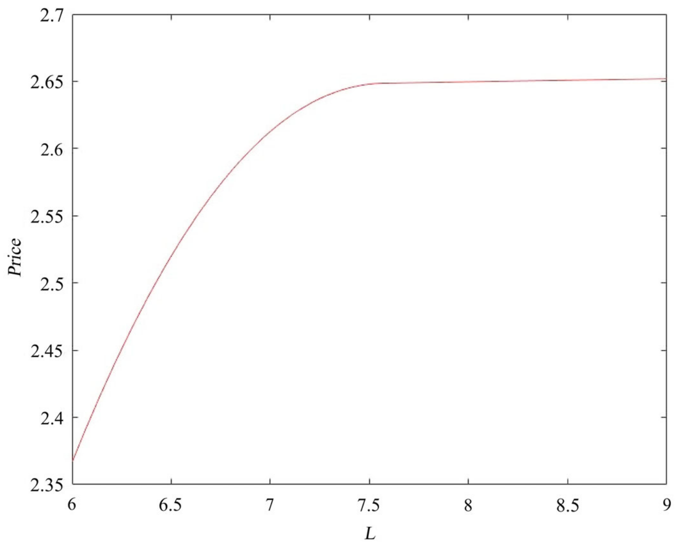

Figure 2 The price f_{di} in regard to barrier level L in Example 3.8

4 Knock-Out Options

In this section, we investigate another kind of barrier option, the knock-out option which is divided into up-and-out scenario and down-and-out scenario. Knock-out option refers to the option is expire worthless only in which the market price of the underlying asset exceeded a certain price level within a specified period. In addition, we derive and give their concerning pricing formulas.

For one thing, we study the up-and-out scenario that is of no effect in which the exchange rate is beneath the trigger point L and goes up within its maturity time. In the uncertain finance market, the currency model for the American up-and-out put option whose exercise price is K, maturity time is T and trigger point is L was defined by Model (2).

Assume that the contract price is f_{uo} in domestic currency. At time 0, the buyer pays f_{uo} to purchase the contract. And the buyer possesses a present value of the revenue in domestic currency at time t which is

$$\sup_{0\le t\le T}\left(1 - I_L\left(\sup_{0\le s\le t} Z_s\right)\right)(K - Z_t)^+.$$

So at time 0, the net revenue of the buyer is

$$-f_{uo} + \sup_{0\le t\le T}\exp(-at)\left(1 - I_L\left(\sup_{0\le s\le t} Z_s\right)\right)(K - Z_t)^+.$$

In contrast, at time 0, the seller obtains f_{uo}. And the seller obtains $(K/Z_t - 1)^+$ in foreign currency at time t, then the seller need to pay

$$\sup_{0 \le t \le T} \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) Z_0(K/Z_t - 1)^+.$$

So at time 0, the net revenue of the seller is

$$f_{uo} - \sup_{0 \le t \le T} \exp(-\beta t) \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) Z_0(K/Z_t - 1)^+.$$

The derived contract price should make the expected revenues of the buyer and the seller equal, which means that,

$$\begin{aligned} &- f_{uo} + E\left[\sup_{0 \le t \le T} \exp(-at) \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) (K - Z_t)^+\right] \\ =& f_{uo} - E\left[\sup_{0 \le t \le T} \exp(-\beta t) \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) Z_0(K/Z_t - 1)^+\right]. \end{aligned}$$

Through the above analysis, we give the pricing formula as follows.

Definition 4.1 Assuming that there exists an American up-and-out put option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American up-and-out put option for Model (2) is

$$\begin{aligned} f_{uo} =& \frac{1}{2} E\left[\sup_{0 \le t \le T} \exp(-at) \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) (K - Z_t)^+\right] \\ &+ \frac{1}{2} E\left[\sup_{0 \le t \le T} \exp(-\beta t) \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) Z_0(K/Z_t - 1)^+\right]. \end{aligned}$$

Theorem 4.2 *Assuming that there exists an American up-and-out put option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American up-and-out put option is controlled by the following equality*

$$\begin{aligned} f_{uo} =& \frac{1}{2} \int_0^b \sup_{0 \le t \le T} \exp(-at) \left(K - Z_0 \exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi} \ln \frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha \\ &+ \frac{1}{2} \int_0^b \sup_{0 \le t \le T} \exp(-\beta t) \left(K / \exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi} \ln \frac{\alpha}{1-\alpha}\right) - Z_0\right)^+ d\alpha, \end{aligned}$$

in which

$$b = \left(1 + \exp\left(\frac{\pi(\mu t + \ln Z_0 - \ln L)}{\sqrt{3}\upsilon t}\right)\right)^{-1}.$$

Proof In the first place, we verify that the following uncertain variable

$$\sup_{0 \le t \le T} \exp(-at) \left(1 - I_L\left(\sup_{0 \le s \le t} Z_s\right)\right) (K - Z_t)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-at)\left(1 - I_L\left(\sup_{0\le s\le t} Z_s^{1-\alpha}\right)\right)(K - Z_t^{1-\alpha})^+,$$

in which

$$Z_t^{1-\alpha} = Z_0 \exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\begin{aligned}
&\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\
\supseteq&\left\{\sup_{0\le s\le t}Z_s \ge \sup_{0\le s\le t}Z_s^{1-\alpha}, Z_t \ge Z_t^{1-\alpha}, t\in[0,T]\right\}\\
\supseteq&\left\{Z_t \ge Z_t^{1-\alpha}, t\in[0,T]\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ > \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\
\supseteq&\left\{\sup_{0\le s\le t}Z_s < \sup_{0\le s\le t}Z_s^{1-\alpha}, Z_t < Z_t^{1-\alpha}, t\in[0,T]\right\}\\
\supseteq&\left\{Z_t < Z_t^{1-\alpha}, t\in[0,T]\right\},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\
\ge&\mathcal{M}\left\{Z_t \ge Z_t^{1-\alpha}, t\in[0,T]\right\}\\
=&\alpha
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ > \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\
\ge&\mathcal{M}\left\{Z_t < Z_t^{1-\alpha}, t\in[0,T]\right\}\\
=&1-\alpha
\end{aligned}\tag{18}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ \le \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\
+&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+ > \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}\\
=&1.
\end{aligned}\tag{19}$$

It follows form Equations (17)–(19) that

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+\le\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+\right\}=\alpha.$$

Then the uncertain variable

$$\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K-Z_t)^+$$

possesses the inverse uncertain distribution which is

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+.$$

So the following equality

$$f_{uo}=\int_0^1\sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+d\alpha$$

holds due to Theorem 2.5.

In the second place, we verify that the following uncertain variable

$$\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+,$$

in which

$$Z_t^{1-\alpha}=Z_0\exp\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$\supseteq\left\{\sup_{0\le s\le t}Z_s\ge\sup_{0\le s\le t}Z_s^{1-\alpha},Z_t\ge Z_t^{1-\alpha},t\in[0,T]\right\}$$
$$\supseteq\left\{Z_t\ge Z_t^{1-\alpha},t\in[0,T]\right\}$$

and

$$\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+>\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}$$
$$\supseteq\left\{\sup_{0\le s\le t}Z_s<\sup_{0\le s\le t}Z_s^{1-\alpha},Z_t<Z_t^{1-\alpha},t\in[0,T]\right\}$$
$$\supseteq\left\{Z_t<Z_t^{1-\alpha},t\in[0,T]\right\},$$

we obtain

$$
\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}\\
\ge&\mathcal{M}\left\{Z_t\ge Z_t^{1-\alpha},st\in[0,T]\right\}\\
=&\alpha
\end{aligned}\tag{20}
$$

and

$$
\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+>\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}\\
\ge&\mathcal{M}\left\{Z_t<Z_t^{1-\alpha},t\in[0,T]\right\}\\
=&1-\alpha
\end{aligned}\tag{21}
$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$
\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}\\
+&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+>\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}\\
=&1.
\end{aligned}\tag{22}
$$

It follows from Equations (20)–(22) that

$$
\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+\le\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+\right\}=\alpha.
$$

Then the uncertain variable

$$
\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s\right)\right)(K/Z_t-1)^+
$$

possesses the inverse uncertain distribution which is

$$
\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+.
$$

So the following equality

$$
f_{uo}=\int_0^1\sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)\right)(K/Z_t^{1-\alpha}-1)^+d\alpha
$$

holds due to Theorem 2.5.

Moreover, the equation

$$
I_L\left(\sup_{0\le s\le t}Z_s^{1-\alpha}\right)=0
$$

holds if and only if

$$\sup_{0\le t\le T} Z_t^{1-\alpha} = \sup_{0\le t\le T} Z_0\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right) < L \tag{23}$$

holds for arbitrary given $t\in[0,T]$.

Because of $Z_0 < L$, (23) can be reduced to

$$Z_0\left(\mu T + \frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) < L,$$

which shows that

$$\alpha < \left(1+\exp\left(\frac{\pi(\mu T+\ln Z_0-\ln L)}{\sqrt{3}\upsilon T}\right)\right)^{-1} \triangleq b.$$

Consequently, the pricing formula of American up-and-out put option is

$$\begin{aligned}
f_{uo} &= \frac{1}{2}\int_0^1 \sup_{0\le t\le T}\exp(-at)\left(1-I_L\left(\sup_{0\le s\le t} Z_s^{1-\alpha}\right)\right)(K-Z_t^{1-\alpha})^+ d\alpha \\
&\quad + \frac{1}{2}\int_0^1 \sup_{0\le t\le T}\exp(-\beta t)\left(1-I_L\left(\sup_{0\le s\le t} Z_s^{1-\alpha}\right)\right) Z_0(K/Z_t^{1-\alpha}-1)^+ d\alpha \\
&= \frac{1}{2}\int_0^b \sup_{0\le t\le T}\exp(-at)(K-Z_t^{1-\alpha})^+ d\alpha + \frac{1}{2}Z_0\int_0^b \sup_{0\le t\le T}\exp(-\beta t)(K/Z_t^{1-\alpha}-1)^+ d\alpha \\
&= \frac{1}{2}\int_0^b \sup_{0\le t\le T}\exp(-at)\left(K-Z_0\exp\left(\mu T+\frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha \\
&\quad + \frac{1}{2}\int_0^b \sup_{0\le t\le T}\exp(-\beta t)\left(K/\exp\left(\mu T+\frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-Z_0\right)^+ d\alpha.
\end{aligned}$$

Thus, the proof is finished. ∎

Corollary 4.3 *Let f_{uo} be the price of American up-and-out put option under uncertain currency Model* (2). *Then,*

1) *f_{uo} is a decreasing function of a.* 2) *f_{uo} is a decreasing function of β.*

3) *f_{uo} is a decreasing function of μ.* 4) *f_{uo} is a decreasing function of υ.*

5) *f_{uo} is a decreasing function of Z_0.* 6) *f_{uo} is an increasing function of K.*

Example 4.4 Considering Model (2) where we set $a=0.03,\ \beta=0.025,\ \mu=-0.02,\ \upsilon=0.05$. And assuming that $Z_0=10,\ K=7,\ T=10,\ L=12$. Then $f_{uo}=2.6565$ according to Theorem 4.2.

In Figure 3, we directly obtain that f_{uo} is monotonously increasing with respect to L providing the other parameters are unchanged.

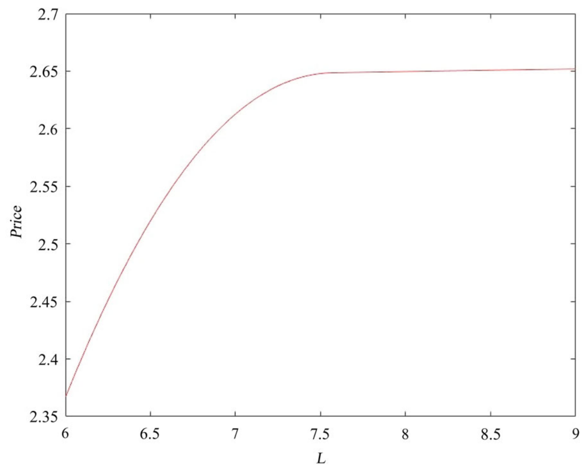

Figure 3 The price f_{uo} in regard to barrier level L in Example 4.4

For another thing, we study the down-and-out scenario that is of no effect in which the exchange rate is above the trigger point L and goes down within its maturity time. In the uncertain finance market, the currency model for the American down-and-out call option whose exercise price is K, maturity time is T and trigger point is L was defined by Model (2).

Assume that the contract price is f_{do} in domestic currency. At time 0, the buyer pays f_{do} to purchase the contract. And the buyer possesses a present value of the revenue in domestic currency at time t which is

$$\sup_{0\le t\le T} I_L\left(\inf_{0\le s\le t} Z_s\right)(Z_t-K)^+.$$

So at time 0, the net revenue of the buyer is

$$-f_{do}+\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t} Z_s\right)(Z_t-K)^+.$$

In contrast, at time 0, the seller obtains f_{do}. And the seller obtains $(1-K/Z_t)^+$ in foreign currency at time t, then the seller need to pay

$$\sup_{0\le t\le T} I_L\left(\inf_{0\le s\le t} Z_s\right)Z_0(1-K/Z_t)^+.$$

So at time 0, the net revenue of the seller is

$$f_{do}-\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t} Z_s\right)Z_0(1-K/Z_t)^+.$$

The derived contract price should make the expected revenues of the buyer and the seller equal, which means that,

$$\begin{aligned}&-f_{do}+E\left[\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t} Z_s\right)(Z_t-K)^+\right]\\=&f_{do}-E\left[\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t} Z_s\right)Z_0(1-K/Z_t)^+\right].\end{aligned}$$

Through the above analysis, we give the pricing formula as follows.

Definition 4.5 Assuming that there exists an American down-and-out call option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American down-and-out call option for Model (2) is

$$\begin{aligned} f_{do} =& \frac{1}{2} E\left[\sup_{0\le t\le T} \exp(-at) I_L\left(\inf_{0\le s\le t} Z_s\right)(Z_t - K)^+\right] \\ &+ \frac{1}{2} E\left[\sup_{0\le t\le T} \exp(-\beta t) I_L\left(\inf_{0\le s\le t} Z_s\right) Z_0(1 - K/Z_t)^+\right]. \end{aligned}$$

Theorem 4.6 *Assuming that there exists an American down-and-out call option whose trigger point is L, exercise price is K and maturity time is T. Then the price of this American down-and-out call option is controlled by the following equality*

$$\begin{aligned} f_{do} =& \frac{1}{2}\int_b^1 \sup_{0\le t\le T} \exp(-at)\left(Z_0 \exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ d\alpha \\ &+ \frac{1}{2}\int_b^1 \sup_{0\le t\le T} \exp(-\beta t)\left(Z_0 - K/\exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha \end{aligned}$$

in which

$$b = \left(1 + \exp\left(\frac{\pi(\mu t + \ln Z_0 - \ln L)}{\sqrt{3}\upsilon t}\right)\right)^{-1}.$$

Proof In the first place, we verify that the following uncertain variable

$$\sup_{0\le t\le T} \exp(-at) I_L\left(\inf_{0\le s\le t} Z_s\right)(Z_t - K)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha) = \sup_{0\le t\le T} \exp(-at) I_L\left(\inf_{0\le s\le t} Z_s^\alpha\right)(Z_t^\alpha - K)^+,$$

in which

$$Z_t^\alpha = Z_0 \exp\left(\mu t + \frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\begin{aligned} &\left\{\sup_{0\le t\le T} \exp(-at) I_L\left(\inf_{0\le s\le t} Z_s\right)(Z_t - K)^+ \le \sup_{0\le t\le T} \exp(-at) I_L\left(\inf_{0\le s\le t} Z_s^\alpha\right)(Z_t^\alpha - K)^+\right\} \\ \supseteq& \left\{\inf_{0\le s\le t} Z_s \le \inf_{0\le s\le t} Z_s^\alpha, Z_t \le Z_t^\alpha, t\in[0,T]\right\} \\ \supseteq& \{Z_t \le Z_t^\alpha, t\in[0,T]\} \end{aligned}$$

and

$$\begin{aligned}
&\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+ > \sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\
&\supseteq\left\{\inf_{0\le s\le t}Z_s>\inf_{0\le s\le t}Z_s^\alpha, Z_t>Z_t^\alpha, t\in[0,T]\right\}\\
&\supseteq\left\{Z_t>Z_t^\alpha, t\in[0,T]\right\},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+ \le \sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\
\ge&\mathcal{M}\left\{Z_t\le Z_t^\alpha, t\in[0,T]\right\}\\
=&\alpha
\end{aligned}\tag{24}$$

and

$$\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+ > \sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\
\ge&\mathcal{M}\left\{Z_t> Z_t^\alpha, t\in[0,T]\right\}\\
=&1-\alpha
\end{aligned}\tag{25}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+ \le \sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\
+&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+ > \sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}\\
=&1.
\end{aligned}\tag{26}$$

It follows from Equations (24)–(26) that

$$\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+ \le \sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+\right\}=\alpha.$$

Then the uncertain variable

$$\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s\right)(Z_t-K)^+$$

possesses the inverse uncertain distribution which is

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+.$$

So the following equality

$$f_{do}=\int_0^1\sup_{0\le t\le T}\exp(-at)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(Z_t^\alpha-K)^+d\alpha$$

holds due to Theorem 2.5.

In the second place, we verify that the following uncertain variable

$$\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+$$

possesses the inverse uncertain distribution

$$\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+,$$

in which

$$Z_t^\alpha=Z_0\exp\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Since

$$\begin{aligned}&\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+\le\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ &\supseteq\left\{\inf_{0\le s\le t}Z_s\le\inf_{0\le s\le t}Z_s^\alpha, Z_t\le Z_t^\alpha, t\in[0,T]\right\}\\ &\supseteq\left\{Z_t\le Z_t^\alpha, t\in[0,T]\right\}\end{aligned}$$

and

$$\begin{aligned}&\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+>\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ &\supseteq\left\{\inf_{0\le s\le t}Z_s>\inf_{0\le s\le t}Z_s^\alpha, Z_t>Z_t^\alpha, t\in[0,T]\right\}\\ &\supseteq\left\{Z_t>Z_t^\alpha, t\in[0,T]\right\},\end{aligned}$$

we obtain

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+\le\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ \ge&\mathcal{M}\left\{Z_t\le Z_t^\alpha, st\in[0,T]\right\}\\ =&\alpha\end{aligned}\tag{27}$$

and

$$\begin{aligned}&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+>\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\ \ge&\mathcal{M}\left\{Z_t>Z_t^\alpha, t\in[0,T]\right\}\\ =&1-\alpha\end{aligned}\tag{28}$$

by Theorem 2.2 and Theorem 2.10.

According to duality axiom, we have

$$
\begin{aligned}
&\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+\le\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\
&+\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+>\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}\\
&=1. \qquad (29)
\end{aligned}
$$

It follows from Equations (27)–(29) that

$$
\mathcal{M}\left\{\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+\le\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+\right\}=\alpha.
$$

Then the uncertain variable

$$
\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s\right)(1-K/Z_t)^+
$$

possesses the inverse uncertain distribution which is

$$
\phi^{-1}(\alpha)=\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+.
$$

So the following equality

$$
f_{do}=\int_0^1\sup_{0\le t\le T}\exp(-\beta t)I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)(1-K/Z_t^\alpha)^+d\alpha
$$

holds due to Theorem 2.5.

Moreover, the equation

$$
I_L\left(\inf_{0\le s\le t}Z_s^\alpha\right)=1
$$

holds if and only if

$$
\inf_{0\le s\le t}Z_s^\alpha=\inf_{0\le t\le T}Z_0\left(\mu t+\frac{\sqrt{3}\upsilon t}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\ge L \qquad (30)
$$

holds for arbitrary given $t\in[0,T]$.

Because of $Z_0\ge L$, (30) can be reduced to

$$
Z_0\left(\mu T+\frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\ge L,
$$

which shows that

$$
\alpha\ge\left(1+\exp\left(\frac{\pi(\mu T+\ln Z_0-\ln L)}{\sqrt{3}\upsilon T}\right)\right)^{-1}\triangleq b.
$$

Consequently, the pricing formula of American down-and-out call option is

$$
\begin{aligned}
f_{do} = & \frac{1}{2}\int_0^1 \sup_{0\le t\le T} \exp(-at) I_L\left(\inf_{0\le s\le t} Z_s^\alpha\right)(Z_t^\alpha - K)^+ d\alpha \\
& + \frac{1}{2}\int_0^1 \sup_{0\le t\le T} \exp(-\beta t) I_L\left(\inf_{0\le s\le t} Z_s^\alpha\right) Z_0(1 - K/Z_t^\alpha)^+ d\alpha \\
= & \frac{1}{2}\int_b^1 \sup_{0\le t\le T} \exp(-at)(Z_t^\alpha - K)^+ d\alpha + \frac{1}{2} Z_0 \int_b^1 \sup_{0\le t\le T} \exp(-\beta t)(1 - K/Z_t^\alpha)^+ d\alpha \\
= & \frac{1}{2}\int_b^1 \sup_{0\le t\le T} \exp(-at)\left(Z_0 \exp\left(\mu T + \frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ d\alpha \\
& + \frac{1}{2}\int_b^1 \sup_{0\le t\le T} \exp(-\beta t)\left(Z_0 - K/\exp\left(\mu T + \frac{\sqrt{3}\upsilon T}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ d\alpha.
\end{aligned}
$$

Thus, the proof is finished. ▮

Corollary 4.7 *Let f_{do} be the price of American down-and-out call option under uncertain currency Model* (2). *Then,*

1) *f_{do} is a decreasing function of a.* 2) *f_{do} is a decreasing function of β.*

3) *f_{do} is an increasing function of μ.* 4) *f_{do} is an increasing function of υ.*

5) *f_{do} is an increasing function of Z_0.* 6) *f_{do} is a decreasing function of K.*

Example 4.8 Considering Model (2) where we set $a = 0.03$, $\beta = 0.025$, $\mu = 0.04$, $\upsilon = 0.05$. And assuming that $Z_0 = 10$, $K = 15$, $T = 10$, $L = 8$. Then $f_{do} = 4.8205$ according to Theorem 4.6.

In Figure 4, we directly obtain that f_{do} is monotonously decreasing with respect to L providing the other parameters are unchanged.

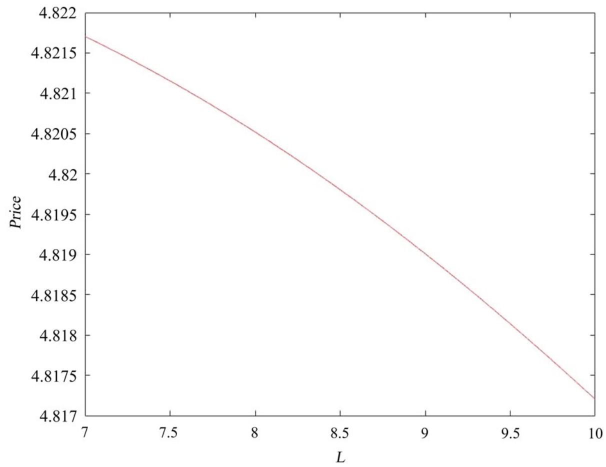

Figure 4 The price f_{do} in regard to barrier level L in Example 4.8

5 Conclusions

Based on the study of American barrier options for a currency model in uncertain finance market, our paper obtains some pricing formulas of American barrier options, including four types: up-and-in, down-and-in, up-and-out and down-and-out. Under the foreshadowing of this paper, in the future, we can continue to research multi-asset barrier options and derive the corresponding pricing formulas.

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