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AU Introspection and Symmetry under non-trivial unawareness

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Abstract
This note discusses the relationship between AU Introspection (i.e., an agent is unaware of some event, then she is unaware of that she is unaware of the event) and Symmetry (i.e., an agent is unaware of some event if and only if she is unaware of the complement set) for non-trivial unawareness (i.e., there is an event an agent is unaware of). without Negative Introspection using a set-theoretical approach in standard state-space models. Previous studies have explored the equivalence between Negative Introspection and AU Introspection, or the equivalence between Negative Introspection and Symmetry, by assuming Necessitation of the knowledge operator. As a corollary, AU Introspection is equivalent to Symmetry. However, no studies have shown the relationship between AU Introspection and Symmetry without Necessitation. Therefore, we explore this issue. Our main result shows that if the knowledge operator satisfies Monotonicity, Truth, and Positive introspection, then Modica and Rustichini's definition of unawareness leads to the equivalence of AU Introspection and Symmetry. In other words, we show that both AU Introspection and Symmetry hold without clashing with non-trivial unawareness.

Keywords Unawareness · Necessitation · Negative introspection · Symmetry · AU introspection · KU introspection

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1 Introduction

Focusing on standard state-space models using a set-theoretical approach, this note discusses the relationship between AU Introspection and Symmetry for non-trivial unawareness (i.e., there is an event an agent is unaware of). As pointed out by previous studies, in standard state-space models, several assumptions leads to Triviality (i.e., an agent is aware of the whole state space). (Modica & Rustichini, 1994) show the equivalence between Negative Introspection and Symmetry (Theorem 1). (Dekel et al., 1998) show that, if state-space models satisfy Necessitation, Plausibility, KU Introspection, and AU Introspection, then there is no event that some agent is unaware of Chen et al. (2012) investigate the relationship between Negative Introspection and AU Introspection. They show that Negative Introspection is equivalent to AU Introspection when assuming Necessitation (Theorem 3). From their results, it is evident that AU Introspection is equivalent to Symmetry. In fact, Chen et al. (2012) show a generalization of Dekel et al. (1998) and the aforementioned equivalence (Theorem 4) (below, such results are called Triviality Theorems.)

To avoid such an issue, (Heifetz et al., 2006) propose unawareness structure models. Their models assume that different agents perceive different disjoint subjective state spaces by defining the (generalized) state space that is a union set of disjoint state spaces as a lattice structure. If some agent's subjective state space is "less expressive" than other state space, then we can say that the agent (denoted as "she" for convenience) is unaware of such state space. In other words, we can represent non-trivial unawareness. Since (Heifetz et al., 2006), mainstream research has focused on unawareness structure models.

However, the above-mentioned research results do not mean that we cannot discuss non-trivial unawareness in standard state-space models. Ewerhart (2001) proposes models of non-trivial unawareness by assuming that an agent is aware of her subjective state spaces to be a proper subset of the objective state space, and that the unaware agent does not know all states in the complementaries set. Fukuda (2021) suggests that we can discuss non-trivial unawareness with Necessitation by excluding AU Introspection. We believe non-trivial unawareness should be reconsidered in the standard state-space models. Common to all Triviality Theorems is the assumption of Necessitation. In other words, Necessitation may lead to trivial unawareness. In fact, as pointed out by Dekel et al. (1998), if we do not assume Necessitation, then we can discuss non-trivial unawareness.

This note attempts to exclude the assumption of Necessitation. Mathematically, Necessitation is $K(\Omega) = \Omega$, that is, in any state, the agent knows the whole state space[1]. "Necessitation corresponds to knowing tautologies. This property is satisfied in any decision model in economics featuring a standard state space. E.g., the moment, [we] consider a possibility correspondence, the knowledge operator satisfies necessitation. The moment [we] have a probability measure on a space, the

[1] For any state, a meaning that the agent knows the whole state space is different to a meaning that the agent knows everything. In the latter, for any state, the agent always knows true state, that is, mathematically, for any $\omega \in \Omega$, $K(\{\omega\}) = \{\omega\}$. The former does not mean it. The author thanks an anonymous referee for pointing this out.

belief operator satisfies necessitation. The same holds for Choquet capacities or sets of probability measures used in modeling ambiguity. The assumption of necessitation is always in the background. However, in reality, we clearly do not know all tautologies otherwise there would not be a labor market for mathematicians and we would not task students of computer science to program tautology checkers. So to model more realistically an agent with logical non-omniscience, giving up necessitation may be the right way to go. But we also like to keep other properties of belief like positive introspection."[2] This note supposes excluding Necessitation and that Monotonicity, Truth and Positive Introspection hold.

Let us use Modica and Rustichini's definition of unawareness. Under several assumptions, Modica and Rustichini (1994) show equivalence between Symmetry[3] and Negative Introspection (Theorem 1), and Chen et al. (2012) show equivalence between AU Introspection[4] and Negative Introspection (Theorem 3). From these two theorems, it is clear that the equivalence of the three properties follows (Theorem 4). The two theorems assume Necessitation. In contrast, because we remove Necessitation, unlike (Modica & Rustichini, 1994), Symmetry might not be equivalent to Negative Introspection, and unlike (Chen et al., 2012), AU Introspection might not be equivalent to Negative Introspection. However, since there are no studies on equivalence between AU Introspection and Symmetry yet, it remains to be seen whether equivalence holds even if the assumption of Necessitation is removed. This note aims to analyze whether the equivalence of AU Introspection and Symmetry holds when the assumption of Necessitation is relaxed. Our main result shows that if the knowledge operator satisfies Monotonicity, Truth, and Positive introspection, then Modica and Rustichini's definition of unawareness leads to the equivalence of AU Introspection and Symmetry (Theorem 5).

We cannot directly prove the equivalence between AU Introspection and Symmetry. To do so, several properties are required, for example, KU Introspection and AA-Self Reflection of the unawareness operator. Therefore, these properties (Lemmas 1, 2, 3, and 4) must be shown before proving Theorem 5. In the proofs of these lemmas, we find that Necessitation is not required, that is, the equivalence of AU Introspection and Symmetry holds without Necessitation. Therefore, when excluding Necessitation, Negative Introspection is equivalent to neither AU Introspection nor Symmetry; however, AU Introspection and Symmetry are equivalent (Corollary 1). Our result implies that the non-triviality of unawareness consists of both AU Introspection and Symmetry, because non-triviality is equivalent to Negative Introspection. However, note that Corollary 1 is a generalization of Theorem 4, but not Theorems 2 and 3. We use Modica and Rustichini's definition of unawareness and Positive Introspection, whereas Theorems 2 and 3 are based on plausible

[2] The anonymous reviewer has more clearly stated the author's motivation for this paper in the peer review report. The text in quotation marks is the text suggested by that reviewer in the peer review report. The author is very grateful to the anonymous reviewer for elaborating.

[3] Symmetry means that the agent is aware of some event if and only if the agent is aware of the complement set.

[4] AU Introspection means that the agent if the agent is unaware of some event, then the agent is unaware that she or he is unaware of it.

unawareness relaxing Modica and Rustichini's definition, and do not suppose Positive Introspection. Hence, our main corollary does not generalize all Triviality Theorems.

Finally, we point out the relationship between this study and *non-normal modal logics*[5]. This study essentially discusses knowledge and unawareness in non-normal modal logics, because this note assumes to exclude Necessitation. In awareness structures proposed by Fagin and Halpern (1988) that is a special version of non-normal world's semantics, we need to distinguish between two knowledge operators, the *explicit knowledge operator* and the *implicit knowledge operator*. The relationship between the explicit and implicit knowledge operators is as follows: an agent explicitly knows some event if and only if she or he implicitly knows the event and she or he is aware of the event. Halpern and Rêgo (2013) points out that the two knowledge operators lead to different properties for knowledge and unawareness, respectively. Given the implicit knowledge operator, Necessitation and Monotonicity hold, but Plausibility and KU Introspection based on the implicit knowledge operator do not hold. By contrast, given the explicit knowledge operator, Necessitation and Monotonicity *may not* hold, but Plausibility and KU Introspection based on the explicit knowledge operator. In this note, our knowledge and unawareness operators satisfy Monotonicity, Plausibility, and KU Introspection, but the knowledge operator does not satisfies Necessitation. Hence, our knowledge operator is the explicit knowledge operator.[6]

The rest of this note is organized as follows. The next subsection highlights related works in the literature. Section 2 introduces standard state-space models following the studies of Dekel et al. (1998) and Chen et al. (2012) and Properties of the knowledge/unawareness operator. Section 3 overviews the Triviality Theorems of Modica and Rustichini (1994), Dekel et al. (1998), and Chen et al. (2012). Section 4 provides and proves our main theorem that AU Introspection is equivalent to Symmetry and generalizes a proof of Theorem 4. The last section concludes.

1.1 Related Literature

Pioneering works on higher-order lack of knowledge include those of Geanakoplos (2021). Heifetz et al. (2006) are the first to introduce unawareness structures. They assume that the family of state spaces is a lattice structure and that there is a difference in expressive power among various state spaces. Heifetz et al. (2013) and Galanis (2013, 2018) use unawareness structures and discuss and generalize Aumann's agreement theorem ((Aumann, 1976)) and the No-Trade Theorem ((Milgrom & Stokey, 1982)). Heifetz et al. (2008) propose canonical models of unawareness. Galanis (2011) considers unawareness of theorems using a logical approach, while (Galanis, 2013) discusses unawareness of theorems via a set-theoretical approach. Galanis (2013) provides a property named Awareness Leads to Knowledge and shows that a knowledge operator in a more expressive state-space leads to a better description of an agent's knowledge than a knowledge operator in

[5] (Rantala, 1982a) (Rantala, 1982b) is one of studies in non-normal modal logics.

[6] This paragraph is written in response to comments from an anonymous referee regarding non-normal modal logics. The author thanks the referee for this comment.

less expressive state space. This result means that Galanis' model allows agents to disagree on whether the opponents know about some event. Li (2009) proposes a product of the state-space model, called an information structure with unawareness. Heinsalu (2012) discusses the relationship between the works of Fagin and Halpern (1988) and Li (2009).

Fagin and Halpern (1988) are among the early studies of unawareness in modal logics and they model awareness structures. Wansing (1990) shows that an awareness structure is a special version of non-normal worlds semantics. In other words, non-normal model logics is sufficiently expressible to model unawareness. Since their researches, Halpern and Rêgo (2009) (Halpern & Rêgo, 2013) and Sillari (2008a) (Sillari, 2008b) discuss knowledge and unawareness on awareness structures.

2 Preliminaries

Let us consider a standard state-space model, such as that of Dekel et al. (1998) or Chen et al. (2012), $\langle \Omega, K, U \rangle$, where

- Ω is a state space. Any $E \subseteq \Omega$ is an event, and $\neg E = \Omega \setminus E$.
- $K : 2^{\Omega} \to 2^{\Omega}$ is the knowledge operator. Given any event $E \subseteq \Omega$, a set $K(E)$ is interpreted as the agent possessing K knows that event E occurs.
- $U : 2^{\Omega} \to 2^{\Omega}$ is the unawareness operator. Given any event E, a set $U(E)$ is interpreted as the agent possessing U is unaware of the event E.

In a partitional state-space model, it is well known that the knowledge operator K satisfies the following properties:

K1 Necessitation: $K(\Omega) = \Omega$;
K2 Monotonicity: if $E \subseteq F$, then $K(E) \subseteq K(F)$;
K3 Truth: $K(E) \subseteq E$;
K4 Positive Introspection: $K(E) \subseteq KK(E)$; and
K5 Negative Introspection: $\neg K(E) \subseteq K\neg K(E)$.

Here, by K5, $\neg K \neg K(E) = \emptyset$ in a partitional state-space model. This means that the agent must know some event. In other words, any higher-order lack of knowledge does not hold.

Previous studies on unawareness attempt to relax Negative Introspection and provide the following axioms of the unawareness operator:

U0 Modica and Rustichini's definition: $U(E) = \neg K(E) \cap \neg K\neg K(E)$;
U1 Plausibility: $U(E) \subseteq \neg K(E) \cap \neg K\neg K(E)$;
U2 KU Introspection: $KU(E) = \emptyset$;
U3 AU Introspection: $U(E) \subseteq UU(E)$; and
U4 Symmetry: $U(E) = U(\neg E)$.

U0 and U4 are proposed by Modica and Rustichini (1994) and U1–3 are provided by Dekel et al. (1998).

Following (Chen et al., 2012), we name and define trivial and non-trivial unawareness as follows:

U5 Triviality: $\forall E \subseteq \Omega, U(E) = \emptyset$; and
U6 Non-Triviality: $\exists E \subseteq \Omega$ subject to $U(E) \neq \emptyset$.

Remark 1 Under U0, K5 if and only if U5.

Finally, we define the awareness operator as $A(E) = \neg U(E)$.

3 Triviality Theorems

Modica and Rustichini (1994), (Dekel et al., 1998), and Chen et al. (2012) present the following theorems about trivial unawareness:

Theorem 1 (Modica and Rustichini (1994)) *If* $\langle \Omega, K, U \rangle$ *satisfies* $K1$–4 *and* $U0$, *then* $K5$ *and* $U4$ *are equivalent.*

Theorem 2 (*Dekel et al*. (1998))

If $\langle \Omega, K, U \rangle$ *satisfies* $K1$ *and* $U1$–3, *then U5 is satisfied.*

Theorem 3 (*Chen et al*. (2012))

If $\langle \Omega, K, U \rangle$ *satisfies* $K1$-3 *and* $U1$, $K5$ *if and only if* $U3$

Theorem 4 (*Chen et al*. (2012))

If $\langle \Omega, K, U \rangle$ *satisfies* $K1$–4 *and* $U0$, $K5$ *if and only if* $U3$ *if and only if* $U4$.

Note that Theorems 1 and 4 use Modica and Rustichini's definition, U0, whereas, Theorems 2 and 3 use plausible unawareness, U1. Moreover, Theorems 2 and 3 do not need Positive Introspection, K4.

The following sketch provides an outline of a proof of Theorem 4 provided by Chen et al. (2012):

Proof (The outline of the proof of Theorem 4.)

1. By Theorem 1, K5 and U4 are equivalent.
2. By Theorem 3, K5 and U3 are equivalent.
3. By 1 and 2, U3 and U4 are equivalent.

Hence, K5, U3, and U4 are equivalent. □

Theorem 4 is a generalization of Theorem 1 and 2, and we use Theorems 1 and 3 to prove Theorem 4. Theorem 1 suggests equivalence between Negative Introspection and Symmetry; Theorem 3 suggests equivalence between Negative Introspection and AU Introspection; and Theorem 4 suggests equivalence between AU Introspection and Symmetry. In proof of Theorem 4, AU Introspection and Symmetry are not directly equivalent. This proof is related to Necessitation. However, is Negative Introspection necessary to prove the equivalence between AU Introspection and Symmetry? Can we directly prove this equivalence without Negative Introspection? We explore this issue in the next section.

4 Main Theorem

In this section, we explore the proof of equivalence between AU Introspection and Symmetry without Negative Introspection. We show the following theorem.

Theorem 5 *If $\langle \Omega, K, U \rangle$ satisfies $K2$–4 and $U0$, then $U3$ is equivalent to $U4$.*

This theorem does not use Necessitation. In other words, Necessitation is not necessary for this theorem. Theorem 5 implies that AU Introspection is equivalent to Symmetry. Put differently, Negative Introspection is not necessary for this equivalence. In other words, an equivalent pair of AU Introspection and Symmetry is not equivalent to Negative Introspection even when Necessitation is not used.

Before proving this theorem, we show the following lemmas.

Lemma 1 *If $\langle \Omega, K, U \rangle$ satisfies $K2$, then*

$$K2^* \qquad K(E \cap F) \subseteq (K(E) \cap K(F)).$$

Proof Suppose that $\langle \Omega, K, U \rangle$ satisfies K2. It is evident that $(E \cap F) \subseteq E$ and $(E \cap F) \subseteq F)$. By K2, $K(E \cap F) \subseteq K(E)$ and $K(E \cap F) \subseteq K(F)$. Hence, $K(E \cap F) \subseteq (K(E) \cap K(F))$. □

This property K2* is the relaxing Conjunction ($K(E \cap F) = K(E) \cap K(F)$), which is one of the standard properties of the knowledge operator. Theorem 5 needs only K2*, not Conjunction. See proofs of Lemmas 2 and 4.

As the following proof of Lemma 2 shows, K4 is not necessary.

Lemma 2 *If $\langle \Omega, K, U \rangle$ satisfies $K2$–3 and $U1$, then $U2$ is satisfied.*

Proof Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–3 and U1. Then,

$$KU(E) \overset{\text{K2, U1}}{\subseteq} K(\neg K(E) \cap \neg K \neg K(E))$$

$$\overset{\text{K2}^*}{\subseteq} K\neg K(E) \cap K\neg K\neg K(E)$$
$$\overset{\text{K3}}{\subseteq} K\neg K(E) \cap \neg K\neg K(E) = \emptyset. \qquad \square$$

Lemma 2 suggests that if a standard state-space model satisfies Monotonicity, Truth, and Plausibility, then KU Introspection is satisfied.

Lemma 3 *If* $\langle \Omega, K, U \rangle$ *satisfies* $K2$–3 *and* $U0$, *then*

*U3** *Reverse AU Introspection*: $UU(E) \subseteq U(E)$.

Proof Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–3 and U0. By Lemma 2, U2 holds. Then, $UU(E) \overset{\text{U0}}{=} \neg KU(E) \cap \neg K\neg KU(E) \overset{\text{U2}}{=} \Omega \cap \neg K(\Omega) = \neg K(\Omega)$ Here, by K2, if $E \subseteq \Omega$, then $K(E) \subseteq K(\Omega)$, and if $\neg K(E) \subseteq \Omega$, then $K\neg K(E) \subseteq K(\Omega)$. Then, $\neg K(\Omega) \subseteq \neg K(E)$ and $\neg K(\Omega) \subseteq \neg K\neg K(E)$, that is, $UU(E) = \neg K(\Omega) \subseteq \neg K(E) \cap \neg K\neg K(E) = U(E)$. $\square$

U3* is proposed by Fukuda (2021). By Lemma 3, the following property holds.

Remark 2 Suppose that Lemma 3. If U3 holds, then

U3** $U(E) = UU(E)$.

The following properties are shown by Modica and Rustichini (1994).[7]

Lemma 4 (Modica and Rustichini (1994)) *If* $\langle \Omega, K, U \rangle$ *satisfies* $K2$–4 *and* $U0$, *then it also satisfies the following*:

A1 AK-Self Reflection: $AK(E)=A(E)$;
A2 AA-Self Reflection: $AA(E)=A(E)$; *and*
A3 A-Introspection: $KA(E)=A(E)$.

Proof Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–4 and A1.

Proof of A1. $AK(E) = KK(E) \cup K\neg KK(E) \overset{\text{K4}}{=} K(E) \cup K\neg K(E) = AE$.

Proof of A3. First, given $K(E)$, by K2 and K4, because $K(E) \subseteq A(E)$, $K(E) = KK(E) \subseteq KA(E)$ $(*)$. Next, given $K\neg K(E)$, $K\neg K(E) \subseteq A(E)$ and $K\neg K(E) \subseteq \neg K(E)$ by K3, that is, $K\neg K(E) \subseteq (\neg K(E) \cap A(E))$. Then, $K\neg K(E) \overset{\text{K4}}{=} KK\neg K(E) \overset{\text{K2}}{\subseteq} K(\neg K(E) \cap A(E))$. $K(\neg K(E) \cap A(E)) \overset{\text{K6}}{\subseteq} K\neg K(E) \cap KA(E)$. That

[7] Those properties are proposed in other literature. A1 and A2 are proved by Modica and Rustichini (1999) and Halpern (2001), respectively, and A3 is proved by Heifetz et al. (2006).

is, $K\neg K(E) \subseteq KA(E)$. Then, $A(E) = K(E) \cup K\neg K(E) \subseteq K(E) \cup KA(E)$. Because $K(E) \subseteq KA(E)$ $(*)$, $K(E) \cup KA(E) = KA(E)$, that is, $A(E) \subseteq KA(E)$. By K3, because $KA(E) \subseteq A(E)$, $KA(E) = A(E)$.

Proof of A2. $AA(E) = KA(E) \cup K\neg KA(E) \overset{\text{A3}}{=} A(E) \cup K\neg A(E) = A(E) \cup KU(E) \overset{\text{U2}}{=} A(E) \cup \emptyset = A(E)$. □

Those properties can be proved in set-theoretical approaches as follows: In contrast to the proofs of Lemmas 1 and 2, The proof of Lemma 4 needs Positive Introspection, K4.

By the above lemmas, we can prove our main theorem.

Proof (Proof of Theorem 5)

Suppose that $\langle \Omega, K, U \rangle$ satisfies K2–4 and U0.

First, assume U3; then, by Remark 2, $U(E) = UU(E)$. Next, by a definition of the awareness operator, for any $E \subseteq \Omega$, $A(E) \overset{\text{U3}^{**}}{=} AU(E) \overset{\text{U0}}{=} KU(E) \cup K\neg KU(E) \overset{\text{U2}}{=} \emptyset \cup K(\neg\emptyset) = \emptyset \cup K(\Omega) = K(\Omega)$. Because E is arbitrary, $A(E) = A(\neg E) = K(\Omega)$. Therefore, $U(E) = \neg A(E) = \neg A(\neg E) = U(\neg E)$.[8]

Next, assume U4, that is, $U(E) = U(\neg E)$. By Lemma 4, because A2, that is, $AA(E) = A(E)$, is satisfied, $UA(E) = U(E)$. By U4, $U(E) = UA(E) = UU(E)$. Hence, $U(E) \subseteq UU(E)$. □

By Theorem 5, we can generalize Theorem 4.

Proof (Proof of Theorem 4.)

Suppose that $\langle \Omega, K, U \rangle$ satisfies K1–4 and U0.

First, assume U4. By Theorem 5, U3 holds.

Next, assume U3. By Lemma 3 and Remark 2, $U(E) = UU(E)$ holds. Then, $U(E) \overset{\text{U3}^{**}}{=} UU(E) \overset{\text{U0}}{=} \neg KU(E) \cap \neg K\neg KU(E) \overset{\text{U2}}{=} \neg\emptyset \cap \neg K(\neg\emptyset) = \Omega \cap \neg K(\Omega) = \neg K(\Omega) \overset{\text{K1}}{=} \neg\Omega = \emptyset$. By Remark 1, K5 holds.

Finally, assume K5. By Remark 1, U5 holds, that is, $U(E) = \emptyset$ for any $E \subseteq \Omega$. Because E is arbitrary, $U(E) = U(\neg E) = \emptyset$. That is, U4 holds. □

Theorems 1, 2, and 3 are evident from Theorem 4.

Note that Theorem 5 generalizes Theorems 1 and 4, but not Theorems 2 and 3. Theorems 1 and 4 require K4, whereas Theorems 2 and 3 do not require K4.

The relationship between Theorems 4 and 5 implies the following corollary.

Corollary 1 *In $\langle \Omega, K, U \rangle$, if K2–4 and U0 hold, but K1 does not hold, then K5 equivalent to neither U3 nor U4, but U3 and U4 are equivalent.*

[8] If we use U1 and not U0, $AU(E) \supseteq KU(E) \cup K\neg KU(E)$. Then, Symmetry might not hold, because $A(E) = K(\Omega)$ might not hold. See Example 1.

Table 1 Example 1

E	$K(E)$	$\neg K(E)$	$K\neg K(E)$	$\neg K\neg K(E)$	$U(E)$	$UU(E)$	$U(\neg E)$
a	a	b, c	b	a, c	c	c	c
b	b	a, c	a	b, c	c	c	c
c	$\emptyset$	a, b, c	a, b	c	c	c	c
a, b	a, b	c	$\emptyset$	a, b, c	c	c	c
a, c	a	b, c	b	a, c	c	c	c
b, c	b	a, c	a	b, c	c	c	c
a, b, c	a, b	c	$\emptyset$	a, b, c	c	c	c
$\emptyset$	$\emptyset$	a, b, c	a, b	c	c	c	c

Example 1 Let us consider a state space $\Omega = \{a, b, c\}$. Suppose that the knowledge operator K satisfies the following: $K(\Omega) = \{a, b\}$, $K(\{a\}) = \{a\}$, $K(\{b\}) = \{b\}$, $K(\{c\}) = \emptyset$, $K(\{a, b\}) = \{a, b\}$, $K(\{a, c\}) = \{a\}$, $K(\{b, c\}) = \{b\}$, and $K(\emptyset) = \emptyset$. In this example, the knowledge operator satisfies Monotonicity (K2), Truth (K3), and Positive Introspection (K4), but only Necessitation (K1) does not hold. Based on this formulation, $\neg K(E)$, $K\neg K(E)$, $\neg K\neg K(E)$, and the unawareness operator U based on Modica and Rustichini's definition can be described as in Table 1. It is clear that Negative Introspection (K5) (or Triviality, U5) does not hold, but AU Introspection (U3) and Symmetry (U4) hold.

Here, let us use plausible unawareness $U^* : 2^\Omega \to 2^\Omega$, that is, U^* satisfies U1 but not U0, and suppose $U^*(\{a\}) = \emptyset$, $U^*(\{b, c\}) = \{c\}$, $U^*U^*(\{a\}) = \{c\}$, and $U^*U^*(\{b, c\}) = \{c\}$. Then, U4 holds, but U3 does not hold. In other words, plausible unawareness might not lead equivalence between AU Introspection and Symmetry.

□

5 Concluding remarks

This note (i) shows that AU Introspection and Symmetry for unawareness are equivalent when relaxing Necessitation; and (ii) generalizes a proof of Theorem 4 proposed by Chen et al. (2012).

This study excludes Necessitation. Given $\langle \Omega, K, U \rangle$, let $\omega \in \Omega$ be a state. Then, Necessitation is redefined as follows: for any state $\omega \in \Omega$, $\omega \in K(\Omega)$. This means that in any state, the agent knows the whole state space. In other word, she or he knows tautologies. When we relax Necessitation, there exists some state ω such that $\omega \notin K(\Omega)$; that is, in some state, the agent does not know the whole state space even if in other state $\omega' \in \Omega$, $\omega' \in K(\Omega)$. In other words, by excluding Necessitation, depending on the given state, the agent might or might not know the whole state space. This may seem contradictory. However, depending on the specific situation, it can be said that there is no contradiction. Let us consider the example of COVID-19 as follows. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. ω_1 is interpreted as "The agent is infected with COVID-19 and gets a fever," ω_2 is interpreted as "The agent gets a fever, but is not infected with COVID-19," ω_3 is interpreted as "The agent is infected with COVID-19 but does not get a fever," and ω_4 is interpreted as "The agent is not infected with COVID-19." Here, let $\omega_1 \in K(\Omega)$, $\omega_2 \in K(\Omega)$, $\omega_3 \notin K(\Omega)$, and $\omega_4 \notin K(\Omega)$. In other words, at ω_1 and ω_2, the agent knows the possibility about her COVID-19 status, whereas she does not know the possibility about it at ω_3 and ω_4. This formulation can be interpreted as follows. When an agent infected with COVID-19 develops a fever, she suspects that she is infected with COVID-19. However, if an agent is infected with COVID-19 but does not have a fever, she does not realize that she is infected with COVID-19 because it is the same condition as not being infected with COVID-19. Moreover, she might have forgotten what she even knew about COVID-19. This can be rephrased as follows. The agent is not usually "aware" of COVID-19 and forgets about its existence when she does not have a fever, but when the agent does have a fever, she recalls her knowledge of COVID-19 and considers the possibility of infection. It means that "unawareness" can be interpreted as "Forgetting." It seems that by relaxing Necessitation, a more realistic situation can be modeled. As it progresses that a study excluding Necessitation, new suggestions may be made.

This study has one limitation. We exclude only Necessitation, because our focus is on axioms of the knowledge operator. However, as well known, in standard information structures that may be non-partitional, both Necessitation and Monotonicity must hold.[9] Hence, the knowledge operator based on the standard information function or the standard possibility correspondence cannot exclude only Necessitation. In other words, AU Introspection and Symmetry must be equivalent to "trivial" unawareness in standard information structures. In future work, we aim to define a novel knowledge operator that excludes only Necessitation in standard information structures.[10]

Recent studies related to the present one include those by Fukuda (2021) and Tada (2021). Fukuda (2021) proposes generalized state-space models that nest both unawareness structures and non-partitional state-space models. He posits that AU Introspection is not consistent with Necessitation, relaxes AU Introspection, and replaces AU Introspection with Reverse AU Introspection ($UU(E) \subseteq U(E)$). Tada (2021) discusses multi-attribute state spaces with complete lattices. This contrasts with (Heifetz et al., 2006), in which the family of disjoint spaces is a lattice structure, his state space is a lattice structure, and the interpretation of all state spaces is the same. His knowledge operator is closer to that of Heifetz et al. (2006) than to that of standard models. In his study, the Symmetry of the unawareness operator is equivalent to the Necessitation of the knowledge operator; that is, if Necessitation is relaxed, Symmetry does not hold, although AU Introspection holds. He names this impossibility Reverse Symmetry. By contrast, we show that Symmetry holds even if

[9] See ((Dekel et al., 1998): 164).

[10] Recently, (Tada, 2021) proposes a definition of the novel knowledge operator based on the information function allowing some information set to be empty such that Monotonicity holds, but Necessitation does not hold. However, his interpretation that the information set is empty is different from the standard interpretation. Hence, future research should reconsider his interpretation.

Necessitation is relaxed. Our result is different from his. This finding means that his models are different from standard state-space models.

Data avalability This paper does not generate or analyze data.

Declarations

Conflict of interest The author declares no conflicts of interest associated with this manuscript.

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