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Expected utility, independence, and continuity

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Abstract
In this paper, we provide two novel expected utility theorems by suitably adjusting the independence and continuity axioms. Our first theorem characterizes expected utility preferences using weak versions of the independence axiom (with varying mixture weights) and a new weak continuity axiom. Our second theorem characterizes these preferences using weaker versions of the independence axiom (with mixture weights fixed at 1/2) and a strong topological continuity axiom. We provide useful examples to illustrate the tightness of these characterizations.

Keywords Expected utility · Independence · Continuity · Even-chance mixtures

1 Introduction

The expected utility model has been part of the standard toolkit of economics ever since (von Neumann & Morgenstern, 1947)'s seminal work on the theory of games. Two key implications of the expected utility preferences are well-known, the independence and continuity axioms. In this paper, we derive a number of expected utility theorems using different forms of independence and continuity axioms.

In satisfying above objectives, two fundamental representation results are illuminating: one by von Neumann and Morgenstern (1947) and another by Herstein and Milnor (1953). These two results characterize the same expected utility model, but with a certain tradeoff between them: the von Neumann and Morgenstern (1947) result uses an order-theoretic Archimedean continuity, but requires a stronger independence (IND), while the Herstein and Milnor (1953) result uses a weaker independence, but requires a topological mixture continuity. This suggests that while weakening or strengthening the continuity axiom, we may consider a suitable independence axiom to characterize the expected utility model.

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We start our analysis by noting that the independence axiom can be decomposed into a number of weaker axioms (Proposition 1): translation independence (tIND) together with scale independence (sIND) or betweenness (BET). While violations of tIND reflect the common consequence effect and violations of sIND coincide with the common ratio effect, the BET axiom is compatible with these two effects.[1] Moreover, although sIND or BET together with tIND imply IND, we note that the two of them together (sIND and BET) do not necessarily imply IND (Example 1). However, we also show that when these two axioms are equipped with a novel weak continuity (wCON) axiom, then the preference order has an expected utility representation (Theorem 1). This result shows that under wCON, the IND axiom is equivalent to any two of the three weaker independence axioms, tIND, sIND, and BET.

Next, we consider a strong continuity (sCON) axiom, which is a topological mixture continuity that implies wCON. By Proposition 1 and Theorem 1, we know that any two of the weaker independence conditions, tIND, sIND, and BET, can be equivalent to IND under the weak continuity axiom. By employing sCON, we further show that tIND must be equivalent to sIND and BET together. That is, under sCON, the tIND axiom becomes equivalent to IND implying that we can weaken the IND axiom using tIND (or sIND and BET together) to obtain an expected utility representation. In fact, we show that when sCON is assumed, a substantially weaker form of these axioms can be used, in which the mixture weight is fixed at 1/2 (Theorem 2).

The tIND axiom is related to the weak certainty independence axiom used by Maccheroni et al. (2006) who appeal to Herstein and Milnor (1953) to obtain a linear representation. A weaker form of the sIND axiom was used by Safra and Segal (1998) and Diecidue et al. (2009) to characterize rank-dependent utility models. The sIND axiom is also related to the best-outcome independence axiom of Maccheroni (2002) who derives a non-expected utility model that takes the minimum of a set of expected utilities. The BET axiom is used by Chew (1983), Dekel (1986), and Gul (1991) to obtain non-expected utility models allowing for common consequence and ratio effects. Shapley and Baucells (1998) and Dubra et al. (2004) use the sCON axiom together with IND while dropping completeness to obtain a multi-expected utility representation. Our approach of obtaining an expected utility representation differs from these works, especially for Theorem 1, since we use wCON and BET axioms to obtain an indifference set and use the sIND axiom to conclude that all possible indifference sets must be parallel to each other implying that there is an expected utility representation.

The rest of the paper is organized as follows. In Sect. 2, we introduce the framework. Section 3 provides a brief overview of some of the earlier expected utility theorems. In Sect. 4, we provide our analysis of the expected utility by employing different forms of independence and continuity axioms. In Sect. 5, we discuss the relation of independence and continuity axioms in varying degrees of

[1] In fact, a strand of literature on non-expected utility theory has utilized BET to offer choice models compatible with the common consequence and ratio effects, which are known as Allais (1953) paradoxes. For earlier accounts of this literature, see, e.g., survey articles by Machina (1987) and Starmer (2000).

strength together with a diagram depicting all implications discussed in this paper. Proofs of all results including observations noted by examples are provided in an Appendix.

2 Framework

Let I denote the set $\{1, 2, ..., n\}$ and let $I_0 = I \cup \{0\}$, where $n \geq 2$. In the following, X is a finite set of $n + 1$ prizes, with typical elements $x_i \in X$ for $i \in I_0$ called *outcomes*; P is the set of all probability distributions on X with typical elements $p, q, r \in P$ called *lotteries.*[2] With slight abuse of notation, we denote a lottery yielding an outcome $x \in X$ for sure by $x \in P$. We denote by p_i the probability of outcome x_i under lottery p. For any $\alpha \in [0, 1]$, let $p\alpha q$ denote a *mixed-lottery*, which is the mixture of lotteries p and q. That is, $p\alpha q$ is the lottery $r \in P$ such that $r_i = \alpha p_i + (1 - \alpha)q_i$ for all $i \in I_0$.

Our primitive is a binary relation $\succsim$ on the set of lotteries, with asymmetric part denoted as $\succ$ and symmetric part denoted as $\sim$. We interpret this binary relation $\succsim$ as the DM's risk preferences and assume that it is a preference order (i.e., a complete and transitive binary relation). We also assume that the outcomes are ordered such that $x_n \succ p \succ x_0$ for any $p \in P$ with $p_i > 0$ for some $i \in I \setminus \{n\}$. Without loss of generality, assume that $x_i \succsim x_j$ if and only if $i \geq j$. Let U denote the set of normalized utilities that are monotone with respect to the DM's preferences $\succsim$ over X; that is, $x_i \succ x_j$ if and only if $u_i > u_j$. More formally, let $U = \{u \in \mathbb{R}^{n+1} : u_0 = 0, u_n = 1, u_i > u_j \text{ iff } x_i \succ x_j\}$.

For any $p \in P$ and $u \in \mathbb{R}^{n+1}$, let $u(p)$ denote the product $u \cdot p \in \mathbb{R}$. In particular, when $u \in U$, let $u(p) \in [0, 1]$ denote the expected utility of p under u. We say that $\succsim$ has a representation if there exists a function $f : P \to \mathbb{R}$ such that $p \succ q$ if and only if $f(p) > f(q)$. We say f is an *expected utility representation* if there exists $u \in U$ such that for all $p \in P$, we have $f(p) = u(p)$. Note that whenever there is a normalized utility $u \in U$ providing an expected utility representation, then it must be unique in U.

3 Basic axioms and expected utility theorems

In this section, we briefly review some of the well-known expected utility theorems.

3.1 von Neumann–Morgenstern expected utility theorem

The following axiom is the key behavioral implication of the expected utility model.

Axiom (Independence, IND) For any $p, q, r \in P$ and $\alpha \in (0, 1)$, $p \succ q$ (resp. $p \sim q$) implies $p\alpha r \succ q\alpha r$ (resp. $p\alpha r \sim q\alpha r$).

[2] We consider a finite outcome setting for ease of exposition and its practicality for experimental work. Our results will easily extend to an infinite outcome setting as long as there is a worst or a best outcome.

Independence says that mixing two lotteries with a common lottery should not alter the preference for any mixture weight or common lottery used. In addition to IND, a continuity axiom is needed to establish an expected utility representation.[3] The following continuity axiom is arguably the simplest one used in the literature for this purpose.

Axiom (Archimedean continuity, aCON) For any $p, q, r \in P$, $p \succ q \succ r$ implies $p\alpha r \succ q$ and $q \succ p\beta r$ for some $\alpha, \beta \in (0, 1)$.

Archimedean continuity states that there is no lottery so good (resp. bad) that when mixed with a lottery worse (resp. better) than another lottery, the mixture is always better (resp. worse) than the intermediate lottery. As is well-known, these two axioms imply an expected utility representation.

Theorem (von Neumann–Morgenstern) *A preference order $\succsim$ satisfies IND and aCON if and only if it has an expected utility representation.*

This result, given by von Neumann and Morgenstern (1947), provided the first axiomatic foundation for the expected utility model.[4]

3.1.1 Herstein–Milnor expected utility theorem

Another well-known expected utility theorem uses a topological mixture continuity axiom instead (rather than the order-theoretic aCON) while it weakens the independence requirement.

Axiom (Mixture continuity, mCON) For any $p, q, r \in P$, the sets $\{\alpha : p\alpha r \succsim q\}$ and $\{\alpha : q \succsim p\alpha r\}$ are closed.

Mixture continuity implies that the preference ordering is continuous in probability distributions by requiring above two sets to be closed with respect to the standard topology. To establish an expected utility representation, mCON is associated with an independence condition.

Axiom (Herstein–Milnor independence, hm-IND) For any $p, q, r \in P$, $p \sim q$ implies $p1/2r \sim q1/2r$.

The Herstein–Milnor independence axiom fixes the mixture weight at 1/2 and requires independence to hold only for the indifference relation. Herstein and Milnor (1953) showed that these two axioms imply an expected utility representation.

[3] IND alone is not enough to guarantee a representation for a preference order. For instance, let $\succsim$ be a preference order such that $p \succsim q$ if $(u(p), v(p)) \geq_L (u(q), v(q))$ for any $p, q \in P$, where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, and $u(.)$ and $v(.)$ are two distinct expected utility functions in U. Clearly, this preference order satisfies IND but it has no real-valued representation, expected utility or not.

[4] Although von Neumann and Morgenstern (1947) provided the first expected utility representation result, the independence axiom was implicit in their result. The first explicit statement of the independence axiom appeared in Marschak (1950), Nash (1950), and Malinvaud (1952); see, e.g., Hammond (1998) and Bleichrodt et al. (2016) for details. Also see our discussions in Sect. 5 on independence and continuity. I thank an anonymous reviewer for their observation on the history of the independence axiom used for characterizing expected utility preferences.

Theorem (Herstein–Milnor) *A preference order $\succsim$ satisfies hm-IND and mCON if and only if it has an expected utility representation.*

Although these two results characterize the same expected utility model, we see a certain tradeoff between them: the von Neumann and Morgenstern (1947) result uses an order-theoretic continuity, but requires a stronger independence, while the Herstein and Milnor (1953) result uses a weaker independence, but requires a topological continuity. This suggests that by strengthening or weakening the continuity requirement, we can obtain a suitable independence axiom to establish an expected utility representation. In the next section, we will investigate these possibilities in more detail.

4 Analysis

In this section, we provide our analysis of the expected utility model. We first discuss how IND can be decomposed into weaker conditions. We then show how one can obtain the expected utility model using these weaker conditions under a weak and a strong continuity axiom.

4.1 Shades of independence

The IND axiom has been extensively scrutinized in the literature given that it is the signifying behavioral implication of the expected utility model. In fact, starting with the Allais (1953) thought experiments, numerous experimental results show that the IND axiom can often be violated by the experimental subjects. Two such experimental results are well-known: *common consequence* and *ratio effects*.

The common consequence (CC) effect can be summarized as the violation of the following weak independence axiom.

Axiom (Translation independence, tIND) For any $p, q, r, s \in P$ and $\alpha \in (0,1)$, $p\alpha r \succsim q\alpha r$ implies $p\alpha s \succsim q\alpha s$.

Translation independence reflects the idea that the preferences between mixtures of two lotteries with a common lottery should stay the same as long as the weight of the two lotteries is fixed across comparisons while the common lottery can vary. Unlike IND, the tIND axiom allows for a change of preferences whenever the weights also vary, and so it is a weaker independence condition than IND.

The common ratio (CR) effect, on the other hand, is about the failure of the following weak independence axiom.

Axiom (Scale independence, sIND) For any $p, q \in P$ and $\alpha \in (0,1)$, $p \succ q$ (resp. $p \sim q$) implies $p\alpha x_0 \succ q\alpha x_0$ (resp. $p\alpha x_0 \sim q\alpha x_0$).

Scale independence says that when mixing two lotteries with the worst-outcome lottery, the preference between the lotteries do not change whenever the mixture weights vary.[5] Unlike IND, the sIND axiom allows the preferences to change when the common lottery is different than the worst-outcome lottery, and so it is a weaker independence condition than IND.

To rationalize the CC and CR effects, a strand of literature on decision-making under risk proposed models which can violate both tIND and sIND, but satisfy the following weak form of independence.[6]

Axiom (Betweenness, BET) For any $p, q \in P$ and $\alpha \in (0,1)$, $p \succ q$ (resp. $p \sim q$) implies $p \succ p\alpha q \succ q$ (resp. $p \sim p\alpha q \sim q$).

Betweenness reflects the idea that the mixture of two lotteries should stay in between them in terms of preference order; that is, the better (resp. worse) lottery should be deemed better (resp. worse) against the mixture of lotteries no matter what mixing weight is used. Clearly, each of the axioms above, tIND, sIND, and BET, are implied by the IND axiom. We note that together these axioms are equivalent to the IND axiom.

Proposition 1 *A preference order $\succsim$ satisfies (i) IND if and only if (ii) it satisfies tIND and sIND if and only if (iii) it satisfies tIND and BET.*

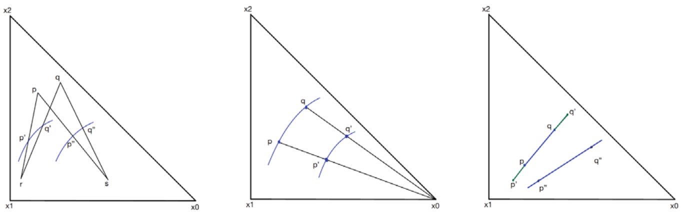
Proposition 1 shows that we can decompose IND into two weaker axioms; either tIND together with sIND or tIND together with BET.

4.1.1 Discussion

We now demonstrate some implications of tIND, sIND, and BET for the case $n = 2$. Consider Fig. 1a and suppose that tIND holds. Suppose we have $p' \sim q'$ and p, q, r are such that $p' = p\alpha r$ and $q' = q\alpha r$ for some $\alpha \in (0,1)$. Suppose also that p'', q'', s are such that $p'' = p\alpha s$ and $q'' = q\alpha s$. Then tIND implies that $p'' \sim q''$. Notice that the distance between p' and p'', and q' and q'' are equal to each other. Thus, given that all these parameters (i.e., r, s, and α) are arbitrary, indifference curves must be parallel to each other. Moreover, since these curves must be parallel to each other along any direction, the indifference curves must be straight lines. But notice also that these straight lines do not need to be solid lines. That is, there could be gaps in indifference sets. To see this, suppose p, q, r are such that $p' = p1/2r$ and $q' = q1/2r$. Then, by tIND, we must have $p \sim q1/2p$ and $p1/2q \sim q$ implying that p, q and their midpoint $p1/2q$ must be on the same indifference curve. This argument, however, does not need to hold for an arbitrary point in between p and q. The preference order given in Example 4 below shows that this can indeed be the case; that is, indifference sets can have arbitrarily many gaps in them.

[5] An analogous-scale independence axiom can be defined using the best-outcome x_n. In general, when the set of outcomes is bounded below (resp. above) such that there is a worst (resp. best) outcome, then the worst (resp. best) outcome can be used to define the scale independence axiom.

[6] See, e.g., Karni and Safra (1989), Crawford (1990), and Epstein and Zin (2001) for betweenness applications, and Camerer and Ho (1994) and Hey and Orme (1994) for experimental evidence on betweenness and related axioms.



(a) tIND (b) sIND (c) BET

Fig. 1 Illustration of independence axioms

Now consider Fig. 1b above, and suppose that sIND holds. Suppose we have $p \sim q$ and p', q' are such that $p' = p\alpha x_0$ and $q' = q\alpha x_0$ for some $\alpha \in (0,1)$. Then, by sIND, we must have $p' \sim q'$. Notice that the slope of the line passing through p and q is the same as the slope of the line passing through p' and q'. Since all these parameters are arbitrary, we conclude that indifference curves must be parallel to each other along the rays starting from x_0.

Finally, consider Fig. 1c and suppose that BET holds. Suppose we have $p \sim q$. Then, by BET, clearly we have $p \sim r$ for any $r = p\alpha q$ for some $\alpha \in (0,1)$. Now suppose also that p', q' are such that $p = p'\beta q$ and $q = q'\gamma p$ for some $\beta, \gamma \in (0,1)$. Then, by BET, we must have $p' \sim p$ because otherwise, by BET, we will have $p'\beta q \succ q$ or $q \succ p'\beta q$, a contradiction. A similar argument applies for q', and so $q' \sim q$. In sum, BET implies that indifference sets must be straight lines (but not necessarily parallel to each other as depicted in Fig. 1c above), and whenever they are not singletons, they must be solid.

We see that two key implications of tIND, parallel and straight indifference sets, are also implied by sIND and BET together. Given this, it might seem plausible to expect that sIND and BET together imply tIND. Example 1, however, shows that in general sIND and BET together do not necessarily imply tIND.

Example 1 Suppose $n = 2$ and let $\succsim$ be a preference order over P such that for all $p, q \in P$,

$$p \succsim q \text{ if } (\psi(p), \gamma(p)) \geq_L (\psi(q), \gamma(q)),$$

where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, $\gamma(r) = r_2 + r_1$ for any r in P, and $\psi(r) = \frac{r_2}{r_2 + r_1}$ whenever $r_2 + r_1 > 0$ and $\psi(r) = 0$ otherwise for any r in P. $\diamondsuit$

In the Appendix, we show that the preference order defined in Example 1 satisfies both sIND and BET, yet it fails to satisfy tIND, and therefore IND.[7]

4.2 Independence with weak continuity

In this section, we establish an expected utility theorem by weakening independence by requiring only sIND and BET, while also employing a novel weak continuity

[7] We have assumed $n = 2$ in this example for simplicity. The example can be easily extended to any finite outcome space with $n \geq 3$.

axiom. We also give counter examples demonstrating that further weakening or replacing these axioms will not guarantee an expected utility representation.

4.2.1 A weak continuity axiom

Example 1 shows that sIND and BET together are not strong enough to imply IND. In fact, this preference order does not even have a representation because it does not allow for substitutions; some of the outcomes are infinitely desirable over others. To avoid these type of preference orders, we do propose the following weak continuity axiom.

Axiom (Weak continuity, wCON) For any distinct $i,j \in I$, there exist $p,q \in P$ with $p \sim q$ and $p_k = q_k = 0$ for all $k \in I \backslash \{i,j\}$ such that $(p_i - q_i)(q_j - p_j) > 0$.

Weak continuity axiom allows for compensation between likelihoods of any two different outcomes $x_i, x_j \in X$ for $i,j \in I$, while the worst outcome's likelihood is used to balance the accounting.[8] It is clear that the mCON axiom directly implies wCON. Next, we will use this weak continuity axiom to obtain an expected utility representation.

4.2.2 An expected utility theorem

The following result shows that an expected utility representation can be obtained by requiring only a weaker set of independence axioms and a weak continuity axiom.

Theorem 1 *A preference order $\succsim$ satisfies sIND, BET, and wCON if and only if it has an expected utility representation.*

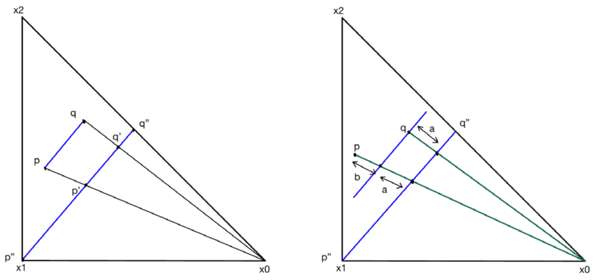
The proof we provide for this result is relatively short while we believe it is also instructive. To prove Theorem 1, we first construct an indifference set passing through outcome x_1. The indifference set is a convex hyperplane such that any point in P can be projected onto it using outcome x_0.[9] We then define all utility weights $u \in U$, and by considering the projections and using the two axioms sIND and BET, we show that $u \in U$ provides a representation for the preference order.

To demonstrate the intuition, suppose $n = 2$ and consider Fig. 2a above. By wCON, there exist some $p,q \in P$ such that $p \sim q$ and $(p_1 - q_1)(q_2 - p_2) > 0$. Without loss of generality, suppose that $(p_1 - q_1), (q_2 - p_2) > 0$ as depicted in Fig. 2a above. Consider line A which passes through $p'' = x_1$ and $q'' = x_2 \frac{q_2 - p_2}{p_1 - q_1} x_0$. When $\alpha = \frac{q_2 - p_2}{p_1 q_2 - p_2 q_1} > 0$, both $p' = p\alpha x_0$ and $q' = q\alpha x_0$ must be on A. By sIND, we must have $p\alpha x_0 \sim q\alpha x_0$. As we argued before (with the help of Fig. 1c above), we

[8] The wCON axiom is a type of solvability axiom. There are many studies in the literature which use solvability axioms to derive an expected utility representation, such as Dekel (1986). For a recent work on expected utility representation using solvability (in a Savage framework), see Abdellaoui and Wakker (2020).

[9] In applying our proof method, requiring scale independence with respect to the worst outcome is crucial since we need to be able to project any lottery onto the indifference set passing through outcome x_1. A common lottery by which this is possible is the worst outcome x_0; another possibility is the best outcome x_n.

Fig. 2 Illustration of Theorem 1 proof



(a) Indifference set (b) Representation

must have $x_1 \sim p\alpha x_0$ and $q\alpha x_0 \sim x_2 \frac{q_2-p_2}{p_1-q_1} x_0$ by BET, and so $x_1 \sim x_2 \frac{q_2-p_2}{p_1-q_1} x_0$ implying that A is an indifference set passing through x_1. Since $x_2 \succ x_1 \succ x_0$, we must have $\frac{q_2-p_2}{p_1-q_1} \in (0,1)$ by BET. Let $u \in U$ such that $u_0 = 0$, $u_2 = 1$, and $u_1 = \frac{q_2-p_2}{p_1-q_1}$. By definition, we have $u(r) = u_1$ if and only if $r \in A$. Moreover, for any $r \in P$ with $u(r) > u_1$, we have $r \frac{u_1}{u(r)} x_0 \in A$ and for any $r \in P$ with $u(r) < u_1$, we have $r = s \frac{u(r)}{u_1} x_0$ for some $s \in A$. Using sIND and BET, we then show that $p \succ q$ if and only if $u(p) > u(q)$ for any $p, q \in P$. For instance, consider Fig. 2b above and let $u(p) > u(q) > u_1$ as depicted. Then we have $(pbx_0)ax_0, qax_0 \in A$, where $b = \frac{u(q)}{u(p)}$ and $a = \frac{u_1}{u(q)}$, and so $p \frac{u(q)}{u(p)} x_0 \sim q$ by sIND. Since $p \succ x_0$, by BET, we have $p \succ q$. By applying similar arguments, we complete the proof.

Notice that either sIND or BET can be replaced with tIND in Theorem 1 given that any two of these axioms imply the third one when wCON holds.

Corollary 1 *A preference order* $\succsim$ *satisfies* (*i*) *tIND, BET, and wCON if and only if* (*ii*) *it satisfies sIND, tIND, and wCON if and only if* (*iii*) *it has an expected utility representation.*

4.2.3 Counter examples

We know by Example 1 that when obtaining an expected utility representation in Theorem 1, we cannot drop wCON. Can we weaken any of the two axioms, sIND and BET, or replace them with tIND, and still establish an expected utility representation? The following three examples show that this is not possible. Thus, our axioms used in Theorem 1 are tight.

Example 2 given below shows that the sIND axiom cannot be replaced with the following weak scale independence axiom to obtain an expected utility representation.

Axiom (Weak scale independence, wsIND) For any $p, q \in P$ and $\alpha \in (0,1)$, $p \succ q$ implies $p\alpha x_0 \succ q\alpha x_0$.

Unlike sIND, the wsIND axiom allows to have $p \sim q$ and $p\alpha x_0 \succ q\alpha x_0$ (or $q\alpha x_0 \succ p\alpha x_0$) for some $p, q \in P$ and $\alpha \in (0,1)$.

Example 2 Let $\succsim$ be a preference order such that for any $p, q \in P$,

$$p \succsim q \text{ if } v(p) \geq v(q),$$

where for any $r \in P$, $v(r) = \sum_{i \in I_0} u(x_i, v(r)) r_i$ for some function $u(.,.) : X \times [0,1] \to \mathbb{R}$ which is continuous in its arguments, and increasing in the preference ordering on X such that $u(x_0, a) = 0$ and $u(x_n, a) = 1$ for any $a \in [0,1]$. $\diamondsuit$

The preference order defined in Example 2, which is also called an implicit expected utility, was first proposed and axiomatically characterized by Dekel (1986) to allow behavior compatible with Allais (1953) paradoxes. In the Appendix, we show that the preference order defined in Example 2 satisfies wsIND, BET, and wCON, but satisfies sIND only when it has an (explicit) expected utility representation; that is, whenever it has a "proper" implicit expected utility representation, then it fails to satisfy sIND. Thus, we cannot replace sIND with wsIND in Theorem 1.

Example 3 given below shows that the BET axiom cannot be replaced in Theorem 1 with the following weak betweenness axiom.

Axiom (Weak betweenness, wBET) For any $p, q \in P$ and $\alpha \in (0,1)$, $p \succsim q$ implies $p \succsim p\alpha q \succsim q$.

Notice that unlike BET, the wBET axiom allows to have $p \succ q$ and $p \sim p\alpha q$ (or $p\alpha q \sim q$) for some $p, q \in P$ and $\alpha \in (0,1)$.

Example 3 Let $\succsim$ be a preference order such that for any $p, q \in P$,

$$p \succsim q \text{ if } (\, u(p),\ \lambda(p)\,) \geq_L (\, u(q),\ \lambda(q)\,),$$

where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, $u(.)$ is an expected utility function defined on P, and $\lambda(.)$ is an indicator function defined on $\mathbb{R}$ such that for all $p \in P$, $\lambda(p) = 1$ if $v(p) \geq v(x_0)$ and $\lambda(p) = 0$ if $v(p) < v(x_0)$ for some $v \in \mathbb{R}^n$ with $v(x_i) < v(x_0) < v(x_j)$ for some $i, j \in I$. $\diamondsuit$

In the Appendix, we show that the preference order defined in Example 3 satisfies sIND, wBET, and wCON, yet fails to satisfy BET, and therefore does not have a representation. Thus, we cannot replace BET with wBET in Theorem 1 to obtain an expected utility representation.

Can tIND imply the two weak independence axioms, sIND and BET, when wCON also holds? The following example shows that this is not true. Thus, we cannot replace sIND and BET with tIND in Theorem 1.

Example 4 Let $\succsim$ be a preference order such that for any $p, q \in P$,

$$p \succsim q \text{ if } (\, u(p),\ \varphi(p_1 - q_1)\,) \geq_L (\, u(q),\ 0\,),$$

where $\geq_L$ is the lexicographic order defined on $\mathbb{R}^2$, $u(.)$ is an expected utility function defined on P, and $\varphi(.)$ is an indicator function defined on $\mathbb{R}$. Specifically, $\varphi(r) = 1$ if $r \in \mathbb{A}$, $\varphi(r) = 0$ if $r \in \mathbb{Q}$, and $\varphi(r) = -1$ if $r \in \mathbb{B}$, where $\mathbb{Q}$ is the set of rational numbers while $\mathbb{A}$ and $\mathbb{B}$ decompose the set of irrationals $\mathbb{I}$ into two sets satisfying the following properties: (i) $\mathbb{A} = -\mathbb{B}$, (ii) $a, a' \in \mathbb{A}$ implies $a + a' \in \mathbb{A}$, and (iii) $a \in \mathbb{A}$ and $r \in \mathbb{Q}$ implies $a + r \in \mathbb{A}$.[10] $\diamondsuit$

In the Appendix, we show that the preference order given in Example 4 satisfies tIND and wCON, but violates both sIND and BET because of the second criterion in its definition. Clearly, this preference order has no representation due to its lexicographic nature.

4.3 Independence with strong continuity

In this section, we further consider the relation between tIND, and sIND, and BET. We first discuss that whenever we employ a strong form of mixture continuity axiom, tIND becomes equivalent to sIND and BET combined. We then provide an alternative expected utility theorem, which is obtained using the stronger mixture continuity axiom but weaker form of independence axioms, tIND, or sIND and BET together, with mixture weights fixed at 1/2.

4.3.1 A strong continuity axiom and equivalence result

The following is a form of mixture continuity axiom that we will use.

Axiom (Strong continuity, sCON) For any $p, q, r, s \in P$, the set $\{\alpha : p\alpha r \succsim q\alpha s\}$ is closed.

In contrast to mCON, strong continuity allows both sides of the preference comparison to vary as the mixture weight varies.[11] In fact, it is clear that sCON implies mCON. The following result shows that there is a direct relation between tIND and sIND together with BET whenever the preference order satisfies sCON instead of wCON.

Lemma 1 *Let $\succsim$ be a preference order satisfying sCON. Then $\succsim$ satisfies tIND if and only if it satisfies sIND and BET.*

Lemma 1 shows that the tIND axiom becomes equivalent to the sIND and BET axioms combined whenever sCON holds.

4.3.2 An alternative expected utility theorem

We have seen in Sect. 3 that for a given continuous preference order, IND is both sufficient and necessary to have an expected utility representation. On the other hand, Theorem 1 shows that the full strength of IND is not needed to characterize expected utility preferences when the wCON axiom is assumed; verifying only sIND and BET

[10] For a proof of the existence of these two sets decomposing the set of irrationals, see Mitra and Ozbek [2020, Theorem 3].

[11] Strong continuity axiom was first used by Shapley and Baucells (1998) to study a model of multi-expected utility. Dubra et al. (2004) call this axiom weak continuity and show that this axiom is equivalent to having closed upper and lower contour sets whenever the outcome space is finite and the preference relation satisfies a weak form of independence axiom (see the sa-IND axiom defined in footnote 14).

is enough. Moreover, Lemma 1 implies that both sIND and BET can be replaced with tIND whenever the stronger continuity axiom, sCON, is assumed. In fact, in that case, we can consider even weaker versions of the weak independence axioms, tIND, sIND, and BET, by requiring that the mixture weights $\alpha \in (0, 1)$ to be fixed at 1/2.

We call these weaker form independence axioms, respectively, even-chance translation independence (ec-tIND), even-chance scale independence (ec-sIND), and even-chance betweenness (ec-BET). To be more precise, $\succsim$ satisfies (i) ec-tIND if for any $p, q, r, s \in P$, we have $p\frac{1}{2}r \succsim q\frac{1}{2}r$ implies $p\frac{1}{2}s \succsim q\frac{1}{2}s$, (ii) ec-sIND if for any $p, q \in P$, we have $p \succ q$ (resp. $p \sim q$) implies $p\frac{1}{2}x_0 \succ q\frac{1}{2}x_0$ (resp. $p\frac{1}{2}x_0 \sim q\frac{1}{2}x_0$), and (iii) ec-BET if for any $p, q \in P$, we have $p \succ q$ (resp. $p \sim q$) implies $p \succ p\frac{1}{2}q \succ q$ (resp. $p \sim p\frac{1}{2}q \sim q$).[12]

The following theorem shows that the expected utility preferences can be characterized with the use of above weaker even-chance independence axioms together with the sCON axiom.

Theorem 2 *A preference order $\succsim$ satisfies (i) ec-tIND and sCON if and only if (ii) it satisfies ec-sIND, ec-BET, and sCON if and only if (iii) it has an expected utility representation.*

Theorem 2 shows that, as long as the sCON axiom holds, we can substantially weaken the independence requirement when characterizing the expected utility model by considering only equal-chance mixtures. In proving this result (in the Appendix), we first strengthen the relations given in Lemma 1. We show that whenever the preference order satisfies ec-tIND, then it satisfies sIND and BET, and likewise, whenever it satisfies ec-sIND and ec-BET, then it satisfies tIND (Lemma 2). We then invoke the fact that sCON directly implies wCON and finally appeal to Theorem 1 to establish an expected utility representation.

5 Discussion

In this section, we briefly review some independence and continuity axioms used in the literature, and provide a discussion about their relation to the independence and continuity axioms that we used for our expected utility characterizations.

5.1 Independence axioms

Many modern textbooks use the IND axiom to characterize expected utility preferences. For instance, Mas-Colell et al. (1995) use mCON, while (Gilboa,

[12] In a finite Savage space framework, Mackenzie (2020) considers a set of axioms on preferences over acts and shows that an induced preference relation over subjective lotteries does satisfy a midpoint-independence property, a counterpart to the equal-chance independence axiom that we consider in our framework. Mackenzie (2020) then uses this property together with a topological strong continuity axiom to obtain an expected utility representation. This suggests that our Theorem 2 can be utilized in establishing similar theories, especially when there is lack of richness in the setting.

2009, 2010) use aCON together with IND to establish a von Neumann–Morgenstern expected utility theorem.[13] The IND is clearly comprise the following weaker independence axioms, first axiom due to Marschak (1950), Nash (1950), and Malinvaud (1952), and second axiom due to Jensen (1967).

Axiom (Marschak–Nash–Malinvaud independence, mnm-IND) For any $p, q, r \in P$ and $\alpha \in (0,1)$, $p \sim q$ implies $p\alpha r \sim q\alpha r$.

The Marschak–Nash–Malinvaud independence axiom requires independence for the symmetric part of the preference order. This axiom clearly implies the hm-IND axiom that Herstein and Milnor (1953) used. The strict counterpart of mnm-IND is formulated as below.

Axiom (Jensen independence, j-IND) For any $p, q, r \in P$ and $\alpha \in (0,1)$, $p \succ q$ implies $p\alpha r \succ q\alpha r$.

The Jensen independence axiom requires independence for the asymmetric part of the preference order.[14] Kreps (1988) uses j-IND together with aCON to establish an expected utility representation. In particular, Kreps (1988) shows that j-IND and aCON imply mnm-IND, and therefore, IND.

In our first theorem, our objective was to keep the independence axiom as strong as possible, while having a continuity axiom as weak as possible. It turns out that even the wCON axiom permits some weakening of the IND axiom by replacing IND with sIND and BET. However, we have also shown that sIND cannot be further weakened to wsIND, or BET cannot be further weakened to wBET. It is an open question whether IND can be replaced with j-IND in Theorem 1.[15] Notice that j-IND directly implies wsIND and the following strict betweenness axiom.

Axiom (Strict betweenness, sBET) For any $p, q \in P$ and $\alpha \in (0,1)$, $p \succ q$ implies $p \succ p\alpha q \succ q$.

Figure 3 below shows how various independence and continuity axioms are related to each other, which helps put in to perspective some of our contributions in this paper. In particular, continuity axioms are given within brackets next to the independence implication they are needed for. Arrows with two bases show that the two axioms at each base together form the implication.

[13] Marschak (1950), Samuelson (1952), and Malinvaud (1952) pointed out that in von Neumann and Morgenstern (1947), characterization use of an independence axiom was implicit. Since (von Neumann & Morgenstern, 1947)'s seminal work, many expected utility theorems have been given by employing different independence and continuity axioms.

[14] Another independence axiom, which is related to both mnm-IND and j-IND, is due to Samuelson (1983). The Samuelson independence (sa-IND) axiom states that for any $p, q, r \in P$ and $\alpha \in (0,1)$, $p \succsim q$ implies $p\alpha r \succsim q\alpha r$. Clearly, sa-IND directly implies mnm-IND. Moreover, since $\succsim$ is complete, sa-IND is equivalent to the opposite implication given in j-IND; that is, sa-IND can be equivalently stated as for any $p, q, r \in P$ and $\alpha \in (0,1)$, $p\alpha r \succ q\alpha r$ implies $p \succ q$.

[15] A similar question can be raised for mnm-IND or sa-IND. However, our hunch is that this is not possible for mnm-IND or sa-IND unless a topological continuity axiom is assumed.

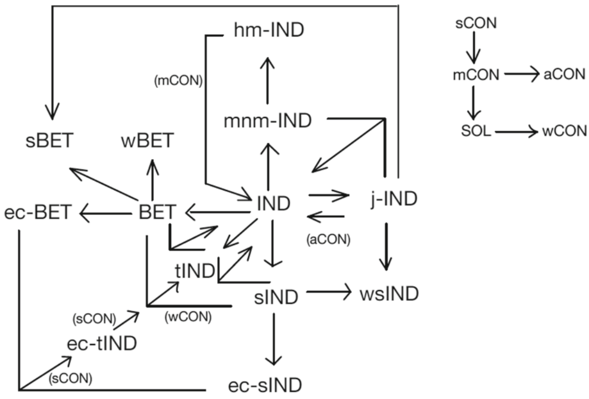


Fig. 3 Independence and continuity axioms

5.1.1 Continuity axioms

So far, we have mentioned many continuity axioms used in the literature for decision-making under risk. It will be helpful to clarify their relation. Basically, there are two types of continuity axioms, either topological or order theoretic. The following solvability axiom belongs to the latter category.

Axiom (Solvability, SOL) For any $p, q, r \in P$, $p \succ q \succ r$ implies $p \alpha r \sim q$ for some $\alpha \in (0, 1)$.

Dekel (1986) uses the solvability axiom in characterizing the class of preference orders that we used in Example 2. Given that we assume $x_i \succ x_j$ for all $i > j$, it is immediate to see that SOL implies our wCON axiom. In fact, the following proposition sets the relation of continuity axioms that we have utilized so far.

Proposition 2 *Let $\succsim$ be a preference order. Then (i) if $\succsim$ satisfies sCON, then it satisfies mCON, (ii) if $\succsim$ satisfies mCON, then it satisfies aCON and SOL, and (iii) if $\succsim$ satisfies SOL, then it satisfies wCON.*

Proposition 2 shows that while sCON is the strongest, wCON is relatively the weakest continuity axiom as summarized also in Fig. 3 above.[16]

[16] We note that while all preference orders given in Sect. 4.2 satisfy wCON, the orders given in Examples 3 and 4 fail to satisfy aCON. This provides evidence that wCON axiom is somewhat weaker than aCON, at least in the presence of weaker independence axioms, tIND, or sIND, and wBET together. It is an open question whether in general aCON does imply wCON.

A Appendix

A.1 Preliminary results

In this section, we provide some preliminary results that we use to prove our main results.

ec-tIND implies milder forms of sIND and BET

We can define a milder form of sIND and BET by requiring mixture weights to be only binary rationals, which we call them b-sIND and b-BET, respectively.[17] A binary (or dyadic) rational is a rational number that can be expressed as a fraction whose denominator is a power of two (e.g., $\frac{1}{2}$, or $\frac{5}{8}$, or $\frac{27}{32}$, etc.). The set of binary rationals is dense in reals. Moreover, a binary rational number has a finite binary representation. In particular, a binary rational between 0 and 1 can be expressed as a convex combination of 0 and 1 in a finite number of steps with equal weights (on each end) at each step. This means for any binary rational $\alpha \in (0,1)$, the mixture lottery $p\alpha q$ can be expressed as a convex combination of p and q in a finite number of steps with equal weights (on each end) at each step.

The following result shows that ec-tIND implies both the b-sIND and b-BET axioms.

Proposition 3 *If a preference order $\succsim$ satisfies ec-tIND, then it must satisfy b-sIND and b-BET.*

Proof Let $\succsim$ be a preference order satisfying ec-tIND. Let $p, q \in P$ and let $\alpha \in (0,1)$ be a binary rational.

ec-tIND implies b-sIND: We want to show that $p \succsim q$ if and only if $p\alpha x_0 \succsim q\alpha x_0$. By above arguments on binary rationals, if we show that $p \succsim q$ if and only if $p\frac{1}{2}x_0 \succsim q\frac{1}{2}x_0$, then by employing a recursive argument, we would be done. To prove that $p \succsim q$ if and only if $p\frac{1}{2}x_0 \succsim q\frac{1}{2}x_0$, first notice that we must have $p \succsim q$ implies $p \succsim p\frac{1}{2}q \succsim q$. To see this, first let $p \succsim q$ and suppose for contradiction that (i) $p\frac{1}{2}q \succ p$ or (ii) $q \succ p\frac{1}{2}q$. In case (i), we must have $q \succ q\frac{1}{2}p$ by ec-tIND, and so $q \succ p$ by transitivity, a contradiction; in case (ii), we must have $q\frac{1}{2}p \succ p$ by ec-tIND, and so $q \succ p$ by transitivity, a contradiction. Thus, we have $p \succsim q$ if and only if $p \succsim p\frac{1}{2}q \succsim q$. By ec-tIND, we have $p\frac{1}{2}q \succsim q$ if and only if $p\frac{1}{2}x_0 \succsim q\frac{1}{2}x_0$ (if and only if $p \succsim p\frac{1}{2}q$). Combining all these equivalences, we conclude by transitivity that we must have $p \succsim q$ if and only if $p\frac{1}{2}x_0 \succsim q\frac{1}{2}x_0$ as desired.

ec-tIND implies b-BET: We want to show that $p \succsim q$ (resp. $p \succ q$) implies $p \succsim p\alpha q \succsim q$ (resp. $p \succ p\alpha q \succ q$). From above arguments, we know that $p \succsim q$ if and only if $p\alpha x_0 \succsim q\alpha x_0$. Thus, if $p \succsim q$ (resp. $p \succ q$), then by ec-tIND, we have $p\alpha q \succsim q$

[17] To be more precise, a preference order $\succsim$ satisfies (i) b-sIND if for all $p, q \in P$ and for all binary rationals $\alpha \in (0,1)$, we have $p \succ q$ (resp. $p \sim q$) implies $p\alpha x_0 \succ q\alpha x_0$ (resp. $p\alpha x_0 \succ q\alpha x_0$) and (ii) b-BET if for all $p, q \in P$ and for all binary rationals $\alpha \in (0,1)$, we have $p \succ q$ (resp. $p \sim q$) implies $p \succ p\alpha q \succ q$ (resp. $p \sim p\alpha q \sim q$).

(resp. $p\alpha q \succ q$). Since $1-\alpha$ is a binary rational whenever α is a binary rational, if $p \succsim q$ (resp. $p \succ q$), then $p(1-\alpha)x_0 \succsim q(1-\alpha)x_0$ (resp. $p(1-\alpha)x_0 \succ q(1-\alpha)x_0$) by the above argument. But then, if $p \succsim q$ (resp. $p \succ q$), by ec-tIND, we have $p \succsim q(1-\alpha)p = p\alpha q$ (resp. $p \succ p\alpha q$). Combining $p\alpha q \succsim q$ (resp. $p\alpha q \succ q$) and $p \succsim p\alpha q$ (resp. $p \succ p\alpha q$), we have $p \succsim p\alpha q \succsim q$ (resp. $p \succ p\alpha q \succ q$), completing the proof. □

Relation of independence conditions under sCON

The following result shows that whenever sCON holds, much weaker even-chance independence conditions do imply relatively stronger forms of independence.

Lemma 2 *Let $\succsim$ be a preference order satisfying sCON. Then (i) if $\succsim$ satisfies ec-tIND, it also satisfies sIND and BET and (ii) if $\succsim$ satisfies ec-sIND and ec-BET, it also satisfies tIND.*

Proof Let $\succsim$ be a preference order that satisfies sCON.

ec-tIND implies sIND and BET: Suppose $\succsim$ satisfies ec-tIND. We want to show that $\succsim$ satisfies sIND and BET. By Proposition 3, we know that $\succsim$ satisfies b-sIND and b-BET. Thus, let $p, q \in P$ and $\alpha \in (0,1)$ be an arbitrary number that is not a binary rational. Since the set of binary rationals are dense in reals, we can find a sequence $\{\alpha_n\} \subset (0,1)$ of binary rationals converging to α.

By b-sIND, for all α_n, we have $p \succsim q$ if and only if $p\alpha_n x_0 \succsim q\alpha_n x_0$. Thus, by sCON, we must have $p \succsim q$ if and only if $p\alpha x_0 \succsim q\alpha x_0$ showing that $\succsim$ satisfies sIND. By b-BET, for all α_n, we have $p \succsim q$ (resp. $p \succ q$) implies $p \succsim p\alpha_n q \succsim q$ (resp. $p \succ p\alpha_n q \succ q$). Thus, by sCON, we must have $p \succsim q$ (resp. $p \succ q$) implies $p \succsim p\alpha q \succsim q$ (resp. $p \succ p\alpha q \succ q$) showing that $\succsim$ satisfies BET.

ec-sIND and ec-BET imply tIND: Assume that the preference order $\succsim$ satisfies ec-sIND and ec-BET. By the proof of Proposition 3, we know that $\succsim$ satisfies b-sIND and b-BET. Since $\succsim$ satisfies sCON, by the arguments we employed in previous part, $\succsim$ satisfies sIND and BET. Since $\succsim$ satisfies sCON, it satisfies mCON, and so wCON by Proposition 2. Since $\succsim$ satisfies sIND, BET, and wCON, by Theorem 1, $\succsim$ has an expected utility representation. Thus, clearly $\succsim$ satisfies IND, and therefore tIND as we noted by Proposition 1. □

A.2 Proofs of main results

In this section, we provide proofs for our results given in the main text.

Proof of Proposition 1

Let $\succsim$ be a preference order. Clearly, when $\succsim$ satisfies IND, then it satisfies tIND, sIND, and BET. As such, first suppose that $\succsim$ satisfies tIND and sIND. We want to show that $\succsim$ satisfies IND. Let $p, q, r \in P$ and $\alpha \in (0,1)$. Then by sIND $p \succsim q$ if and only if $p\alpha x_0 \succsim q\alpha x_0$. By tIND, $p\alpha x_0 \succsim q\alpha x_0$ if and only if $p\alpha r \succsim q\alpha r$. Combining these two, we obtain $p \succsim q$ if and only if $p\alpha r \succsim q\alpha r$ showing that IND holds.

Now suppose that $\succsim$ satisfies tIND and BET. We want to show that $\succsim$ satisfies IND. Let $p, q, r \in P$ and $\alpha \in (0, 1)$. First suppose that $p \succsim q$. Then, by BET, we have $p \succsim p\alpha q \succsim q$. Using tIND, we have $p\alpha r \succsim q\alpha r$ showing one direction of IND. Now for the opposite direction, suppose that $p\alpha r \succsim q\alpha r$, but also suppose, for contradiction, that $q \succ p$. Then, by BET, $q \succ q(1-\alpha)p \succ p$, and so, by tIND, $q\alpha r = r(1-\alpha)q \succ r(1-\alpha)p = p\alpha r$, a contradiction, which completes the proof. □

Proof of statements about Example 1

Let $\succsim$ be a preference order defined as in Example 1. We want to show that $\succsim$ satisfies sIND and BET but fails to satisfy tIND and wCON. Let $p, q \in P$ and $\alpha \in (0, 1)$. If $p = q$ or $q = x_0$, then all implications below trivially hold. Thus, assume that $p \neq q$ and $q \neq x_0$.

By definition, we have (i) $\frac{p_2}{p_2+p_1} \geq \frac{q_2}{q_2+q_1}$ if and only if $\frac{\alpha p_2}{\alpha p_2+\alpha p_1} \geq \frac{\alpha q_2}{\alpha q_2+\alpha q_1}$. Moreover, if (ii) $\frac{p_2}{p_2+p_1} = \frac{q_2}{q_2+q_1}$, we have $p_2 + p_1 \geq q_2 + q_1$ if and only if $\alpha p_2 + \alpha p_1 \geq \alpha q_2 + \alpha q_1$. Combining (i) and (ii), by definition, we obtain $p \succsim q$ if and only if $p\alpha x_0 \succsim q\alpha x_0$ showing that sIND holds.

Now we want to show that BET holds. Suppose $p \succ q$. We must have either (i) $\frac{p_2}{p_2+p_1} > \frac{q_2}{q_2+q_1}$ or (ii) $\frac{p_2}{p_2+p_1} = \frac{q_2}{q_2+q_1}$ and $p_2 + p_1 > q_2 + q_1$. In case (i), we have

$$\frac{p_2}{p_2+p_1} > \frac{\alpha p_2 + (1-\alpha)q_2}{\alpha(p_2+p_1) + (1-\alpha)(q_2+q_1)} > \frac{q_2}{q_2+q_1},$$

and so $p \succ p\alpha q \succ q$; in case (ii), we have $\frac{p_2}{p_2+p_1} = \frac{\alpha p_2+(1-\alpha)q_2}{\alpha(p_2+p_1)+(1-\alpha)(q_2+q_1)} = \frac{q_2}{q_2+q_1}$ and $p_2 + p_1 > \alpha(p_2 + p_1) + (1-\alpha)(q_2 + q_1) > q_2 + q_1$ and so $p \succ p\alpha q \succ q$ showing that BET holds.

We now show that tIND fails. To see this, suppose $p_1 = kq_1$ and $p_2 = kq_2$ for some $k > 1$. By definition, we have $p \succ q$. Since $\succsim$ satisfies sIND, we must have $p\alpha x_0 \succ q\alpha x_0$. Let $r = p\alpha x_2$ and $s = q\alpha x_2$. We have $\frac{r_2}{r_2+r_1} = \frac{\alpha p_2+(1-\alpha)}{\alpha p_2+\alpha p_1+(1-\alpha)}$ and $\frac{s_2}{s_2+s_1} = \frac{\alpha q_2+(1-\alpha)}{\alpha q_2+\alpha q_1+(1-\alpha)}$. Notice that we have $\frac{r_2}{r_2+r_1} < \frac{s_2}{s_2+s_1}$ if and only if $k > 1$, which is true by our supposition. Thus, by definition, we have $q\alpha x_2 \succ p\alpha x_2$ showing that $\succsim$ violates tIND.

Finally, notice that whenever $p \sim q$, by definition, we must have $p = q$ implying immediately that wCON cannot hold. □

Proof of Theorem 1

It is clear that whenever a preference order $\succsim$ has an expected utility representation for some $u \in U$, then it satisfies sIND, BET, and wCON. Thus, we omit this part of the proof. Now let $\succsim$ be a preference order that satisfies sIND, BET, and wCON. We want to show that $\succsim$ has an expected utility representation for some $u \in U$.

Let $i,j \in I$ such that $i<j$. By wCON, there exist some $p,q \in P$ such that $p \sim q$ with $p_k = q_k = 0$ for all $k \neq i,j,0$ and $(p_i - q_i)(q_j - p_j) > 0$. Similar to our discussion in Sect. 4.2 using Fig. 2, by applying sIND and then BET, we derive that for all $i,j \in I$ such that $i<j$, we have $x_i \sim x_j \alpha_{ij} x_0$ where $\alpha_{ij} = \frac{q_j - p_j}{p_i - q_i}$. Since $x_j \succ x_i \succ x_0$ for all $i<j$, by BET we must have $\alpha_{ij} \in (0,1)$. Let $r^{k,i} = x_k (\prod_{m=i}^{k-1} \alpha_{m(m+1)}) x_0$ for all $k > i$, for all $i \in I$. By iterative application of sIND, we have $r^{k,i} \sim x_i$ for all $k > i$ and $i \in I$. By BET, we must have $\alpha_{in} = \prod_{m=i}^{n-1} \alpha_{m(m+1)}$ for all $i \in I$.

Now let $u \in \mathbb{R}^n$ such that $u_0 = 0$, $u_n = 1$, and $u_i = \alpha_{in}$ for all $i \in I \backslash \{n\}$. Since $\alpha_{in} = \prod_{m=i}^{n-1} \alpha_{m(m+1)}$ for all $i \in I$, we have $0 < u_i < u_j < 1$ for all $i<j$ and so $u \in U$. Let $r^{1,1} = x_1$ and $A = co(\{r^{k,1} : k \in I\})$. By BET, we have $r \sim x_1$ for any $r \in A$. Moreover, by definition of u, we have $u(r) = u_1 > 0$ if and only if $r \in A$. Clearly, for any $p \in P$, we have either (i) $u(p) > u_1$, or (ii) $u(p) = u_1$, or (iii) $u(p) < u_1$. Note that in case (i), there exists some $r \in A$ such that $r = p(\frac{u_1}{u(p)})x_0$, while in case (iii), there exists some $r \in A$ such that $p = r(\frac{u(p)}{u_1})x_0$.

Now let $p,q \in P$ such that $p \succ q$. Suppose, for contradiction, that $u(q) \geq u(p)$. If $u(q) = u(p) > u_1$, we have $q(\frac{u_1}{u(q)})x_0, p(\frac{u_1}{u(p)})x_0 \in A$ and so $q(\frac{u_1}{u(q)})x_0 \sim p(\frac{u_1}{u(p)})x_0$ by definition. Then, by sIND, we must have $q \sim p$, a contradiction. Similarly, if $u(q) = u(p) < u_1$, we have $p = r(\frac{u(p)}{u_1})x_0$ and $q = s(\frac{u(q)}{u_1})x_0$ for some $r,s \in A$. By sIND, we must have $q \sim p$, a contradiction. And clearly, when $u(q) = u(p) = u_1$, we have $p \sim q$ by definition, a contradiction.

Thus, suppose that $u(q) > u(p)$. First, if $u(p) \geq u_1$, then we have $(q\frac{u(p)}{u(q)}x_0)\frac{u_1}{u(p)}x_0 \in A$ and $p(\frac{u_1}{u(p)})x_0 \in A$. By definition, we have $(q\frac{u(p)}{u(q)}x_0)\frac{u_1}{u(p)}x_0 \sim p(\frac{u_1}{u(p)})x_0$, and so by sIND, $q\frac{u(p)}{u(q)}x_0 \sim p$. Since we have $q \succ x_0$, by BET $q \succ q\frac{u(p)}{u(q)}x_0$ implying that $q \succ p$, a contradiction. Now suppose $u(q) \geq u_1 > u(p)$. By similar arguments as above, we have $q \succsim x_1$. Moreover, since $u_1 > u(p)$, we have $p = r(\frac{u(p)}{u_1})x_0$ for some $r \in A$. Since we have $r \succ x_0$, by BET $r \succ r(\frac{u(p)}{u_1})x_0$ and so $q \succ p$, a contradiction. Finally, when $u_1 \geq u(q) > u(p)$, we have $q = s(\frac{u(q)}{u_1})x_0$ and $p = (r\frac{u(p)}{u(q)}x_0)\frac{u(q)}{u_1}x_0$ for some $r,s \in A$. By BET, we have $r \succ r\frac{u(p)}{u(q)}x_0$ and so $s \succ r\frac{u(p)}{u(q)}x_0$ implying by sIND $q \succ p$, a contradiction. Hence, we must have $u(q) > u(p)$. Clearly, by above arguments, whenever $u(p) > u(q)$, we must have $p \succ q$, which shows that $u \in U$ provides an expected utility representation for the given preference order. □

Proof of statements about Example 2

Let $\succsim$ be a preference order defined as in Example 2. We want to show that $\succsim$ satisfies wsIND, BET, and wCON, but fails to satisfy sIND whenever v is not an explicit, but an implicit expected utility; that is, for some $i \in I$, $u(x_i, a) \neq u(x_i, b)$ for some $a \neq b$. Dekel (1986) shows that $\succsim$ satisfies wsIND, BET, and SOL. By Proposition 2, $\succsim$ satisfies wCON. Now suppose that $\succsim$ satisfies sIND. Let $i<j$. By our assumption, we have $x_j \succ x_i \succ x_0$. By SOL, we have $x_i \sim x_j \beta x_0$ for some $\beta \in (0,1)$.

Using the representation, we derive $v(x_i) = u(x_i, v(x_i)) = \beta u(x_j, v(x_i))$. Now let $\alpha \in (0,1)$. By sIND, we have $x_i \alpha x_0 \sim x_j \alpha \beta x_0$. This implies that $v(x_i \alpha x_0) = \alpha u(x_i, v(x_i \alpha x_0)) = \alpha \beta u(x_j, v(x_i \alpha x_0))$. Since $\alpha \in (0,1)$ is arbitrary and we have $u(x_i, v(x_i \alpha x_0)) = \beta u(x_j, v(x_i \alpha x_0))$, we must have $v(x_i \alpha x_0) = \alpha v(x_i)$ for all $\alpha \in (0,1)$ implying that v is an expected utility. □

Proof of statements about Example 3

Let $\succsim$ be a preference order defined as in Example 3. We want to show that $\succsim$ satisfies (i) sIND, (ii) wCON, and (iii) wBET, but fails to satisfy (iv) BET, (v) tIND, and (vi) aCON. Let $p, q, r \in P$ and $\alpha \in (0,1)$.

(i) Suppose $p \succsim q$. Then, by definition $u(p) \geq u(q)$ and $\lambda(p) \geq \lambda(q)$ if $u(p) = u(q)$. Thus, either $u(p\alpha r) \geq u(q\alpha r)$, or $u(p\alpha r) = u(q\alpha r)$ and $\lambda(p\alpha r) \geq \lambda(q\alpha r)$. Hence, by definition, we have $p\alpha r \succsim q\alpha r$ showing one direction of sIND. Now, suppose that $p\alpha r \succsim q\alpha r$. Then, by definition, $u(p\alpha r) \geq u(q\alpha r)$ and $\lambda(p\alpha r) \geq \lambda(q\alpha r)$ if $u(p\alpha r) = u(q\alpha r)$. Thus, either $u(p) \geq u(q)$, or $u(p) = u(q)$ and $\lambda(p) \geq \lambda(q)$. Hence, by definition, we have $p \succsim q$ showing the other direction of sIND.

(ii) Let $i, j \in I$ and let $p \in P$ such that $v(p) > v(x_0)$ and $p_i, p_j > 0$. Clearly, we can find such a lottery. Let $q \in P$ such that $q_k = p_k$ for all $k \in I \backslash \{i, j\}$, and $q_i = p_i + \frac{\epsilon}{u_i}$ and $q_j = p_j - \frac{\epsilon}{u_j}$ where $\epsilon(\frac{v_j}{u_j} - \frac{v_i}{u_i}) < v(p) - v(q)$. Note that such an $\epsilon > 0$ always exists. Thus, by construction, we have $u(p) = u(q)$ and $v(q) > v(x_0)$, and so by definition, we derive $p \sim q$ showing that wCON holds.

(iii) Suppose $p \succsim q$. Since $u(p) \geq u(q)$, we have $u(p) \geq u(p\alpha q) \geq u(q)$. In particular, when $u(p) = u(q)$, then $\lambda(p) \geq \lambda(q)$, and so $v(p) \geq v(q)$. In that case, we have $v(p) \geq v(p\alpha q) \geq v(q)$ implying that $\lambda(p) \geq \lambda(p\alpha q) \geq \lambda(q)$. Combining these, by definition, we have $p \succsim p\alpha q \succsim q$ showing that wBET holds.

(iv) Suppose that $u(p) = u(q)$, but $v(p) \geq v(x_0) > v(q)$ and so $\lambda(p) = 1 > 0 = \lambda(q)$ implying that $p \succ q$. Clearly, $v(x_0) > \alpha v(p) + (1-\alpha) v(q) = v(p\alpha q)$ for some $\alpha \in (0,1)$. Since we have $u(p\alpha q) = u(q)$, we deduce that $p\alpha q \sim q$ showing that BET does not hold.

(v) Suppose that $p \sim q$; that is, suppose $u(p) = u(q)$ and $\lambda(p) = \lambda(q)$. Since $u \neq kv$ for some $k > 0$, we can pick p, q such that $v(p) > v(q)$. If $v(q) \geq v(x_0)$, then let $\alpha \in (\frac{v(x_0) - v(x_i)}{v(p) - v(x_i)}, \frac{v(x_0) - v(x_i)}{v(q) - v(x_i)})$, where $x_i \in X$ is such that $v(x_i) < v(x_0)$; by definition, we then have $p\alpha x_i \succ q\alpha x_i$, violating tIND. If, on the other hand, $v(x_o) > v(p)$, then let $\alpha \in (\frac{v(x_j) - v(x_0)}{v(x_j) - v(p)}, \frac{v(x_j) - v(x_0)}{v(x_j) - v(q)})$, where $x_j \in X$ is such that $v(x_j) > v(x_0)$; by definition, we then have $p\alpha x_j \succ q\alpha x_j$, violating tIND.

(vi) Suppose that $u(p) = u(q) > u(r)$ and $\lambda(p) > \lambda(q) = \lambda(r)$. This means we have $p \succ q \succ r$. Now notice that for any $\alpha \in (0,1)$, we have $u(q) > u(p\alpha r)$ and so $q \succ p\alpha r$ showing that aCON fails. Notice also that the preference order $\succsim$ is of a lexicographic type. As is well-known, a lexicographic order cannot have a representation, and so we conclude that $\succsim$ does not have a representation. □

Proof of statements about Example 4

Let $\succsim$ be a preference order defined as in Example 4. We want to show that $\succsim$ satisfies (i) tIND and (ii) wCON, but fails to satisfy (iii) sIND and (iv) wBET, and it does not have a representation.

(i) Let $p, q, r, s \in P$ and $\alpha \in (0,1)$. By definition, $p\alpha r \succsim q\alpha r$ if and only if (a) $u(p\alpha r) - u(q\alpha r) > 0$ or (b) $u(p\alpha r) - u(q\alpha r) = 0$ and $\varphi(p_1\alpha r_1 - q_1\alpha r_1) \geq 0$. Thus, $p\alpha r \succsim q\alpha r$ if and only if (a) $u(p\alpha s) - u(q\alpha s) > 0$ or (b) $u(p\alpha s) - u(q\alpha s) = 0$ and $\varphi(p_1\alpha s_1 - q_1\alpha s_1) \geq 0$. Hence, by definition, $p\alpha r \succsim q\alpha r$ if and only if $p\alpha s \succsim q\alpha s$, implying tIND.

(ii) Let $i, j \in I$ and let $p, q \in P$ such that $q_k = p_k$ for all $k \in I \setminus \{i, j\}$, and $q_i = p_i + \frac{\epsilon}{u_i}$ and $q_j = p_j - \frac{\epsilon}{u_j}$ where $\epsilon > 0$. In particular, let $\frac{\epsilon}{u_i} \in \mathbb{Q}$ if $i = 1$ and $\frac{\epsilon}{u_j} \in \mathbb{Q}$ if $j = 1$. Then, by construction, we have $u(p) = u(q)$ and $p_1 - q_1 \in \mathbb{Q}$ implying that $p \sim q$ by definition. This shows that wCON holds.

(iii) Now let $p, q \in P$ such that $u(p) = u(q)$ and $p_1 - q_1$ is a rational number. By definition, we have $p \sim q$. Let $\alpha \in (0,1)$ be an irrational number. Then we have $\alpha(p_1 - q_1) \notin \mathbb{Q}$, and so either $p\alpha x_0 \succ q\alpha x_0$ or $q\alpha x_0 \succ p\alpha x_0$, showing that $\succsim$ violates sIND.

(iv) Now let $p, q \in P$ and $\alpha \in (0,1)$ as in previous part (iii) Then we have $p \sim q$, but clearly $p\alpha q \not\sim q$ showing that $\succsim$ violates wBET.

Finally, notice that the preference order $\succsim$ is of a lexicographic type. As is well-known, a lexicographic order cannot have a representation, and so we conclude that $\succsim$ does not have a representation. □

Proof of Lemma 1

The proof directly follows from Lemma 2. □

Proof of Theorem 2

Clearly, whenever the preference order $\succsim$ has an expected utility representation, then it satisfies all of the axioms: ec-tIND, ec-sIND, ec-BET, and sCON showing that (iii) expected utility representation implies (i) ec-tIND and sCON and (ii) ec-sIND, ec-BET, and sCON. We now show the other directions.

(i) implies (iii): Let $\succsim$ be a preference order which satisfies ec-tIND and sCON. By Lemma 2 $\succsim$ satisfies sIND and BET and by Proposition 2, it satisfies wCON. Thus, by Theorem 1, $\succsim$ has an expected utility representation.

(ii) implies (i): Let $\succsim$ be a preference order which satisfies ec-sIND, ec-BET, and sCON. By Lemma 2, $\succsim$ satisfies tIND, and so ec-tIND. □

Proof of Proposition 2

Let $\succsim$ be a preference order. We want to show that (i) sCON implies mCON, (ii) mCON implies aCON and SOL, and (iii) SOL implies wCON.

(i) The proof is immediate since we can write $q = q\alpha q$ for all $q \in P$ and $\alpha \in [0,1]$.

(ii) Let $p, q, r \in P$ such that $p \succ q \succ r$. Let $A = \{\alpha : p\alpha r \succsim q\}$ and $B = \{\alpha : q \succsim p\alpha r\}$. By mCON, both sets are closed. First, we show that $\succsim$ satisfies aCON. Since A and B are closed, A has a minimum element $a > 0$ and B has a maximum element $b < 1$. Let $\alpha \in (b, 1)$ and $\beta \in (0, a)$. Since $1 \notin B$ and $0 \notin A$, we have $p\alpha r \succ q$ and $q \succ p\beta r$, as desired. We now want to show that $\succsim$ satisfies SOL. Since $\succsim$ is complete, we have $A \cup B = [0, 1]$. Since A and B are closed and [0, 1] is a connected set, we must have $A \cap B \neq \emptyset$. Since $1 \notin B$ and $0 \notin A$, there must be some $\alpha \in (0, 1)$ such that $p\alpha r \sim q$ showing that SOL holds.

(iii) Let $i, j \in I$ and suppose $j > i$. By our assumption, we have $x_j \succ x_i \succ x_0$. By SOL, we have $x_i \sim x_j \alpha x_0$ for some $\alpha \in (0, 1)$ showing that wCON holds. □

Declarations

Conflict of interest The author declares that there is no conflict of interest for this manuscript, "Expected Utility, Independence, and Continuity".

References

Abdellaoui, M., & Wakker, P. P. (2020). Savage for Dummies and Experts, Journal of Economic Theory, 1–20.

Allais, M. (1953). Le Comportement de l'Homme Rationnel Devant le Risque: Critique des Postulats et Axiomes de l'École amÉricaine, Econometrica, 503–546.

Bleichrodt, H., Li, C., Moscati, I., & Wakker, P. P. (2016). Nash was a first to axiomatize expected utility, Theory and Decision.

Camerer, C. F. and T.-H. Ho (1994): Violations of the Betweenness Axiom and Nonlinearity in Probability, Journal of Risk and Uncertainty, 8, 167–196.

Chew, S. H. (1983): A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox, Econometrica, 51, 1065–1092.

Crawford, V. P. (1990): Equilibrium without independence, Journal of Economic Theory, 50, 127–154.

Dekel, E. (1986): An Axiomatic Characterization of Preferences under Uncertainty: Weakening the Independence Axiom, Journal of Economic Theory, 40, 304–318.

Diecidue, E., U. Schmidt, and H. Zank (2009): Parametric Weighting Functions, Journal of Economic Theory, 144, 1102–1118.

Dubra, J., F. Maccheroni, and E. A. Ok (2004): Expected utility theory without the completeness axiom, Journal of Economic Theory, 115, 118–133.

Epstein, D., & Zin, S. E. (2001). The Independence Axiom and Asset Returns. *Journal of Empirical Finance,8*, 537–572.

Gilboa, I. (2009): Theory of Decision under Uncertainty, Cambridge University Press.

—— (2010): Rational Choice, The MIT Press.

Gul, F. (1991): A Theory of Disappointment in Decision Making under Uncertainty, Econometrica, 59, 667–686.

Hammond, P. J. (1998). Handbook of Utility Theory, Springer, chap. Chapter 5 - Objective Expected Utility: A Consequentialist Perspective.

Herstein, I. N., & Milnor J. (1953). An axiomatic approach to measurable utility. Econometrica, 291–297.

Hey, J. and C. Orme (1994): Investigating Generalizations of Expected Utility Theory Using Experimental Data, Econometrica, 62, 1291–1326.
Jensen, N. E. (1967). An Introduction to Bernoullian Utility Theory, I: Utility Functions, Swedish Journal of Economics.
Karni, E., & Safra, Z. (1989). Dynamic consistency, revelations in auctions and the structure of preferences. *The Review of Economic Studies,56*, 421–433.
Kreps, D. (1988): Notes on the Theory of Choice, Westview Press.
Maccheroni, F. (2002): Maxmin under risk, Economic Theory, 19, 823–831.
Maccheroni, F., M. Marinacci, and A. Rustichini (2006): Ambiguity Aversion, Robustness, and the Variational Representation of Preferences, Econometrica, 74, 1447–1498.
Machina, M. J. (1987): Choice under Uncertainty: Problems Solved and Unsolved, Economic Perspectives, 1, 121–154.
Mackenzie, A. (2020). On atoms and event richness in Savage's model, Working paper.
Malinvaud, E. (1952). Note on von Neumann-Morgenstern's strong independence axiom. *Econometrica*.
Marschak, J. A. (1950). Rational behavior, uncertain prospects, and measurable utility. *Econometrica*.
Mas-Colell, A., M. D. Whinston, and J. R. Green. (1995): Microeconomic Theory, Oxford University Press.
Mitra, T. & Ozbek, K. (2020). Ranking by Weighted Sum, Economic Theory.
Nash, J. F. (1950). The bargaining problem. *Econometrica*.
Safra, Z., & Segal, U. (1998). Constant Risk Aversion. *Journal of Economic Theory,83*, 19–42.
Samuelson, P. A. (1952). Probability, utility, and the independence axiom. *Econometrica*.
Samuelson, P. A. (1983): Foundations of Economic Analysis: Enlarged Edition, Cambridge, Mass.: Harvard University Press.
Shapley, L. S. & Baucells, M. (1998) Multiperson Utility, Tech. rep.
Starmer, C. (2000): Developments in Non-Expected Utility Theory: The Hunt for a Descriptive Theory of Choice under Risk, Journal of Economic Literature, 38, 332–382.
von Neumann, J. & Morgenstern, O. (1947). The theory of games and economic behaviour. Princeton: Princeton University Press, 2nd edn.

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