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# A new axiomatization of discounted expected utility

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## Abstract

We present a new axiomatization of the classical discounted expected utility model, which is primarily used as a decision model for consumption streams under risk. This new axiomatization characterizes discounted expected utility as a model that satisfies natural extensions of standard axioms as in the one-period case and two additional axioms. The first axiom is a weak form of time separability. It only requires that the choice between certain constant consumption streams and lotteries should be made by just taking into account the time periods where the consumption is different. The second axiom, *the time–probability equivalence*, requires that risk and time preferences basically work in the same way. Moreover, we prove that preferences satisfying the natural extensions of the standard axioms as well as the first axiom can be represented in a simple form relying on three functions linked to the risk or time preferences in simple situations. Finally, we illustrate that several examples that are not fully time separable satisfy all our axioms except for the time–probability equivalence.

**Keywords** Expected utility theory · Discounted utility · Choice under risk · Intertemporal choice · Axiomatization

## 1 Introduction

In many economic problems, an agent has to choose between alternatives that yield payoffs in several periods of time. Examples for this include household saving and investment decisions as well as corporate decisions on project investments. In most applications, discounted expected utility is used to model "sensible" preferences of an agent. In the static one-period case, expected utility has been characterized by natural normative axioms Von Neumann & Morgenstern (1953). In the intertemporal case, several axiomatizations of expected utility with exponential (Koopmans, 1960; Epstein, 1983; Bleichrodt et al., 2008; Anchugina, 2017), hyperbolic (Hayashi, 2003; Olea &

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Strzalecki, 2014; Anchugina, 2017) or general discounting (Blavatskyy, 2020) in the non-stochastic as well as the stochastic case have been introduced. All these axiomatizations require some form of time separability. This time separability is achieved by independence conditions as in Debreu (1959) that require that the preferences for outcomes in time periods from a set $A$ are independent of the outcomes in the other periods. This requirement is not as compelling from a normative perspective as the analogous requirement for states in the one-period case. Indeed, whereas states are mutually exclusive, time periods are not. In particular, any formulation relying on such a condition cannot capture substitution or smoothing effects. In this light, axiomatizations of preferences with such properties have been introduced: Gilboa (1989) introduced an axiomatization of preferences that can capture preferences regarding the variation of outcomes in the time dimension and Wakai (2008) proposed an axiomatization that allows to prefer the spread of good and bad outcomes over time.

Another important and related question is how risk and time preferences in intertemporal decisions can be disentangled. This question has received much attention starting with Kreps and Porteus (1978). However, most preferences that have been considered so far in this regard take not only risk and time preferences into account, but also preferences about the timing of the resolution of uncertainty (Kreps & Porteus, 1978; Epstein & Zin, 1989). The axiomatization of Kreps and Porteus (1978) yields that whenever the agent is non-indifferent to the resolution of uncertainty, the preferences can be represented by a recursive expected utility function that combines the previous consumption, the current consumption and a certainty equivalent of the possible future consumptions. However, in the case that the agent is indifferent with respect to the resolution of uncertainty, the preferences of the agent are represented by the expectation of a utility function on the space of all intertemporal consumption streams, which does not yield any insights into the relation of risk and time preferences.

In this paper, we propose a new axiomatization of general discounted utility that provides additional insights regarding both questions raised above. We assume that all uncertainty is resolved at time zero to investigate only the relation of risk and time preference with no role for dynamic decision principles. Note that this assumption is often reasonable in applications. For example, consider irreversible investment decisions where the agent can only take an action before the first payment. In this context, assuming sensible extensions of the classical axioms (completeness, transitivity, independence, continuity, dominance), we show that discounted expected utility is characterized by two additional axioms. The first axiom requires that the agent's preferences when choosing between a constant consumption stream $(x_1, \ldots, x_{t-1}, a, x_{t+1}, \ldots)$ and a lottery over the two streams $(x_1, \ldots, x_{t-1}, I_+, x_{t+1}, \ldots)$ and $(x_1, \ldots, x_{t-1}, I_-, x_{t+1}, \ldots)$ are independent of the consumption in other periods, i.e., only depend on what happens at time point $t$. This axiom is a weak form of time separability compared to a Debreu-type independence condition, utility independence as in Keeney and Raiffa (1976) and also risk independence as in Epstein (1983) and Hayashi (2003): indeed, all these conditions require that even for deterministic consumption streams, the choice between two consumption streams that only differ in one period should be independent of past and future consumption, which is not covered by our relatively mild assumption. Nonetheless, our axiom does still not cover all effects that might be reasonable. Indeed, our assumption cannot cover the effect that agents might be more risk loving when the payoffs in the other periods are low than if they are high. Note that recursive utility preferences (Backus et al., 2005) do not satisfy our assumption because there the utility depends on the certainty equivalent of the future consumption.

This first additional axiom alone yields a handy characterization of a utility function characterizing the preferences. This characterization is similar to the decomposition results in multiattribute utility theory (see Keeney and Raiffa (1976), Chapter 6 and Farquar (1977) for an overview), where under various independence assumptions, decompositions of the utility function into simpler parts are obtained. Our characterization is more general than the decomposition resulting from their utility independence. Moreover, our parts have a clear interpretation in the context of inter-temporal decision making. This characterization allows us to describe several examples of preferences that satisfy all the axioms described so far, but which are not representable by discounted expected utility. This changes when we introduce an additional second axiom, the *time–probability equivalence*, which basically states that risk and time preferences work in the same way. Indeed, in this case we obtain that any preference satisfying the classical axioms as well as our two additional axioms is of discounted expected utility type.

Our new axiomatization now includes a weak form of time separability. Indeed, we can describe preferences that satisfy all axioms except for the time–probability equivalence and that are not fully time separable (Example 4.2 and 4.3). In this sense, we provide a new axiomatization of discounted expected utility that in particular emphasizes that the strong link of risk and time preferences is a characteristic property of preferences described by discounted expected utility.

Let us finally compare our approach to Blavatskyy (2020), which also provides an axiomatization of general discounted expected utility for lotteries. Blavatskyy (2020) first derives a set of axioms (including a Debreu-type independence condition) that yield a general time-additive utility function. Thereafter, also an axiom linking the time and risk preference is described and it is shown that this additional axiom yields a discounted expected utility formulation. The approach of Blavatskyy (2020) and our approach can therefore be seen as addressing two different sides of the problem: he characterizes discounted expected utility as the only time-additive preference relation that links time and risk preferences in a certain way. We characterize discounted expected utility (for more general lotteries) as the only preference relation where first the risk preference in certain situations only depends on the consumption in periods that are affected by the choice and second time and risk preferences are linked in a certain way.

The remainder of the paper is organized as follows: In Sect. 2, we present the new axiomatization and state our main results. Section 3 provides the proofs of the main results and in Sect. 4 we describe three examples that satisfy all axioms except for the time–probability equivalence. Section 5 concludes the study.

## 2 Axiomatization and main results

We consider consumption streams $x = (x_1, x_2, \ldots, x_T)$ of length $T$, where $x_t$ is from the compact interval $I = [I_-, I_+] \subset \mathbb{R}$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an arbitrary probability space. Let $\mathcal{X}$ denote the set of all $(I^T, \mathcal{B}(I^T))$-valued random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$, where $\mathcal{B}(I^T)$ is the Borel-$\sigma$-algebra on $I^T$, and let $\mathcal{X}_l$ denote the set of all lotteries over consumption streams (i.e., random variables with finitely many values). As usual, we equip the space $\mathcal{X}$ with the topology of weak convergence, which makes the space $\mathcal{X}$ a compact metric space. We will denote with small letters $x = (x_1, \ldots, x_T)$ the deterministic outcome $x \in I^T$ and with capital letters we denote the lottery or general random variable $X \in \mathcal{X}$. For $t \in \{1, \ldots, T\}$ we will write $X_t$ for the projection of $X$ onto the $t$-th coordinate.

Let $\succeq$ describe a preference relation on $\mathcal{X}$. As usual we write $X \sim Y$ if $X \succeq Y$ and $Y \succeq X$ and $X \succ Y$ if $X \succeq Y$ and not $X \sim Y$. In what follows, we will formulate the axioms necessary for our characterization of discounted expected utility. At first, we will introduce extensions of the usual axioms in the one-period model. Thereafter, we set up the two axioms that characterize discounted expected utility.

Let us start with a simple monotonicity axiom that states that an agent prefers a deterministic consumption stream with a larger outcome in each time period to a deterministic consumption stream with a smaller outcome in each time period.

**Axiom 0** (Monotonicity) *For all $x, y \in I^T$ such that $x_t \geq y_t$ for all $t \in \{1, \ldots, T\}$ we have $x \succeq y$.*

Now, let us introduce the natural multi-period counterparts of completeness, transitivity, independence, dominance and continuity.

**Axiom 1** (Completeness) *Let $X, Y \in \mathcal{X}$, then $X \succeq Y$ or $Y \succeq X$.*

**Axiom 2** (Transitivity) *Let $X, Y, Z \in \mathcal{X}$ and $X \succeq Y$ and $Y \succeq Z$, then $X \succeq Z$.*

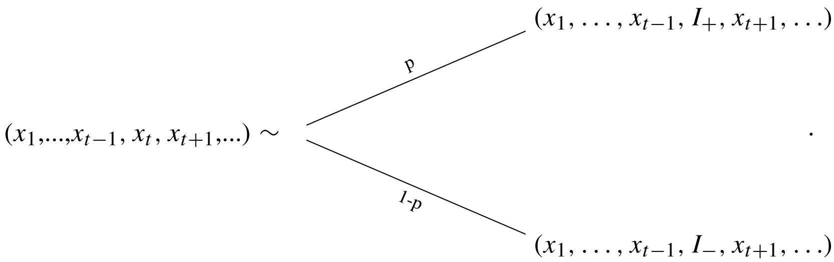
**Axiom 3** (Independence) *Let $X, Y \in \mathcal{X}$ such that $X \succeq Y$. Then for all $Z \in \mathcal{X}$ and all $p \in (0, 1)$, we have $pX + (1-p)Z \succeq pY + (1-p)Z$.*

**Axiom 4** (Dominance) *Let $X, Y, Z \in \mathcal{X}$ and $A \in \mathcal{B}(I^T)$ such that $\mathbb{P}(X \in A) = 1$. If $Y \succ x$ for all $x \in A$, then we have $Y \succeq X$. Analogously, if $x \succ Z$ for all $x \in A$, then we have $X \succeq Z$.*

**Axiom 5** (Continuity) *Let $X \in \mathcal{X}$, then $\{Y : Y \succeq X\}$ and$\{Y : X \succeq Y\}$ are closed.*[1]

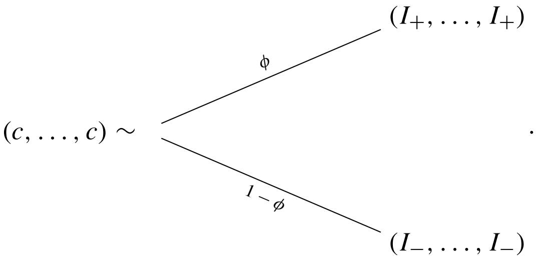
If we would restrict our attention to the space $\mathcal{X}_l$ of lotteries, then as in the one-period case, Axiom 4 would no longer be necessary. Moreover, in this case we could weaken the continuity condition by just requiring the following axioms:
**Axiom 5a** (First intertemporal continuity axiom) For any $t \in \{1, \ldots T\}$ and $x \in I^T$, there exists $p := p(x_1, \ldots, x_{t-1}, x_{t+1}, \ldots) \in$ [0,1] such that:

[1] Note that this condition is equivalent to the following. Let $X_n, X, Y_n, Y \in \mathcal{X}$ for all $n \in \mathbb{N}$ with $X_n \to X$ and $Y_n \to Y$ and $X_n \succeq Y_n$ for all $n \in \mathbb{N}$. Then $X \succeq Y$.




**Axiom 5b** (Second intertemporal continuity axiom) Let $a \in I^T$, then there exists $c \in I$ such that $(a_1, \ldots, a_T) \sim (c, \ldots, c)$.

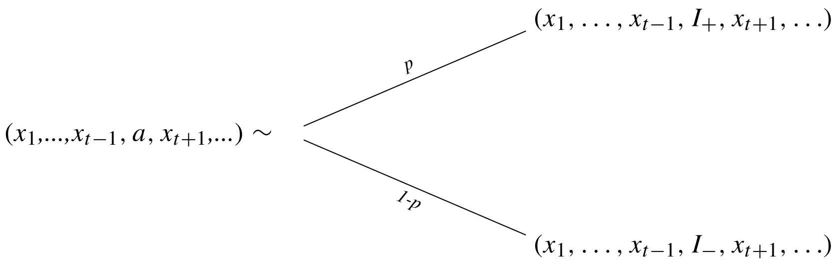
**Axiom 5c** (Global continuity) For every $c \in I = [I_-, I_+]$, there exists $\phi \in [0,1]$ such that




***Remark 2.1*** Note that Axiom 5 implies Axioms 5a, 5b and 5c.

With these natural extensions of the standard axioms, we can now turn to the characterization of discounted expected utility. A first natural requirement is that the risk preference in one time period does not depend on the payoffs from previous or future time periods. Indeed, it will turn out that it suffices to require this only for the choices as in Axiom 5a. We remark that here in Axiom 6, the choice of $p$ does not depend on $x_1, \ldots, x_{t-1}, x_{t+1}, \ldots, x_T$, whereas in Axiom 5a such a dependence is allowed. In particular, we note that Axiom 6 implies Axiom 5a.

**Axiom 6** *(Uniform first intertemporal continuity axiom) For any $t \in \{1, \ldots, T\}$ and $a \in I$, there is a $p \in [0, 1]$ such that*




*for all $x \in I^T$.*

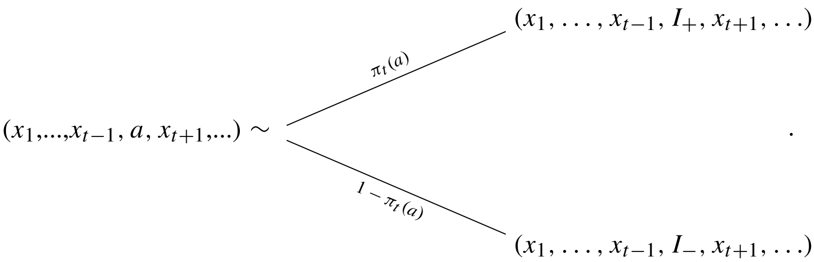
Before we move on, let us briefly emphasize that the axiom is a weak form of time separability. Here, we only require that preferences between lotteries between maximal and minimal outcomes in one period, and constant consumptions in this period are independent of the fixed levels of wealth in the other components. Classical utility independence would mean that preferences for all lotteries randomizing over outcomes in one period are independent of the consumption levels in all other periods. In particular, classical utility independence means that the preferences between two constant consumption streams $(x_1, \dots, x_{t-1}, a, x_{t+1}, \dots)$ and $(x_1, \dots, x_{t-1}, b, x_{t+1}, \dots)$ are independent of the choice $(x_1, \dots, x_{t-1})$ and $(x_{t+1}, \dots, x_T)$, which does not have to hold in our setting.

To motivate our last axiom, let us consider the following example showing that Axiom 6 is not sufficient to characterize discounted expected utility preferences (see Sect. 4 for details and further examples):

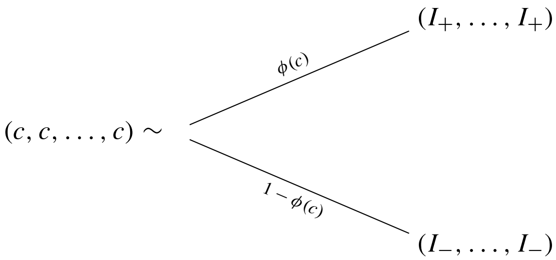
***Example 2.2*** Take $T = 2$, $I = [0,1]$. Assume a risk averse person with $(a, 1-a) \sim (b, 1-b)$ for all $a, b \in I$. Then this person cannot be described by standard EUT. To see this, recall that risk aversion in the classical model implies a preference for consumption smoothing, thus indifference over all distributions in time is not possible.

More generally, standard EUT does not allow the elasticity of intertemporal substitution to be independent of risk aversion: risk aversion and time preferences are coupled in the classical model. This observation leads us to the definition of an axiom which we call *time–probability equivalence*. To formally define this axiom, we need the following definitions:

**Definition 2.3** We define $\pi_t : I \to [0,1]$ such that for all $x \in I^T$ and all $a \in I = [I_-, I_+]$:




**Definition 2.4** We define $\phi : I \to [0, 1]$ such that for all $c \in I$:



**Definition 2.5** We further define $F : I^T \to I$ such that for all $a \in I^T$:

$$(a_1, \ldots, a_T) \sim (F(a), \ldots, F(a)).$$

We can interpret the functions $\pi_t$, $\phi$ and $F$ as follows: The function $\pi_t$ describes the risk preference in time period $t$, the function $\phi$ describes the risk preference for all time periods and $F$ describes the time preference for constant consumption streams.

***Remark 2.6*** By Axioms 5a–5c and 6, the functions $\pi_t$, $F$ and $\phi$ exist. By Axioms 0 and 3, the functions are weakly increasing.

We are now able to define the central axiom of our axiomatization:

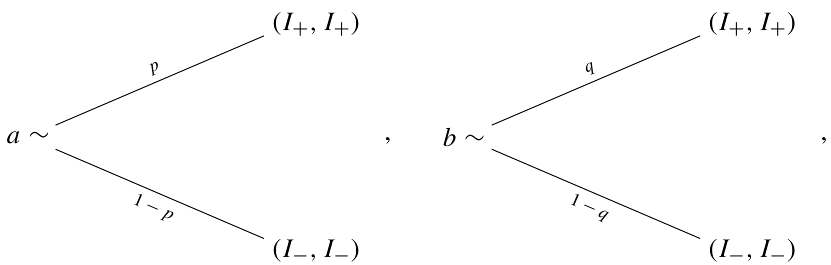
**Axiom 7** *(Time–probability equivalence) If $a, b \in \{I_-,I_+\}^T$ and if $\{t \in \{1, \ldots, T\} | a_t = I_+$ and $b_t = I_+\} = \emptyset$, then*

$$\phi(F(a)) + \phi(F(b)) = \phi(F(\max(a_1, b_1), \max(a_2, b_2), \ldots, \max(a_T, b_T))).$$

Let us first emphasize that also this axiom only makes a statement for a small class of choices, namely, only for pairs of consumption streams $a$ and $b$ such that in each time period the agent either gets the minimal ($I_-$) or the maximal outcome ($I_+$) and, moreover, he gets the maximum outcome either in the consumption stream $a$ or in the consumption stream $b$, but not in both.

With this in mind, let us look at the axiom from a different perspective to make it clearer. First note that, by Definitions 2.4 and 2.5, the function $\phi(F(a))$ is a transformation of a deterministic time-varying consumption stream into an equally preferred lottery over the two consumption streams $(I_-, \ldots, I_-)$ and $(I_+, \ldots, I_+)$. Let us write $A = \{t \in \{1, \ldots, T\} : a_t = I_+\}$ and $B = \{t \in \{1, \ldots, T\} : b_t = I_+\}$ for the set of time points, at which the consumption stream $a$ and $b$, respectively, gives $I_+$. The axiom now states that the sum of the lotteries $\phi(F(a))$ and $\phi(F(b))$, which are the risk–time transformations of the consumption streams with maximal outcomes at the (disjoint) time points in $A$ and $B$, respectively, should be equally preferred to the lottery $\phi(F(\max(a_1, b_1), \ldots, \max(a_T, b_T)))$, which is the risk– time transformation of the consumption stream that gives the maximal outcome at the time points in $A \cup B$. This means that Axiom 7 requires that the time–risk transformation $\phi(F(\cdot))$ respects the additivity in time points for this special type of lotteries.

To get a better intuition for this axiom, let us now focus on the case $T = 2$: Indeed, the axiom states that for $a = (I_+,I_-)$ and $b = (I_-,I_+)$ and



it has to hold that $p + q = 1$. This requirement is indeed natural. Let us write $(p; I_+)$ for the lottery where with probability $p$, we get $I_+$ in both periods and otherwise $I_-$ in both periods. Then we can rewrite the indifference relations as $(I_+, I_-) \sim (p; I_+)$ and $(I_-, I_+) \sim (q; I_+)$ and we note that the two indifference relations state that we are indifferent between the outcome fluctuation in time on the left side of the equations and the risky lottery on the right side (i.e., fluctuations of the outcomes between states). Now, consider the sum of the outcomes on the left. It is $(I_+, I_+) = (1; I_+)$. The sum of the outcomes on the right, however, is $(p + q; I_+)$. If we assume that adding the outcomes that differ over time and adding risky lotteries have the same effect, then we should also be indifferent between these two sums, i.e., $(1; I_+) \sim (p + q; I_+)$. This is, however, only the case when $p + q = 1$. In the general case, the same idea holds: equivalence of our preferences between time and state–risk fluctuations implies Axiom 7.

Let us finally compare Axiom 7 to the axiom in Blavatskyy (2020) that links risk and time preferences: Whereas the risk-time reversal in Blavatskyy (2020) requires a link of risk and time preferences for all possible lotteries, it is in our case sufficient to have a similar condition for all extreme lotteries randomizing over minimal and maximal outcomes.

We can now state the main result of this paper:

**Theorem 2.7** *Assume that Axioms 0–6 hold. Then the functions $F$, $\phi$ and $\pi_t$ defined in Definitions 2.2, 2.3 and 2.4 are (weakly) increasing and satisfy*

$$\begin{aligned} 0 &= \pi_t(I_-) = \phi(I_-) & 1 &= \pi_t(I_+) = \phi(I_+) \\ I_- &= F(I_-, \ldots, I_-) & I_+ &= F(I_+, \ldots, I_+) \end{aligned}$$

*for $t = 1, \ldots, T$. Moreover, a utility function $U : \mathcal{X} \to \mathbb{R}$ exists and is given by*

$$U(X) = \sum_{\sigma \in \{I_-, I_+\}^T} \mathbb{E}\left[\prod_{t=1}^{T} \Big(\sigma_t' \pi_t(X_t) + (1 - \sigma_t')(\pi_t(X_t))\Big)\phi(F(\sigma))\right], \qquad (1)$$

*where $\sigma_t' = \frac{\sigma_t - I_-}{I_+ - I_-}$ (for $I = [0, 1]$ we have $\sigma_t' = \sigma_t$). Conversely, assume that $\pi_t : I \to [0, 1]$ ($t \in \{1, \ldots, T\}$) are continuous and (weakly) increasing functions and that $F : I^T \to I$ and $\phi : I \to [0, 1]$ are (weakly) increasing functions such that*

$$\begin{aligned} 0 &= \pi_t(I_-) = \phi(I_-) & 1 &= \pi_t(I_+) = \phi(I_+) \\ I_- &= F(I_-, \ldots, I_-) & I_+ &= F(I_+, \ldots, I_+) \end{aligned}$$

*for $t = 1, \ldots, T$. Moreover, let $\succeq_U$ be given by $X \succeq_U Y$ if and only if $U(X) \geq U(Y)$ with $U$ given as in (1). Then $\succeq_U$ satisfies Axioms 0–6.*

This first result means that by describing functions $\pi$, $\phi$ and $F$, it is possible to describe a utility function satisfying Axioms 0–6. Note, however, that not the full functions $\phi$ and $F$ show up in the utility representation (1), but only the values $\phi(F(\sigma))$, which means that the shapes of $F$ and $\phi$ only partly influence the behavior of the utility function.

**Theorem 2.8** *Assume that Axioms 0–7 hold. Then a utility function $U : \mathcal{X} \to \mathbb{R}$ exists and is given by*

$$U(X) = \sum_{\tau=1}^{T} \delta_\tau \mathbb{E}[u_\tau(X_\tau)] \tag{2}$$

*with $\delta_\tau = \phi(F(e_\tau))$ ($e_\tau$ is the $\tau$-th unit vector of length $T$) and $u_\tau = \pi_T$ being (weakly) increasing. Conversely, let $(\delta_t)_{t\in\{1,\ldots,T\}} \in \mathbb{R}^T_{\geq 0}\backslash\{0\}$ and $u_t : I \to \mathbb{R}$ ($t \in \{1, \ldots, T\}$) be continuous and (weakly) increasing. Then the preferences $\succeq_U$ given by $X \succeq_U Y$ if and only if $U(X) \geq U(Y)$ with $U$ as in (2) satisfy Axioms 0–7.*

We highlight that this result makes the idea precise that any preferences admitting an discounted expected utility representation fixes risk and time preferences. Namely, any preference admitting an discounted expected utility representation satisfies that $F$ (capturing the time preference) and $\phi$ (capturing the risk preference) are linked in a certain way.

***Remark 2.9*** Both theorems are only almost "if and only if" statements, since for the second implication we additionally need to assume that $\pi_t$ and $u_t$ ($t \in \{1, \ldots, T\}$) are continuous, respectively, which does not follow from the axioms. Indeed, it might happen that a preference relation where $\pi$ exhibits jump discontinuities exists (but these are the only possible discontinuities since $\pi$ and $u$ are weakly increasing).

***Remark 2.10*** If one would consider the space $\mathcal{X}_l$ consisting of lotteries and not the space $\mathcal{X}$ of general probability measures on $I^T$, then it is not necessary to assume Axioms 4 and 5, instead it suffices to require Axioms 5a–5c.

## 3 Proofs

In this section, we prove the two main results. As a first step, we will prove that the axioms imply the utility representation for lotteries. Thereafter, we will generalize the statement for general probability measures. Finally, we prove that preferences induced by a utility function given in (1) or (2), respectively, satisfy the Axioms 0–6 or 0–7, respectively.

**Proposition 3.1** *Let the Axioms 0–3, 5a–5c as well as Axiom 6 hold. Then a utility $U(X)$ exists for all $X \in \mathcal{X}_l$ and is given by*

$$U(X) = \sum_{i=1}^{n} \left( p_i \sum_{\sigma\in\{I_-,I_+\}^T} \prod_{t=1}^{T} \Big(\sigma_t' \pi_t(x_{ti}) + (1-\sigma_t')(1-\pi_t(x_{ti})\Big)\phi(F(\sigma)) \right) \tag{3}$$

$$= \sum_{\sigma\in\{I_-,I_+\}^T} \mathbb{E}\left[\prod_{t=1}^{T} \Big(\sigma_t' \pi_t(X_t) + (1-\sigma_t')(1-\pi_t(X_t)\Big)\phi(F(\sigma))\right] \tag{4}$$

*where $F$, $\phi$ and $\pi_t$ are given as in Definition 2.3–2.5.*

*Assuming additionally time–probability equivalence (Axiom 7), we obtain the classical time-separated expected utility*

$$U(X) = \sum_{\tau=1}^{T} \delta_\tau \mathbb{E}\left[u_\tau(X_\tau)\right], \tag{5}$$

*where* $\delta_\tau = \phi(F(e_\tau))$*,* $e_\tau$ *is the* $\tau$*-th unit vector of length* $T$ *and* $u_\tau = \pi_\tau$*.*

***Proof*** Without loss of generality, we assume for simplicity that $I = [0,1]$.

The lottery $X$ can be written as a lottery over $n$ deterministic consumption streams $x^i = (x_{1i}, \ldots, x_{Ti})$ for $i = 1, \ldots, n$, each with respective probability $p_i$, i.e.,

$$x = \begin{array}{l} \xrightarrow{\;p_1\;} (x_{11}, \ldots, x_{T1}) \\ \\ \xrightarrow{\;p_n\;} (x_{1n}, \ldots, x_{Tn}) \end{array} .$$

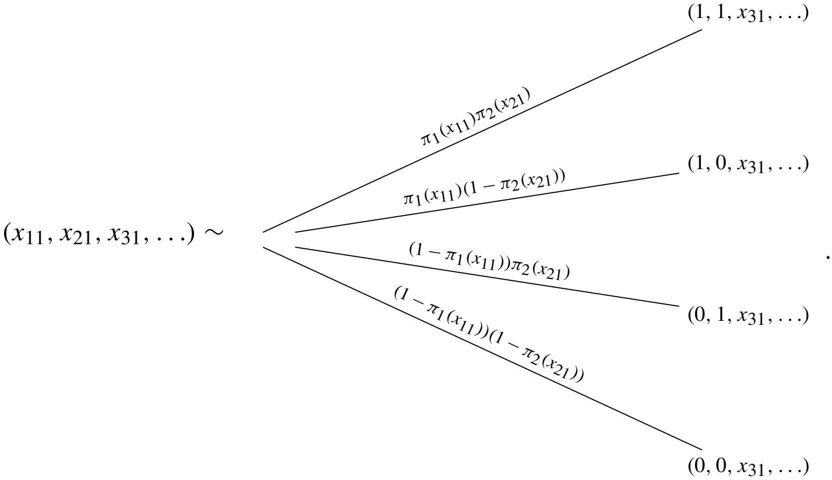
Let $\pi_t : I \to [0,1]$, $\phi : I \to [0,1]$ and $F : I^T \to I$ be given as in Definitions 2.3–2.5. Using Axioms 5a and 6, we obtain

$$(x_{11}, x_{21}, x_{31}, \ldots) \sim \begin{array}{l} \xrightarrow{\;\pi_1(x_{11})\;} (1, x_{21}, x_{31}, \ldots) \\ \\ \xrightarrow{\;1-\pi_1(x_{11})\;} (0, x_{21}, x_{31}, \ldots) \end{array} .$$

Again, applying Axioms 5a and 6 now together with the Independence Axiom 3, we obtain

$$(x_{11}, x_{21}, x_{31}, \ldots) \sim \begin{array}{l} \xrightarrow{\;\pi_1(x_{11})\pi_2(x_{21})\;} (1, 1, x_{31}, \ldots) \\ \xrightarrow{\;\pi_1(x_{11})(1-\pi_2(x_{21}))\;} (1, 0, x_{31}, \ldots) \\ \xrightarrow{\;(1-\pi_1(x_{11})\;} (0, x_{21}, x_{31}, \ldots) \end{array} .$$

and another application of the Axioms 3, 5a and 6 yields




Proceeding in this fashion for all $n$ possible outcomes and combining the results using Axiom 3, we obtain that $X$ is equivalent to a lottery with possible outcomes $\sigma \in \{0, 1\}^T$ where each outcome $\sigma$ has the probability $q(\sigma)$ given by

$$\begin{aligned} q(1, \ldots, 1) &= (\pi_1(x_{11}) \cdot \pi_2(x_{21}) \cdot \cdots \cdot \pi_T(x_{T1})) \cdot p_1 \\ &\quad + (\pi_1(x_{12}) \cdot \pi_2(x_{22}) \cdot \cdots \cdot \pi_T(x_{T2})) \cdot p_2 \\ &\quad + \ldots \\ &\quad + (\pi_1(x_{1n}) \cdot \pi_2(x_{2n}) \cdot \cdots \cdot \pi_T(x_{Tn})) \cdot p_n \\ &= \sum_{i=1}^{n} p_i \prod_{t=1}^{T} \pi_t(x_{ti}) \\ q(1, \ldots, 1, 0) &= \sum_{i=1}^{n} p_i \left( \prod_{t=1}^{T-1} \pi_t(x_{ti}) \right) (1 - \pi_T(x_{Ti})) \\ &\ldots \end{aligned}$$

For an arbitrary $\sigma \in \{0, 1\}^T$ we therefore get:

$$q(\sigma) = \sum_{i=1}^{n} p_i \left( \prod_{\{t|\sigma_t=1\}} \pi_t(x_{ti}) \right) \left( \prod_{\{t|\sigma_t=0\}} (1 - \pi_t(x_{ti})) \right).$$

Using Axiom 5b, we have

$$\begin{aligned} &(1, \ldots, 1) \sim (F(1, \ldots, 1), \ldots, F(1, \ldots, 1)), \\ &(1, \ldots, 1, 0) \sim (F(1, \ldots, 1, 0), \ldots, F(1, \ldots, 1, 0)), \\ &\text{etc.} \end{aligned}$$

or more generally

$$\sigma \sim (F(\sigma), \ldots, F(\sigma)).$$

Applying this and Axiom 3, we obtain that $x$ is equivalent to a lottery between constant consumption streams, where with probability $q(\sigma)$ the constant consumption stream $(F(\sigma), F(\sigma), \ldots, F(\sigma))$ is obtained.

Using global continuity (Axiom 5c) and independence (Axiom 3), we therefore obtain the existence of a probability $q \in [0,1]$ such that

$$\begin{tikzcd} & (1,\ldots,1) \\ X \sim \arrow[ur, dash, "q"] \arrow[dr, dash, "1\text{-}q"'] & \\ & (0,\ldots,0) \end{tikzcd} .$$

In fact, this probability $q$ can be computed as follows:

$$\begin{aligned} q &= \sum_{\sigma \in \{0,1\}^T} q(\sigma)\phi(F(\sigma)) \\ &= \sum_{\sigma \in \{0,1\}^T} \sum_{i=1}^{n} p_i \left( \prod_{\{t|\sigma_t=1\}} \pi_t(x_{ti}) \right) \left( \prod_{\{t|\sigma_t=0\}} (1-\pi_t(x_{ti})) \right) \phi(F(\sigma)) \\ &= \sum_{i=1}^{n} p_i \sum_{\sigma \in \{0,1\}^T} \prod_{t=1}^{T} \Big( \sigma_t \pi_t(x_{ti}) + (1-\sigma_t)(1-\pi_t(x_{ti})) \Big) \phi(F(\sigma)). \end{aligned} \tag{6}$$

By independence (Axiom 3) and monotonicity (Axiom 0), we moreover have that the lotteries

$$\begin{tikzcd} & (1,\ldots,1) \\ L(p) = \arrow[ur, dash, "p"] \arrow[dr, dash, "1\text{-}p"'] & \\ & (0,\ldots,0) \end{tikzcd} , \; p \in [0, 1],$$

satisfy $L(p) \succeq L(p')$ if and only if $p \geq p'$. Since the previous argument shows that $X \sim L(q)$, this proves that $U(X) = q$ is indeed a utility function.

Assuming additionally time–probability equivalence (Axiom 7), we have

$$\phi(F(\sigma)) = \sum_{\tau=1}^{T} \sigma_\tau \phi(F(e_\tau)),$$

where $e_\tau \in \{0, 1\}^T$ is the $\tau$-th unit vector of length $T$, i.e., given by

$$(e_\tau)_t := \begin{cases} 1 & t = \tau \\ 0 & t \neq \tau \end{cases}.$$

Therefore, we can rewrite (6) as

$$q = \sum_{i=1}^{n} p_i \left[ \sum_{\sigma \in \{0,1\}^T} \left( \sum_{\tau=1}^{T} \prod_{t=1}^{T} \Big( \sigma_t \pi_t(x_{ti}) + (1-\sigma_t)(1-\pi_t(x_{ti})) \Big) \phi(F(e_\tau)) \sigma_\tau \right) \right].$$

This can be simplified as follows:

$$\begin{aligned} q &= \sum_{i=1}^{n} \left[ p_i \sum_{\tau=1}^{T} \sum_{\sigma \in \{0,1\}^T} \prod_{t=1}^{T} \Big( \sigma_t \pi_t(x_{ti}) + (1-\sigma_t)(1-\pi_t(x_{ti})) \Big) \phi(F(e_\tau)) \sigma_\tau \right] \\ &= \sum_{i=1}^{n} \left[ p_i \sum_{\tau=1}^{T} \left( \phi(F(e_\tau)) \cdot \sum_{\substack{\sigma \in \{0,1\}^T \\ \sigma_\tau = 1}} \prod_{t=1}^{T} \Big( \sigma_t \pi_t(x_{ti}) + (1-\sigma_t)(1-\pi_t(x_{ti})) \Big) \right) \right]. \end{aligned} \tag{7}$$

Before we conclude our proof, we need the following lemma:

**Lemma 3.2** *Let $T \in \mathbb{N}$ and $s \in \mathbb{R}^T$, then*

$$\sum_{\sigma \in \{0,1\}^T} \prod_{t=1}^{T} \Big( \sigma_t s_t + (1-\sigma_t)(1-s_t) \Big) = 1.$$

***Proof*** (of the lemma) We prove the statement by induction. The statement is obviously true for $T = 1$:

$$\sum_{\sigma_1=0}^{1} \Big( \sigma_1 s_1 + (1-\sigma_1)(1-s_1) \Big) = s_1 + (1-s_1) = 1.$$

Assuming that the statement holds for $T - 1$, we prove that it also holds for $T$:

$$\sum_{\sigma \in \{0,1\}^{T-1}} \prod_{t=1}^{T} \Big( \sigma_t s_t + (1-\sigma_t)(1-s_t) \Big)$$

$$
\begin{aligned}
&= \sum_{\sigma \in \{0,1\}^{T-1}} \sum_{\sigma_T=0}^{1} \left( \prod_{t=1}^{T-1} (\sigma_t s_t + (1-\sigma_t)(1-s_t)) \right) \cdot \Big( \sigma_T s_T + (1-\sigma_T)(1-s_T) \Big) \\
&= \sum_{\sigma \in \{0,1\}^{T-1}} \prod_{t=1}^{T-1} \Big( \sigma_t s_t + (1-\sigma_t)(1-s_t) \Big) \underbrace{(s_T + (1-s_T))}_{=1} \\
&= \sum_{\sigma \in \{0,1\}^{T-1}} \prod_{t=1}^{T-1} \Big( \sigma_t s_t + (1-\sigma_t)(1-s_t) \Big) = 1,
\end{aligned}
$$

where the last equality holds by assumption for $T-1$. □

Using this lemma (on the relabelled indices) and defining $u_\tau = \pi_\tau$ and $\delta_\tau := \phi(F(e_\tau))$, we finally obtain

$$
\begin{aligned}
q &= \sum_{i=1}^{n} p_i \sum_{\tau=1}^{T} \phi(F(e_\tau))\pi_\tau(x_{\tau i}) \underbrace{\sum_{\substack{\sigma \in \{0,1\}^T \\ \sigma_\tau = 1}} \prod_{\substack{t=1 \\ t \neq \tau}}^{T} \sigma_t \pi_t(x_{ti}) + (1-\sigma_t)(1-\pi_t(x_{ti}))}_{=1} \\
&= \sum_{i=1}^{n} p_i \sum_{\tau=1}^{T} \delta_\tau u_\tau(x_{\tau i}).
\end{aligned}
$$

In other words, this yields

$$
q = \sum_{\tau=1}^{T} \delta_\tau \sum_{i=1}^{n} p_i u_\tau(x_{\tau i}) = \sum_{\tau=1}^{T} \delta_\tau \mathbb{E}\left[u_\tau(X_\tau)\right].
$$

Defining our utility by $U(X) := q$ completes the proof of the theorem. □

As in the one-period case (see Fishburn (1970), Ch.10 or Savage (1972), Sect. 5.4), we now extend our result for lotteries to arbitrary random variables:

**Proposition 3.3** *Let Axioms 0–6 hold. Then a utility $U(X)$ exists for all $X \in \mathcal{X}$ and is given by*

$$
U(X) = \sum_{\sigma \in \{I_-, I_+\}^T} \mathbb{E}\left[ \prod_{t=1}^{T} \Big( \big( \sigma_t' \pi_t(X_t) + (1-\sigma_t')(1-\pi_t(X_t)) \big) \phi(F(\sigma)) \Big) \right],
$$

*where $F$, $\phi$ and $\pi_t$ are given as in Definition 2.3, 2.4 and 2.5.*

***Proof*** In the following, we construct for each $X \in \mathcal{X}$ two sequences $(X_n^l)_{n\in\mathbb{N}}$ and $(X_n^u)_{n\in\mathbb{N}}$ in $\mathcal{X}_l$ such that

- $X_n^l \preceq X_{n+1}^l \preceq X \preceq X_{n+1}^u \preceq X_n^u$ for all $n \in \mathbb{N}$ and
- $U(X) = \lim_{n\to\infty} U(X_n^l) = \lim_{n\to\infty} U(X_n^u)$.

The existence of these sequences then yields the desired claim:

Assume first that $U(X) > U(Y)$. Then, by the previous construction, there is an $N \in \mathbb{N}$ such that $U(X_n^l) > U(Y_n^u)$ for all $n \geq N$. Thus, again by construction,

$$X \succeq X_n^l \succ Y_n^u \succeq Y.$$

Now, let $U(X) = U(Y)$. Then we find subsequences $\left(X^l_{n^1_k}\right)_{k\in\mathbb{N}}$ and $\left(Y^l_{n^2_k}\right)_{k\in\mathbb{N}}$ such that

$$X^l_{n^1_k} \preceq Y^l_{n^2_k} \preceq X^l_{n^1_{k+1}} \preceq Y^l_{n^2_{k+1}}$$

for all $k \in \mathbb{N}$. Indeed, given $X^l_{n^1_k}$, we can define $\epsilon = (U(X) - U(X^l_{n^1_k}))/2$ and obtain that there is a $N \in \mathbb{N}$ such that $U(Y) - U(Y_n) < \epsilon$ for all $n \geq N$, which in particular yields that $U(X^l_{n^1_k}) < U(Y^l_N)$ and thus $X^l_{n^1_k} \preceq Y^l_N$. Now, we have

$$\lim_{k\to\infty} X^l_{n^1_k} \preceq \lim_{k\to\infty} Y^l_{n^2_k} \preceq \lim_{k\to\infty} X^l_{n^1_{k+1}},$$

which, by Axiom 5, means that $X \sim Y$.

So let us construct the sequences: for each $n \in \mathbb{N}$ and each $i \in \mathbb{N}^T$ such that $1 \leq i_t \leq 2^n$ for all $t \in \{1, \ldots, 2^n\}$ define the set

$$M_{i_1,\ldots,i_T,n} = \Big\{x \in I^T : I_- + (i_t - 1) \cdot \frac{(I_+ - I_-)}{2^n} < x_t < I_- + i_t \cdot \frac{(I_+ - I_-)}{2^n} \text{ for all } t \in \{1, \ldots, T\}\Big\}.$$

Then independence (Axiom 3) yields that

$$X \sim \sum_{i_1,\ldots,i_T=1}^{2^n} \mathbb{P}(X \in M_{i_1,\ldots,i_T,n}) X_{M_{i_1,\ldots,i_T,n}},$$

where $X_{M_{i_1,\ldots,i_T,n}}$ is the random variable with distribution $\mathbb{P}(X_{M_{i_1,\ldots,i_T,n}} \in A) = \mathbb{P}(X \in A | X \in M_{i_1,\ldots,i_T,n})$. Utilizing dominance (Axiom 4), we obtain that

$$\begin{aligned}
&\sum_{i_1,\ldots,i_T=1}^{2^n} \mathbb{P}(X \in M_{i_1,\ldots,i_T,n}) X_{M_{i_1,\ldots,i_T,n}} \\
&\succeq \sum_{i_1,\ldots,i_T=1}^{2^n} \mathbb{P}(X \in M_{i_1,\ldots,i_T,n}) \left(I_- + (i_k - 1)\frac{(I_+ - I_-)}{2^n}\right)_{k\in\{1,\ldots,T\}}
\end{aligned}$$

and that

$$\begin{aligned}&\sum_{i_1,\ldots,i_T=1}^{2^n} \mathbb{P}(X \in M_{i_1,\ldots,i_T,n}) X_{M_{i_1,\ldots,i_T,n}} \\ &\preceq \sum_{i_1,\ldots,i_T=1}^{2^n} \mathbb{P}(X \in M_{i_1,\ldots,i_T,n}) \left( I_- + i_k \cdot \frac{(I_+ - I_-)}{2^n} \right)_{k\in\{1,\ldots,T\}}.\end{aligned}$$

Let us now define $X_n^l$ as the lottery over the consumption streams $\left(I_- + (i_k - 1)\frac{(I_+ - I_-)}{2^n}\right)_{k\in\{1,\ldots,T\}}$ $(i_1, \ldots, i_T \in \{1, \ldots, 2^n\})$, which are each chosen with probability $\mathbb{P}(X \in M_{i_1,\ldots,i_T,n})$, and $X_n^u$ as the lottery over the consumption streams $\left(I_- + i_k \frac{(I_+ - I_-)}{2^n}\right)_{k\in\{1,\ldots,T\}}$ $(i_1, \ldots, i_T \in \{1, \ldots, 2^n\})$, which are each chosen with probability $\mathbb{P}(X \in M_{i_1,\ldots,i_T,n})$. By construction and dominance (Axiom 4), we immediately obtain $X_n^l \preceq X_{n+1}^l$ and $X_n^u \succeq X_{n+1}^u$. Moreover, by monotone convergence, we obtain $U(X) = \lim_{n\to\infty} U(X_n^l)$ and $U(X) = \lim_{n\to\infty} U(X_n^u)$, which proves the claim. □

Exactly the same proof also yields the desired extension for lotteries $X \in \mathcal{X}$ to general probability measures $X \in \mathcal{X}$ for the second part of Proposition 3.1:

**Proposition 3.4** *Let Axioms 0–7 hold. Then a utility $U(X)$ exists for all $X \in \mathcal{X}$ and is given by*

$$U(X) = \sum_{\tau=1}^{T} \delta_\tau \mathbb{E}[u_\tau(X_\tau)].$$

Let us now turn to prove the second parts of both statements:

**Proposition 3.5** *Let $\pi_t : I \to [0, 1]$ $(t = 1, \ldots, T)$ be (weakly) increasing and continuous functions and $F : I^T \to I$ and $\phi : I \to [0, 1]$ be (weakly) increasing functions such that*

$$\begin{aligned} 0 &= \pi_t(I_-) = \phi(I_-) & 1 &= \pi_t(I_+) = \phi(I_+) \\ I_- &= F(I_-, \ldots, I_-) & I_+ &= F(I_+, \ldots, I_+) \end{aligned}$$

*for $t = 1, \ldots, T$. Let $\succeq$ be a preference relation on $\mathcal{X}$ given by $X \succeq Y$ if and only if $U(X) \geq U(Y)$ with $U$ as in (3), i.e.,*

$$U(X) = \mathbb{E}\left[ \sum_{\sigma\in\{I_-,I_+\}^T} \prod_{t=1}^{T} \left( \sigma_t' \pi_t(x_{ti}) + (1 - \sigma_t')(1 - \pi_t(x_{ti})) \right) \phi(F(\sigma)) \right].$$

*Then, $\succeq$ satisfies Axioms 0–6.*

***Proof*** Axiom 1 is satisfied, since any two numbers $U(X), U(Y)$ are comparable in $\mathbb{R}$. Axiom 2 holds by the transitivity of $\leq$ on $\mathbb{R}$. Axiom 5 directly follows from the continuity of $U$.

To prove that Axiom 0 is satisfied, it suffices to show that the function $(a_1, \ldots, a_n) \mapsto U(a_1, \ldots, a_n)$ with

$$U(a_1, \ldots, a_n) = \sum_{\sigma \in \{I_-, I_+\}^T} \prod_{t=1}^{T} \Big( \sigma_t' \pi_t(a_t) + (1 - \sigma_t')(1 - \pi_t(a_t)) \Big) \phi(F(\sigma))$$

is increasing in $a_i$ for each $i = 1, \ldots, T$. Indeed, the more $I_+$ are in $\sigma$, the larger is $\phi(F(\sigma))$. Thus, the function itself is increasing when permutations $\sigma \in \{I_-, I_+\}^T$ with more $I_+$ get a larger weight. This happens when $\pi_t(a_t)$ is larger, which happens when $a_t$ is increasing. This monotonicity and the monotonicity of the expectation moreover directly yield that Axiom 4 holds.

To prove Axiom 3, let $X, Z \in \mathcal{X}_l$ be arbitrary. Write

$$U(X) = \sum_{i=1}^{n_x} p_i^x \sum_{\sigma \in \{I_-, I_+\}^T} \prod_{t=1}^{T} \Big( \sigma_t' \pi_t(x_{ti}) + (1 - \sigma_t')(1 - \pi_t(x_{ti})) \Big) \phi(F(\sigma)),$$

$$U(Z) = \sum_{i=1}^{n_z} p_i^z \sum_{\sigma \in \{I_-, I_+\}^T} \prod_{t=1}^{T} \Big( \sigma_t' \pi_t(z_{ti}) + (1 - \sigma_t')(1 - \pi_t(z_{ti})) \Big) \phi(F(\sigma)).$$

Then for $p \in (0, 1)$, we have

$$\begin{aligned} &U(pX + (1-p)Z) \\ &= \sum_{i=1}^{n_x} (p \cdot p_i^x) \sum_{\sigma \in \{I_-, I_+\}^T} \prod_{t=1}^{T} \Big( \sigma_t' \pi_t(x_{ti}) + (1 - \sigma_t')(1 - \pi_t(x_{ti})) \Big) \phi(F(\sigma)) \\ &\quad + \sum_{i=1}^{n_y} ((1-p) \cdot p_i^z) \sum_{\sigma \in \{I_-, I_+\}^T} \prod_{t=1}^{T} \Big( \sigma_t' \pi_t(z_{ti}) + (1 - \sigma_t')(1 - \pi_t(z_{ti})) \Big) \phi(F(\sigma)) \\ &= pU(X) + (1-p)U(Z). \end{aligned}$$

Thus, if $X, Y \in \mathcal{X}_l$ are such that $U(X) \geq U(Y)$ and $Z \in \mathcal{X}_l$ as well as $p \in (0, 1)$ are chosen arbitrarily, we obtain

$$\begin{aligned} U(pX + (1-p)Z) = pU(X) + (1-p)U(Z) &\geq pU(Y) + (1-p)U(Z) \\ &= U(pY + (1-p)Z). \end{aligned}$$

The construction in the proof of Theorem 3.3 now immediately yields that Axiom 3 also holds for $X, Y, Z \in \mathcal{X}$.

To show that Axiom 6 is satisfied, assume that $x \in I^T$, $t \in \{1, \ldots, T\}$ and $a \in I$. Let us write

$$L_t(p) = \begin{cases} \xrightarrow{p} (x_1, \ldots, x_{t-1}, I_+, x_{t+1}, \ldots) \\ \xrightarrow{1-p} (x_1, \ldots, x_{t-1}, I_-, x_{t+1}, \ldots) \end{cases}.$$

Now, direct calculations yield

$$\begin{aligned} &U(x_1, \ldots, x_{\tau-1}, a, x_{\tau+1}, \ldots, x_T) \\ &= \sum_{\sigma \in \{I_-, I_+\}^T} \phi(F(\sigma)) \Bigg( \prod_{t \neq \tau} \Big( \sigma_t' \pi_t(x_t) + (1 - \sigma_t')(1 - \pi_t(x_t)) \Big) \\ &\quad \cdot (\sigma_\tau' \pi_\tau(a) + (1 - \sigma_\tau')(1 - \pi_\tau(a)) \Bigg) \end{aligned}$$

and

$$\begin{aligned} &U(L_t(p)) \\ &= p \sum_{\sigma \in \{I_-, I_+\}^T} \phi(F(\sigma)) \Bigg( \prod_{t \neq \tau} \Big( \sigma_t' \pi_t(x_t) + (1 - \sigma_t')(1 - \pi_t(x_t)) \Big) \\ &\quad \cdot (\sigma_\tau' \pi_\tau(1) + (1 - \sigma_\tau')(1 - \pi_\tau(1))) \Bigg) \\ &\quad + (1 - p) \sum_{\sigma \in \{I_-, I_+\}^T} \phi(F(\sigma)) \Bigg( \prod_{t \neq \tau} \Big( \sigma_t' \pi_t(x_t) + (1 - \sigma_t')(1 - \pi_t(x_t)) \Big) \\ &\quad \cdot (\sigma_\tau' \pi_\tau(0) + (1 - \sigma_\tau')(1 - \pi_\tau(0))) \Bigg) \\ &= \sum_{\sigma \in \{I_-, I_+\}^T} \phi(F(\sigma)) \Bigg( \prod_{t \neq \tau} \Big( \sigma_t' \pi_t(x_t) + (1 - \sigma_t')(1 - \pi_t(x_t)) \Big) \\ &\quad \cdot (p \sigma_\tau' + (1 - p)(1 - \sigma_\tau')) \Bigg), \end{aligned}$$

which shows that for $p = \pi_t(a)$ the two utilities coincide. $\square$

**Proposition 3.6** *Let $(\delta_t)_{t=1,\ldots,T} \in \mathbb{R}^T_{\geq 0} \backslash \{0\}$ and $u_t : I \to \mathbb{R}$ for all $t = 1, \ldots, T$ be continuous and (weakly) increasing functions. Let $\succeq$ be a preference relation on $\mathcal{X}$ defined by $X \succeq Y$ if and only if $U(X) \geq U(Y)$ for*

$$U(X) = \sum_{\tau=1}^{n} \delta_\tau \mathbb{E}[u_\tau(X_\tau)].$$

*Then $\succeq$ satisfies Axioms 0–7.*

***Proof*** Axiom 1, 2 and 5 follow as in the previous proof. Axiom 3 directly follows from the linearity of the expectation. For Axiom 0, note that $u_\tau$ is (weakly) increasing for all $t \in \{1, \dots, T\}$. Since the function $U$ is a weighted sum of these functions with non-negative weights, it is also (weakly) increasing. Thus, Axiom 0 holds. This and the monotonicity of the expectation directly yield that Axiom 4 is satisfied.

To prove that Axiom 6 holds, let $x \in I^T$, $\tau \in \{1, \dots, T\}$ and $a \in I$. Then for $p = (u_\tau(a) - u_\tau(I_-))/(u_\tau(I_+) - u_\tau(I_-))$, we have

$$\begin{aligned}
&U(x_1, \dots, x_{\tau-1}, a, x_{\tau+1}, \dots, x_T)\\
&= \sum_{t \neq \tau} \delta_t u_t(x_t) + \delta_\tau u_\tau(a)\\
&= \sum_{t \neq \tau} \delta_t u_t(x_t) + \delta_\tau (p u_\tau(I_+) + (1-p) u_\tau(I_-))\\
&= U(L_t(p)),
\end{aligned}$$

where $L_t(p)$ is as in the proof of Proposition 3.5.

To prove Axiom 7, let $a \in \{I_-, I_+\}^T$ be arbitrary. Then, by definition of $F(a)$, we have

$$U(a_1, \dots, a_T) = \sum_{\tau=1}^{T} \delta_\tau u_\tau(a_\tau) = \sum_{\tau=1}^{T} \delta_\tau u_\tau(F(a)) = U(F(a), \dots, F(a)).$$

Moreover, by definition of $\phi(F(a))$, we have

$$\begin{aligned}
U(L(\phi(F(a)))) &= \phi(F(a)) \sum_{\tau=1}^{T} \delta_\tau u_\tau(I_+) + (1 - \phi(F(a))) \sum_{\tau=1}^{T} \delta_\tau u_\tau(I_-)\\
&= \sum_{\tau=1}^{T} \delta_\tau u_\tau(F(a)) = U(F(a), \dots, F(a)).
\end{aligned}$$

Thus,

$$\sum_{\tau=1}^{T} \delta_\tau u_\tau(a_\tau) = \Phi(F(a)) \sum_{\tau=1}^{T} \delta_\tau u_\tau(I_+) + (1 - \phi(F(a))) \sum_{\tau=1}^{T} \delta_\tau u_\tau(I_-),$$

and therefore

$$\phi(F(a)) = \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau=1}^{T} \delta_\tau (u_\tau(a_\tau) - u_\tau(I_-)). \tag{8}$$

Now, let $a, b \in \{I_-, I_+\}^T$ such that $\{t : a_t = I_+ \text{ and } b_t = I_+\} = \emptyset$. Then by (8), we have

$$
\begin{aligned}
&\phi(F(a)) + \phi(F(b)) \\
&= \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau=1}^{T} \delta_\tau \left(u_\tau(a_\tau) - u_\tau(I_-)\right) \\
&\quad + \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau=1}^{T} \delta_\tau \left(u_\tau(b_\tau) - u_\tau(I_-)\right) \\
&= \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau : a_\tau = I_+} \delta_\tau \left(u_\tau(I_+) - u_\tau(I_-)\right) \\
&\quad + \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau : b_\tau = I_+} \delta_\tau \left(u_\tau(I_+) - u_\tau(I_-)\right) \\
&= \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau : \max\{a_\tau, b_\tau\} = I_+} \delta_\tau \left(u_\tau(I_+) - u_\tau(I_-)\right) \\
&= \frac{1}{\sum_{\tau=1}^{T} \delta_\tau (u_\tau(I_+) - u_\tau(I_-))} \sum_{\tau=1}^{T} \delta_\tau \left(u_\tau(\max\{a_\tau, b_\tau\}) - u_\tau(I_-)\right) \\
&= \phi(F(\max\{a_1, b_1\}, \ldots, \max\{a_T, b_T\})).
\end{aligned}
$$

□

## 4 Examples

Let us conclude by listing some examples that satisfy Axioms 0–6, but not the time–probability equivalence (Axiom 7). Note that to define a preference relation that satisfies Axioms 0–6, it suffices, by Theorem 2.7, to describe continuous and (weakly) increasing functions $\pi_t : I \to [0, 1]$, $t \in \{1, \ldots, T\}$ satisfying $\pi_t(I_-) = 0$ and $\pi_t(I_+) = 1$ for all $t \in \{1, \ldots, T\}$ as well as values $\phi(F(\sigma))$ for all $\sigma \in \{I_-, I_+\}^T$ such that $\phi(F(I_+, \ldots, I_+)) = 1$, $\phi(F(I_-, \ldots, I_-)) = 0$ and $\phi(F(\sigma)) \geq \phi(F(\sigma'))$ whenever $\sigma_t \geq \sigma'_t$ for all $t \in \{1, \ldots, T\}$.

***Example 4.1*** Let us reconsider Example 2.2. Indeed, let $\phi(F(1, 1)) = 1$, $\phi(F(0, 0)) = 0$ and $\phi(F(1, 0)) = F(\phi(0, 1)) = c$ with $c \in (1/2, 1)$ and let $\pi_1 : [0, 1] \to [0, 1]$ be an arbitrary continuous and (weakly) increasing function. Moreover, set $\pi_2 : [0, 1] \to [0, 1]$ such that

$$\pi_2(x) = \frac{c - c\pi_1(1 - x)}{\pi_1(1 - x)(1 - 2c) + c}.$$

Then these functions and values describe, according to Theorem 2.7, a preference relation satisfying Axioms 0–6. Its utility function is

$$U(X) = \mathbb{E}\left[\frac{\pi_1(X_1)\,(c - c\pi_1(1-X_2))}{\pi_1(1-X_2)(1-2c)+c}(1-2c) \right. \\ \left. +c\left(\pi(X_1) + \frac{c - c\pi_1(1-X_2)}{\pi_1(1-X_2)(1-2c)+c}\right)\right],$$

and thus direct calculations yield that $U(a, 1-a) = c$ for all $a \in [0, 1]$. Finally, note that

$$\phi(F(1,1)) = 1 \neq 2c = \phi(F(1,0) + \phi(F(0,1)),$$

thus, Axiom 7 is not satisfied.

***Example 4.2*** As a next example, we propose a preference relation on $[0, 1]^T$ such that $(a_1, \dots, a_T) \sim (0, \dots, 0)$ if and only if $a_i = 0$ for some $i \in \{1, \dots, T\}$. Indeed, let $\pi_t : [0, 1] \to [0, 1]$, $t \in \{1, \dots, T\}$, be arbitrary continuous and strictly increasing functions such that $\pi_t(0) = 0$ and $\pi_t(1) = 1$. Moreover, choose $\phi(F(1, \dots, 1)) = 1$ and $\phi(F(\sigma)) = 0$ for $\sigma \neq (1, \dots, 1)$. By Theorem 2.7, these functions and values describe a preference relation satisfying Axioms 0-6, whose utility function reads

$$U(X) = \mathbb{E}\left[\pi_1(X_1) \cdot \dots \cdot \pi_T(X_T)\right].$$

It is immediate that $U(a_1, \dots, a_T) = 0$ if and only if $a_t = 0$ for $t \in \{1, \dots, T\}$. Moreover, since

$$F(\phi(1, \dots, 1)) = 1 \neq 0 = F(\phi(1, 0, \dots, 0)) + F(\phi(0, 1, \dots, 1)),$$

Axiom 7 is not satisfied.

***Example 4.3*** Let us finally propose an example where, with $I = [0, 1]$, $T = 4$, we have

$$(1, 1, 0, 0) \sim (0, 0, 1, 1) \succ (1, 0, 1, 0) \sim (0, 1, 0, 1),$$

which are preferences that previously occurred in the analysis of preference that incorporates consumption smoothing (i.e., Gilboa (1989)). Namely, choose $\pi_1(x) = \pi_2(x) = x$ for all $x \in [0, 1]$ and

$$\phi(F(a_1, a_2, a_3, a_4)) = \begin{cases} 0 & \text{if } |\{i : a_i = 1\}| = 0 \\ 0.25 & \text{if } |\{i : a_i = 1\}| = 1 \\ 0.5 & \text{if } |\{i : a_i = 1\}| = 2 \text{ and } a_i \neq a_{i-1} \text{ for all i} \\ 0.6 & \text{if } |\{i : a_i = 1\}| = 2 \text{ and } a_i = a_{i-1} \text{ for some i} \\ 0.75 & \text{if } |\{i : a_i = 1\}| = 3 \\ 1 & \text{if } |\{i : a_i = 1\}| = 4 \end{cases}.$$

Then,

$$U(1, 1, 0, 0) = 0.6 \qquad U(0, 0, 1, 1) = 0.6$$
$$U(1, 0, 1, 0) = 0.5 \qquad U(0, 1, 0, 1) = 0.5$$

as desired. Moreover, we directly note that Axiom 7 is not satisfied since

$$\phi(F(1, 1, 0, 0)) + \phi(F(0, 0, 1, 1)) = 1.2 \neq 1 = \phi(F(1, 1, 1, 1)).$$

## 5 Conclusion

This article has provided an alternative way to axiomatize multi-period expected utility. Moreover, it describes new classes of decision models where the risk preference in certain situations only depends on the consumption periods that are affected by the choice. In particular, these new classes describe preferences that might not be fully time separable. Full time separability is not a compelling requirement in intertemporal decision theory, since it ignores consumption smoothing and variation aversion. Therefore, the axiomatization and the corresponding characterization of the new modeling class allows for a larger degree of flexibility that will be appropriate in certain situations.

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