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Social evaluation functionals with an arbitrary set of alternatives

Juan C. Candeal[1]

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Abstract

This paper explores the concept of a social evaluation functional in the case of an arbitrary set of alternatives. In the first part, a characterization of projective social evaluations functionals is shown whenever the common restricted domain is the set of all bounded utility functions equipped with the supremum norm topology. The result makes a crucial use, among others, of a continuity axiom. In the second part, a comparison meaningful property is introduced for a social evaluation functional which allows us for obtaining a more general result with no continuity requirements. Finally, an impossibility theorem, which is reminiscent of that is obtained by Chichilnisky in (Q J Econ 97:337–352, 1982) but without using topological conditions, is offered.

Keywords Social evaluation functionals · Continuity · Comparison meaningfulness · Projective rules

1 Introduction

This paper deals with aggregation of utility functions. In particular, we examine what is called a Social Evaluation Functional (SEF) which is, basically, a specification of a Social Welfare Functional (SWF) in which the range is a subset of the set of all real-valued functions defined on the alternative set, henceforth denoted by X. Thus, we pay attention to the social choice problem consisting in the aggregation of individual utility functions into social utility functions. In a series of classical works by Sen (1970a, 1970b, 1977) and others (see, e.g., Roberts 1980a, b; d'Aspremont and Gevers 2002; Bossert and Weymark 2004), the range includes ordinal preferences, not real-valued functions (see Roberts 1983). Of course, one can get a real-valued function from a preference as long as the preference is continuous. However, having a numerical function from the beginning is giving us the possibility of formulating axioms that cannot be treated without a numerical function in the range. As will be

Extended author information available on the last page of the article

seen later, the ordinal-scale-preserving property, the continuity and the projectivity are some of such axioms. Thus, for a SEF (social) numerical evaluations, associated to profiles of individual utilities, of the alternatives do matter whereas for a SWF do not. It should be also remarked that this approach is closely related to a line of works started by Chichilnisky where, usually, the set of alternatives has certain (topological) structure.

Many of the SWFs that appear in the literature are implicitly defined in terms of corresponding SEFs. For example, the Borda rule and other scoring rules are defined in terms of numerical evaluations of the candidates which yields the corresponding social ordering among them. Note that, in elections, the number of votes obtained by each party not only determines the (possible) winner but also the number of seats or representatives got in the parliament. Thus, numerical evaluations have relevance in this context. The same happens with social rules that allow for certain kind of inter/intra-personal comparability such as the utilitarian rule, the Rawlsian one and other rules that arise in consumer theory and welfare economics [for instance, those related to inequality measurement as can be seen in Fleurbaey and Hammond (2004)]. Moreover, if one intends to reflect the intensity among distinct alternatives in both individual and social preferences, a scenario built upon SEFs seems to be more plausible than the one based on merely SWFs.

For a more concrete example, think of a situation where a finite number of individuals have to express their utilities derived from establishing a facility or a public service in their neighborhoods. Significant variables to determine such a utility may include those of the distance to the facility, the kind of services offered, the environmental impact and so on. Thus, we are facing with a spatial location problem. In this context, it is natural to demand that the social ranking among the vectors of the distinct characteristics be carried out as a certain average of the individual utilities over these vectors. Therefore, numerical evaluations of the alternatives provide a more appropriate framework to collect individuals' assessments than merely ordinal rankings among them.

Given a finite set X of alternatives (of size m) and a finite number n of individuals, a SEF is a mapping $F : \mathcal{U}^n \longrightarrow \mathcal{U}$, where $\mathcal{U} = \mathbb{R}^X$ is the set of all real-valued functions from X into the reals $\mathbb{R}$. A SEF can be interpreted as a function mapping $m \times n$ real-valued matrices into column vectors with m components. This view allows us for introducing the concept of continuity in a natural way in the social aggregation problem and, in particular, enables us to study the trade-off betweeen continuity and certain classical axioms of the social choice literature such as the axiom of the Independence of the Irrelevant Alternatives (IIA). An account of all this was done in Candeal (2015) for X finite. However, it is also natural to extend the study for an arbitrary set X because in many economic models both the social alternatives and the individual utilities exhibit considerable structure. For example, alternatives could be vectors of public goods with individual utility functions required to be continuous, monotonic and quasi-concave. For other significant examples in economic environments see LeBreton and Weymark (2002) and Campbell and Kelly (2002).

Three main results are shown in the paper. The first one (Theorem 1) is a characterization of the SEFs that are projective; i.e., functionals F for which there is an individual, say i, such that $F(u_1, \ldots, u_j, \ldots, u_n) = u_i$, for all u_j in a certain common subdomain of $\mathbb{R}^X$. Because individual i dictates over all profiles, this result entails the existence of a (single) strong dictator. This conclusion strongly depends upon the continuity axiom but it should be noted that IIA is not required as an assumption. It is assumed that all individuals have a common domain restriction. Indeed, the domain should include all real-valued bounded functions defined on X and be equipped with the supremum norm topology.

The second main result (Theorem 2) introduces the concept of comparison meaningfulness with respect to (henceforth, w.r.t.) certain scales as a key assumption to deal with the aggregation problem. For this concept to be consistent, the codomain of the social rule must be formed of real-valued functions. So, the framework of SEFs really justifies its use. Theorem 2 characterizes those SEFs which have a very particular functional form; to wit, for each alternative there is an individual so that, for any profile of utility functions, the value that society assigns to that alternative coincides with the value that the referred individual gives to that alternative. In particular, this means that the individual may, or may not, vary with the alternative but not with the profile. In addition to comparison meaningfulness, the result requires two more axioms; namely, unanimity and monotonicity. An interpretation of such rules in the context of social choice theory would tell about the existence of a committee of experts taking all social evaluations. Although the result is, in a sense, more flexible than the (negative) conclusion stated in the classical Arrow's theorem, it is somewhat reminiscent of both Gibbard's oligarchy theorem (see Gibbard 2014; Weymark 2014) and certain results about the existence of local dictators in the context of social welfare functions. The domain is also restricted; in concrete, it has to content all constant functions and has to be stable under the set of all strictly increasing real-valued transformations of one real variable. It is well known that comparison meaningfulness is a core concept in measurement theory (see, e.g., Krantz et al. 1971; Roberts 1979). Moreover, it also has great significance in social choice theory because its interpretation in terms of utility measurability and inter/intra-comparability of well-being among individuals (see Luce and Raiffa 1957; Sen 1970a, b, 1977; Roberts 1980a, b; d'Aspremont and Gevers 2002; Bossert and Weymark 2004).

The third result shown (Theorem 4) is linked to Chichilnisky's topological model of preference aggregation. Specifically, it is reminiscent of Chichilnisky's theorem about the nonexistence of social aggregation rules that are continuous, anonymous and respect unanimity in certain preference domains. After presenting the main features of Chichilnisky's model, an alternative framework, which makes no use of topological assumptions, is proposed. The result obtained turns out to be an impossibility theorem provided that the common domain is a superset of the set of all affine real-valued functions defined on a bounded set of $\mathbb{R}^m$. Because Theorem 4 requires a strong form of interpersonal noncomparability, based on cardinal utilities, it can also be related to certain variants of Arrow's impossibility theorem in this context (see, e.g., Sen 1970a, b; d'Aspremont and Gevers 2002).[1]

All proofs of the results stated in the paper are relegated to an "Appendix".

2 Preliminaries

In this section some basic background is presented in connection with SEFs.

Let $N := \{1, \ldots, n\}$ and $M := \{1, \ldots, m\}$, where $n, m \in \mathbb{N}$. Throughout the paper N refers to the set of individuals and M to the set of alternatives provided that it is finite. As usual $\mathbb{R}^n$ is the n-dimensional Euclidean space, i.e., $\mathbb{R}^n = \{a = (a_j)_{j \in N} : a_j \in \mathbb{R}\}$.

Let X be a nonempty set. A typical utility function from X to $\mathbb{R}$ will be denoted by v. The notation $\mathcal{V}$ refers to the set of all utility functions (also denoted by $\mathbb{R}^X$).

A *profile* (or an *n-tuple*) of utility functions will be denoted by $V = (v_j)_{j \in N}$, and $\mathcal{V}^n$ will be the set of all possible profiles. A profile $V = (v_j)_{j \in N} \in \mathcal{V}^n$ can also be viewed as a real-valued map defined on $X \times N$ in the following manner: $(x, j) \in X \times N \longrightarrow V(x, j) = v_j(x) \in \mathbb{R}$. Note that $V(i, j)$ can be interpreted as the value that individual j assigns to alternative i. When $X = M$ a profile $V = (v_j)_{j \in N}$ can be identified with an $m \times n$ matrix whose entries are the real numbers $(v_j(i))_{i \in M, j \in N}$.

We now present some basic notations that will be useful later. For a given $x \in X$ and $V = (v_j)_{j \in N} \in \mathcal{V}^n$, $V(x)$ will denote the following vector $V(x) := (v_j(x))_{j \in N} \in \mathbb{R}^n$. Let $\Omega := \mathbb{R}^{\mathbb{R}} = \{\phi : \mathbb{R} \to \mathbb{R}\}$ and $a = (a_j)_{j \in N} \in \mathbb{R}^n$. If $\phi \in \Omega$, then $\phi(a)$ is the vector $\phi(a) := (\phi(a_j))_{j \in N} \in \mathbb{R}^n$. In a similar way, if $\Phi = (\phi_j)_{j \in N} \in \Omega^n$, then $\Phi(a) := (\phi_j(a_j))_{j \in N} \in \mathbb{R}^n$. For now on, and if no confusion is possible, subscripts will be usually omitted.

Let there be given $\Phi = (\phi_j) \in \Omega^n$ and $V = (v_j) \in \mathcal{V}^n$. Then $\Phi \circ V := (\phi_j \circ v_j) \in \mathcal{V}^n$, where "$\circ$" stands for the usual composition operation, i.e., $\phi_j \circ v_j(x) = \phi_j(v_j(x))$, $\forall\, x \in X$, $\forall\, j \in N$. Also, and to make the notation as easy as possible, sometimes it will be used $\Phi(V)$ instead of $\Phi \circ V$. Note that, with the notation introduced, it holds that $\Phi(V)(x) = \Phi(V(x))$, $\forall\, x \in X$. Moreover, if $\Phi = (\phi_j) \in \Omega^n$ is such that $\phi_j = \phi$, for some $\phi \in \Omega$, $\forall\, j \in N$, then we will write $\phi(V)$ instead of $\Phi \circ V$.

For a given $v \in \mathcal{V}$, V_v will stand for the following profile $V_v := (v, \ldots, v) \in \mathcal{V}^n$. A constant utility function, with value $c \in \mathbb{R}$, will be denoted by c_X (i.e., $c_X(y) = c$, $\forall y \in X$).

The subdomain of Ω which consists of all strictly increasing real-valued functions defined on $\mathbb{R}$ will be denoted by Δ. An important subdomain of Δ is Δ_{ia}, the set of all increasing affine real-valued functions. That is, $\Delta_{ia} = \{\phi \in \Delta : \phi(t) = at + b, a > 0, b \in \mathbb{R}\}$. In economics and social sciences a function $\phi \in \Delta_{ia}$ is

[1] The relationship of our work with Gibbard's oligarchy theorem, with the existence of local dictators and with the contributions of Sen and d'Aspremont and Gevers mentioned above, when individuals' utilities are cardinally measurable and interpersonal noncomparable, were suggested us by two anonymous referees.

usually referred to as a *cardinal scale* while the term *interval scale* is more frequently used in measurement theory.

Let there be given $\Theta \subseteq \Omega$ and $\mathcal{U} \subseteq \mathcal{V}$. Then $\mathcal{U}$ is said to be Θ-*stable* provided that $\theta \circ u \in \mathcal{U}, \forall\, u \in \mathcal{U}, \forall\, \theta \in \Theta$.

Given a permutation σ of N and a profile $V = (v_j) \in \mathcal{V}^n$, we will denote by $V^\sigma \in \mathcal{V}^n$ the following profile: $V^\sigma := (v_{\sigma(j)})$. The set of all permutations from N onto N will be denoted by $S(N)$.

Let $\mathcal{U}$ be a subset of $\mathcal{V}$. A *Partial Social Evaluation Functional* or, for brevity, a *Social Evaluation Functional (SEF)* is a rule $F : \mathcal{U}^n \to \mathcal{V}$ that assigns a real-valued function $F(U) \in \mathcal{V}$, viewed as a social evaluation, to any profile of individual utilities U in the domain $\mathcal{U}^n$.

Let there be given a SEF F and $x \in X$. Then the pair (F, x) induces a real-valued function defined on $\mathcal{U}^n$, which will be denoted by F^x, in the following way: $U \in \mathcal{U}^n \to F^x(U) = F(U)(x) \in \mathbb{R}$.

The following definition presents some of the main concepts that will appear later.

Definition 1 Assume $\mathcal{U} \subseteq \mathcal{V}$ is a Δ-stable domain. A SEF $F : \mathcal{U}^n \to \mathcal{V}$ is said to be, or satisfies:

(i) *Ordinal-Scale-Preserving* (*OSP*) if $F(\phi(U)) = \phi(F(U)), \quad \forall\, \phi \in \Delta, \forall\, U \in \mathcal{U}^n$,

(ii) *Ordinal-Measurability-Interpersonal-Noncomparability* (*OMIN*) if $\forall\, U \in \mathcal{U}^n, \forall\, \Phi \in \Delta^n$, there is $\varphi \in \Delta$, which depends on U and Φ, such that $F(\Phi(U)) = \varphi(F(U))$,

(iii) *Unanimous* (*Una*) if $F(U_u) = u, \forall\, u \in \mathcal{U}$,

(iv) *Anonymous* (*A*) if $F(U^\sigma) = F(U), \forall\, U \in \mathcal{U}^n, \forall\, \sigma \in S(N)$,

(v) *Projective* (*Pro*) if it is a projection, i.e., if there is an individual $i \in N$ such that $F(U) = u_i, \forall\, U = (u_j) \in \mathcal{U}^n$.

Remark 1

(i) If cardinality of X is one (i.e., if $\#X = 1$), then a SEF reduces to a real-valued function of n real variables. In this case, the terms unanimity and anonymity are often referred to as idempotency and symmetry, respectively.

(ii) If in the definition of OMIN above $\Phi \in \Delta^n_{ia}$, then F is said to satisfy *Cardinal-Measurability-Interpersonal-Noncomparability (CMIN)*.

In our context, Pro axiom turns out to be the functional form of a strong dictatorial social rule because it requires the social utility function to be exactly the utility function of one individual (the dictator). Now, whereas Una and A conditions are clear for a normative point of view, both OSP and OMIN need a more detailed justification. Indeed, OSP means that if all individuals use the same scale to measure the corresponding utilities, then so does the social utility associated with each profile. Thus, it can be viewed as a unanimity principle over the type of scale used to measure both individual and collective utilities. Note that OSP allows for interpersonal comparability of utilities across individuals. OMIN conveys that a

SEF is consistent in terms of preferences. Recall that two profiles $U_1, U_2 \in \mathcal{U}^n$ are said to be *informationally equivalent* provided that $U_2 = \Phi \circ U_1$, for some $\Phi \in \Delta^n$. A similar concept applies to utility functions. Then, OMIN tells us that informationally equivalent profiles are mapped into informationally equivalent social utilities. In the social choice theory setting, and from an ordinal point of view, this is often an appealing property.

3 Projective SEFs

In this section there is a common domain restriction for both individuals and society. Indeed, the set of utility functions considered is the set of all bounded real-valued functions defined on X, henceforth, denoted by $\mathcal{B}$. Thus, and according to the previous notations, in this section $\mathcal{U} = \mathcal{V} = \mathcal{B}$. It should be noted that, a priori, this is not a severe restriction because, viewed as preference relations defined on X, both individual and collective utility functions can be assumed to have the range on the open unit interval (0, 1). The main reason for doing this is due to the fact of introducing a topology on the corresponding set of functions. We will consider that $\mathcal{B}$ is equipped with the *supremum norm topology* which is given by the metric induced by the supremum norm. That is, $d(u,v) = ||u - v||_\infty = \sup_{x\in X}|u(x) - v(x)|$, $(u, v \in \mathcal{B})$. With this norm, actually, $\mathcal{B}$ becomes a Banach space with the usual binary operations of addition and multiplication by scalars defined pointwise. Note that if $X = M$, then $\mathcal{B}$ can be identified with $\mathbb{R}^m$. On the Cartesian $\mathcal{B}^n$ we will consider the product topology.

Let $m \in \mathbb{N}$. A partition of X is a finite collection of pairwise disjoint subsets of X whose union is X. A typical partition of X will be denoted by $E = (E_i)_{i\in M}$. Let there be given $E = (E_i)_{i\in M}, F = (F_j)_{j\in N}$ two partitions of X. Then F is said to be *finer than* E, denoted by $E \precsim F$, whenever for every F_j there is E_i such that $F_j \subseteq E_i$. Equivalently, for each $i \in M$, $E_i = \bigcup_{j\in K\subseteq N} F_j$. It is easy to see that $\precsim$ is a reflexive and transitive binary relation defined on the set of all the partitions of X.

The indicator function of a subset $E \subseteq X$ will be denoted by 1_E. That is, $1_E(x) = 1$, whenever $x \in E$, and $1_E(x) = 0$, otherwise. Let $m \in \mathbb{N}$. A *simple function* $s \in \mathcal{B}$ is one of the form $s = \sum_{i\in M} a_i 1_{E_i}$, where, for each $i \in M$, $a_i \in \mathbb{R}$, $E_i \subseteq X$, and $E = (E_i)_{i\in M}$ is a partition of X. So, a simple function is identified by means of a natural number m, a vector $a = (a_i) \in \mathbb{R}^m$, and a partition $E = (E_i)_{i\in M}$ of X. The subsets of the partition are said to be the support of the corresponding simple function. The subset of $\mathcal{B}$ which consists of all simple functions will be denoted by $\mathcal{S}$. Given a partition $E = (E_i)_{i\in M}$ of X the subset of $\mathcal{S}$ consisting of all simple functions supported on E will be denoted by $\mathcal{S}_E$. In formula, $\mathcal{S}_E = \{s_a = \sum_{i\in M} a_i 1_{E_i} : a = (a_i) \in \mathbb{R}^m\}$. Note that $\mathcal{S} = \bigcup_E \mathcal{S}_E$, where E runs over the set of all partitions of X. Thus, for every partition E of X, it holds that $\mathcal{S}_E \subseteq \mathcal{S} \subseteq \mathcal{B}$.

The natural concept of continuity in this framework is now in order.

Definition 2 A SEF $F : \mathcal{B}^n \to \mathcal{B}$ is *Continuous* (C) if, for all open set B of $\mathcal{B}$, $F^{-1}(B) := \{U \in \mathcal{B}^n : F(U) \in \mathcal{B}\}$ is an open set of $\mathcal{B}^n$.

The continuity axiom is a very technical requirement. Roughly speaking, if the values of two profiles of individual utility functions are close to each other and continuity is satisfied, then the corresponding social utilities are also close to one another. Moreover, and because $\mathcal{B}$ is endowed with the supremum norm topology, this occurs uniformly on all alternatives. Although certain preference spaces can also be topologized, hence the continuity axiom could also be introduced in the context of SWFs, is in the framework of SEFs where is defined in a more natural way. As far as I know, Chichilnisky (1982) pioneered the introduction of certain topologies in spaces of utility functions (see also Sect. 5).

Now the main result of this section is shown.

Theorem 1 *Assume* $\#X > 1$ *and* $n > 1$*. Then a SEF* $F : \mathcal{B}^n \to \mathcal{B}$ *fulfils OMIN, OSP, C and Una iff it is Pro.*

Remark 2

(i) It can be shown that all the conditions given in the statement of Theorem 1 are, in fact, independent.

(ii) The assumptions $\#X > 1$ and $n > 1$ are essential in the statement of Theorem 1. Indeed, if $\#X = 1$, then $\mathcal{B} = \mathbb{R}$. If, in addition, $n = 1$, then a SEF F is Pro if and only if it satisfies OSP. In the case that $\#X = 1$ and $n > 1$ OMIN axiom is obviously satisfied and the fulfilment of the other axioms does not guarantee the rule to be Pro. This is the case for the SEF $F : \mathbb{R}^2 \to \mathbb{R}$ defined by $F(a, b) = \max\{a, b\}$, $a, b \in \mathbb{R}$. If $\#X > 1$ and $n = 1$, then $\mathcal{B}$ can be identified with the space of column vectors of one component. In addition, note that Una turns out to be a superfluous assumption. Moreover, OMIN follows from OSP. In this case there are SEFs satisfying OSP and C which are not Pro. For example, let $X = \{x, y\}$, $n = 1$, and $F : \mathcal{B} \to \mathcal{B}$ defined by $F(U)(x) = \max\{U(x), U(y)\}$, and $F(U)(y) = \min\{U(x), U(y)\}$, $\forall\, U \in \mathcal{B}$. Anyway, the cases $\#X = 1$ or $n = 1$ lack of interest in the social choice setting [for more details, see De Miguel et al. (2017)].

4 Comparison meaningful SEFs

We begin this section by introducing the axioms of monotonicity and comparison meaningfulness for SEFs. It should be noted that axioms similar to these ones had already been formulated for real-valued functions of several real variables and aggregation operators in the context of measurement theory (see, e.g., Marichal 2002; Marichal and Mesiar 2009; Ovchinnikov and Dukhovny 2002; Candeal and Induráin 2015). Further, and in contrast to what was stated in Theorem 2 of the previous section, here the assumptions $\#X > 1$ and $n > 1$ can be dispensed with.

Definition 3 Assume $\mathcal{U} \subseteq \mathcal{V}$ is a Δ-stable domain. A SEF $F : \mathcal{U}^n \to \mathcal{V}$ is said to be:

(i) *Monotonic* (*M*) if $\forall\, U, V \in \mathcal{U}^n$, $U(z,j) \le V(z,j)$, $\forall\, z,j \Rightarrow F(U)(x) \le F(V)(x)$, $\forall\, x$,
(ii) *Comparison Meaningful w.r.t. independent ordinal scales* or, for brevity, (*CM*), if $\forall\, U, V \in \mathcal{U}^n$, $\forall\, x$, $F(U)(x) \le F(V)(x) \Rightarrow F(\Phi(U))(x) \le F(\Phi(V))(x)$, $\forall\, \Phi \in \Delta^n$.

Remark 3

(i) Note that the two conditions above can be equivalently stated by saying that, for all $x \in X$, each component F^x is monotonic or comparison meaningful w. r.t. independent ordinal scales, respectively.
(ii) If, in Definition 3(ii) above, $\Phi \in \Delta^n_{ia}$, then F is said to be comparison meaningful w.r.t. independent interval scales.

Monotonicity is a very natural condition which is often encountered in the social choice literature. It tells us that the dominance, according to the large-small relation of the numbers assigned, between profiles entails the same kind of dominance between the corresponding social utilities. That is, for all pair of profiles $U, V \in \mathcal{U}^n$ such that for any individual $j \in N$ and any alternative $z \in X$, the individual j, by choosing the alternative z, is better in V than she is in U, then the social value that $F(V)$ assigns to any alternative is greater than the corresponding assigned by $F(U)$.

CM states that the large-small relation of the values assigned to any two social utilities at each alternative is preserved for informationally equivalent profiles. From the point of view of utility comparisons, CM entails that utilities are measured into (independent) ordinal scales and, therefore, only intrapersonal comparisons of utility are allowed.

The main result of this section is now presented. Note that the statement below has a direct application into group decision-making. Indeed, it provides a characterization of those SEFs having a very particular functional form; namely, there is a set of individuals, say $\psi(X) \subseteq N$, who take all social evaluations. That is, attached to each alternative $x \in X$ there is a member of this set, say $\psi(x)$, so that, for any profile of individual utilities, the value that the society assigns to the alternative x coincides with the value that this member $\psi(x)$ gives to x. The subset $\psi(X)$ could be interpreted as a certain expert committee being each one of its members the corresponding expert attached to each alternative. The result could be somehow reminiscent of Gibbard's oligarchy theorem and certain results in the social choice literature related to the existence of local dictators. However, our framework largely differs from these two. Indeed, Gibbard's oligarchy theorem is obtained by weakening the collective rationality assumption which is implicitly assumed in our setting. Local dictators for social welfare functions appear whenever the corresponding social rule is Arrow-consistent over some restricted preference domains. In our context, the members of the expert committee do not dictate over alternatives simply provide numerical evaluations of the alternatives.

Theorem 2 *Assume $\mathcal{U} \subseteq \mathcal{V}$ is a Δ-stable set which includes all constant functions. Then, for a SEF $F : \mathcal{U}^n \to \mathcal{V}$ the following statements are equivalent*:

(i) F satisfies CM, M and Una,
(ii) There is a function $\psi : X \to N$ such that $F(U)(x) = U(x, \psi(x))$, $\forall\, U \in \mathcal{U}^n$, $\forall\, x \in X$.

Remark 4

(i) The conclusion of Theorem 2 remains true if comparison meaningfulness refers only to interval scales and $\mathcal{U}$ is a subset of $\mathcal{B}$. This assertion follows from the proof of Theorem 2 given in the "Appendix". Moreover, if $\#X = 1$, then the monotonicity assumption can be ruled out and F is, in fact, Pro.
(ii) As a direct consequence of Theorem 2 and Remark 4(i) the following impossibility result holds true: Assume $\mathcal{U} \subseteq \mathcal{B}$. Then, no SEF F can exist that satisfies M, Una, A and CM w.r.t. independent interval scales.
(iii) No topological conditions are required in the statement of Theorem 2. As was already argued in the Introduction, if X is finite, say $X = M$, then a SEF can be viewed as a vector-valued function defined on the space of all $m \times n$ real matrices having, as codomain, the space of all real column matrices of size $m \times 1$. In this context, a natural topology to consider is the Euclidean one. Note that, in this case, the SEFs provided in Theorem 2 are continuous.[2]

We finish this section by including some examples which show that Una, M and CM are independent assumptions. Note that, in none of the three cases presented below, F fits the functional form established in Theorem 2. For simplicity, the examples are given for $\#X = 3$ and $n = 2$.

Examples Let $X = \{x, y, z\}$, $F : \mathcal{U}^2 \to \mathcal{V}$, and $U = (u_1, u_2) \in \mathcal{U}^2$.

1. Define the SEF as follows: $F(U) = c_X = c1_X$, $\forall\, U \in \mathcal{U}^2$, where $c \in \mathbb{R}$. Obviously, F satisfies M and CM but it is not Una.
2. Consider the SEF given by $F(U) = u_1$, provided that the vectors $U(\cdot, 1) := (u_1(x), u_1(y), u_1(z))$ and $U(\cdot, 2) := (u_2(x), u_2(y), u_2(z))$ are comonotonic,[3] and $F(U)(x) = u_1(y)$, $F(U)(y) = u_1(x)$, $F(U)(z) = u_1(z)$, otherwise. F satisfies Una and CM. However, it is not M.
3. Define the following SEF: $F(U)(x) = \max\,\{u_1(x), u_2(x)\}$, and $F(U)(y) = \min\,\{u_1(y), u_2(y)\}$, and $F(U)(z) = u_1(z)$. It is clear that F satisfies M and Una but fails to be CM.

[2] The assumption X finite is essential for this claim. To this respect, it can be proved that if $X \subseteq \mathbb{R}^n$ is a compact and convex subset with at least two points (hence an infinite set), and $\mathcal{U}$ is the space of all continuous functions defined on X equipped with the supremum norm topology, then the only continuous SEFs that satisfy Una, M and CM are the projections.

[3] That is, $u_1(r) \le u_1(k) \Leftrightarrow u_2(r) \le u_2(k)$, $\forall\, r, k \in X$. Alternatively, it is also said that $U(\cdot, 1)$ and $U(\cdot, 2)$ belong to the same rank-ordered set.

5 Chichilnisky's topological social choice model revisited

In the late seventies, Chichilnisky initiated the topological approach to social choice theory. In order to present her model some basic definitions and concepts are needed. Suppose that there are a finite number of individuals, say n, in the society. The first important issue to consider is the description of the space of individual preferences in Chichilnisky's model. To that end, let X be a cube in $\mathbb{R}^m_{++}$ which is called in this context the *choice space*. There is a common restricted domain in Chichilnisky's setting. Indeed, the space of individual and collective preferences, henceforth denoted by $\mathcal{P}$, is the set of all C^1 (continuously differentiable) *integrable unit vector fields* defined on the choice space X (see Chichilnisky 1982, 1983). A *social aggregation rule* is then a map $F : \mathcal{P}^n \to \mathcal{P}$ assigning to each profile of individual preferences $(p_1, \ldots, p_n) \in \mathcal{P}^n$ another preference $F(p_1, \ldots, p_n) \in \mathcal{P}$ which is interpreted as a social preference. A social aggregation rule $F : \mathcal{P}^n \to \mathcal{P}$ is said to be *anonymous* if $F(p_1, \ldots, p_n) = F(p_{\sigma(1)}, \ldots, p_{\sigma(n)})$, $\forall (p_1, \ldots, p_n) \in \mathcal{P}^n$, $\sigma \in S(N)$. It respects *unanimity* if $F(p, \ldots, p) = p$, $\forall\, p \in \mathcal{P}$.

Under this framework Chichilnisky established the following result.

Theorem 3 (*Chichilnisky* 1982) *There is no social aggregation rule $F : \mathcal{P}^n \to \mathcal{P}$ that is continuous,*[4] *anonymous, and respects unanimity.*

We now comment on the argument supporting the proof of this result as appears in Chichilnisky (1982). A key fact used in Chichilnisky (1982) is the following identification. A preference $p \in \mathcal{P}$ can be viewed as a map assigning to each alternative $x \in X$, a vector $p(x)$, which is the gradient of a (differentiable) utility function defined on the choice space X, i.e., it is a vector field on X. The direction of the vector $p(x)$ is interpreted as the "most desirable" direction because it is the direction of the largest increase in utility. Thus, it is required that each vector be normalized with unit length, which implies that the gradient vector field before normalization should not be vanished in the interior of X because, otherwise, the normalized gradient would be left undefined.[5] Therefore, each preference $p \in \mathcal{P}$ must be a unit vector field on X. Under these considerations, the proof runs, by contradiction, as follows. Suppose that there is a continuous social aggregation rule $F : \mathcal{P}^n \to \mathcal{P}$ that is anonymous and respects unanimity. Let $x \in X$ be fixed. Then, the existence of such a rule would induce a continuous, symmetric and idempotent function $\psi : (S^{m-1})^n \to S^{m-1}$, where S^{m-1} denotes the $m-1$-dimensional sphere of $\mathbb{R}^m$. Continuity refers to the (relative) Euclidean topology on the sphere and the product topology on the Cartesian. Symmetry and idempotency have the usual meanings. However, by using certain arguments of algebraic topology (specifically, of degree theory) such a function ψ cannot exist. Therefore, a contradiction is reached and the result follows.

[4] Continuity refers to the uniform convergence in $\mathcal{P}$ and the corresponding product topology in $\mathcal{P}^n$ [for details, see Chichilnisky (1982)].

[5] This technical condition can be somehow relaxed (see Section IV in Chichilnisky 1982). See also Jones et al. (2003) for a detailed discussion of some difficulties that appear in Theorem 2 of Chichilnisky (1982).

The fact of using normalized gradient vectors of unit length is essential in Chichilnisky's proof. However, this entails that the model does not allow for considering *intensity* of preferences. Recall that a preference model is responsive to intensity of preferences provided that the utilities are measured in an interval scale.

We now present a variation of Chichilnisky's model which is inspired by the ideas of the previous section and does allow for introducing preference intensities. Quite surprisingly, no topological conditions are required in its formulation. It is just based on the approach followed in Sect. 4. In Remark 5 below both similarities as well as differences between the two approaches are briefly discussed.

Let $X \subset \mathbb{R}^m$ be a bounded subset. Denote by $\mathcal{A}$ the set of all affine real-valued functions defined on X; i.e., $\mathcal{A} = \{u : X \to \mathbb{R} : u(x) = ax + b = \sum_{j\in M} a_j x_j + b, \forall\, x = (x_j) \in X,$ where $a = (a_j) \in \mathbb{R}^m, b \in \mathbb{R}\}$. Note that $\mathcal{A}$ contains all constant functions defined on X and is Δ_{ia}-stable. Moreover, $\mathcal{A}$ is responsive to intensities.

Before presenting the next theorem a new concept, which strengthens that of CMIN, is introduced. Indeed, this strong form of CMIN entails that CMIN should be satisfied uniformly on all profiles of individual affine utilities.

Definition 4 Assume $\mathcal{A} \subseteq \mathcal{D}$. A SEF $F : \mathcal{D}^n \to \mathcal{V}$ satisfies *Strong Cardinal-Measurability-Interpersonal-Noncomparability* (*SCMIN*) provided that there is a map $R : \Delta^n_{ia} \to \Delta$ such that the following functional equation is met: $F(\Phi(A)) = R(\Phi)(F(A)), \forall\, A \in \mathcal{A}^n, \forall\, \Phi \in \Delta^n_{ia}$.[6]

Because Arrow's impossibility theorem extends to the setting where individuals' utilities are cardinally measurable and interpersonally noncomparable (see, e.g., Sen 1970a, b, Theorem 8*2); d'Aspremont and Gevers (2002, Theorem 3.16)) our next result not only is reminiscent of Chichilnisky's but also of variants of Arrow's impossibility theorem established based on cardinal utilities.

Theorem 4 *Assume $\mathcal{A} \subseteq \mathcal{D}$. Then there is no SEF $F : \mathcal{D}^n \to \mathcal{V}$ that satisfies SCMIN, M, Una and A.*

Remark 5 It is instructive to compare Theorem 4 to both Theorem 3 above and Theorem 2 of Chichilnisky (1982) where the assumption of nonvanishing gradients is slightly relaxed. First of all, the model presented here is established in terms of SEFs instead of social aggregation rules like Chichilnisky is. Specifically, the common individual domain in our result is restricted and includes the set of all affine funtions. Thus, preferences are exactly utility functions and both *linear* preferences as well as the *trivial* one take part of the model.[7] In contrast, in Chichilnisky's setting preferences are unit vector fields and this requires that they are normalized gradients at each point of the choice space X. As a consequence, the model shown here is more general than Chichilnisky's. Moreover, unlike Chichilnisky's framework, it allows for considering preference intensities. Further, Theorem 4 is stated without assuming

[6] Note that if a SEF $F : \mathcal{D}^n \to \mathcal{V}$ satisfies SCMIN, then, when restricted to $\mathcal{A}^n$, it satisfies CMIN and the function $\phi \in \Delta$ that appears in Definition1(ii) only depends on Φ.

[7] A preference defined on X is said to be linear if it can be represented by a (non-null) linear utility function. In terms of gradients they agree with those preferences having non-null constant gradients. The trivial preference is the one for which all alternatives have the same utility.

any topological condition. In what concerns the assumptions imposed, note that unanimity and anonymity are present in both statements. The key difference can be found in the continuity condition (together with the Pareto condition given in Chichilnisky (1982, Theorem 2) to cover certain cases where the gradients could vanish) given in Chichilnisky's result in contrast to the assumptions of monotonicity and SCMIN appearing in our Theorem 4.

Appendix

This section includes the proofs of the results stated in the paper. We begin with two technical lemmata which will be useful for the proof of Theorem 1. Let $n, m, k \in \mathbb{N}$. Recall that $N =: \{1, \ldots, n\}$, $M =: \{1, \ldots, m\}$, and $K =: \{1, \ldots, k\}$.

Lemma 1 *Let there be given* $E = (E_i)_{i \in M}$, $F = (F_j)_{j \in N}$ *two partitions of X. Then there is a partition* $G = (G_k)_{k \in K}$ *such that* $E, F \precsim G$.

Proof Let $K \subset \mathbb{N}$ be an enumeration of $\{(i,j) : i \in M, j \in N\}$. Then, for each $k \in K$, it is sufficient to consider $G_k = E_i \cap F_j$. □

The second lemma is folklore and is included here for the sake of completeness. Recall that $\mathcal{S} \subseteq \mathcal{U}$ denotes the set of all simple functions, i.e., $\mathcal{S} = \{s = \sum_{i \in M} a_i 1_{E_i} : (a_i) \in \mathbb{R}^m, (E_i) \text{ is a partition of } X, m \in \mathbb{N}\}$.

Lemma 2 *S is dense in* $\mathcal{U}$.

Proof Let $u \in \mathcal{U}$ and $\epsilon > 0$ be fixed. Then there are reals a, b such that $a \leq u(x) < b$, for all $x \in X$. Take a positive integer n such that $\frac{b-a}{n} < \epsilon$. Define, for each $j = 0, \ldots, n-1$, the numbers $\alpha_j = a + j\frac{b-a}{n}$, and consider the subsets of X defined as follows: $E_j = \{x \in X : \alpha_j \leq u(x) < \alpha_{j+1}\}$, for every j. Note that (E_j) is a partition of X. Finally, define the simple function $s = \sum_j \alpha_j 1_{E_j}$. Then, $d(u, s) = ||u - s||_\infty = \sup_{x \in X} |u(x) - s(x)| \leq \frac{b-a}{n} < \epsilon$, which proves Lemma 2. □

Proof of Theorem 1 The "if" part is easy to prove. So, we will only show the "only if" part which will be carried out in several steps.

Step (1). Let $E = (E_i)_{i \in M}$ be a partition of X. Consider the subset of $\mathcal{U}$ consisting of all simple functions supported on (E_i), i.e., $\mathcal{S}_E = \{s_a = \sum_{i \in M} a_i 1_{E_i} : a = (a_i) \in \mathbb{R}^m\}$. Note that $\mathcal{S}_E$ can be identified, both algebraically and topologically, with $\mathbb{R}^m$. The first step consists in proving that, when restricted to $\mathcal{S}_E^n$, F turns out to be a projection. To that end, let $C = (x_i) \subseteq X$ be a collection of points such that $x_i \in E_i$, for every $i \in M$. Define the function $F_E^C : \mathcal{S}_E^n \longrightarrow \mathcal{S}_E$ as follows: $F_E^C(s_{a^1}, \ldots, s_{a^n}) = \sum_{i \in M} F(s_{a^1}, \ldots, s_{a^n})(x_i) 1_{E_i}$, where $a^1, \ldots, a^n \in \mathbb{R}^m$. Let us see that this definition is consistent in the sense that, actually, it does not depend on the collection C chosen. Indeed, note that since F satisfies OMIN, Una, C and OSP, so does F_E^C. So, by Theorem 2 in Candeal (2015), there is $j \in N$ such that $F_E^C(s_{a^1}, \ldots, s_{a^n}) = s_{a^j}$, $\forall\ (s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}_E^n$. In particular, it holds that $F(s_{a^1}, \ldots, s_{a^n})(x_i) = s_{a^j}(x_i)$, for all $i \in M$. Let $D = (y_i) \subseteq$

X be a collection of points such that $y_i \in E_i$, for every $i \in M$. Define $D_1 = (z_i)_{i \in M} \subseteq X$ as follows: $z_1 = y_1$, and $z_i = x_i$, $\forall\ i > 1$. Then, by the same argument as above, $F_E^{D_1}$ turns out to be a projection. Thus, there is $k \in N$ such that $F_E^{D_1}(s_{a^1}, \ldots, s_{a^n}) = s_{a^k}$, $\forall\ (s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}_E^n$. In particular, it holds that $F(s_{a^1}, \ldots, s_{a^n})(y_1) = s_{a^k}(y_1)$ and $F(s_{a^1}, \ldots, s_{a^n})(x_i) = s_{a^k}(x_i)$, for $i > 1$. The latter equality clearly entails that $k = j$. So, $F(s_{a^1}, \ldots, s_{a^n})(y_1) = s_{a^j}(y_1) = s_{a^j}(x_1)$, the latter equality being true because $y_1 \in E_1$ and s_{a^j} is constant on E_1. Thus, $F_E^{D_1} = F_E^C$. Now, by letting $D_2 = (t_i)_{i \in M} \subseteq X$ be defined as: $t_1 = y_1$, $t_2 = y_2$, and $t_i = x_i$, $\forall\ i > 2$ and arguing as above, it follows that $F_E^{D_2} = F_E^C$. By repeating the process $(m-2)$-times it holds that $F_E^D = F_E^C$ and so $F(s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}_E$, $\forall\ (s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}_E^n$. Moveover, we have proved that the restriction of F to $\mathcal{S}_E^n$, $F|_{\mathcal{S}_E^n}$, turns out to be a projection. From now on we will denote such a restriction by F_E.

Step (2). Let E, E' be two partitions of X such that $E \precsim E'$. Then $F_{E'}|_{\mathcal{S}_E^n} = F_E$. To see this, note that each profile $(s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}_E^n$ can be viewed as a profile in $\mathcal{S}_{E'}^n$. Now, by Step (1), both $F_{E'}$, and F_E are projections. Since, by Step (1) again, $F_{E'}(\mathcal{S}_E^n) \subset \mathcal{S}_E$, it follows that $F_{E'}|_{\mathcal{S}_E^n} = F_E$.

Step (3). Let E, E' be two arbitrary partitions of X. Then, for the $j \in N$ existing from Step (1), it holds that $F(s_{a^1}, \ldots, s_{a^n}) = s_{a^j}$, $\forall\ (s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}_E^n$, and $F(s_{b^1}, \ldots, s_{b^n}) = s_{b^j}$, $\forall\ (s_{b^1}, \ldots, s_{b^n}) \in \mathcal{S}_{E'}^n$. Indeed, consider a partition E'' such that $E \precsim E''$, and $E' \precsim E''$. Lemma 1 above ensures the existence of such a partition. Then, by Step (1), $F_{E''}$, is a projection. In addition, by Step (2), it holds that $F_{E''}|_{\mathcal{S}_E^n} = F_E$, and $F_{E''}|_{\mathcal{S}_{E'}^n} = F_{E'}$. Therefore, the conclusion clearly follows.

Step (4). Let $\mathcal{S}$ be the set of all simple functions. Then, for the $j \in N$ of Step (1), it holds that $F(s_{a^1}, \ldots, s_{a^n}) = s_{a^j}$, $\forall\ (s_{a^1}, \ldots, s_{a^n}) \in \mathcal{S}^n$. Indeed, by Step (3), the latter equality holds true provided that all the entries of the profile are supported over the same partition. Suppose now that the simple functions $s_{a^1}, \ldots, s_{a^n}$ are supported over partitions $E^1, \ldots, E^n$, respectively. Then, by considering a partition J such that $E^1, \ldots, E^n \precsim J$ it follows that $F(s_{a^1}, \ldots, s_{a^n}) = F_J(s_{a^1}, \ldots, s_{a^n}) = s_{a^j}$, where the latter equality holds true by Step (1).

Step (5) For the $j \in N$ of Step (1), it holds that $F(u_1, \ldots, u_n) = u_j$, $\forall\ (u_1, \ldots, u_n) \in \mathcal{U}^n$. Indeed, let $(u_1, \ldots, u_n) \in \mathcal{U}^n$ an arbitrary profile. Consider corresponding sequences $(s_{a_l^1})_{l \in \mathbb{N}}, \ldots, (s_{a_l^n})_{l \in \mathbb{N}} \subseteq \mathcal{S}$ such that $(s_{a_l^1}) \to u_1, \ldots, (s_{a_l^n}) \to u_n$. The existence of such sequences is guaranteed by Lemma 2 above. Then, clearly, $(s_{a_l^1}, \ldots, s_{a_l^n}) \to (u_1, \ldots, u_n)$. Hence, since F satisfies C, $F(s_{a_l^1}, \ldots, s_{a_l^n}) \to F(u_1, \ldots, u_n)$. Now, by Step (4), $F(s_{a_l^1}, \ldots, s_{a_l^n}) = s_{a_l^j}$, $\forall\ l \in \mathbb{N}$, and, by hypothesis, $(s_{a_l^j}) \to u_j$. Therefore, $F(u_1, \ldots, u_n) = u_j$, as desired, and the proof is over. □

Proof of Theorem 2 [8] (ii) entails (i) is easy to check. So, we will focus on (i) implies (ii) which is developed in five steps. Denote by $\mathcal{Q} := \mathcal{U} \cap \mathcal{B}$ and note that $\mathcal{Q}$ is a nonempty $\varDelta$-stable subset of $\mathcal{U}$. We first show that the implication holds whenever F is restricted to $\mathcal{Q}^n$. Thus, we have to prove that, for each $x \in X$, the function $U \in \mathcal{Q}^n \rightarrow F^x(U) = F(U)(x) \in \mathbb{R}$ is a projection. So, and in order to simplify the notation, let us denote such a function by W; i.e., $W(U) = F^x(U)$, where $x \in X$ is fixed.

Step (1). We first compute the value of W in the following subset of $\mathcal{Q}^n$, $\mathcal{T} := \{U \in \mathcal{Q}^n : \text{ there is } a = (a_j) \in \mathbb{R}^n \text{ such that } U(z,j) = a_j, \quad \forall\, z \in X, \forall\, j \in N\}$. For that purpose, consider the real-valued function of n variables, $h_x : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by letting, for each $a = (a_j) \in \mathbb{R}^n$, $h_x(a) = W(U^a) = F^x(U^a)$, where $U^a \in \mathcal{T}$ is such that $U^a(z,j) = a_j$, $\forall\, z \in X$, $\forall\, j \in N$. Note that the domain of h_x is $\mathbb{R}^n$ because $\mathcal{Q}$ contains the constant functions. Now, since F satisfies Una and CM so does h_x. Hence, by Corollary 6.1 in Marichal and Mesiar (2009) (see also Proposition 8.55 in Grabisch et al. (2009)), there is $\psi(x) \in N$ such that $h_x(a) = a_{\psi(x)}$, $\forall\, a = (a_j) \in \mathbb{R}^n$. So, $W(U^a) = a_{\psi(x)}$, $\forall\, a = (a_j) \in \mathbb{R}^n$.

Step (2). Consider the set $\mathcal{T}_{\psi(x)} \subseteq \mathcal{Q}^n$ defined as follows $\mathcal{T}_{\psi(x)} := \{U = (u_j) \in \mathcal{Q}^n : \text{ there is } r \in \mathbb{R} \text{ such that } u_j(t) = r, \quad \forall\, t \in X, \forall\, j \in N \backslash \{\psi(x)\}\}$. We show that $W(U) = W(V)$, for all $U = (u_j)$, $V = (v_j) \in \mathcal{T}_{\psi(x)}$ for which $u_{\psi(x)} = v_{\psi(x)}$. That is, when restricted to $\mathcal{T}_{\psi(x)}$, W only depends upon the values of the function $u_{\psi(x)}$. Let then $U = (u_j)$, $V = (v_j) \in \mathcal{T}_{\psi(x)}$ such that $u_{\psi(x)} = v_{\psi(x)}$, and let $r, s \in \mathbb{R}$ be such that $u_j = r_X, v_j = s_X$, $\forall\, j \in N \backslash \{\psi(x)\}$. First, note that, by Una, $W(U) = W(U_{W(U)_X})$, (remember that $W(U)_X(t) = W(U)$, $\forall\, t \in X$). Let now $\varPhi = (\phi_j) \in \varDelta^n$ where $\phi_{\psi(x)}$ is the identity function (i.e., $\phi_{\psi(x)}(c) = c$, $\forall c \in \mathbb{R}$) and, for each $j \in N \backslash \{\psi(x)\}$, $\phi_j(c) = c + (s - r)$, $\forall\, c \in \mathbb{R}$. Then, since F satisfies CM, it holds that $W(\varPhi \circ U) = W(\varPhi \circ U_{W(U)_X})$. But $\varPhi \circ U = (\phi_j \circ u_j)$, hence $\phi_{\psi(x)} \circ u_{\psi(x)} = u_{\psi(x)}$, and, for each $j \in N \backslash \{\psi(x)\}$, $(\phi_j \circ u_j)(t) = \phi_j(u_j(t)) = \phi_j(r) = s$, $\forall\, t \in X$. So, $\varPhi \circ U = V$ and $W(V) = W(\varPhi \circ U) = W(\varPhi \circ U_{W(U)_X})$. Now, $\varPhi \circ U_{W(U)_X} = (\phi_j \circ W(U)_X)$. But $\phi_{\psi(x)} \circ W(U)_X = W(U)_X$, and, for each $j \in N \backslash \{\psi(x)\}$, $(\phi_j \circ W(U)_X)(t) = \phi_j(W(U)_X(t)) = \phi_j(W(U)) = W(U) + s - r$, $\forall\, t \in X$. Hence, $\varPhi \circ U_{W(U)_X} \in \mathcal{T}$. So, $W(\varPhi \circ U_{W(U)_X}) = h_x(a)$, where $a = (a_j)$ is such that $a_{\psi(x)} = W(U)$ and $a_j = W(U) + s - r$, $\forall\, j \in N \backslash \{\psi(x)\}$. Hence, by Step (1), $W(\varPhi \circ U_{W(U)_X}) = h_x(a) = a_{\psi(x)} = W(U)$. Therefore, $W(V) = W(U)$.

Step (3). The conclusion of Step (2) is now extended to all $U \in \mathcal{Q}^n$ by using a proof by contradiction. If W does not only depend on the values of the function $u_{\psi(x)}$, there must exist profiles $U_1 = (u^1_j)$, $U_2 = (u^2_j) \in \mathcal{Q}^n$ such that $u^1_{\psi(x)} = u^2_{\psi(x)} = u$, for some $u \in \mathcal{B}$, and $W(U_2) < W(U_1)$. Define now two profiles

[8] This proof closely follows that appearing in Candeal (2016, Theorem 4.12) [see, also Khmelnitskaya and Weymark (2000)]. However, it is more general because the common individual and collective domain consists not only of bounded utilities.

$\tilde{U}_1,\ \tilde{U}_2 \in \mathcal{T}_{\psi(x)}$ by letting $\tilde{u}^1_{\psi(x)} = \tilde{u}^2_{\psi(x)} = u$, and, for each $j \in N\backslash\{\psi(x)\}$ and $t \in X$, $\tilde{u}^1_j(t) = \sup\ \{u^1_j(z) : z \in X, j \in N\backslash\{\psi(x)\}\}$, and $\tilde{u}^2_j(t) = \inf\ \{u^2_j(z) : z \in X, j \in N\backslash\{\psi(x)\}\}$. Note that, so-defined, $\tilde{U}_1,\ \tilde{U}_2 \in \mathcal{T}_{\psi(x)}$. Moreover, clearly, $U_1 \leq \tilde{U}_1$ and $\tilde{U}_2 \leq U_2$. Now, since F satisfies M, so does W, hence $W(\tilde{U}_2) = F(\tilde{U}_2)(x) \leq F(U_2)(x) = W(U_2) < W(U_1) = F(U_1)(x) \leq F(\tilde{U}_1)(x) = W(\tilde{U}_1)$. Therefore, $W(\tilde{U}_2) < W(\tilde{U}_1)$, which contradicts the conclusion of Step (2).
Step (4). Let now $U \in \mathcal{Q}^n$. Then, by Step (3), $W(U) = W(U_{u_{\psi(x)}})$. But, by Una again, $W(U_{u_{\psi(x)}}) = u_{\psi(x)}(x) = U(x, \psi(x))$. So, $W(U) = F^x(U) = F(U)(x) = U(x, \psi(x))$, $\forall\ U \in \mathcal{Q}^n$. Therefore, $F(U)(x) = U(x, \psi(x))$, $\forall\ U \in \mathcal{Q}^n$.
Step (5). It remains to show that the conclusion also holds true for every $U \in \mathcal{U}^n$. To that end, let $U \in \mathcal{U}^n$ be fixed and note that, by Una, $F(U) = F(U_{F(U)})$. Thus, by CM, it follows that $F(\Phi \circ U) = F(\Phi \circ U_{F(U)})$, $\forall\ \Phi = (\phi_j) \in \Delta^n$. Take a profile $\Phi = (\phi_j) \in \Delta^n$ of bounded functions. Then $\Phi \circ U,\ \Phi \circ U_{F(U)} \in \mathcal{Q}^n$. So, by Step (4), $\phi_{\psi(x)}(U(x, \psi(x))) = \phi_{\psi(x)}(F(U)(x))$, for every $x \in X$. Now, since $\phi_{\psi(x)} \in \Delta$, it follows that $U(x, \psi(x)) = F(U)(x)$, for every $x \in X$. Therefore, $U(x, \psi(x)) = F(U)(x)$, holds true $\forall\ U \in \mathcal{U}^n$, $\forall\ x \in X$ which concludes the proof. □

Proof of Theorem 4 Suppose, by contradiction, that such a SEF F does exist. Consider the restriction of F to $\mathcal{A}^n$, which is denoted by G. Clearly, G satisfies M, Una and A. We now prove that G is CM w.r.t. independent interval scales. Indeed, let there be given $x \in X$, $A, B \in \mathcal{A}^n$ such that $G(A)(x) \leq G(B)(x)$, and $\Phi \in \Delta^n_{ia}$. Then, because F satisfies SCMIN, it holds that $G(\Phi(A))(x) = F(\Phi(A))(x) = R(\Phi)(F(A))(x) \leq R(\Phi)(F(B))(x) = F(\Phi(B))(x) = G(\Phi(B))(x)$, the inequality being true because $R(\Phi)$ is increasing. Thus, G is CM w.r.t. independent interval scales. Now, by Remark 4(iii), such a function cannot exist. Therefore, a contradiction is reached and the proof is ended. □

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Authors and Affiliations

Juan C. Candeal[1]

✉ Juan C. Candeal
candeal@unizar.es

[1] Departamento de Análisis Económico, Facultad de Economía y Empresa, Universidad de Zaragoza, Gran Vía, 2, 50005 Zaragoza, Spain