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Timescale standard to discriminate between hyperbolic and exponential discounting and construction of a nonadditive discounting model

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Abstract

Under the presupposition that human time perception is distorted in intertemporal choice, this study constructs a time scale in the framework of axiomatic measurement. First, the conditions (homogeneity of degree one or two) to identify the form of a time scale are proposed so that one can determine whether the hyperbolic or exponential is a more suitable function for modeling people's discounting. Homogeneity of degree one implies that subjective time delay is measured by a power scale and its discount function becomes a power-exponential type, whereas homogeneity of degree two implies that subjective time delay is measured by a log scale and its discount function becomes a hyperbolic type. Second, a method is shown for constructing a model that can reflect subadditive and superadditive discounting. A non-commutative and non-associative concatenation operation is provided to generate a time string consisting of subdivided durations that incorporates the effect of repeated delay with a subdivided duration. The discounting model is yielded by substituting these strings in an exponential model equipped with a generalized time scale, and the determination of subadditivity or superadditivity depends on whether the discount function value of a product by the non-commutative and non-associative operation is, respectively, smaller or greater than that of a product by an operation of the usual extensive structure.

Keywords Extensive structure · Hyperbolic discounting · Subadditivity · Intertemporal choice · Time perception

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1 Introduction

Many descriptive models of delay discounting have been developed in the field of economics and psychology, and most of them have revised the previous models to explain people's behavior on trade offs between time and outcomes more faithfully. The generalized hyperbolic discounting model (Loewenstein & Prelec, 1992) was proposed to reflect decreasing impatience, addressing the problem that the exponential discounting model (Samuelson, 1937) can deal only with constant impatience.[1] However, the detection of issues that the generalized hyperbolic discounting model cannot address has encouraged construction of a new discounting model by introducing the distortion of human time perception. Ebert and Prelec (2007), noting the relationship between time-insensitivity and non-constant impatience, proposed the constant-sensitivity model, which is an exponential model in which subjective time measured by a log scale is introduced. Furthermore, Bleichrodt et al. (2009) extended the constant-sensitivity model so that it can accommodate increasing impatience (Attema et al., 2010) and strong degrees of decreasing impatience. Meanwhile, Read (2001) advocated the importance of subadditivity (which means that the total discounting is greater when an interval is divided into subintervals) and demonstrated strong evidence of subadditive discounting, but no evidence of decreasing impatience, through intertemporal choice experiments. Furthermore, Scholten and Read (2006) presented the concept of "subadditive discounting model[2]" (a discount function dependent on the interval between two adjacent outcomes, not between the present and when an outcome occurs), and proposed two parameterized discounting models (one of which is similar to the constant-sensitivity model, and the other is the generalized hyperbolic model in which a subjective time interval measured by a power scale is substituted) so that they could explain subadditive and superadditive discounting properties.

In contrast, there is little work conducted on strictly axiomatizing the models. Let us review some major, typical approaches to the axiomatization. Initially, an exponential type of discounting model was axiomatized based on constant impatience. Using the topological framework of Debreu (1960); Koopmans (1972) constructed a utility model for an infinite outcome sequence. To overcome the difficulty in testing topological conditions (except for money outcomes), Hübner and Suck (1993) adapted Koopmans's result to a general algebraic framework using an infinite-dimensional additive conjoint structure. Subsequently, by generalizing constant impatience to the Thomsen condition, and also using Debreu's topological results, Fishburn and Rubinstein (1982) constructed a multiplicative utility model $u(x, t) = \varphi(t)u(x)$ for a dated outcome, where (x, t) denotes the receipt of outcome x at time t and u and φ are a utility and a weight function, respectively. Furthermore, Loewenstein and Prelec (1992), introducing the concept of distorted time perception

[1] Constant impatience is also called stationarity (Koopmans, 1960), which for our domain of dated outcomes means that the preference between two dated outcomes is invariant as long as the difference between the receipt times of these outcomes are the same.

[2] Ok and Masatlioglu (2007), utilizing a result of a functional equation, constructed a utility model for which the discount is assessed by a function of any two points in time, which corresponds to the subadditive discounting model.

(a person might feel time passage at different speeds in accordance with the time distance from the present to the occurrence of delay) into this multiplicative utility model, derived the generalized hyperbolic discounting model. However, a condition has not yet been proposed that is written in such an explicit form that one can examine how a person's time passage is distorted through experiments.

It is worthwhile to pay attention to the work that investigated the relationship between a discount function and "subjective time perception." Zauberman et al. (2009) and Bradford et al. (2019) assessed subjective duration by asking the subjects directly how long they felt each calendar time frame (e.g., 1 day, 1 week, 4 weeks) was, and showed that perception of subjective time closely follows a log-shaped curve and that the estimated discount rates were indistinguishable between most of the subjective time intervals.[3] This result suggests that an exponential function is suitable to explain discounting if the delay time is allowed to assess subjectively, which can be expressed by the formula:

$$\exp(-vw(t)) \cdot \exp(-vw(s)) = \exp(-v(w(s) + w(t))), \tag{1}$$

where v is a discount rate and w is a function that transforms objective time into subjective time. As stated by Read (2001), (1) might represent the preferences of a "rational" decision maker because the discounting is assessed with a constant discount rate and the time perception adjusted to being inherent in a decision maker. It should be noted (Takahashi et al., 2012) that setting w as a logarithmic function (i.e., logarithmic time perception) yields the generalized hyperbolic discounting model. However, it seems strange to acknowledge the existence of a log or power scale over the time horizon a priori. An explicit reason for the existence of a nonlinear scale on the set of time has not been given, although the Weber-Fechner and Stevens' power laws are often cited for its validity. From a practical viewpoint, estimation of the form of w based on questionnaires, which is performed independently of experiments regarding time preference (Zauberman et al., 2009; Bradford et al., 2019), might deprive us of a chance to grasp a dynamic relation between subjective time perception and discounting. Indeed, their approach might evaluate time perception separately from the context of time preference. In contrast, the condition of "temporal scale invariance" (Ebert & Prelec, 2007; Bleichrodt et al., 2009) is suitable because its validity can be verified by an experiment regarding time preference and one can know distorted time perception implicitly through a person's preferences. Therefore it is desirable to derive a scale from an empirical relational structure according to the approach of axiomatic measurement (Krantz et al., 1971, p.1).

In view of not a small gap between the normative and descriptive models, this study aims at deriving conditions to decide the form of w so that one can identify whether a hyperbolic or exponential function is more suitable for modeling people's discounting. Another aim is to formulate conditions to construct a discounting model that reflects subadditive and superadditive discounting. This is also conducted associated with the measurement of subjective time. The existence of an exponential discount function with a generalized time scale w of (1) is essential for realizing these

[3] However, Bradford et al. (2019) showed that the discount rates were much different between 1-day delay and other time delays.

aims. Hence the exponential function must be constructed assuming that a generalized concatenation operation, not the usual addition, exists on the set of time, and this will follow the result of Matsushita (2017). Precisely, the positive closed extensive structure (Krantz et al., 1971) is introduced in the set of time for the first aim, but a generalized extensive structure (Matsushita, 2014) for the second aim. That is, a commutative and associative concatenation operation is used for the first aim, whereas a non-commutative and non-associative operation for the second aim. The latter operation involves the order of each duration of a time string in time perception.

The rest of this paper is organized as follows. Section 2 shows the construction of the exponential discount function of (1) using a delay operator and a generalized concatenation operation and explains a basic approach to generating all discounting models in this paper from this exponential function. Section 3 deals with the case of additive discounting and proposes a standard to determine whether a log or power scale is suitable for measuring the passage of time. The standard is expressed as two types of homogeneity of time unit: one is close to a property of multiplication, corresponding to a log scale, and the other is close to a property of addition, corresponding to a power scale. In addition, a comment is made on a procedure for testing the proposed standard experimentally. Section 4 considers the case of nonadditive discounting but the additivity can be revived by introducing a non-commutative and non-associative operation; eventually, a function that reflects subadditive or superadditive discounting is obtained by incorporating the order effect of delay with each duration in a time string. Section 5 provides some conclusions. The proofs of the propositions are given in Sect. 6. The appendix shows the reason that the multiplicative utility model induces a restrictive condition.

2 Basic concepts and axioms

2.1 Expression of additive and non-additive discounting

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}_0^+$ denote the sets of all real numbers and all nonnegative real numbers, respectively. Let X be a nonempty set of outcomes (which at this stage may be general) and let $\succsim_X$ be a binary relation on X, which is interpreted as a preference relation. As usual, $\succ_X$ denotes the asymmetric part, $\sim_X$ the symmetric part, and $\precsim_X$ and $\prec_X$ reversed relations. The binary relation $\succsim_X$ on X is a *weak order* if and only if it is connected and transitive. An antisymmetric weak order is called a *simple order*. The usual order $\geqslant$ on $\mathbb{R}$ is a simple order. Let $T = \mathbb{R}_0^+$, denoting the set of all delay durations.

When expressing additive and non-additive discounting, this paper provides two new devices: an operator to delay the receipt of outcomes and a generalized concatenation operation on T. Repeated delay is described by multiplying outcomes by this delay operator from the right multiple times (see below), without using any time scale. The generalized operation reflects human time perception and hence extends an expression frame for additive discounting. These devices make the exponential discount function of (1) work as a generating function that yields all discount functions in this paper. In the following, the definition and efficiency of the devices will be shown.

Right multiplication by elements of T is provided to define an operator expressing the delayed receipt of outcomes. That is, xt denotes that the receipt of x is delayed by duration t, but the quantity is itself defined as the *present value* of a delayed outcome[4], implying a value possessed in the present, equivalent to the outcome received at time t. To be more precise, an element (x, t) in $X \times T$ indicates the receipt of x at time t. Here $(x, 0)$ implies the receipt of x at the present. Then xt is defined as a solution[5] to the following equivalence:

$$(xt, 0) \sim (x, t). \tag{2}$$

Note that xt is an element of X.

For simplicity, we will now restrict our concern to monetary amounts ($X = \mathbb{R}_0^+$) and consider a situation where delay occurs repeatedly. It will be valid that additive discounting: $(xs)t \sim_X x(s+t)$ holds under time consistency. A multiplicative utility model $u(xt) = \varphi(t)u(x)$ is applied to this equivalence to obtain $(\varphi(s)\varphi(t))u(x) = (\varphi(s+t))u(x)$, so that $\varphi(s)\varphi(t) = \varphi(s+t)$ because $u(x)$ is arbitrary. Hence it follows from Theorem 2.1.2.1 (Aczél, 1966) that $\varphi(t) = \exp(-vt)$ if φ is continuous and non-constant for $t > 0$. Recall here that the violation of additive discounting was found through experiments (Read, 2001; Scholten & Read, 2006), which invalidates the premise to derive the exponential function. However, the introduction of a generalized concatenation operation $\cdot_T$ on T (from $+$) might solve this problem. Indeed, even if $(xs)t \not\sim_X x(s+t)$, it is likely that $(xs)t \sim_X x(s \cdot_T t)$. There are several methods to construct an additive scale w with respect to $\cdot_T$, i.e., $w(s \cdot_T t) = w(s) + w(t)$. Thus substituting $w(t)$ for t in the expression $\varphi(t) = \exp(-vt)$ yields the exponential discount function of (1). Moreover, it is possible to clarify the form of w by finding an algebraic property of $\cdot_T$ in the context of axiomatic measurement. In Sect. 3, concrete forms of w will be determined according to the property of "homogeneity." (Using the property, one can decide whether w is a log or a power scale.)

Let $t_1, \ldots, t_n \in T$ be time durations and let $t(n) = (\cdots(t_1 \cdot_T t_2)\cdots) \cdot_T t_n$. Clearly, $t(n)$ is not necessarily equal to the usual sum $\sum_{i=1}^{n} t_i$ because subjectivity is reflected in it. The use of right multiplication of time durations allows for the expression of a step-by-step postponement of the receipt of x as $(\cdots(xt_1)\cdots)t_n$. However, $(\cdots(xt_1)\cdots)t_n$ is not always equivalent to the one-step delay of x by duration $t(n)$. Hence the following three cases arise according to the inequalities between $(\cdots(xt_1)\cdots)t_n$ and $xt(n)$.

Subadditive discounting: $(\cdots(xt_1)\cdots)t_n \prec_X xt(n)$. This implies that the n-step delay reduces the value of x more than the one-step delay.

[4] Conversely, the notation xt was used to imply the "advanced" receipt of x by duration t in Matsushita (2017).

[5] Restricted solvability (Matsushita, 2017) along with other axioms guarantees the existence of a solution $y = xt$ to $(y, 0) \sim (x, t)$.

Superadditive discounting: $(\cdots(xt_1)\cdots)t_n \succ_X xt(n)$. This implies that the n-step delay does not reduce the value of x more than the one-step delay.
Additive discounting: $(\cdots(xt_1)\cdots)t_n \sim_X xt(n)$. This implies that the step-by-step delay has no effect.

In Sect. 4, the operation $\cdot_T$ will be further generalized so that sub- and superadditive discounting above can be transformed into additive discounting. This study basically takes the approach to yielding discounting models by defining a generalized time concatenation operation, reviving additive discounting, and enabling the use of the exponential function of (1).

2.2 The base structure

This subsection presents the definition of an extensive structure because it is given to X and T as a base structure. However, if the reader is to focus on monetary amounts and market pricing, then it is unnecessary to assume X to be the extensive structure; instead, it may be regarded as the algebraic structure of nonnegative real numbers, $\langle \mathbb{R}_0^+, \geqslant, + \rangle$.

Let $\cdot$ be a "partial" binary operation on X, meaning a function from a subset B of $X \times X$ into X. The expression $x \cdot y$ is said to be *defined* (in X) if and only if $(x, y) \in B$. Henceforth, the system $\langle X, \succsim_X, \cdot \rangle$ is an *extensive structure* if and only if the following conditions hold for all $x, y, z \in X$ for which the products are defined:

A1. *Weak order:* $\succsim_X$ is a weak order.
A2. *Local definability:* if $x \cdot y$ is defined, $x \succsim_X x'$,and $y \succsim_X y'$, then $x' \cdot y'$ is defined.
A3. *Monotonicity:* $x \succsim_X y \Leftrightarrow x \cdot z \succsim_X y \cdot z \Leftrightarrow z \cdot x \succsim_X z \cdot y$.
A4. *Weak associativity:* $(x \cdot y) \cdot z \sim_X x \cdot (y \cdot z)$.
A5. *Solvability:* whenever $x \succ_X y$, there exists $z \in X$ such that $x \succsim_X z \cdot y$.
A6. *Positivity:* $x \cdot y \succ_X x$.
A7. *Archimedeanness:* every *bounded* sequence $\{nx\}$, which means $y \succ_X nx$ for some $y \in X$, is finite. Here the sequence $\{nx\}$ is defined inductively by $1x = x$, $2x = x \cdot x$, and $nx = x \cdot (n-1)x$ if the right-hand side is defined.

Assume that $\langle T, \geqslant, \cdot_T, 0 \rangle$ is a positive *closed* extensive structure (Krantz et al., 1971) with an identity element 0; that is, $T \backslash [0]$ is a positive closed extensive structure in which $\geqslant$ is a simple order. This can be regarded as an extensive structure possessing a binary operation.[6] Attention should be paid to the fact (Krantz et al. 1971, Lemma 3.5) that T is uniquely extended to an Archimedean simply ordered group $\langle \mathbb{R}, \geqslant, \cdot_{\mathbb{R}} \rangle$. The symbols t and $-t$ are used when it is necessary to distinguish positive from negative elements. The symbol t^{-1} is used to denote the inverse of t.

[6] In the formal definition of a "closed extensive structure" by Krantz et al. (1971), a complicated Archimedean axiom is used instead of A7 with the elimination of A5. However, it is well-known (Roberts & Luce, 1968, p. 321) that Axioms A1–A7 imply the complicated Archimedean axiom in the positive system.

Note here that neither $t \cdot_{\mathbb{R}} (-t) = 0$ nor $(-t) \cdot_{\mathbb{R}} t = 0$ is always satisfied, while $t \cdot_{\mathbb{R}} t^{-1} = t^{-1} \cdot_{\mathbb{R}} t = 0$.

2.3 Extensive structure with right action

In a similar manner, xt^{-1} denotes that the receipt of x is advanced by duration t. This is defined as a solution to $(xt^{-1}, t) \sim (x, 0)$. Here xt^{-1} is an outcome received at time t that is equivalent to the outcome x received at the present, which means a markup value of x that is assessed "at time t" when the receipt of x proceeds from t to the present.

Some possible properties of right multiplication by elements of $\mathbb{R}$ on the extensive structure X are listed as follows.

- A8. $x \succsim_X y \Leftrightarrow xt \succsim_X yt \Leftrightarrow xt^{-1} \succsim_X yt^{-1}$.
- A9. $s \leqslant t \Leftrightarrow xs \succsim_X xt \Leftrightarrow xs^{-1} \precsim_X xt^{-1}$.
- A10. $(xt)t^{-1} \sim_X (xt^{-1})t \sim_X x$.
- A11. For any $x \in X$ and any $t \in T$, there exists $y, z \in X$ such that $yt \sim_X x$ and $zt^{-1} \sim_X x$.
- A12. Whenever $x \cdot y$ is defined in X, then $(x \cdot y)t \sim_X (xt) \cdot (yt)$ and $(x \cdot y)t^{-1} \sim_X (xt^{-1}) \cdot (yt^{-1})$.
- A13. $x0 \sim_X x$.

The above properties can be obtained by introducing several conditions (Matsushita, 2017) into a weak order $\succsim$ on the product set $X \times \mathbb{R}$. Axiom A12 implies right-homogeneity of the right multiplication, and it is trivial because $x \cdot y$ can be interpreted as the simultaneous receipt of x and y (the reader may regard $x \cdot y$ as $x + y$ in the case of monetary amounts). Indeed, A12 only claims that the delayed (or advanced) receipt of a joint outcome $x \cdot y$ by duration t is equivalent to the simultaneous receipt of outcomes xt and yt (or xt^{-1} and yt^{-1}), the receipt of which are delayed (or advanced) by duration t.

Definition 1 Let $\langle \mathbb{R}, \geqslant, \cdot_{\mathbb{R}} \rangle$ be an Archimedean simply ordered group. An *extensive structure equipped with a right action of* $\mathbb{R}$ is an extensive structure $\langle X, \succ_X, \cdot \rangle$ equipped with the action of right multiplication for which A8–A13 are satisfied.

From Theorem 1 (Matsushita, 2017) a weighted additive model is obtained for the delay and advance operators.

Proposition 1 *Let $\langle X, \succ_X, \cdot \rangle$ be an extensive structure equipped with a right action of $\mathbb{R}$, and let u be an additive representation of X. Then there exists a function $\varphi : T \to (0, 1]$ with $\varphi(0) = 1$ such that*

(i) $u(xt) = \varphi(t)u(x)$,
(ii) $u(xt^{-1}) = \frac{1}{\varphi(t)}u(x)$,
(iii) $s \geqslant t \Leftrightarrow \varphi(s) \leqslant \varphi(t)$.

Here φ is an absolute scale.

Remark 1 Since $u((xs)t) = u((xt)s)$ by Proposition 1, it follows that $(xs)t \sim_X (xt)s$, implying that the delay operator is commutative. The delay action is also written for dated outcomes in the right-multiplication form $(x, t)s$ and it is defined that $(x, t)s = (x, t \cdot_T s)$ (where $0 \cdot_T s = s$). Assume further that monotonicity holds, i.e., $(x, t) \sim (xt, 0) \Leftrightarrow (x, t)s \sim (xt, 0)s$. Then the commutativity, along with (2), induces

$$(x, 0)s \sim (xs, 0) \Leftrightarrow (x, t)s \sim (xs, t).$$

This is called *time invariance* (Halevy, 2015) and a very restrictive condition because it says that the present value of each period t (offset against delay) is invariant regardless of when delay occurs. The commutativity of the delay operator is a constraint arising from a multiplicative utility accompanied by the assumption $xt \in X$ (by (2)).[7] Although time invariance is in itself not assumed in the original formulation (Matsushita 2017, Eq. (11)), it is given rise to by A4 (regarding $\cdot$), A12, and the assumption $xt \in X$ (see the appendix for the detail). This problem will be resolved by introducing a non-associative operation in Sect. 4.

Henceforth, the focus is on additive discounting to obtain a discount function of the exponential form.

A14. $(xs)t \sim_X x(s \cdot_T t)$.

Inductive use of A14 along with A8 guarantees the additive discounting: $(\cdots(xt_1)\cdots)t_n \sim_X xt(n)$. Mathematically, the introduction of A14 makes the right multiplication by t a representation of T on X (i.e., a homomorphism of T into the set of homomorphisms on X) in the sense of monoids (i.e., semigroups with identity). Therefore it is appropriate to refer to the right multiplication by t as a *right action*. The following is a key principle in this study.

Proposition 2 *Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a positive closed extensive structure with an identity element 0, and let w be an additive representation of T. Let $\langle X, \succsim_X, \cdot\rangle$ be an extensive structure equipped with a right action of $\mathbb{R}$. Let $\varphi : T \to (0, 1]$ be the weight function of Proposition 1. If A14 is satisfied, then φ is of the multiplicative form $\varphi(s \cdot_T t) = \varphi(s)\varphi(t)$, where*

$$\varphi(t) = \exp(-vw(t)) \textit{ for some real constant } v > 0.$$

[7] This can be verified by Theorems 5 and 6 in Bleichrodt et al. (2015). Indeed, the present value is equal to tomorrow's present value in a constant discount utility (Theorem 5), but not in the usual additive utility (Theorem 6).

If A14 is assumed, then the repeated use of A8 yields

$$y \sim_X xt_1 \Leftrightarrow yt(n-1) \sim_X xt(n) \text{ for any } n \geqslant 2.$$

This is a generalized stationarity axiom in the sense that an indifference between two (delayed) outcomes is invariant under the assumption that the duration is summed based on the operation $\cdot_T$ of the extensive structure.

3 Two types of discount functions derived from time perception

Introducing a binary operation on T, we have so far derived an exponential discount function equipped with a generalized time scale. However, the binary operation is merely a generalization of the addition and is not shown to have a close connection with human time perception. This section connects the operation to time perception and expresses the additive representation w of Proposition 2 in an explicit form. For this purpose, a *continuous* binary operation $\circ_T$ on $T \backslash \{0\}$ is provided; regarding $\circ_T$ as a function of two variables, it is continuous in each variable. Then conditions for $\circ_T$ are extracted in the context of time perception such that w can be a log scale or exponential scale. We shall now imagine a situation where a person is faced with a discount evaluation when a short-term delay occurs repeatedly. In this case, he/she might recognize each term either as the difference between the added and the cumulative duration or as the ratio of the added to the cumulative duration. Let α denote a unit of time (e.g., a month, 6 months, or a year) for delay. Let s, t be arbitrary time durations. Two types of standards regarding time perception are provided.

B1. *Left-homogeneity:* $\alpha(s \circ_T t) = (\alpha s) \circ_T t$.
B2. *Homogeneity* (*of degree* 1): $\alpha(s \circ_T t) = (\alpha s) \circ_T (\alpha t)$.

Axiom B1 means that the concatenation of increments, whose unit is turned into a multiple of α, is identical to the concatenation of increments, any one of which is a multiple of α, indicating that $\circ_T$ is close to multiplication. Meanwhile, B2 means that the concatenation of increments, whose unit is turned into a multiple of α, is identical to the concatenation of the increments that are multiples of α, indicating that $\circ_T$ is close to addition.

In addition, if $\circ_T$ is assumed to be commutative (i.e., $s \circ_T t = t \circ_T s$), then B1 induces also right-homogeneity, and hence *homogeneity of degree* 2 is obtained as

$$\alpha^2(s \circ_T t) = (\alpha s) \circ_T (\alpha t).$$

The next proposition is a restricted version of Theorem 1 of Marchant and Luce (2003).

Proposition 3 *Assume that $\circ_T$ is a continuous binary operation on $T \backslash \{0\}$ for which associativity, commutativity, and monotonicity are satisfied. Then the operation $\circ_T$ and an additive representation v of $T \backslash \{0\}$ are of one of the following forms, depending on whether either B1 or B2 is satisfied.*

(i) *If B1 is satisfied, then*

$$s \circ_T t = \eta\ st \text{ with } \eta > 0, \tag{3}$$

$$v(t) = c \ln(\eta t) \text{ with } c > 0. \tag{4}$$

(ii) *If B2 is satisfied, then*

$$s \circ_T t = (s^b + t^b)^{\frac{1}{b}} \text{ with } b > 0, \tag{5}$$

$$v(t) = \rho t^b \text{ with } \rho > 0. \tag{6}$$

Proposition 3 states that human subjective perception of time follows the "Weber-Fechner law" (Bradford et al., 2019) if B1 holds, and follows "Stevens' power law" (Stevens, 1957) if B2 holds. Bradford et al. (2019) showed that the predicted subjective time was close to a log-shaped curve. Hence it is suitable to determine that $0 < b < 1$ in the case of B2.

We will consider the possibility of substituting the two types of additive functions v for w in Proposition 2. For this purpose, we may verify whether $\circ_T$ can be identified with $\cdot_T$. In the case of Proposition 3(ii), it can be seen from (5) that $\langle T \backslash \{0\}, \geqslant, \circ_T \rangle$ satisfies all the conditions of a positive closed extensive structure. Moreover, 0 is added to $T \backslash \{0\}$ to supply an identity element for $\circ_T$. Therefore it is permissible to assume that $v = w$. On the contrary, in the case of Proposition 3(i), it is impossible to assume that $v = w$ because 0 cannot be an identity with respect to $\circ_T$ by (3). Therefore an alternative identity element $\varepsilon > 0$ is prepared for $\circ_T$ of (3).

B3. *Identity*: $t \circ_T \varepsilon = \varepsilon \circ_T t = t$.

It is clear from (3) that this condition causes $\eta = 1/\varepsilon$ in (3). Furthermore, an equation is provided to transfer the roles of the identity elements of $\circ_T$ and $\cdot_T$:

$$(\varepsilon + \varepsilon s) \circ_T (\varepsilon + \varepsilon t) = \varepsilon + \varepsilon(s \cdot_T t). \tag{7}$$

Substitution of $s = 0$ (or $t = 0$) into both sides of (7) activates the respective identity elements ε and 0 for $\circ_T$ and $\cdot_T$; that is, $\varepsilon \circ_T (\varepsilon + \varepsilon t) = \varepsilon + \varepsilon t,\ 0 \cdot_T t = t$. It must be verified here that (7) does not invalidate the assumption of $\langle T, \geqslant, \cdot_T, 0 \rangle$ being a positive closed extensive structure. From (3) with $\eta = 1/\varepsilon$ and (7) it follows that $s \cdot_T t = s + t + st$. Therefore $\langle T, \geqslant, \cdot_T, 0 \rangle$ satisfies all the conditions of a closed extensive structure (see Luce et al. (1990, pp. 28–29) for an analogous case). Furthermore, it is worthwhile to note that (7) gives the formal consistency between the product and its variables. Indeed, after restricting $\circ_T$ to the set $\varepsilon + \varepsilon T = \{\varepsilon + \varepsilon t \mid t \in T\}$, we will interpret the meaning of this multiplication. The left-hand side of (7) implies the multiplication of deviations $\varepsilon s,\ \varepsilon t$ from the identity element ε, and the right-hand side means that their product is also a deviation from ε, in which the deviation is expressed as $\varepsilon(s \cdot_T t)$. Hence it seems proper to regard the identity ε and a deviation as the present and an increment of duration, respectively (Fig. 1).

Example 1 In the case of (3), since $s \cdot_T t = s + t + st$ as shown above, $t(n)$ (which is defined in Sect. 2.1) varies in accordance with the value of t_n, which is shown in Table 1. In contrast, in the case of (5) with $b = 1$, the constant $t_n = 1/2$ gives the same variation of $t(n)$ as in Table 1.

Example 1 shows that in the case of B1, the increment becomes shorter to guarantee an equal interval between adjacent delays as the cumulative delay gets longer. Since the discount factor is assessed as a ratio of the present value to the value of an outcome, it is likely that a person assesses each increment of delay as the ratio as above. Thus, although preceding work (Zauberman et al., 2009; Bradford et al., 2019) reported that people overestimate short delays and underestimate long delays, this might be explained just by assuming that human subjective perception of time is assessed based on the ratio, without daring to bring up the Weber-Fechner or Stevens' power law.

Proposition 4 *Let $\langle T, \succcurlyeq, \cdot_T, 0\rangle$ be a positive closed extensive structure with an identity element 0, and assume that $\cdot_T$ is continuous. Let w be an additive representation of T. Assume that $\circ_T$ is a binary operation on $\varepsilon + \varepsilon T$ satisfying all the conditions of Proposition 3(i) and B3. Let v be an additive representation of the form (4). Then (7) holds if and only if $w(t) = v(\varepsilon + \varepsilon t)$ for all $t \in T$. Hence w is expressed as*

$$w(t) = c \ln(1 + t) \text{ with } c > 0.$$

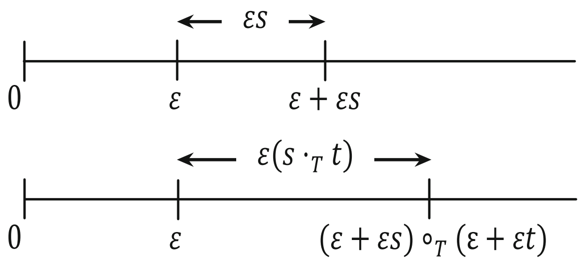

Fig. 1 Relation between $\circ_T$ and $\cdot_T$

Table 1 Cumulative duration in accordance with increments

n	1	2	3	4	5	...
t_n	1/2	1/3	1/4	1/5	1/6	...
$t(n)$	1/2	1	3/2	2	5/2	...

Given v of (4), Proposition 4 says that w can be identified with v as long as a variable of v is written in the form $\varepsilon + \varepsilon t$ and that w is expressed as a log scale having a variable of the form $1 + t$. Since w is a function that puts 0 at the present, the log scale is considered to be a function that puts 1 at the present by analogy with the discussion on Fig. 1.

When the unit is multiplied by α in the case of B1, we consider how an increment in time duration from the present is expressed. By B1 and (7),

$$(\alpha(\varepsilon + \varepsilon s)) \circ_T (\varepsilon + \varepsilon t) = \alpha\varepsilon + \alpha\varepsilon(s \cdot_T t).$$

The right-hand side indicates that $\alpha\varepsilon$ is regarded as a starting point of delay and hence $\alpha\varepsilon(s \cdot_T t)$ as an increment in duration from the present. In view of the variable of w being an increment from the present, it follows from Proposition 4 that $w(\alpha(s \cdot_T t)) = c \ln(1 + \alpha(s \cdot_T t))$. The following is a corollary to Proposition 2; the first and the second case correspond to Propositions 3(i) and (ii), respectively:

Proposition 5 *Assume that the hypotheses of Propositions 2 and 3 hold and that $\cdot_T$ is continuous. Let αt, $t > 0$ denote a delay duration from the present.*

(i) *If B1, B3, and (7) are satisfied, then*

$$\varphi(\alpha t) = \frac{1}{(1 + \alpha t)^{\beta/\alpha}}, \ \beta = vc\alpha. \tag{8}$$

(ii) *If B2 is satisfied, then*

$$\varphi(\alpha t) = \exp[-v\rho(\alpha t)^b]. \tag{9}$$

Remark 2 The right-hand side of (8) is the (generalized) hyperbolic discount function by Loewenstein and Prelec (1992). The right-hand side of (9) is similar to a discount function by Ebert and Prelec (2007)[8], which was derived by applying the condition of temporal scale invariance (i.e., a preference invariance condition under the conversion of a time unit) to the multiplicative utility model (Fishburn & Rubinstein, 1982). Since Ebert and Prelec (2007) set the exponential function as an underlying function, their discount function was derived in a form similar to the model of (9).

The condition of temporal scale invariance (Ebert & Prelec, 2007; Bleichrodt et al., 2009) is simple to check experimentally because it is written based on preferences between dated outcomes. It is desirable that the empirical test on the validity of B2 or B1 be also run in the framework of time preference. The following procedure might be one of the candidates. Subjects are asked to assess a present value each time the receipt of an outcome x is delayed by duration t_i $(i = 1, \ldots, n)$ with the use of a time unit α. This gives a present value, denoted $x_n^{(\alpha)}$, of the n-step

[8] Furthermore, Bleichrodt et al. (2009) showed that the discount function with $0 < \beta < 1$ can only address moderately decreasing impatience.

delay of x; formally, $x_n^{(\alpha)} \sim_X (\cdots((x(\alpha t_1))t_2)\cdots)t_n$. Furthermore, it is possible to determine a duration s such that $x_n^{(\alpha)} \sim_X xs$ by asking the one-step preference questions with a varied delay duration. The duration s is written as $s_n^{(\alpha)}$, which can be regarded as equivalent to $\alpha((\cdots(t_1 \cdot_T t_2)\cdots) \cdot_T t_n)$. Similarly, such a duration $s_n^{(\alpha')}$ can be obtained with another unit α'. If B1 is satisfied, then $s_n^{(\alpha)}/s_n^{(\alpha')}$ ought to equal α/α' for each $n \geqslant 1$. Thus it will be proper to establish this equality as a validity condition of B1. The validity of B2 can be verified in the same manner by replacing the first equivalence with $x_n^{(\alpha)} \sim_X (\cdots((x(\alpha t_1))(\alpha t_2))\cdots)(\alpha t_n)$.

In addition, define a function $\imath$ to give correspondence between a duration t and its inverse t^{-1} by $\imath(t) = -t^{-1}$. The form of $\imath$ is determined according to whether B1 or B2 is being satisfied.

Proposition 6 *Assume that the hypotheses of Propositions 2 and 3 hold and that $\cdot_T$ is continuous. Then $\imath$ is written as follows.*

(i) *If B1, B3, and (7) are satisfied, then $\imath(t) = \frac{t}{1+t}$.*
(ii) *If B2 is satisfied, then $\imath(t) = \gamma^{1/b} t$ for some $\gamma > 0$.*

Testing the form of $\imath$ might allow us to examine whether the hyperbolic or the exponential function is more suitable, in the context of time perception of advancement.

4 A discount function reflecting subadditivity or superadditivity

This section shows a way to construct a function that reflects subadditive or superadditive discounting. For this purpose, a "generalized" extensive structure (Matsushita, 2014) is provided. Let $*$ be a concatenation operation of elements of $T = \mathbb{R}_0^+$, and assume that $*$ is not associative. Let $T^{(0)} = \mathbb{R}_0^+$, let $T^{(1)} = \{s * t | s, t \in T^{(0)}\}$ and proceeding recursively, let $T^{(n+1)} = \{s * t | s, t \in T^{(n)}\}$. Note here that $T^{(n+1)} \supset T^{(n)}$ for $n \geqslant 0$ because a left identity element is included in $T^{(n)}$ (C1 below). Hence $T^* = \cup_{n=0}^{\infty} T^{(n)}$ is closed with respect to $*$. Let $\succsim_*$ be a binary relation on T^*.

C1. *Left identity: o* is a *left identity element*; that is, $o * s \sim_* s$ for all $s \in T^*$.
C2. *R-positivity: $s * t \succ_* s$* unless $t \sim_* o$.
C3. *Left solvability:* whenever $s \succ_* t$, there exists $r \in T^*$ such that $s \sim_* r * t$.

The system $\langle T^*, \succsim_*, *, o\rangle$ is a *left nonnegative concatenation structure with left identity* if and only if A1–A3, A7, and C1–C3 are satisfied with $\succsim_*$ and $*$. Note here that A7 is satisfied only in relation to concatenation from the left side. It will be seen (the proof of Proposition 8 below) that $o \sim_* 0$. In view of non-associativity of $*$, the well-definedness of a product demands a specification of the order of multiplication. Let $t^*(n)$ denote a product of the form $(\cdots(t_1 * t_2) * \cdots) * t_n$. Each $t^*(n)$ can be regarded as a string of elements $t_1, \ldots, t_n$ of $\mathbb{R}_0^+$; that is, $t^*(n)$ is identified with an n-

tuple $(t_1, \ldots, t_n)$. The arrangement order of elements indicates the order in which the delay with duration t_i occurs. A period in which the most recent delay has occurred is called a "reference period." It is important to note the following distinguishing feature of the product (not n-tuple) expression: the right multiplication and division by left identity o serves as operators moving the occurrence of each delay away from and closer to the reference period, respectively. For example, for $s * t$, a reference lies in the period where a delay with duration t occurred. Then $s * o$ means that a delay with duration s occurred earlier than the reference period by one period, and s/o implies that the occurrence of a delay with duration s is pulled back into the reference period. Hence $(s/o) * t$ denotes a delay in the reference period whose duration is expressed by the concatenation of s and t, say $s \cdot_T t$. Thus it is satisfied that $(s * o)/o \sim_* (s/o) * o \sim_* s$. The functions of the operators by o are shown as follows.

Example 2 The next correspondences hold:

$$(((t_1/o)/o) * (t_2/o)) * t_3 \leftrightarrow (o,\, o,\, (t_1 \cdot_T t_2) \cdot_T t_3).$$
$$(t_1 * (t_2 * o)) * ((t_3 * o) * o) \leftrightarrow ((t_1 \cdot_T t_2) \cdot_T t_3,\, o,\, o).$$

Since the products on the left-hand side have the terms $(t_1/o)/o$ and $(t_3 * o) * o$ in the former and latter cases, respectively, it is seen that the reference period lies in the third period, implying that the reference period is located in the third component on the right-hand side in both the cases. The former case indicates that a delay with duration $t(3)$ occurs after a two-period passage, while the latter demonstrates that the delay occurs in the present.

The following knowledge is basic to the subsequent consideration.

- Assume that the next conditions hold for $r, s, t \in T^*$.

C4. $s * (t * r) \sim_* t * (s * r)$.
C5. $(s * t) * o \sim_* (s * o) * (t * o)$.

Define a binary operation on T^* by

$$s \circledast t = (s/o) * t.$$

If C4 and C5 hold[9], then $E(T^*) = \langle T^*, \succsim_*, \circledast, o \rangle$ is a positive closed extensive structure with identity (Matsushita, 2014, Lemma 2). Let w^* be an additive representation of $E(T^*)$.[10] Then

$$w^*(s * o) = \lambda w^*(s),\ \ \lambda > 0. \tag{10}$$

Since $s * t \sim_* ((s * o)/o) * t \sim_* (s * o) \circledast t$ by A3 regarding $*$, it follows that

[9] Condition C4 says that if the latest delay with duration r occurs, then looking back from the reference period, the perceived time length is indifferent to the occurrence order of the previous two delays with durations s and t.

[10] In Matsushita (2014), $w^*(s * o) = \lambda w(s),\ \lambda \geqslant 1$ because it was assumed that $s * o \succsim_* s$ for $s \succ_* o$. This paper invalidates the assumption to consider the case of $s * o \prec_* s$.

$$w^*(s * t) = \lambda w^*(s) + w^*(t) \tag{11}$$

For simplicity, the notation $o^{(n)}$ is provided to express the product:

$$s * o^{(n)} = (\cdots(s * \underbrace{o) * \cdots * o) * o}_{n \text{ times}} \; (n \geqslant 1).$$

Here $s * o^{(n)}$ means that after the delay with duration s, the n-times delay continues without specifying each duration. Each product $t^*(n)$ is expressed as the *standard form* $(t_1 * o^{(n-1)}) \circledast (t_2 * o^{(n-2)}) \circledast \cdots \circledast t_n$.

Since $s \circledast t$ $(s, t \in T)$ means the concatenation of durations whose delays occurred simultaneously, it is rational to assume that the restriction of $\circledast$ to T, denoted $\circledast|_T$, is a closed operation, i.e., $s \circledast t \in T$ whenever $s, t \in T$. It is also clear that the restriction of $\succsim_*$ to T, denoted $\succsim_*|_T$, is the simple order $\geqslant$ because of $T = \mathbb{R}_0^+$. In this regard, it would be appropriate to assume the following.

Restriction assumption $\circledast|_T = \cdot_T$ and $\succsim_*|_T = \geqslant$.

Let $s, t \in T^*$ and it is presupposed that $xs \in X$ for all $s \in T^*$. Analogously to Definition 1, it is possible to define an extensive structure with the action of the right multiplication of elements of $E(T^*)$. Since this section deals only with the delayed case, the parts written by the right multiplication by inverse elements should be removed in A8–A12, so that A10 can be eliminated. Since T^* is a weakly ordered set, A9 is rewritten as

A9*. $s \precsim_* t \Leftrightarrow xs \succsim_X xt$.

The next axiom is necessary to make the additivity of the delay operator (the right multiplication by elements of T^*) revival.

A14*. $(xs)t \sim_X x((s * o) \circledast t)$.

The following propositions are an adapted version of Propositions 1 and 2 for the generalized right action, respectively.

Proposition 7 *Let $E(T^*)$ be a positive closed extensive structure with an identity that is derived from the left nonnegative concatenation structure with left identity for which C4 and C5 are satisfied. Let $\langle X, \succ_X, \cdot \rangle$ be an extensive structure equipped with a right action of $E(T^*)$ and let u be an additive representation of X. Then there exists a function $\varphi : T^* \to (0, 1]$ with $\varphi(o) = 1$ such that for $s, t \in T^*$,*

(i) $u(xt) = \varphi(t)u(x)$,
(ii) $s \succsim_* t \Leftrightarrow \varphi(s) \leqslant \varphi(t)$.

Proposition 8 *Assume that the hypotheses of Proposition 7 hold, and let w be an additive representation of T. If A14* is satisfied, then φ is of the multiplicative form*

$\varphi((s*o)\circledast t)=\varphi(s*o)\varphi(t)$ for any $s,t\in T^*$. Furthermore, if the restriction assumption is satisfied, then for $s\in T$,

$$\varphi(s*o^{(n)})=\exp(-v\lambda^n w(s))\ \textit{for some}\ v,\ \lambda>0.$$

Remark 3 Since each product has a standard form, it follows from the equation displayed in Proposition 8 that

$$\varphi(t^*(n))=\exp[-v(\lambda^{n-1}w(t_1)+\lambda^{n-2}w(t_2)+\cdots+w(t_n))].$$

The standard form suggests that the product $t^*(n)$ reflects not only the total delay duration but also the effect of repeated delay with a subdivided duration on time perception; that is, the duration $t_i*o^{(n-i)}$ $(1\leqslant i\leqslant n-1)$ is assessed as longer or shorter than the pure duration t_i, depending on whether the constant λ of (10) is greater or smaller than 1. Meanwhile, the product $t(n)=(\cdots(t_1\cdot_T t_2)\cdots)\cdot_T t_n$ expresses only the total delay duration. Hence it seems proper to judge subadditivity or superadditivity by comparing the values of the discount function of these two products; that is, subadditivity or superadditivity depends on whether $\varphi(t^*(n))$ is smaller or greater than $\varphi(t(n))$. From Propositions 2 and 8 it is seen that the discounting is subadditive if $\lambda>1$ and is superadditive if $\lambda<1$. However, it is problematic to assess the value of λ in (10) because $s*o$ does not show an explicit delay duration. If $s*o$ can be interpreted as a delay with a duration of a time unit subsequent to the delay with duration s (many decision makers may feel so)[11], then it will be possible to extract a duration equivalent to $s*o$. That is,

***T*-equivalence** For any $s\in T^*$, there exists $r\in T$ such that $r\sim_* s$.

This is guaranteed by making C3 a bit stronger: whenever $s\succ_* t$, there exists $r\in T$ such that $s\sim_* r\circledast t$. Repeated use of T-equivalence and A3 regarding $*$ gives $(t_1*o^{(n-1)})\circledast(t_2*o^{(n-2)})\circledast\cdots\circledast t_n\sim_* r_1\cdot_T r_2\cdot_T\cdots\cdot_T t_n$, where $t_i*o^{(n-i)}\sim_* r_i$ $(1\leqslant i\leqslant n-1)$. Thus $t^*(n)$ has been converted into a concatenation of real numbers, so that λ can be assessed by $w(r)/w(s)$ when $s*o\sim_* r$. In fact, the assessment will be based on v regarding $\circ_T$ of Proposition 3. Moreover, by applying the results of the homogeneity condition to Proposition 8, concrete forms of nonadditive discount functions can be obtained.

Example 3 Define $F_{t\to s*t}=\varphi(s*t)/\varphi(t)$ for $s,\ t\in T$. Since $\varphi(s*t)=\varphi(s*o)\varphi(t)$ by Proposition 8, it follows that $F_{t\to s*t}=\exp(-v\lambda w(s))$. Hence

[11] The decision makers assesses the delay length of $s*o$ regarding it as a delay with duration s, which is subsequently followed by a delay with a time unit.

(i) If B1, B3, and (7) are satisfied with $\circ_T$, then

$$F_{t\to s*t} = \frac{1}{(1+\alpha s)^{\lambda\beta/\alpha}}.$$

(ii) If B2 is satisfied with $\circ_T$, then

$$F_{t\to s*t} = \exp[-v\lambda\rho(\alpha s)^b].$$

To explain subadditive or superadditive discounting behavior, Scholten and Read (2006) proposed a generalized discounting model and derived several descriptive models by providing two types of parameters related to the effect of interval length and nonadditivity. Unfortunately, the models in Example 3 are not identical with their descriptive models. This is attributed to the difficulty in introducing a parameter in a power form, as Scholten and Read (2006) did, because a function is constructed based on the admissible transformation of an additive representation in this study.

5 Conclusion

Under the presupposition that human time perception is distorted, this study devised the conditions (homogeneity of degree one or two) to identify the form of a time scale so that one can determine whether the hyperbolic or exponential function is more suitable for modeling people's discounting. Homogeneity of degree one implies that subjective time delay is measured by a power scale and its discount function is a power-exponential type, whereas homogeneity of degree two implies that subjective time delay is measured by a log scale and its discount function is a hyperbolic type. Furthermore, the study showed a way of constructing a function that reflects subadditive or superadditive discounting. The introduction of a non-commutative and non-associative concatenation operation generated a time string consisting of subdivided durations that incorporates the effect of repeated delay with each duration. Eventually, subadditivity or superadditivity was determined depending on whether the discount function value of a product by the non-commutative and non-associative operation was smaller or greater than that of a product by an operation of the usual extensive structure. An important research problem is to examine whether the procedure for testing the homogeneity condition stated immediately after Remark 2 can be applied to actual intertemporal choice problems.

6 Proofs

6.1 Proposition 1

Proof The proof is similar to that of Matsushita (2017). However, since the underlying structure is reduced to an extensive structure, so that the first (key) part of the proof can be much simplified, we will show the summarized proof. For any given

t, let $X_t = \{xt | x \in X\}$. From A1–A7, A8, A11, and A12 it is seen that X_t is an extensive substructure of X. Hence the restriction of u is an additive representation of X_t. Define $u_t(x) = u(xt)$ for all $t \in T$. Since A8 and A12 guarantee that $u_t(x)$ is an order-preserving additive function, it follows from the admissible transformation that $u(xt) = \varphi(t)u(x)$ with $\varphi(t) > 0$. Note in the paper that the role of right multiplication by duration $t > 0$ has been changed from an advance to a delay operator. Hence we must prove that $\varphi(t) < 1$. By A9, $u(xt) = \varphi(t)u(x) < u(x)$ whenever $t > 0$, implying that $\varphi(t) < 1$. Since $u((xt^{-1})t) = \varphi(t)u(xt^{-1}) = u(x)$ by A10, $u(xt^{-1}) = (1/\varphi(t)) u(x)$. By A13, $u(x0) = \varphi(0)u(x) = u(x)$, so that $\varphi(0) = 1$. Part (iii) is obtained similarly to Matsushita (2017). □

6.2 Proposition 2

Proof We must adapt the proof of Proposition 8 in Matsushita (2017) because the role of right multiplication was changed. In view of A14, it follows from (i) of Proposition 1 and order preservation of u that $(\varphi(s)\varphi(t))u(x) = (\varphi(s \cdot_T t))u(x)$. Since $u(x)$ is arbitrary, $\varphi(s)\varphi(t) = \varphi(s \cdot_T t)$. Hence $1/[\varphi(s)\varphi(t)] = 1/\varphi(s \cdot_T t)$. Furthermore, by (iii) of Proposition 1, $s \geqslant t \Leftrightarrow 1/\varphi(s) \geqslant 1/\varphi(t)$. Set $\phi(t) = \ln(1/\varphi(t))$. These two properties of $1/\varphi$ show that $\phi(t)$ is an additive representation of T with $\phi(0) = 0$. Hence the admissible transformation implies that $\phi(t) = vw(t)$ for some $v > 0$, or $\varphi(t) = \exp(-vw(t))$. □

6.3 Proposition 4

Proof Set $h(\varepsilon t) = \varepsilon + \varepsilon t$ for any $t \in T$, and set $a = h(\varepsilon s)$ and $b = h(\varepsilon t)$. Note that $\circ_T$ is strictly increasing because of monotonicity. In addition, since $\circ_T$ is continuous and associative, the theorem in Aczél (1966) guarantees the existence of a function v such that $a \circ_T b = v^{-1}[v(a) + v(b)]$, implying that v is an additive representation with respect to $\circ_T$. Similarly, it holds that $s \cdot_T t = w^{-1}[w(\frac{1}{\varepsilon}h^{-1}(a)) + w(\frac{1}{\varepsilon}h^{-1}(b))]$. Hence

$$\begin{aligned} & h(\varepsilon s) \circ_T h(\varepsilon t) = h(\varepsilon(s \cdot_T t)) \\ & \Leftrightarrow v^{-1}[v(a) + v(b)] = h(\varepsilon(w^{-1}[w(\frac{1}{\varepsilon}h^{-1}(a)) + w(\frac{1}{\varepsilon}h^{-1}(b))])) \\ & \Leftrightarrow v = w \circ L_\varepsilon^{-1} \circ h^{-1}, \end{aligned}$$

where L_ε is a map defined by the rule $L_\varepsilon(t) = \varepsilon t$. Thus $w(t) = v(\varepsilon + \varepsilon t)$. Since $\eta = 1/\varepsilon$ in (4) by B3, it follows that $w(t) = c\ln(1 + t)$. □

6.4 Proposition 5

Proof (i) By Proposition 4, $w(\alpha t) = c\ln(1 + \alpha t)$. Substituting this equality in the equation of Proposition 2 gives the conclusion.

(ii) Since $v = w$ (by the statement after Proposition 3), it follows from (6) that $w(\alpha t) = \rho(\alpha t)^b$. A similar substitution to (i) gives the conclusion. □

6.5 Proposition 6

Proof (i) Let $s \geqslant 0$ be arbitrary and let t be such that $s \cdot_T t = 0$. Since $s \cdot_T t = s + t + st$ (mentioned before Example 1), it follows that $t = -s/(1+s)$. Hence $\imath(s) = s/(1+s)$.

(ii) The domain of $\cdot_T$ is extended to the domain in which $s \geqslant 0 \geqslant t$ with $s \cdot_T t \geqslant 0$. According to Marchant and Luce (2003), $\cdot_T$ is extended as follows:

$$s \cdot_T t = (s^b - \frac{1}{\gamma}(-t)^b)^{1/b}, \ \gamma > 0.$$

Let t be a solution to $s \cdot_T t = 0$. The above equation gives $t = -\gamma^{1/b}s$, and hence $\imath(s) = \gamma^{1/b}s$.

6.6 Proposition 7

Proof The proof is similar to that of Proposition 1. However, we must address the problem that T^* is extended to a weakly ordered set. Let $[s] = \{s' \in T^* | s' \sim_* s.\}$ denote the equivalence class determined by $s \in T^*$ and let $T^*/\sim_*$ be the set of equivalence classes. Since $T^*/\sim_*$ is a simply ordered set, a similar action to Proposition 1 can be introduced by considering the right multiplication by elements $[s]$. An ordering between elements $x[s]$ and $y[t]$ can be determined by setting $x[s] \succsim_X y[t]$ if there exist $s' \in [s]$, $t' \in [t]$ such that $xs' \succsim_X yt'$. This is possible because it is true that $xs' \sim_X xs''$ whenever $s', s'' \in [s]$ by A9* (i.e., a representative of $x[s]$ is uniquely determined up to $\sim_X$). Thus we will use the same symbol as the order on X to express the order among elements $x[s]$. The following properties hold.

$$x \succsim_X y \Leftrightarrow x[s] \succsim_X y[s] \tag{12}$$

$$[s] \succsim_* [t] \Leftrightarrow x[s] \precsim_X x[t] \tag{13}$$

Moreover, a concatenation operation between $x[s]$ and $y[t]$ can be defined to be the equivalence class of $(xs) \cdot (yt)$. The uniqueness of the definition is verified by A3 (Krantz et al., 1971, p. 87). Hence

$$(x \cdot y)[s] \sim_X (x[s]) \cdot (y[s]) \tag{14}$$

where $\cdot$ in the right-hand side denotes the induced operation on the set of elements of the form $x[s]$.

For any $t \in T$, let $X_{[t]} = \{x[t] | x \in X\}$. With the use of (12), (13), and (14), it can be verified that $X_{[t]}$ is an extensive substructure of X. Hereafter, it is only proved that A5 holds for $X_{[t]}$ because the remaining axioms can be proven similarly. By A5 regarding X, we may assume that $x \succsim_X z \cdot y$ for some $z \in X$ whenever $x \succ_X y$. However, (12) guarantees that $x[s] \succ_X y[s]$ and $x[s] \succsim_X (z \cdot y)[s]$. By using (14) and A1 regarding X, the latter inequality is rewritten as $x[s] \succsim_X (z[s]) \cdot (y[s])$, implying A5 regarding $X_{[t]}$. The method of the proof of Proposition 1 gives $u(x[t]) = \varphi([t])u(x)$ where $0 < \varphi([t]) \leqslant 1$; in particular $\varphi([o]) = 1$. Moreover, by A9*,

$[s] \succsim_* [t] \Leftrightarrow \varphi([s]) \leqslant \varphi([t])$. This shows that φ takes the same value for any element s' belonging to the class $[s]$. Hence it is also valid to set $u(x[t]) = \varphi(t)u(x)$. Consequently, we have obtained the properties (i) and (ii) of the proposition. □

6.7 Proposition 8

Proof Let $s, t \in T^*$ be arbitrary and let w^* be an additive representation of $E(T^*)$. Similarly to the proof of Proposition 2, it is shown that φ is of the multiplicative form $\varphi((s * o) \circledast t) = \varphi(s * o)\varphi(t)$, and

$$\varphi(s * o) = \exp(-v' w^*(s * o)),\ \ \varphi(t) = \exp(-v' w^*(t)) \text{ where } v' > 0. \tag{15}$$

The former equation along with (10) yields $\varphi(s * o^{(n)}) = \exp(-v' \lambda^n w^*(s))$. It remains to be shown that the restriction of w^* to T is equivalent to w. The restriction assumption implies that $E(T^*)|_T$ is equivalent to $E(T) = \langle T, \geqslant, \cdot_T, 0 \rangle$ as an extensive structure (so that o is identified with 0). Hence, by the admissible transformation, $w^*(s)|_T = \mu w(s)$ for some $\mu > 0$ whenever $s \in T$. Thus we have $\varphi(s * o^{(n)}) = \exp(-v \lambda^n w(s))$ where $v = v' \mu$. □

Appendix

For a concise explanation, let $X = \mathbb{R}_0^+$ and assume that $\cdot_T$ is continuous. For any given t let f_t be the delay operator by t, i.e., $f_t(x) = xt$, and we denote $F(x, y) = x \cdot y$, where F is requested by A4 (regarding $\cdot$) to satisfy the associativity equation $F[F(x, y), z] = F[x, F(y, z)]$. Recall from the proof of Proposition 4 that F satisfying the associativity equation has an additive representation v with respect to $\cdot$. Here v is a strictly monotonic function; furthermore, it is possible to let v be an increasing function by multiplying it by -1 if necessary. Axiom A12 produces a functional equation, $f_t(F(x, y)) = F(f_t(x), f_t(y))$, whose only solution is written as $f_t(x) = v^{-1}[c_t v(x)]$, or $v(f_t(x)) = c_t v(x)$ (Aczél, 1966, pp. 62–63), where c_t is a constant. Thus it has been shown that A12 and A4 bring a multiplicative form. Hence, in view of the fact that $(xt)s \in X$, use Proposition 1(i) to obtain $u((xt)s) = \varphi(s)u(xt) = (\varphi(s)\varphi(t))u(x)$. This brings time invariance, as was shown in Remark 1.

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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