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The classification of preordered spaces in terms of monotones: complexity and optimization

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Abstract
The study of complexity and optimization in decision theory involves both partial and complete characterizations of preferences over decision spaces in terms of real-valued monotones. With this motivation, and following the recent introduction of new classes of monotones, like injective monotones or strict monotone multi-utilities, we present the classification of preordered spaces in terms of both the existence and cardinality of real-valued monotones and the cardinality of the quotient space. In particular, we take advantage of a characterization of real-valued monotones in terms of separating families of increasing sets to obtain a more complete classification consisting of classes that are strictly different from each other. As a result, we gain new insight into both complexity and optimization, and clarify their interplay in preordered spaces.

Keywords Multi-utility representation · Richter–Peleg function · Injective monotone · Majorization · Uncertainty preorder

1 Introduction

The question of how well a preorder relation can be captured through real-valued functions is an ongoing research topic since the introduction of utility functions in the early days of mathematical economics. The key observation is that sometimes preferences can not only be measured locally to decide between two elements, but there might be a global real-valued preference function that fully captures the

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corresponding order relation. That is, in certain situations, one can not only choose a preferable item between any two items in a given set of options, but one can find a single function, or a family of functions, defined on the decision space whose function values quantify the preference relation, so that one can compare function values to decide about the order relation of the corresponding arguments. Since the existence of such functions only depends on the properties of the corresponding preorder, this idea can naturally be applied in many domains of science. In particular, instead of considering preference relations and utility functions on decision spaces, many systems of interest can be thought of as sets of possible states endowed with an order relation encapsulating the intrinsic tendency of the system to transition from one state to another. The fields where these ideas are relevant include *thermodynamics* (Lieb & Yngvason, 1999; Giles 2016), *general relativity* (Bombelli et al., 1987; Minguzzi, 2010), *quantum physics* (Nielsen, 1999; Brandao et al., 2015) and *economics* (Debreu, 1954; Ok, 2002), among others.

The basic property of these real-valued functions f is that they have to be *monotones* with respect to the corresponding preorder $\preceq$, that is, $x \preceq y$ implies $f(x) \leq f(y)$. There are mainly three types of monotones that appear in this context: strict monotones (Alcantud et al., 2016; Peleg, 1970; Richter, 1966), injective monotones (Hack et al., 2022a, b), and utility functions (Debreu, 1954, 1964). In particular, these different types of monotones are used to classify preordered spaces mostly in two different ways: either by whether a given type of monotone exists, or, by whether there exists a family of such monotones, known as a *multi-utility*, that characterizes the preorder completely (Evren & Ok, 2011; Alcantud et al., 2013, 2016; Hack et al., 2022a, b; Bosi & Herden, 2012, 2016). Equivalently, these spaces can be classified according to either the existence of optimization principles with certain characteristics or the complexity of the preorder, that is, the amount and type of multi-utilities that exist for them. Even though the cardinality of such representing families plays an important role, so far mostly the two cases of *countable* multi-utilities and multi-utilies consisting of a *single* element, that is, utilities, have been considered.

Moreover, several connections between both types of classifications have been pointed out in the literature (Alcantud et al., 2016; Hack et al., 2022a, b; Alcantud et al., 2013; Bosi et al., 2018), but certain gaps in these connections have prevented the presentation of a general classification of preordered spaces through real-valued monotones. One of the aims of this contribution is to reduce this gap, achieving, thus, a more complete classification (see Fig. 1) and, hence, a better understanding of both complexity and optimization, including how they are related, in preordered spaces.

In particular, we take advantage of a characterization of real-valued monotones in terms of families of increasing sets (Alcantud et al., 2013; Hack et al., 2022a, b) that allows to distinguish more classes of preordered spaces than before, both in terms of the cardinality of the multi-utilities and the cardinality of the quotient space of the preorder. Importantly, by providing the corresponding counter examples, we show that certain classes of preorded spaces are, in fact, strictly contained in each other, which, to our knowledge, was not known before.

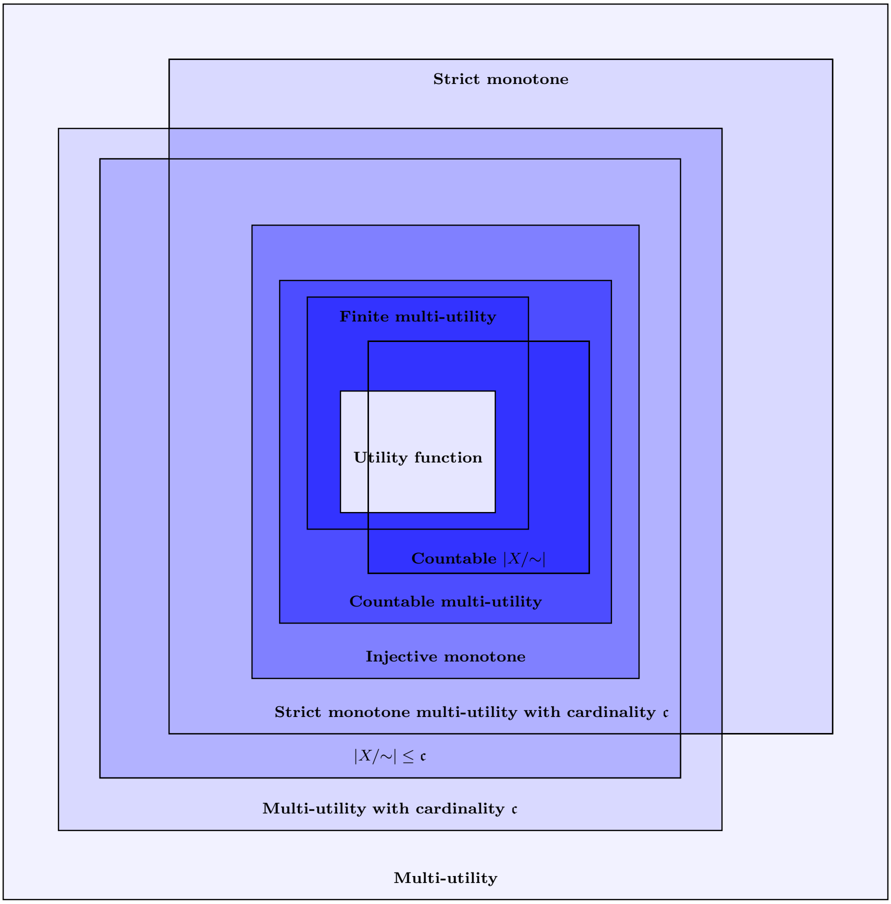

Fig. 1 Classification of preordered spaces according to the existence of various real-valued monotones. A distinction between our contributions here and previously known results can be found in the discussion (Sect. 4). Moreover, our contributions can be visualized in Fig. 2

2 Real-valued monotones and their role in complexity and optimization

In this section, we introduce the classes of real-valued monotones that are relevant throughout this work and relate them to both optimization and complexity. Before entering the general picture, we motivate them through an example with several applications, namely, the uncertainty preorder $\preceq_U$ (Hack et al. 2022a, b).

Consider a casino owner that intends to incorporate a new game to the casino, where all games under consideration follow the same idea, namely, bets are placed on the outcome of a random variable that is subsequently realized. Since the players win whenever they predict the outcome correctly, it is in the owner's interest to make the prediction as difficult as possible, that is, to make the game's outcome as uncertain as possible. For example, the game of rolling a fair die is preferred compared to that of

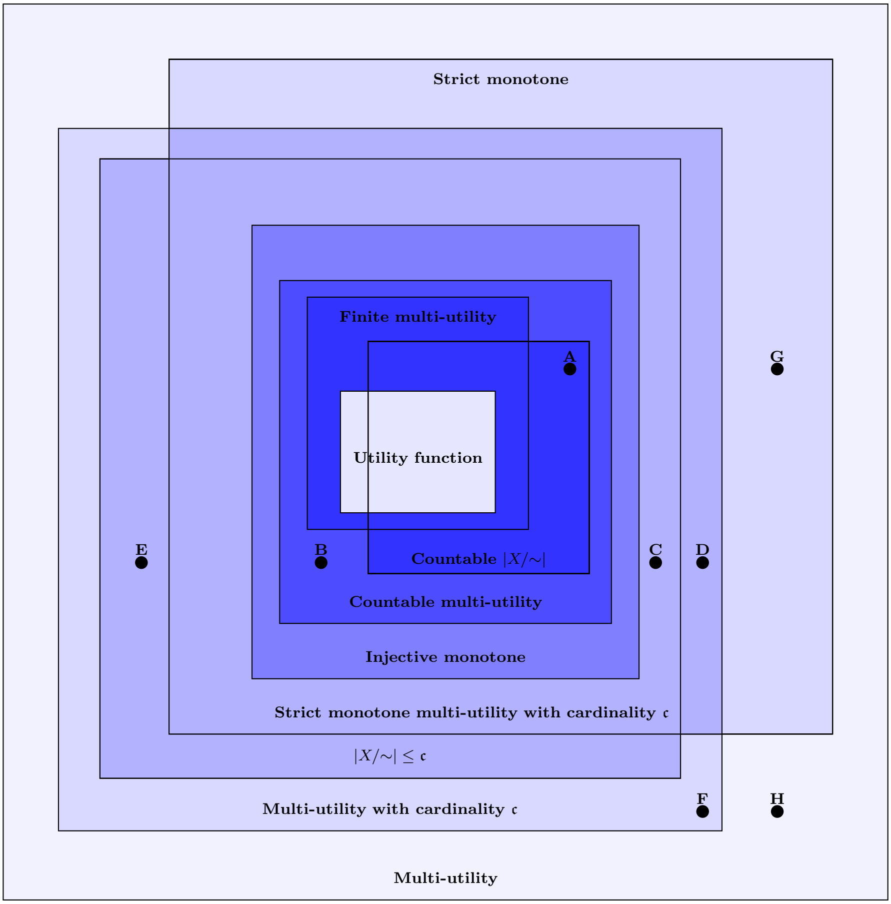

Fig. 2 Contributions of this work to the classification of preordered spaces. We reproduce here Fig. 1, incorporating a point for each preorder we have introduced that has allowed us to distinguish between classes. In particular, **A** stands for the preorder in Proposition 7, **B** for the one in Proposition 8, **C** for the preorder in both Proposition 2 and Corollary 1, **D** for that of Proposition 4 (ii) taking $I = \mathbb{R}$, **E** for the one in Corollary 2, **F** for the preorder in Corollary 5 taking $I = \mathbb{R}$, **G** for the space in Proposition 5 taking $I = \mathbb{R}$ and, lastly, **H** for the one in Proposition 6 taking $I = \mathbb{R}$

rolling a loaded one. When considering the games over some finite set Ω, the preference of the casino owner among them can be modeled on $\mathbb{P}_\Omega$, the space of outcome probability distributions that are associated to the games, by the uncertainty preorder

$$p \preceq_U q \iff u_i(p) \leq u_i(q) \quad \forall i \in \{1, .., |\Omega| - 1\}, \tag{1}$$

where $u_i(p) := -\sum_{n=1}^{i} p_n^{\downarrow}$ and $p^{\downarrow}$ denotes the decreasing rearrangement of p (same components as p but ordered decreasingly). Notice, $\preceq_U$ is known in mathematics, economics, and quantum physics as *majorization* (Hardy et al., 1952; Marshall et al., 1979; Arnold, 2018; Brandao et al., 2015).

When deciding between a certain game and another one, according to Eq. (1), the owner evaluates $|\Omega|-1$ functions. Hence, the larger the number of possible outcomes, the harder it becomes to decide which game to choose. It is in this sense that we can say the number of functions in Eq. (1) measures the *complexity* or *dimension* of the decision space.

In general, because of the gambling regulation, the owner may be required to choose games from some subset $B \subseteq \mathbb{P}_\Omega$ to diminish the gamblers' probability of loosing. To automatize the decision, the owner uses the maximum entropy principle (Jaynes, 1957, 2003), picking, hence, a distribution that maximizes Shannon entropy $H(p) := -\mathbb{E}_p[\log p]$ over B. Although H is not guaranteed to have maxima over every B, whenever it does, its maxima correspond to maximizing the preferences of the owner over B, that is, to distributions $p \in B$ such that there is no $q \in B$ that simultaneously fulfills $p \preceq_U q$ and $\neg(q \preceq_U p)$. Since the owner can make a decision on B by simply optimizing H, we say H is an *optimization principle*. As we will see, the optimization properties of H are closely related to how well H preserves the properties of $\preceq_U$.

We introduce now the general picture in terms of both preorders and real-valued monotones, and return to optimization and complexity at the end of this section. A *preorder* $\preceq$ on a set X is a reflexive ($x \preceq x$ $\forall x \in X$) and transitive ($x \preceq y$ and $y \preceq z$ implies $x \preceq z$ $\forall x, y, z \in X$) binary relation. A tuple $(X, \preceq)$ is called a *preordered space* or *preference* space and X the *ground set* or *decision space*. For example, $(\mathbb{P}_\Omega, \preceq_U)$ is a preordered space. An antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$ $\forall x, y \in X$) preorder $\preceq$ is called a *partial order*. The relation $x \sim y$, defined by $x \preceq y$ and $y \preceq x$, forms an *equivalence relation* on X, that is, it fulfills the reflexive, transitive and symmetric ($x \sim y$ if and only if $y \sim x$ $\forall x, y \in X$) properties. Notice, a preorder $\preceq$ is a partial order on the quotient set $X/\sim\ = \{[x] | x \in X\}$, consisting of all equivalence classes $[x] = \{y \in X | y \sim x\}$. In case $x \preceq y$ and $\neg(x \sim y)$ for some $x, y \in X$ we say y is *strictly preferred* to x, denoted by $x \prec y$. If $\neg(x \preceq y)$ and $\neg(y \preceq x)$, we say x and y are *incomparable*, denoted by $x \bowtie y$. Whenever there are no incomparable elements a preordered space is called *total*. By the Szpilrajn extension theorem (Szpilrajn, 1930; Harzheim, 2006), *every partial order can be extended to a total order*, that is, to a partial order that is total. Notice Szpilrajn extension theorem is a consequence of the axiom of choice, which we assume throughout this work. Equivalently, we assume $I \times I$ and I are equinumerous for any infinite set I and, thus, both $I \times \mathbb{N}$ and $I \cup I$ are also equinumerous to I.

To numerically characterize the relations established in a preordered space, one or several real-valued functions may be used. This results in a classification of preorders according to how well their information can be captured using these functions. We introduce now several classes that have been previously considered. A real-valued function $f : X \to \mathbb{R}$ is called a *monotone* or an *increasing function* if $x \preceq y$ implies $f(x) \leq f(y)$ (Evren & Ok, 2011). If the converse is also true, then f is called a *utility function* (Debreu, 1954). Furthermore, if f is a monotone and $x \prec y$ implies $f(x) < f(y)$, then f is called a *strict monotone*, a *Richter–Peleg function* or an *order-preserving function* (Alcantud et al., 2016). Similarly, a monotone f is called an *injective monotone* if $f(x) = f(y)$ implies $x \sim y$, that is, if f is injective considered as a function on the quotient set $X/\sim$ (Hack et al., 2022a, b). For example, H is a strict monotone but not an injective one on $(\mathbb{P}_\Omega, \preceq_U)$. Whenever a single function is insufficient to capture all the information in a preorder, for example when it is non-total (see (Bridges & Mehta, 2013) Theorem 1.4.8) for the total case), a family of functions may be used instead. A family V of real-valued functions $v : X \to \mathbb{R}$ is called a *multi-utility (representation) of* $\preceq$ (Evren & Ok, 2011) if

$$x \preceq y \iff v(x) \leq v(y) \quad \forall v \in V.$$

Whenever a multi-utility consists of strict monotones it is called a *strict monotone* (or *Richter–Peleg*Alcantud et al., 2016) *multi-utility (representation) of* $\preceq$. For example, $(u_i)_{i=1}^{|\Omega|-1}$ is a multi-utility that is not strict on $(\mathbb{P}_\Omega, \preceq_U)$. Analogously, if the multi-utility consists of injective monotones, we call it an *injective monotone multi-utility (representation) of* $\preceq$. Notice the cardinality plays a key role in the classification when we consider multi-utilities.

The role of the different characterizations via a single function can be clarified alluding to optimization (see Hack et al., 2022a, b, Sect. 4). In this regard, monotones are not interesting in general, since they may carry no information about $\preceq$ (as in the case of constant functions). Strict monotones, however, are interesting from the perspective of optimization. In fact, strict monotones exist if and only if *optimization principles* do (Hack et al., 2022a, b, Proposition 3), where a function $f : X \to \mathbb{R}$ is an optimization principle or *represents maximal elements* of $\preceq$ if, for any $B \subseteq X$, we have $\text{argmax}_B f \subseteq B_M^{\preceq}$, where $\text{argmax}_B f := \{x \in B |\ \nexists y \in B \text{ such that } f(x) < f(y)\}$ and $B_M^{\preceq} := \{x \in B |\ \nexists y \in B \text{ such that } x \prec y\}$.[1] Hence, optimizing f implies optimizing $\preceq$, like optimizing H implies optimizing $\preceq_U$. Following this parallelism, the existence of injective monotones is equivalent to that of *injective optimization principles* (Hack et al., 2022a, b, Proposition 3), where a function $f : X \to \mathbb{R}$ is an injective optimization principle or *injectively represents maximal elements* of $\preceq$ if, for any $B \subseteq X$ such that $\text{argmax}_B f \neq \emptyset$, we have $\text{argmax}_B f = [x_0]|_B$, where $[x_0]|_B$ is the equivalence class of x_0 restricted to B. In particular, if $\preceq$ is a partial order, then optimizing f yields a unique element, which is, ultimately, the goal of any optimization principle. Note that this is not the case in general for H, since it is not an injective monotone whenever $|\Omega| \geq 3$ (Hack et al., 2022a, b, Lemma 4). However, the maximum entropy principle does output a single distribution in the cases where it is usually applied, namely, when $B \subseteq \mathbb{P}_\Omega$ is given by a linear constraint (Jaynes, 1957, 2003). As a final remark, notice strict monotones allow the existence of *local* injective optimization principles, that is, those where injectivity is fulfilled for some subset $B \subseteq X$ (White, 1980), which contrasts with the *global* approach from injective monotones, which works for all subsets $B \subseteq X$. In summary, the existence of both strict and injective monotones can be characterized in terms of optimization principles.

The purpose of the different classes of characterizations via a family of functions or multi-utility can be clarified in terms of complexity, where we consider the sort of

[1] Notice that the equivalence between the existence of optimization principles and that of strict monotones is not exactly what is stated in Hack et al. (2022a, b) (Proposition 3), although it can be easily derived from it. The same holds true for injective monotones and injective optimization principles.

characterizations and the number of functions needed for each characterization as measures of complexity. Multi-utilities form the first class, where the aim is to find the minimal family of monotones V such that any false preference, $\neg(x \preceq y)$, is contradicted, at least, by one monotone, $v(x) > v(y)$ for some $v \in V$. Strict monotone multi-utilities form the second class, which can be characterized by those multi-utilities V which fulfill $x \prec y \iff v(x) < v(y)$ for all $v \in V$. The third class, injective monotone multi-utilities, possesses the properties of the previous two plus the fact $x \sim y \iff v(x) = v(y)$ for *any* $v \in V$. Note, on the contrary, the previous two classes require that $v(x) = v(y)$ for all $v \in V$ to determine that $x \sim y$.

Inside each class, we can distinguish preorders in terms of the minimal amount of monotones that are required to form a multi-utility. We will refer to this minimal amount, in the specific case of multi-utilities, as the *dimension* of the preorder, since it characterizes the minimal number of copies of the real line that are needed, using their natural order, to fully represent a preorder. Note that this definition differs from the usual one by Dushnik and Miller (1941), since we restrict ourselves to products of the real line with its usual ordering instead of the general approach in Dushnik and Miller (1941) (see Hack et al., 2022a, b for a discussion reagarding the notion of dimension for partial orders). In fact, since their definition is restricted to partial orders, the distinction between strict and injective monotone multi-utilities can be improved. Their definition considers a family of *realizers* $(\preceq_i)_{i \in I}$, i.e. partial orders $\preceq_i$ that are total, fulfill that $x \preceq y$ implies $x \preceq_i y$ for all $i \in I$, and such that any false preference, $\neg(x \preceq y)$, is contradicted, at least, by one linear extension, $y \prec_i x$ for some $i \in I$. Such partial orders, however, cannot be defined using a strict monotone that is not injective v, since there exist $x, y \in X$ such that $x \bowtie y$ and $v(x) = v(y)$. As a result, we obtain both $x \sim_v y$, given that we define $x \preceq_v y \iff v(x) \leq v(y)$, and $x \neq y$, contradicting, thus, antisymmetry.

Although several connections between the existence of these real-valued monotones are known (Evren & Ok, 2011; Alcantud et al., 2016; Hack et al., 2022a, b), we further clarify the relation between them throughout the following section. Mainly, using a characterization of these classes in terms of families of increasing sets which separate the elements in a preordered space (Herden, 1989; Alcantud et al., 2013; Bosi & Zuanon, 2013; Hack et al., 2022a, b), we introduce several counterexamples which allow us to distinguish the scope of the different classes and, hence, to improve on the study of both complexity and optimization, and their relation in preordered spaces.

3 Classification of preorders through real-valued monotones

3.1 Characterization of real-valued monotones by families of increasing sets

A subset $A \subseteq X$ is called *increasing* if, for all $x \in A$, $x \preceq y$ implies that $y \in A$ (Mehta, 1986). We say a family $(A_i)_{i \in I}$ of subsets $A_i \subseteq X$ *separates x from y*, if there exists $i \in I$ with $x \notin A_i$ and $y \in A_i$. Families of increasing sets have been used to characterize the existence of several classes of preorders in terms of real-valued representations in the literature. We state these results in Lemma 1 without proof.

Lemma 1 *Let $(X, \preceq)$ be a preordered space.*

(i) *For any infinite set I, there exists a multi-utility with the cardinality of I if and only if there exists a family of increasing subsets $(A_i)_{i \in I}$ that $\forall x, y \in X$ with $x \prec y$ separates x from y, and $\forall x, y \in X$ with $x \bowtie y$ separates both x from y and y from x.*

(ii) *There exists a strict monotone if and only if there exists a countable family of increasing subsets that $\forall x, y \in X$ with $x \prec y$ separates x from y.*

(iii) *There exists an injective monotone if and only if there exists a countable family of increasing subsets that $\forall x, y \in X$ with $x \prec y$ separates x from y and $\forall x, y \in X$ with $x \bowtie y$ separates either x from y or y from x.*

The proof of (i) can be found in Bosi and Zuanon (2013) and Alcantud et al. (2013) and that of (ii) and (iii) in Hack et al. (2022a, b). The characterizations in Lemma 1 can be useful to distinguish certain classes of preorders in terms of real-valued monotones, as we showed in Hack et al. (2022a, b) (Proposition 8), where we used them to build a preorder where injective monotones exist and countable multi-utilities do not. Note that (i) is not true for finite sets *I*, because there are preordered spaces that have a finite multi-utility but do not have a finite separating family of increasing subsets, for example majorization.

The statements in Lemma 1 can be complemented with a characterization of the existence of strict monotone multi-utilities with the cardinality of an infinite set *I*, which we include in the following proposition.

Proposition 1 *If $(X, \preceq)$ is a preordered space and I is a set of infinite cardinality, then the following are equivalent.*

(i) *There exists a strict monotone multi-utility with the cardinality of I.*

(ii) *There exists a strict monotone and a multi-utility with the cardinality of I.*

(iii) *There exists a family of increasing sets $(A_i)_{i \in I}$ which separates x from y if $x \bowtie y$ and a countable set $I' \subseteq I$ such that $(A_i)_{i \in I'}$ separates x from y if $x \prec y$.*

Proof Clearly, (i) implies (ii) by definition. To show (ii) implies (iii), notice, given there is a multi-utility with the cardinality of *I*, we can follow the proof of Lemma 1 (i) in Alcantud et al. (2013) to show there exists a family of increasing sets with the cardinality of $I \times \mathbb{N}$, which is equinumerous to *I*, which separates *x* from *y* whenever $x \bowtie y$. We can similarly follow the proof of Lemma 1 (ii) in Hack et al. (2022a, b) to get, given there exists a strict monotone, there exists a countable family of increasing sets which separates *y* from *x* whenever $x \prec y$. Since $I \cup \mathbb{N}$ is equinumerous to *I*, we get the desired result. In order to show (*iii*) implies (*i*), we can again follow both Alcantud et al. (2013) and Alcantud et al. (2016) (Proposition 7). From the first one, we can construct a multi-utility $(u_i)_{i \in I}$ and, from the second one, we can construct a strict monotone *v*. Finally, we consider, as in Hack et al. (2022a, b) (Theorem 3.1), the family of monotones $(v_{i,n})_{i \in I, n \in \mathbb{N}}$ where $v_{i,n} := u_i + \alpha_n v$, where $(\alpha_n)_{n \in \mathbb{N}}$ is a numeration of the rational numbers which are greater than zero. This family can be

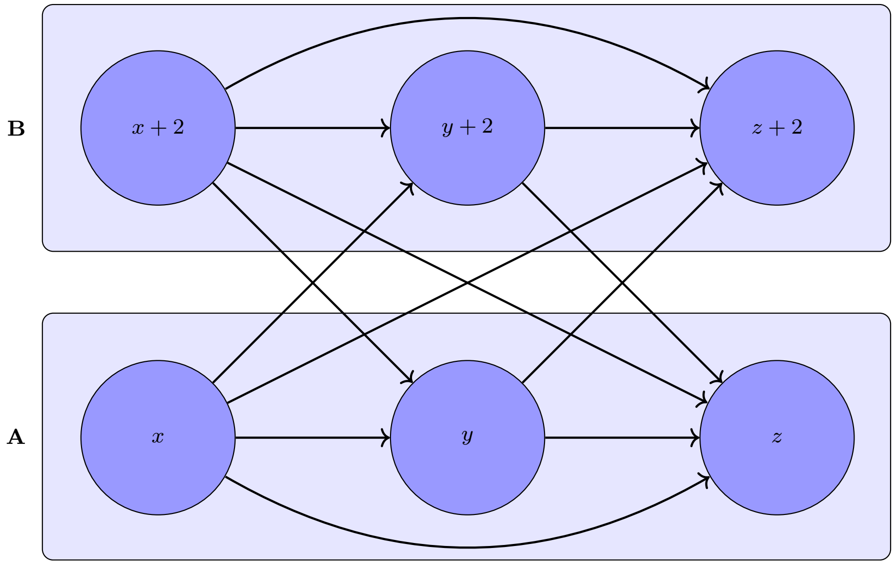

Fig. 3 Graphical representation of a preordered space, defined in Proposition 2, with the cardinality of the continuum and where strict monotones exist while injective monotone do not. In particular, we show $A := [0, 1]$, $B := [2, 3]$ and how $x, y, z \in A$, $x < y < z$, are related to $x + 2, y + 2, z + 2 \in B$. Notice, an arrow from an element w to an element t represents $w \prec t$

shown to be a strict monotone multi-utility and has the cardinality of $I \times \mathbb{N}$, which is the same as that of I. □

Notice, in case the set I is finite, the relation between (*i*) and (*ii*) is addressed in Proposition 10.

3.2 Improving the classification of preorders

In this section, we present several results that improve on the classification of preorderes spaces in terms of real-valued monotones. When applicable, we include, right after the proof of the result, an interpretation in terms of either complexity, optimization or the interplay between them.

Let us begin with the relation between preorders which have strict monotones and those which have injective monotones. Clearly, an injective monotone is also a strict monotone, since $x \prec y$ and $f(x) = f(y)$ contradicts injectivity. There are, however, preordered spaces with strict monotones and without injective monotones, as was shown in Hack et al. (2022a, b) (Proposition 1). The argument there is purely in terms of cardinality, since, whenever injective monotones exist, we have $|X/\sim| \leq \mathfrak{c}$ with $\mathfrak{c}$ the cardinality of the continuum, but there are preordered spaces with strict monotones and $|X/\sim| = |\mathcal{P}(\mathbb{R})|$. We can, however, improve upon this by showing there are preordered spaces where $X/\sim$ has the cardinality of the continuum and strict monotones exist while injective monotones do not. We include such a preordered space in Proposition 2.

Proposition 2 *There are preordered spaces* $(X, \preceq)$ *where* $X/\sim$ *has the cardinality of the continuum* $\mathfrak{c}$ *and strict monotones exist while injective monotones do not.*

Proof Consider $X := [0,1] \cup [2,3]$ equipped with $\preceq$ where

$$x \preceq y \iff \begin{cases} x, y \in [0,1] \text{ and } x \le y, \\ x, y \in [2,3] \text{ and } x \le y, \\ x \in [0,1], y \in [2,3] \text{ and } x + 2 < y, \\ x \in [2,3], y \in [0,1] \text{ and } x - 2 < y \end{cases} \quad (2)$$

$\forall x, y \in X$ (see Fig. 3 for a representation of $\preceq$). Notice $(X, \preceq)$ is a preordered space and $v : X \to \mathbb{R}$ where $x \mapsto x$ if $x \in [0,1]$ and $x \mapsto x - 2$ if $x \in [2,3]$ is a strict monotone. We will show that any family $(A_i)_{i \in I}$, where $A_i \subseteq X$ is increasing $\forall i \in I$ and $\forall x, y \in X$ such that $x \bowtie y$ there exists some $i \in I$ such that either $x \notin A_i$ and $y \in A_i$ or $y \notin A_i$ and $x \in A_i$, is uncountable. Since the existence of some $(A_i)_{i \in I}$ with these properties and countable I is implied by the existence of an injective monotone by Lemma 1 (iii), we obtain that there is no injective monotone for X.

Let $(A_i)_{i \in I}$ be a family with the properties in the last paragraph and, for each $x \in [0,1]$, define $y_x := x + 2$. Since $x \bowtie y_x$ by definition, there exists some $A_x \in (A_i)_{i \in I}$ such that either $x \in A_x$ and $y_x \notin A_x$ or $y_x \in A_x$ and $x \notin A_x$. We fix such an A_x for each $x \in [0,1]$ and consider the map $f : [0,1] \to (A_i)_{i \in I}$, $x \mapsto A_x$. Consider some $x, z \in [0,1]$ such that $A_x = A_z$ and assume $x \neq z$. We show first the case where $z < x$ leads to contradiction. Assume first $x \in A_x$ and $y_x \notin A_x$. Then, since $A_x = A_z$, we either have $z \in A_x$ or $y_z \in A_x$. Both cases lead to contradiction, since we get $y_x \in A_x$ because A_x is increasing and we either have $z \prec y_x$ with $z \in A_x$ or $y_z \prec y_x$ with $y_z \in A_x$. We can proceed analogously if we assume $y_x \in A_x$, relying on the fact both $z \prec x$ and $y_z \prec x$ hold. In case we assume $x < z$, we also achieve a contradiction following the same argument but interchanging the role of x and z.

Thus, $x \neq z$ leads to a contradiction and by injectivity of f we get $|[0,1]| \le |(A_i)_{i \in I}|$. As a consequence, X has no injective monotone. □

Proposition 2 shows there are preference spaces for which optimization principles exist while injective ones do not. In particular, it shows it is sufficient to consider preference spaces with a continuum decision space to find such cases (notice, as shown in Hack et al. (2022a, b) (Proposition 5), preference spaces with a countable ground set always have injective optimization principles). Furthermore, it shows that, unlike when the decision space is countable, the existence of local injective optimization principles and that of global ones are not equivalent. Hence, provided the decision space is sufficiently large, differences between local and global optimization arise. Notice, the proof of Proposition 2 relies on the existence of connections between several elements, which allow us to assure the sets from Lemma 1 that separate the preorder differ when different elements inside certain sets are considered. This is the reason why a related preorder was used in Hack et al. (2022a, b) (Proposition 8) to show countable multi-utilities and injective monotones are not equivalent. One may think the trivial preorder on the real line $(\mathbb{R}, =)$ would have an injective monotone, the identity, and no countable multi-utility.[2] However, due to it being completely disconnected, $(\chi_{\le q}, \chi_{\ge q})_{q \in \mathbb{Q}}$ is a countable multi-utility, where $\chi_{\leq q}(x) := 1$ if $x \leq q$ and $\chi_{\leq q}(x) := 0$ otherwise, and $\chi_{\geq q}(x) := 1$ if $x \geq q$ and $\chi_{\geq q}(x) := 0$ otherwise.

While the existence of injective monotones implies the existence of multi-utilities with the cardinality of the continuum (in particular, composed of injective monotones), as we showed in Hack et al. (2022a, b) (Proposition 4), the converse was unknown up to now. The preordered space in Proposition 2 shows the converse in false. Actually, it shows the stronger statement that the existence of strict monotone multi-utilities with cardinality $\mathfrak{c}$ is still not sufficient for the existance of an injective monotone, as we state in Corollary 1.

Corollary 1 *There are preordered spaces which have strict monotone multi-utilities with cardinality* $\mathfrak{c}$ *and no injective monotone.*

Proof We can use the counterexample from Proposition 2 which has no injective monotone. Moreover, it is straightforward to see that $(\chi_{i(x)})_{x \in X}$ is a multi-utility with cardinality $\mathfrak{c}$ (Evren & Ok, 2011), where χ_A is the indicator function of a set A and $i(x) := \{y \in X | x \preceq y\}$ $\forall x \in X$. Since there exist strict monotones, as we showed in the proof of Proposition 2, we can follow Proposition 1 and get that there exist strict monotone multi-utilities with cardinality $\mathfrak{c}$. □

Corollary 1 states, by Proposition 1, that the existence of optimization principles for preference spaces with continuum dimension is not enough for injective optimization principles to exist. This differs from the case of countable complexity, where injective optimization principles always exist (Hack et al., 2022a, b). Thus, whenever the preference space is sufficiently involved, local and global injective optimization principles do not coincide in general. Notice, Corollary 1 implies the class of preorders with injective monotones is strictly contained inside the class where multi-utilities with cardinality $\mathfrak{c}$ exist. In fact, we can improve upon this modifying the preorder in Proposition 2 to show there are preordered spaces where multi-utilities with cardinality $\mathfrak{c}$ exist while strict monotones do not. We present such a preorder in Proposition 3, which is the same as the one in Proposition 2 with the exception that we have $x \prec y_x$ instead of $x \bowtie y_x$ $\forall x \in [0, 1]$.

Proposition 3 *There are prerordered spaces which have multi-utilities with cardinality* $\mathfrak{c}$ *and no strict monotone.*

Proof Consider $X := [0, 1] \cup [2, 3]$ equipped with $\preceq$ where

$$x \preceq y \iff \begin{cases} x, y \in [0, 1] \text{ and } x \leq y, \\ x, y \in [2, 3] \text{ and } x \leq y, \\ x \in [0, 1], y \in [2, 3] \text{ and } x + 2 \leq y \\ x \in [2, 3], y \in [0, 1] \text{ and } x - 2 < y \end{cases} \tag{3}$$

$\forall x, y \in X$ (see Fig. 4 for a representation of $\preceq$), which differs from Eq. (2) only in $x + 2 \leq y$ instead of $x + 2 < y$ for $x \in [0, 1]$ and $y \in [2, 3]$. Notice $(X, \preceq)$ is a pre-ordered space and there is a multi-utility with cardinality $\mathfrak{c}$ as in the proof of

[2] We say a binary relation $\preceq$ on a set X is a *trivial ordering* if $x \preceq y \iff x = y$ $\forall x, y \in X$.

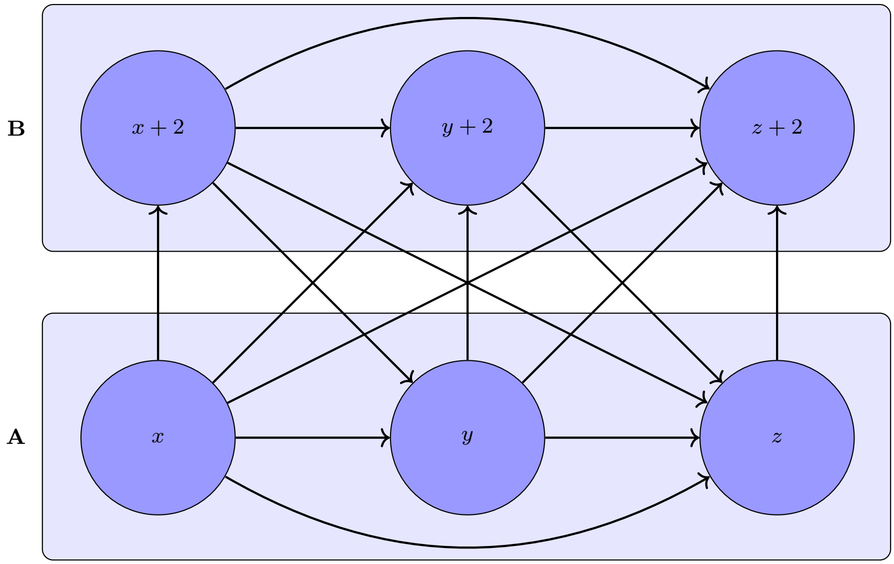

Fig. 4 Graphical representation of a preordered space, defined in Proposition 3, which has multi-utilities with cardinality $\mathfrak{c}$ and no strict monotone. In particular, we show $A := [0,1]$, $B := [2,3]$ and how $x, y, z \in A$, $x<y<z$, are related to $x+2, y+2, z+2 \in B$. Notice, an arrow from an element w to an element t represents $w \prec t$

Corollary 1. We will show that any family $(A_i)_{i\in I}$, where $A_i \subseteq X$ is increasing $\forall i \in I$ and $\forall x, y \in X$ such that $x \prec y$ there exists some $i \in I$ such that $y \in A_i$ and $x \notin A_i$, is uncountable. Since the existence of some $(A_i)_{i\in I}$ with these properties and countable I is implied by the existence of a strict monotone by Lemma 1 (ii), we conclude that there is no strict monotone for X.

Let $(A_i)_{i\in I}$ be a family with the properties in the last paragraph and, for each $x \in [0,1]$, define $y_x := x+2$. Since $x \prec y_x$ by definition, there exists some $A_x \in (A_i)_{i\in I}$ such that both $y_x \in A_x$ and $x \notin A_x$ hold. We fix such an A_x for each $x \in [0,1]$ and consider the map $f : [0,1] \to (A_i)_{i\in I}$, $x \mapsto A_x$. Consider some $x, z \in [0,1]$ such that $A_x = A_z$ and assume $x \neq z$. We show first the case where $z<x$ leads to contradiction. Since $A_x = A_z$, we have $y_z \in A_x$. Given the fact A_x is increasing and $y_z \prec x$ by definition, we get $x \in A_x$, a contradiction. In case we assume $x<z$, we also achieve a contradiction following the same argument but interchanging the role of x and z. Thus, $x \neq z$ leads to contradiction and we get, by injectivity of f, $|[0,1]| \leq |(A_i)_{i\in I}|$. As a consequence, X has no strict monotone. □

The relation between optimization and complexity is improved in Proposition 3, where we show that having a continuum dimension is not sufficient for optimization principles to exist, which contrasts with the case where the dimension is countable (Alcantud et al., 2016). In summary, no optimization principle may exist provided the complexity of the preference space is large enough (see, also, Corollary 4). Notice, essentially, we recover in Proposition 3 the *lexicographic plane*, the classical counterexample used by Debreu (1954, 1959) to show the existence of total preordered spaces without utility functions. Another counterexample, which relies on Szpilrajn extension theorem, can be found in Hack et al. (2022a, b) (A.2.1). Notice, in particular, Proposition 3 implies that the class of preordered spaces with strict monotone multi-utilities with cardinality $\mathfrak{c}$ is strictly contained inside the class with multi-utilities of the same cardinality. This contrasts with the fact that countable multi-utilites and countable strict monotone multi-utilities coincide for any preordered space (Alcantud et al., 2016, Proposition 4.1). In fact, they also coincide with countable injective monotone multi-utilities (Hack et al., 2022a, b, Proposition 6). Notice, also, the preordered space in Proposition 3 shows the stronger fact that strict monotones do not always exist when $X/\sim$ has cardinality $\mathfrak{c}$, as we state in Corollary 2.

Corollary 2 *There are preordered spaces $(X, \preceq)$ where $X/\sim$ has cardinality $\mathfrak{c}$ and strict monotone multi-utilities with cardinality $\mathfrak{c}$ do not exist.*

Proof Consider the preordered space in Proposition 3. Notice, since $(\chi_{i(x)})_{x \in X/\sim}$ is a multi-utility of cardinality $\mathfrak{c}$, strict monotone multi-utilities of cardinality $\mathfrak{c}$ and strict monotones are equivalent, by Proposition 1. Thus, they do not exist. □

Corollary 2 states that we can strengthen the bound in Proposition 3 from the dimension to the decision space and, nonetheless, optimization principles do not exist. That is, optimization principles may not exist provided the amount of alternatives in the decision space is sufficiently large. Notice, if $X/\sim$ is countable, then it has countable multi-utilities (we can follow the proof in Corollary 1) and, by Alcantud et al. (2016) (Theorem 3.1), countable strict monotone multi-utilities. Furthermore, we can follow Corollary 1 and Proposition 1 to conclude that every preorder with strict monotones has strict monotone multi-utilities with the cardinality of some infinite set I if $X/\sim$ has the cardinality of I. As we show in Proposition 4, the converse is not true, that is, whenever multi-utilities with the cardinality of an inifinite set I exist, we have $|X/\sim| \leq |\mathcal{P}(I)|$ with some preorders achieving equality. Furthermore, the bound cannot be improved even when strict monotone multi-utilities with the cardinality of I exist. Equivalently, we show whenever an infinite Debreu upper dense subset $I \subseteq X$ exists, then we have $w(X, \preceq) \leq |\mathcal{P}(I)|$ where $w(X, \preceq)$ is the *width* of $(X, \preceq)$, that is, the maximal cardinality of the antichains[3] in X. Recall we say a subset $Z \subseteq X$ is *upper dense in the sense of Debreu* (or *Debreu upper dense* for short) if $x \bowtie y$ implies that there exists a $z \in Z$ such that $x \bowtie z \preceq y$ (Hack et al., 2022a, b).[4]

Proposition 4 *Let $(X, \preceq)$ be a preordered space and I be an infinite set.*

(i) *If there exist multi-utilities with the cardinality of I, then $|X/\sim| \leq |\mathcal{P}(I)|$, where $\mathcal{P}(I)$ denotes the power set of I. Furthermore, the bound is sharp, i.e. it cannot be improved.*

---

[3] Any two elements in an antichain are incomparable.

[4] Notice, for a fixed pair $x, y \in X$ where $x \bowtie y$ holds, there exist $z_1, z_2 \in Z$ such that $x \bowtie z_1 \preceq y$ and $y \bowtie z_2 \preceq x$.

(ii) *Even if there exist strict monotone multi-utilities with the cardinality of* I, *the bound in* (i) *is sharp.*

(iii) If $I \subseteq X$ *is a Debreu upper dense subset, then* $w(X, \preceq) \leq |\mathcal{P}(I)|$, *where* $w(X, \preceq)$ *is the width of* $(X, \preceq)$. *Furthermore, the bound is sharp.*

Proof (i) For the first statement, notice, by Lemma 1 (i), there exists a family of increasing sets $(A_i)_{i \in I}$ that $\forall x, y \in X$ with $x \prec y$ separates x from y and $\forall x, y \in X$ with $x \bowtie y$ separates both x from y and y from x. Consider the map $f : X/\sim \; \to \mathcal{P}(I)$, $[x] \mapsto B_x$ where $B_x := \{i \in I | [x] \subseteq A_i\}$. If $[x] \neq [y]$, then we either have $x \bowtie y$, $x \prec y$ or $y \prec x$ $\forall x \in [x], y \in [y]$. In any case, there exists some $i \in I$ such that $x \subseteq A_i$ and $y \not\subseteq A_i$ and vice versa. Thus, $B_x \neq B_y$ and f is injective. We get $|X/\sim| \leq |\mathcal{P}(I)|$.

For the second statement, consider the set $X := \mathcal{P}(I)$ equipped with the preorder $\subseteq$, where $\subseteq$ denotes set inclusion. One can see $(f_i)_{i \in I}$ is a multi-utility for X, where $f_i : \mathcal{P}(I) \to \mathbb{R}$, $U \mapsto 1$ if $i \in U$ and $U \mapsto 0$ otherwise. Notice we have $|\mathcal{P}(I)/\sim| = |\mathcal{P}(I)|$. Thus, the bound in the first statement cannot be improved.

(ii) Consider $X := \mathcal{P}(I)$ equipped with the trivial ordering $\preceq$. Notice $(f_i)_{i \in I} \cup (g_i)_{i \in I}$ is a strict monotone multi-utility with the cardinality of $I \cup I$, which is equinumerous to I, where $f_i : X \to \mathbb{R}$, $U \mapsto 1$ if $i \in U$ and $U \mapsto 0$ otherwise and $g_i := -f_i$, and we also have $|\mathcal{P}(I)/\sim| = |\mathcal{P}(I)|$. Thus, the bound in (i) cannot be improved.

(iii) Consider A an antichain of X and, for each $x \in A$, $I_x := \{i \in I | i \preceq x\}$. We will show the map $f : A \to \mathcal{P}(I)$, $x \mapsto I_x$ is injective, proving, thus, any antichain A fulfills $|A| \leq |\mathcal{P}(I)|$ which leads to $w(X, \preceq) \leq |\mathcal{P}(I)|$. Given $x, y \in A$, $x \neq y$, we have $x \bowtie y$ and, by Debreu upper density of I, there exists some $i \in I$ such that $x \bowtie i \preceq y$. As a consequence, $i \in I_y$ and $i \notin I_x$. Resulting in $f(x) = I_x \neq I_y = f(y)$ and, hence, in f being injective.

For the second statement, consider the preorder $(\Sigma^* \cup \Sigma^\omega, \preceq)$ where $\Sigma := \{0, 1\}$, Σ^* is the set of finite sequences over Σ, Σ^ω is the set of infinite sequences over Σ, if $x \in \Sigma^*$ and $y \in \Sigma^\omega$ then $x \preceq_C y$ if x is a prefix of y and $\preceq$ is defined $\forall x, y \in \Sigma^* \cup \Sigma^\omega$ like

$$x \preceq y \iff \begin{cases} x = y \\ x \preceq_C y. \end{cases}$$

Notice Σ^* is a countable Debreu upper dense subset and we have $w(X, \preceq) = |\Sigma^w| = |\mathcal{P}(\Sigma^*)|$. Thus, the bound in the first statement cannot be improved. □

Primarily, Proposition 4 shows how, whenever they are infinite, bounds on the dimension of a preference space result in bounds on the number of commodities in the decision space and, moreover, it shows that these bounds are optimal, since they are achieved by some preference spaces. Notice, while the analog of (iii) remains a question whenever I is a finite set, the bounds in both (i) and (ii) do not hold. Although a trivial example supporting this assertion would be the real line with its usual order $(\mathbb{R}, \leq)$, since the identity is a strict monotone finite multi-utility, we conclude this paragraph including two, perhaps, more interesting counterexamples.

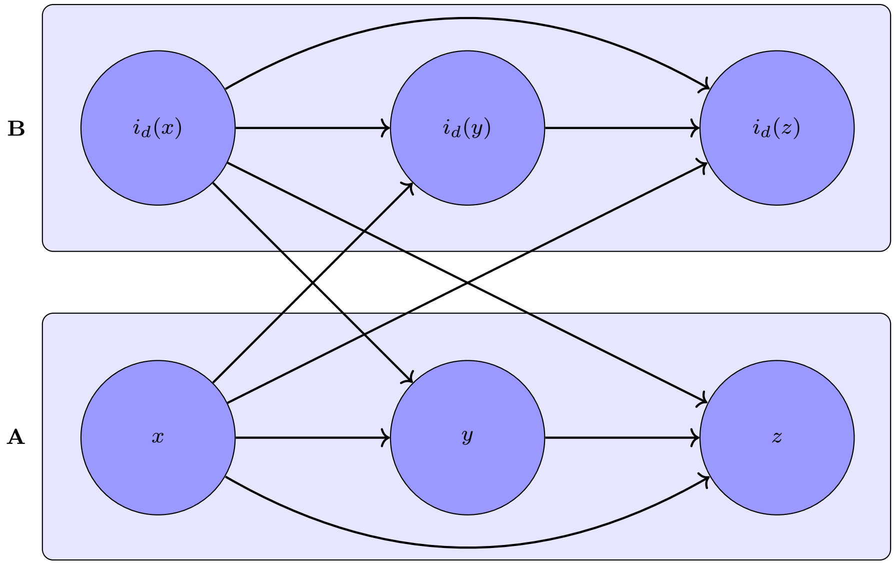

Fig. 5 Representation of a preordered space, defined by Eq. (4), where finite strict monotone multi-utilities exist and $|X/\sim| = \mathfrak{c}$. In particular, we relate three different points $x, y, z \in A$ with $i_d(x), i_d(y), i_d(z) \in C$, where $A, B := \mathbb{R}/\{0\}$, $x<0<y<z$ and $i_d : A \to B$ is the identity on $\mathbb{R} \setminus \{0\}$. Notice an arrow from an element w to an element t represents $w \prec t$

Majorization also proves (i) is false when I is finite, since, although it is defined through Eq. (1), it fulfills $|\mathbb{P}_\Omega/\sim_U| = \mathfrak{c}$. We adapt from Bosi et al. (2020) (Example 2) a preorder which also illustrates that (ii) is false when I is finite. In particular, we take $X := A \cup B$, where A and B are two copies of $\mathbb{R} \setminus \{0\}$, and equip them with $\preceq$ where

$$x \preceq y \iff \begin{cases} x, y \in A \text{ and } x \leq y, \\ x, y \in B \text{ and } x \leq y, \\ x \in A, x<0, y \in B \text{ and } 0<y, \\ x \in B, x<0, y \in A \text{ and } 0<y \end{cases} \tag{4}$$

$\forall x, y \in X$ (see Fig. 5 for a representation of $\preceq$). Note $|X/\sim| = \mathfrak{c}$ and $V := \{v_1, v_2\}$ is a finite strict monotone multi-utility, where $v_1(x) := x - 1$ if $x \in A$ and $x<0$, $v_1(x) := \mathrm{e}^x - 1$ if $x \in B$ and $x<0$, $v_1(x) := 1 - \mathrm{e}^{-x}$ if $x \in B$ and $x > 0$ and $v_1(x) := x + 1$ if $x \in A$ $x > 0$, and $v_2(x) := x - 1$ if $x \in B$ and $x<0$, $v_2(x) := \mathrm{e}^x - 1$ if $x \in A$ and $x<0$, $v_2(x) := 1 - \mathrm{e}^{-x}$ if $x \in A$ and $x > 0$ and $v_2(x) := x + 1$ if $x \in B$ and $x > 0$.

Proposition 4 improves the relation between the existence of multi-utilities and the cardinality of $X/\sim$. In particular, whenever we have $|X/\sim| \leq \mathfrak{c}$, then there exist multi-utilities with cardinality $\mathfrak{c}$ (Evren & Ok, 2011) (see Corollary 1) and, whenever injective monotones exist, we have $|X/\sim| \leq \mathfrak{c}$. However, there are preorders where, although $|X/\sim| \leq \mathfrak{c}$ holds, injective monotones do not exist, like the one in Proposition 2. Furthermore, there are preorders with $|X/\sim| \leq \mathfrak{c}$ where strict

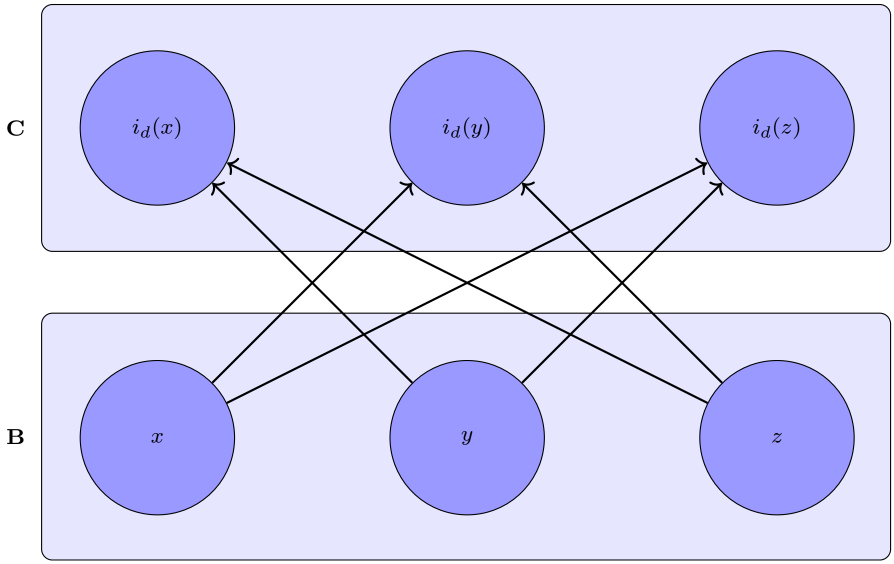

Fig. 6 Representation of a preordered space, defined in Proposition 5, where strict monotones exist and multi-utilities with the cardinality of I, an uncountable set, do not. In particular, we relate three different points $x, y, z \in B$ with $i_d(x), i_d(y), i_d(z) \in C$, where $B, C := \mathcal{P}(I)$ and and $i_d : B \to C$ is the identity on $\mathcal{P}(I)$. Notice an arrow from an element w to an element t represents $w \prec t$. Notice, also, this preorder is, essentially, the one we introduced in Hack et al. (2022a, b) (Proposition 8) with a larger ground set

monotones do not exist, like the example in Corollary 2. Finally, there exist preorders with strict monotone multi-utilities with cardinality $\mathfrak{c}$ and where $|X/\sim| > \mathfrak{c}$, like the one in Proposition 4 (ii).

Returning to Proposition 3, notice its converse also holds, that is, there are preordered spaces where strict monotones exist and multi-utilities with cardinality $\mathfrak{c}$ do not. In general, for any uncountable set I, there exist preordered spaces where strict monotones exist and multi-utilities with the cardinality of I do not, as we show in Proposition 5. Notice the counterexample we present is, essentially, the one we introduced in Hack et al. (2022a, b) (Proposition 8), but for a larger ground set. Despite the large ground set, however, the proof is constructive.

Proposition 5 *If I is an uncountable set, then there exist preordered spaces with strict monotones and without multi-utilities with the cardinality of I.*

Proof Consider $X := B \cup C$, where B and C are two copies of $\mathcal{P}(I)$, equipped with $\preceq$ where

$$x \preceq y \iff \begin{cases} x = y \\ x \in B, y \in C \text{ and } y \neq i_d(x) \end{cases} \tag{5}$$

$\forall x, y \in X$ with $i_d : B \to C$ the identity on $\mathcal{P}(I)$ (see Fig. 6 for a representation of $\preceq$). Notice $(X, \preceq)$ is a preordered space and $v : X \to \mathbb{R}$ $x \mapsto 0$ if $x \in B$ and $x \mapsto 1$ if $x \in C$ is a strict monotone. By Lemma 1 (i) there exists a family $(A_j)_{j \in J}$ of increasing subsets of X such that whenever $x \bowtie y$ there exists some $j \in J$ such that $x \in A_j$ and $y \notin A_j$. It is enough to show that such a family has a larger cardinality than I to see that there is no multi-utility for X with the cardinality of I.

Notice, $x \bowtie i_d(x)\ \forall x \in B$. There exists, thus, some $A_x \in (A_j)_{j\in J}$ such that $x \in A_x$ and $i_d(x) \notin A_x$. We fix such an A_x for each $x \in B$ and consider the map $f : B \to (A_j)_{j\in J}, x \mapsto A_x$. Consider a pair $x, z \in B$ such that $A_x = A_z$ and assume $x \neq z$. Since A_x is increasing, $z \prec i_d(x)$ and $z \in A_x$, we get $i_d(x) \in A_x$, a contradiction. Thus, $A_x = A_z$ implies $x = z$ and we have, by injectivity of f, $|\mathcal{P}(I)| = |B| \leq |(A_j)_{j\in J}|$. As a consequence, X has no multi-utility with the cardinality of I. □

Proposition 5 shows that the existence of optimization principles does not imply any bound on the dimension of the preference space in question. This does vary with respect to the case where injective optimization principles exist since, there, the dimension cannot surpass the continuum. Hence, although injective optimziation is not, optimization is possible in spaces of arbitrarily large complexity.

Notice, in particular, Proposition 5 shows there are preordered spaces with strict monotones and without strict monotone multi-utilities with cardinality $\mathfrak{c}$, which is not true for injective monotones (see (Hack et al., 2022a, b, Proposition 4). It also shows that the class of preordered spaces where multi-utilities with cardinality $\mathfrak{c}$ exist is strictly contained inside the class of preordered spaces with multi-utilities, which consists of all preordered spaces (Evren & Ok, 2011, Proposition 1). In fact, as we

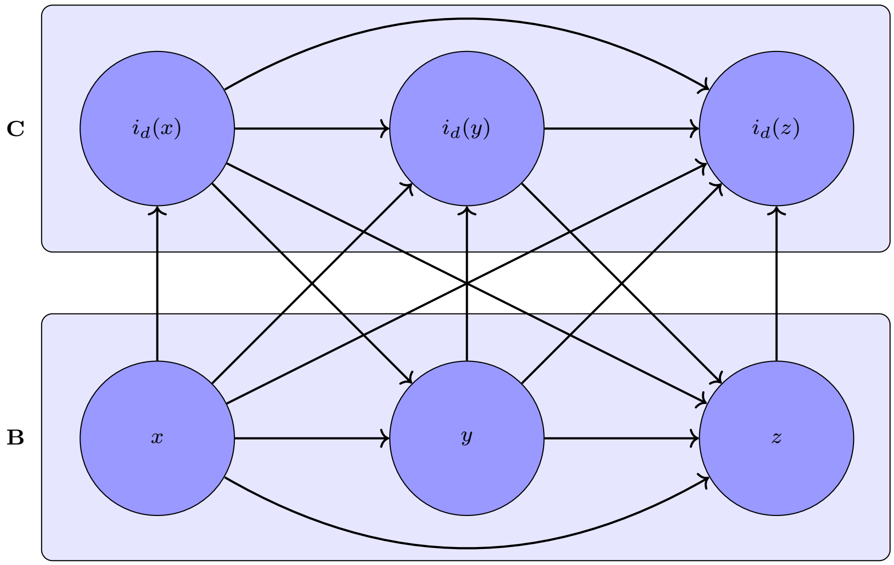

Fig. 7 Representation of a preordered space, defined in Proposition 6, which has no multi-utility with the cardinality of an uncountable set I and no strict monotone. In particular, we show how $x, y, z \in B$, $x \prec y \prec z$, are related to $i_d(x) \prec i_d(y) \prec i_d(z) \in C$ where $B, C := \mathcal{P}(I)$ and $i_d : B \to C$ is the identity on $\mathcal{P}(I)$. Notice an arrow from an element w to an element t represents $w \prec t$. Notice, also, this preorder is, essentially, the same as the one in the proof of Proposition 3 (see Fig. 4) with a larger ground set. As a result, we used a non-constructive argument relying on Szpilrajn extension theorem to define it

show in Proposition 6 through a variation of the preorder in Proposition 3, given an uncountable set I, there exist preordered spaces where neither strict monotones nor multi-utilities with the cardinality of I exist. Notice the proof of Proposition 6 relies on Szpilrajn extension theorem and, thus, is non-constructive.

Proposition 6 *If I is an uncountable set, then there exist preordered spaces where neither multi-utilities with the cardinality of I nor strict monotones exist.*

Proof Consider $X := B \cup C$, where B and C are two copies of $\mathcal{P}(I)$ and consider on both C and B the total order $\preceq_S$ that results from applying Szpilrajn extension theorem (Szpilrajn, 1930) to the partial order defined by set inclusion on $\mathcal{P}(I)$. Furthermore, equip X with $\preceq$ where

$$x \preceq y \iff \begin{cases} x \preceq_S y \text{ and } x, y \in B \\ x \preceq_S y \text{ and } x, y \in C \\ i_d(x) \preceq_S y, x \in B \text{ and } y \in C \\ x \prec_S i_d(y), x \in C \text{ and } y \in B \end{cases} \tag{6}$$

$\forall x, y \in X$ with $i_d : B \to C$ the identity on $\mathcal{P}(I)$ (see Fig. 7 for a representation of $\preceq$). Notice $(X, \preceq)$ is a preordered space. In analogy to Propositions 3 and 5, one can show that any family $(A_j)_{j \in J}$ of increasing subsets $A_j \subseteq X$ that separates x and y wehenver $x \prec y$ has larger cardinality than I. Since the existence of some $(A_j)_{j \in J}$ with those properties and $|J| \leq |I|$ is implied by both the existence of a multi-utility with the cardinality of I by Lemma 1 (i) and the existence of a strict monotone by Lemma 1 (ii), we obtain that there is no multi-utility with the cardinality of I nor a strict monotone for X. □

Proposition 6 shows that, for any cardinal, there are preorders where both optimization principles do not exist and the dimension is larger than the cardinal. On the contrary, whenever the complexity is countable, (injective) optimization principles always exist Hack et al. (2022a, b). Note that the preorder we introduced in Proposition 6 actually supports a stronger statement, which we include in Corollary 3. To prove it, we simply follow Proposition 6 and add the fact that, as in Corollary 1, $(\chi_{i(x)})_{x \in X}$ is a multi-utility with the cardinality of $\mathcal{P}(I)$.

Corollary 3 *If I is an uncountable set, then there exist preordered spaces where multi-utilities with the cardinality of $\mathcal{P}(I)$ exist, although neither strict monotones nor multi-utilities with the cardinality of I do.*

Regarding optimization and complexity, Corollary 3 can be interpreted as Proposition 6. To complement Propositions 5 and 6, we show, in Corollary 4, for any uncountable set I there exist preorders which have multi-utilities with the cardinality of I and no strict monotones. Notice, again, we follow the basic construction in Proposition 3, although we use the same non-constructive approach in Proposition 6.

Corollary 4 *If I is an uncountable set, then there exist preordered spaces which have multi-utilities with the cardinality of I and no strict monotone.*

Proof Consider $X := C \cup B$, where C and B are two copies of I, and equip it with a preorder analogous to the one in Proposition 6. Notice $(\chi_{i(x)})_{x \in X}$ is a multi-utility for X. By slightly modifying the argument in Proposition 6, we conclude there are no strict monotones. □

Corollary 4 shows that, unlike the countability ones (Alcantud et al. 2016; Hack et al. 2022a, b), uncountability restrictions on the dimension of a preorder have no effect in general on the existence of optimization principles. Moreover, as we show in Corollary 5, we can put together the preorders from Proposition 4 (i) and Corollary 4 to improve the relation between multi-utilities and the cardinality of $X/\sim$ even more.

Corollary 5 *If I is an uncountable set, then there exist preordered spaces where $|X/\sim| > |I|$ and multi-utilities with the cardinality of I exist, while strict monotones do not.*

Proof Take $X := A \cup B$ where A and is the ground sets of the preorder in Corollary 4 and B is the ground set of the preorder in Proposition 4 (i) without the empty set. We equip X with the preorder in Corollary 4 on A and that of Proposition 4 (i) on B, leaving $x \bowtie y$ $\forall x, y \in X$ such that $x \in A$ and $y \in B$ or vice versa. Since any strict monotone on X would also be a strict monotone on A, they do not exist by Corollary 4. Notice we have $|X/\sim| > |I|$, since $|B/\sim| > |I|$. Notice, also, $(g_i)_{i \in I} \cup (h_y)_{y \in A}$ is a multi-utility with the cardinality of I for X, where $\forall i \in I$ $g_i(x) := f_i(x)$ if $x \in B$ and $g_i(x) := 0$ if $x \in A$, with $(f_i)_{i \in I}$ defined as in Proposition 4 (i), and $\forall y \in A$ $h_y(x) := \chi_{i(y)}$ if $x \in A$ and $h_y(x) := 0$ if $x \in B$. □

Corollary 5 also deals with the connections between optimization and complexity, showing that, even if the uncountability restrictions on the dimension do not apply to the decision space, there still exist preference spaces with no optimization principle.

To finish this section, since we have been mainly concerned with preordered spaces with infinite multi-utilities and uncountable $X/\sim$, we address both finite multi-utilities and countable $X/\sim$. The first thing to notice is the existence of finite multi-utilities does not imply $X/\sim$ is countable. This is exemplified by majorization (Marshall et al., 1979; Arnold, 2018), since it is defined through a finite multi-utility Eq. (1) but the corresponding quotient space $\mathbb{P}_\Omega/\sim_U$ has the cardinality of the continuum. It is straightforward to see that, whenever $X/\sim$ is finite, there exists a finite multi-utility (see Corollary 1). However, as we show in Proposition 7, there exist preorders where $X/\sim$ is countably infinite and finite multi-utilities do not exist. Notice the preorder that supports this claim is, essentially, the same as the one in Proposition 5. However, in Proposition 7, we follow a simpler proof.

Proposition 7 *There are preordered spaces $(X, \preceq)$ where $X/\sim$ is countably infinite and no finite multi-utilities exist.*

Proof Consider $X := \mathbb{Z} \setminus \{0\}$ equipped with $\preceq$ where

$$n \preceq m \iff \begin{cases} n = m \\ n > 0, m < 0 \text{ and } n \neq -m. \end{cases}$$

Notice $(\mathbb{Z} \setminus \{0\})/\sim$ is countable. Assume there exists a finite multi-utility $(u_i)_{i=1}^k$. Notice for any pair $n, -n$ we have $n \bowtie -n$ and there must be some i_n such that $u_{i_n}(-n) < u_{i_n}(n)$ by definition of multi-utility. If we consider, however, some $m \neq n$, then we have $u_{i_n}(m) \leq u_{i_n}(-n) < u_{i_n}(n) \leq u_{i_n}(-m)$. Thus, $i_m \neq i_n$. Considering w.l.o.g. $i_n = n$, we get $u_i(k+1) < u_i(-(k+1))$ for $i = 1, .., k$. Thus, there is no multi-utility of cardinality k for any $k < \infty$. □

Proposition 7 shows that countability restrictions on the decision space do not necessarily imply finite bounds on the dimension of the preference space and, thus, on its complexity. As a result of Proposition 7, there are preorders where countably infinite multi-utilities exist while finite ones do not. The preorder we used had, however, a countable $X/\sim$. We, therefore, complement this statement by showing in Proposition 8 that there are preorders with the same characteristics but uncountable $X/\sim$.

Proposition 8 *There are preordered spaces $(X, \preceq)$ where $X/\sim$ is uncountable and, although countable multi-utilities exist, finite multi-utilities do not.*

Proof Let $\mathcal{P}_{inf}(\mathbb{N})$ be the set of infinite subsets of $\mathbb{N}$. Consider $X := (\mathbb{N} \cup \mathcal{P}_{inf}(\mathbb{N}), \preceq)$ equipped with the preorder $\preceq$

$$x \preceq y \iff \begin{cases} x = y \\ x \in \mathbb{N}, y \in \mathcal{P}_{inf}(\mathbb{N}) \text{ and } x \in y \end{cases}$$

$\forall x, y \in X$. Clearly, $|X/\sim| = \mathfrak{c}$, thus uncountable.

One can see $U := (u_n, v_n)_{n \geq 0}$ is a countable multi-utility, where $u_n(x) := 1$ if $x = n$ or $n \in x \in \mathcal{P}_{inf}(\mathbb{N})$ and $u_n(x) := 0$ otherwise, and $v_n(x) := 1$ if $n \notin x$ and $x \in \mathcal{P}_{inf}(\mathbb{N})$ and $u_n(x) := 0$ otherwise. Notice if $x \preceq y$ and $x \neq y$ then $x \in \mathbb{N}$ and $x \in y$. Thus, $u(x) \leq u(y)$ $\forall u \in U$. Assume now we have $\neg(x \preceq y)$. If $y \prec x$, then $y \in \mathbb{N}$ and $x \in \mathcal{P}_{inf}(\mathbb{N})$. Thus, there exists $m \in x$ such that $m \neq y$ and $u_m(y) < u_m(x)$. If $x \bowtie y$, then we consider four cases. If $x, y \in \mathbb{N}$, then $u_x(x) > u_x(y)$. If $x, y \in \mathcal{P}_{inf}(\mathbb{N})$, then, if there exists $n \in x/y$, we have $u_n(x) > u_n(y)$. Otherwise, there exists $n \in y/x$ and we have $v_n(x) > v_n(y)$. If $x \in \mathbb{N}$ and $y \in \mathcal{P}_{inf}(\mathbb{N})$, then $x \notin y$ and we have $u_x(x) > U_x(y)$. If $y \in \mathbb{N}$ and $x \in \mathcal{P}_{inf}(\mathbb{N})$, then $y \notin x$ and we have $v_y(x) > v_y(y)$.

To conclude, we show there is no finite multi-utility. Let $A_0 \subseteq \mathcal{P}_{inf}(\mathbb{N})$, fix some $k \in \mathbb{N}$ and consider $(b_i)_{i=1}^{k+1} \subseteq A_0$, where $b_i \neq b_j$ if $i \neq j$, and $(A_i)_{i=1}^{k+1}$, where $A_i := A_0/b_i$ for $i = 1, .., k+1$. Notice $(b_i, A_i)_{i=1}^{k+1}$ is a finite portion of the preorder in Proposition 7, since we have $b_i \preceq A_j$ if and only if $i \neq j$, and we can argue analogously as we did there that no multi-utility with cardinality k exists. Since k is arbitrary, we obtain there is no finite multi-utility. □

Proposition 8 proves we can relax the restriction on the decision space in Proposition 7 from countable to uncountable and still find preference spaces with countably infinite dimension.

The preorder we introduced in Proposition 8 can, in fact, be used to improve the relation between real-valued monotones characterization of preorders and order density properties. A subset $Z \subseteq X$, such that $x \prec y$ implies that there exists $z \in Z$ with $x \preceq z \preceq y$ is called *order dense in the sense of Debreu* (or *Debreu dense* for short) (Ok, 2002; Bridges & Mehta, 2013). Accordingly, we say that $(X, \preceq)$ is *Debreu separable* (Mehta, 1986) if there exists a countable Debreu dense set in $(X, \preceq)$. Similarly, $(X, \preceq)$ is called *Debreu upper separable* if there exists a countable subset which is both Debreu dense and Debreu upper dense (Hack et al., 2022a, b) (we defined Debreu upper dense subsets right before Proposition 4). As was shown in Hack et al. (2022a, b) (Proposition 9), Debreu upper separable preorders have countable multi-utilities. However, there exist preorders which have countable multi-utilities but are not Debreu separable, like majorization for $|\Omega| \geq 3$ (see Hack et al., 2022a, b) (Lemma 5 (ii)). In Proposition 9, we complement these results by showing a preorder where countable multi-utilities exist and countable Debreu upper dense subsets do not. In particular, we show the preorder we introduced in Proposition 8 has no countable Debreu upper dense subsets although, as we showed there, it has countable multi-utilities. Notice, a preorder where the weaker fact that injective monotones exist and countable Debreu upper dense subsets do not can be found in Hack et al. (2022a, b) (Proposition 8). There, an injective monotone was introduced and, although it was shown no countable multi-utility exists, it is easy to see any Debreu upper dense subset would be uncountable.

Proposition 9 *There are preordered spaces where countable multi-utilities exist and every Debreu upper dense subset is uncountable.*

Proof Consider the preorder $X := (\mathbb{N} \cup \mathcal{P}_{inf}(\mathbb{N}), \preceq)$ from Proposition 8. As we showed there, countable multi-utilities exist. Assume there exists a Debreu upper dense subset $D \subseteq X$. Consider $y \in \mathcal{P}_{inf}(\mathbb{N})$, $y \neq \mathbb{N}$. Notice there exists some $n_y \in \mathbb{N}/y$ and $y \cup \{n_y\} \bowtie y$. Since D is Debreu upper dense, there exists some $d \in D$ such that $y \cup \{n_y\} \bowtie d \preceq y$. Since $d \preceq y$ implies either $d \in y$ or $d = y$, and $d \in y$ implies $d \in y \cup \{n_y\}$, thus $d \preceq y \cup \{n_y\}$ contradicting the definition of d, we have $d = y$. As a result, $\mathcal{P}_{inf}(\mathbb{N})/\{\mathbb{N}\} \subseteq D$ and D is uncountable. □

Although usually expressed as an equivalence, for total preorders, between the existence of utility functions and that of countable Debreu dense subsets, the classical result by Debreu (see (Bridges & Mehta, 2013, Theorem 1.4.8) can, alternatively, be stated as the equivalence between the existence of countable multi-utilities and that of countable Debreu separable subsets. From this perspective, the interest in the result lies in the fact that the existence of a countable family of increasing sets that separate (in the sense of Lemma 1) the elements in a preorder results in a countable subset of elements with this separation (in the sense of order density) property. Since the natural extension of Debreu separability to non-total preorders is Debreu upper separability, Proposition 9, together with Hack et al. (2022a, b)

(Proposition 8), shows that the transition from separating sets to elements does not hold in general for non-total preorders.

Notice, although they coincide when they are countable (see (Alcantud et al., 2016, Proposition 4.1; Hack et al., 2022a, b, Proposition 6), it remains an open question how the different sorts of multi-utilities relate to each other when they are finite. As a first result in this direction, we finish with a characterization of preordered spaces with finite injective monotone multi-utilities.

Proposition 10 *If $(X, \preceq)$ is a preordered space, then the following are equivalent:*

(i) *There exists a finite multi-utility $(u_i)_{i \leq N}$ such that the image of the non-injective set*

$$I_{u_i} := \{r \in \mathbb{R} | \exists x, y \in X \text{ such that } x, y \in u_i^{-1}(r) \text{ and } \neg(x \sim y)\} \tag{7}$$

is countable $\forall i \leq N$.

(ii) *There exists a finite injective monotone multi-utility $(v_i)_{i \leq N}$.*

Proof By definition, given an injective monotone multi-utility $(v_i)_{i \leq N}$, we have $I_{v_i} = \emptyset \; \forall i \leq N$. Conversely, consider $u \in (u_i)_{i \leq N}$ a monotone such that the image of its non-injective set I_u is countable. Take $(r_n)_{n \geq 0}$ a numeration of I_u, $(y_n)_{n \geq 0} \subseteq X$ a set such that $u(y_n) = r_n \; \forall n \geq 0$ and, w.l.o.g., an injective monotone $c_0 : X \to (0, 1)$. Notice injective monotones exist under the hypotheses, as we showed in Hack et al. (2022a, b) (Proposition 5). Define, then,

$$w_0(x) := \begin{cases} u(x) & \text{if} \quad u(x) < r_0 \\ u(x) + c_0(x) & \text{if} \quad u(x) = r_0 \\ u(x) + 1 & \text{else.} \end{cases}$$

$\forall x \in X$. Notice $I_{w_0} \subset I_u$, since $x_0 \notin I_{w_0}$, and we have both $u(x) \leq u(y)$ implies $w_0(x) \leq w_0(y)$ and $u(x) < u(y)$ implies $w_0(x) < w_0(y) \; \forall x, y \in X$. Similarly, consider a family of injective monotones $(c_n)_{n \geq 1}$ such that $c_n : X \to (0, 2^{-n})$ for $n \geq 1$ and define, also for $n \geq 1$,

$$w_n(x) := \begin{cases} w_{n-1}(x) & \text{if} \quad w_{n-1}(x) < w_{n-1}(y_n) \\ w_{n-1}(x) + c_n(x) & \text{if} \quad w_{n-1}(x) = w_{n-1}(y_n) \\ w_{n-1}(x) + 2^{-n} & \text{else} \end{cases}$$

$\forall x \in X$. Notice $I_{w_{n-1}} \subset I_{w_n}$ holds $\forall n \geq 1$, since $x_n \notin I_{w_n}$, and we have both $w_{n-1}(x) \leq w_{n-1}(y)$ implies $w_n(x) \leq w_n(y)$ and $w_{n-1}(x) < w_{n-1}(y)$ implies $w_n(x) < w_n(y) \; \forall x, y \in X$. Lastly, consider the pointwise limit $v(x) := \lim_{n \to \infty} w_n(x)$. Noitce v is well-defined and, also, an injective monotone, since $I_v = \emptyset$ by construction.

Following the same procedure for each monotone in $(u_i)_{i \leq N}$, we get a family of injective monotones $(v_i)_{i \leq N}$. To conclude it is a multi-utility, we need to show, $\forall x, y \in X$ with $\neg(x \preceq y)$, there exists some $i \leq N$ such that $v_i(x) > v_i(y)$. If $y \prec x$, then $v_i(x) > v_i(y) \; \forall i \leq N$ by definition of injective monotone. Otherwise, if $x \bowtie y$, there exists some $i \leq N$ such that $u_i(x) > u_i(y)$. Thus, we also have $v_i(x) > v_i(y)$. Hence, $(v_i)_{i \leq N}$ is a multi-utility. □

Notice we can weaken the hypothesis, assuming, instead of (7), that

$$\{r \in \mathbb{R} | \exists x, y \in X \text{ such that } x, y \in u_i^{-1}(r) \text{ and } x \prec y\}$$

is countable $\forall i \leq N$, to conclude, analogously, that the existence of finite multi-utilities and that of finite strict monotone multi-utilities are equivalent. Notice, as a result, we obtain that the existence of finite multi-utilities coincides with that of finite strict monotone multi-utilities and that of finite injective monotone multi-utilities whenever $X/\sim$ is countable. The general case where $X/\sim$ is uncountable (in particular, when $|X/\sim| \leq \mathfrak{c}$ since, otherwise, there are no injective monotones), remains open. This is due to the fact the technique in Proposition 10 cannot be used and I_{u_i} is not necessarily countable $\forall i \leq N$, as one can see in majorization, for example. There, taking u_i as in Eq. (1), we have $(\frac{i}{|\Omega|}, 1) \subseteq I_{u_i}$ $\forall i \leq |\Omega| - 1$ and, thus, I_{u_i} is uncountable $\forall i \leq |\Omega| - 1$. Notice, also, the technique in Proposition 10 is similar to the one we used in Hack et al. (2022a, b) (Proposition 2), where we showed the existence of an injective monotone is equivalent to that of a strict monotone f whose non-injective set

$$\{x \in X | \exists y \in X \text{ such that } f(x) = f(y) \text{ and } x \bowtie y\}$$

is countable. Notice the hypothesis there is stronger, since the assumption that the image of the non-injective set I_f is countable is insufficient, as one can see using the preorder in Hack et al. (2022a, b) (Proposition 1 (i)).

The technique in Proposition 10 can actually be used to prove that countable multi-utilities and countable injective monotone multi-utilities always coincide (see Hack et al., 2022a, b, Proposition 6). The only detail of importance is, whenever a countable multi-utility exists, there exists, by Lemma 1 (i), a countable family of increasing sets $(A_n)_{n \geq 0}$ that $\forall x, y \in X$ with $x \prec y$ separates x from y and $\forall x, y \in X$ with $x \bowtie y$ separates both x from y and y from x. In particular, $(\chi_{A_n})_{n \geq 0}$ is a countable multi-utility with the property that $I_{\chi_{A_n}}$ is finite $\forall n \geq 0$. Since injective monotones exist, we can follow Proposition 10 to construct a countable injective monotone multi-utility.

4 Discussion

In this work, we have improved the classification of preordered spaces through real-valued monotones in terms of the cardinality of multi-utilities and quotient spaces, c. f. Fig. 1, and, as a result, we have contributed to the study of complexity, optimization and their relation in preorderd spaces.

4.1 Classification of preorderd spaces through real-valued monotones

The state of the classification of preordered spaces in terms of real-valued monotones can be found in Fig. 1, whereas our contributions are shown in Fig. 2. In this paragraph, we summarize the relation between the different classes and distinguish between our results and the ones in the literature. We will begin from the innermost class, preorders with utility functions, and finish with the outermost class, which contains all preorder (Evren & Ok, 2011), that is, peorders with multi-utilities.

The relation between utility functions and the subsequent classes, finite multi-utilities and preorders with countable $X/\sim$ is as follows. A utility function is a finite multi-utility, although there are prerodered spaces where finite multi-utilities exist and utilities do not, like majorization (Arnold, 2018; Marshall et al., 1979). We can also use majorization to show there are preorders with a finite multi-utility where $X/\sim$ is uncountable. By Proposition 7, a countable $X/\sim$ does not imply there exists a finite multi-utility. Notice, also, preorders with utilities can have an uncountable $X/\sim$, the easiest example being $(\mathbb{R}, \leq)$, and any non-total preorder with countable $X/\sim$ has no utility function.

The next class of interest are preorders with countable multi-utilities, which are exactly those with countable strict monotone multi-utilities (Alcantud et al., 2016, Proposition 4.1) and countable injective monotone multi-utilities (Hack et al., 2022a, b, Proposition 6). By Proposition 8, there are preorders with countable multi-utilities where $X/\sim$ is uncountable such that no finite multi-utility exists, although finite multi-utilities are, of course, countable. Also, whenever $X/\sim$ is countable, there exists a countable multi-utility, namely, $(\chi_{i(x)})_{[x]\in X/\sim}$ (Evren & Ok, 2011).

The following wider category are preorders with injective monotones, which are equivalent to those with injective monotone multi-utilities with cardinality $\mathfrak{c}$ by Hack et al. (2022a, b) (Proposition 4). As we showed in Hack et al. (2022a, b) (Proposition 5), injective monotones can be constructed from countable multi-utilities. However, again by Hack et al. (2022a, b) (Proposition 8), the converse is false.

Injective monotones are contained inside two classes: preorders with strict monotone multi-utilities of cardinality $\mathfrak{c}$ and preorders where $|X/\sim|\leq\mathfrak{c}$. It is straightforward to see $|X/\sim|\leq\mathfrak{c}$ whenever injective monotones exist. Because of this, since it implies multi-utilities of cardinality $\mathfrak{c}$ exist (Evren & Ok, 2011), and Proposition 1, strict monotone multi-utilities of cardinality $\mathfrak{c}$ exist whenever injective monotones do. However, by Proposition 1 and Corollary 1, there are preordered spaces with strict monotone multi-utilities of cardinality $\mathfrak{c}$ and without injective monotones. Similarly, as Proposition 2 shows, there are preorders where we have $|X/\sim|\leq\mathfrak{c}$ and no injective monotone. Moreover, by Proposition 4(ii), having strict monotone multi-utilities of cardinality $\mathfrak{c}$ does not imply $|X/\sim|\leq\mathfrak{c}$. Conversely, as noticed in Corollary 2, we also get a negative result if we interchange the role of both clauses, that is, there are preorders where $|X/\sim|\leq\mathfrak{c}$ holds and no strict monotone multi-utility of cardinality $\mathfrak{c}$ exists. Notice the preorder in Corollary 2 was, essentially, already introduced by Debreu (1954). As we stated in Proposition 1, having a strict monotone and a multi-utility of cardinality $\mathfrak{c}$, the following class of interest, is equivalent to having a strict monotone multi-utility of that cardinality.

However, by Proposition 3, $|X/\sim| \leq \mathfrak{c}$ does not imply that there exists a strict monotone multi-utility of cardinality $\mathfrak{c}$. If we relax the implication of the statement to *multi-utility of cardinality* $\mathfrak{c}$, then it is indeed true (see Evren & Ok, 2011 or Corollary 1). There are, actually, preorders with a multi-utility of cardinality $\mathfrak{c}$ and no strict monotone multi-utility of that cardinality such that $|X/\sim| > \mathfrak{c}$, as Corollary 5 shows. Finally, by Proposition 5, there are preorders where strict monotones exist and multi-utilities of cardinality $\mathfrak{c}$ do not. In fact, by Proposition 6, there are preorders without both strict monotones and multi-utilities of cardinality $\mathfrak{c}$. This completes the results which are needed to construct Fig. 1. Notice, although we have focused on the case $I = \mathbb{R}$, many of the results hold for a general uncountable set I, as we stated them in Sect. 3.

Aside from those in the last paragraph, there are four more results in Sect. 3. Proposition 10 shows the equivalence between finite multi-utilities and finite injective monotone multi-utilities in well-behaved cases. Notice the only finite case which appears in Fig. 1 is that of multi-utilities, as the relation with the other types remains to be clarified. Proposition 9 improves upon (Hack et al., 2022a, b), where it was shown that Debreu upper separable preorders have countable multi-utilities (Hack et al., 2022a, b, Proposition 9) while there are preorders with countable multi-utilities which are not Debreu separable (Hack et al., 2022a, b, Lemma 5), by showing there exist preorders with countable multi-utilities where every Debreu upper dense subset is uncountable. Lastly, Corollary 3 is slightly stronger than Proposition 6 and uses the same preorder, while Corollary 4 is weaker than Corollary 5.

4.2 Complexity and optimization

Since the minimal cardinality of the existing multi-utilities can be used as a measure of complexity and the existence of optimization principles can be reformulated in terms of strict and injective monotones (Hack et al., 2022a, b), the classification of preorders in terms of real-valued monotones improves our knowledge regading complexity, optimization and the connections between them. Although we omit it here for the sake of brevity, we can interpret the classification of preorders according to monotones (cf. the paragraph above and Fig. 1) in terms of complexity and optimization (as we did right after presenting each result in Sect. 3).

4.3 Debreu dimension

There is a notion of dimension for partial orders which goes back to Dushnik and Miller (1941) and has remained somewhat disconnected from the more intuitive geometrical notion, which corresponds to multi-utilities. In fact, there exist preorders where the classical definition of dimension is finite while the geometrical one is uncountable. In Hack et al. (2022a, b), we propose a variation of the classical notion, called Debreu dimension, and, using results from this work, show that such a disconnection between this definition and the geometrical one does not occur. That is, we show that the geometrical dimension is countable if and only if the Debreu dimension also is.

4.4 Open questions

Several scientific disciplines rely on preordered spaces and their representation via real-valued monotones. Thus, refining the classification via the introduction of new classes and establishing more connections between separated classes in cases of interest would, potentially, improve several areas, like utility theory (Debreu, 1954; Rébillé, 2019) and the study of social welfare relation (Banerjee et al., 2010) in economics, statistical estimation (Hennig & Kutlukaya, 2007) in statistics, equilibrium thermodynamics (Lieb & Yngvason, 1999; Candeal et al., 2001), entanglement theory (Nielsen, 1999; Turgut, 2007) and general relativity (Bombelli et al., 1987; Minguzzi, 2010) in physics and, lastly, multicriteria optimization (Jahn, 2009; Ehrgott, 2005). Specific questions that remain to be solved include, for example, the relation between the different sorts of finite multi-utilities we have introduced. In particular, it is unclear whether Proposition 10 can be improved or preorders with finite multi-utilities and no finite injective multi-utilities exist.

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Declarations

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