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Necessary and sufficient conditions for pairwise majority decisions on path-connected domains

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Abstract
In this paper, we consider choice functions that are unanimous, anonymous, symmetric, and group strategy-proof and consider domains that are single-peaked on some tree. We prove the following three results in this setting. First, there exists a unanimous, anonymous, symmetric, and group strategy-proof choice function on a path-connected domain if and only if the domain is single-peaked on a tree and the number of agents is odd. Second, a choice function is unanimous, anonymous, symmetric, and group strategy-proof on a single-peaked domain on a tree if and only if it is the pairwise majority rule (also known as the tree-median rule) and the number of agents is odd. Third, there exists a unanimous, anonymous, symmetric, and strategy-proof choice function on a strongly path-connected domain if and only if the domain is single-peaked on a tree and the number of agents is odd. As a corollary of these results, we obtain that there exists no unanimous, anonymous, symmetric, and group strategy-proof choice function on a path-connected domain if the number of agents is even.

Keywords Pairwise majority rule · Single-peaked domains on trees · Unanimity · Anonymity · Group strategy-proofness · Strategy-proofness

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1 Introduction

We consider standard social choice problems where a group of agents have to collectively decide an alternative from a set of feasible alternatives. A choice function selects an alternative for every collection of individual preferences.

We impose desirable conditions on choice functions such as unanimity, anonymity, symmetry, and group strategy-proofness. A choice function is unanimous if, whenever all the individuals have the same preference, their common top-ranked alternative is chosen. It is called anonymous if it treats all the individuals equally. Symmetry ensures that if the role of two alternatives (at the top of preferences) is interchanged at certain type of profiles, the outcome is also interchanged accordingly. A choice function is called group strategy-proof if no group of agents can be strictly better off by misrepresenting their preferences and is called strategy-proof if no individual can be better off by misrepresenting his/her preference.

A preference is called single-peaked on a tree if the alternatives can be arranged on a tree[1] so that preference declines as one moves away from the top-ranked alternative. Such preferences are well known in the literature for their usefulness in modelling public good location problems.

We assume a mild structure called path-connectedness (see, Aswal et al. (2003)) on the domains we consider in this paper. Theorem 4.1 shows that a path-connected domain admits unanimous, anonymous, symmetric, and group strategy-proof choice functions if and only if it is single-peaked on a tree and the number of agents is odd. It follows as a corollary of this result that there exists no unanimous, anonymous, symmetric, and group strategy-proof choice function on a path-connected domain if the number of agents is even. When the number of agents is odd, Theorem 4.2 characterizes all unanimous, anonymous, symmetric, and group strategy-proof choice functions on single-peaked domains on trees as the tree-median rule. Finally, we investigate what happens if we replace group strategy-proofness by strategy-proofness. Theorem 5.1 says that if we strengthen the notion of path-connectedness in a suitable manner, then the conclusion of Theorem 4.1 can be achieved with strategy-proofness, that is, a strongly path-connected domain admits unanimous, anonymous, symmetric, and strategy-proof choice functions if and only if it is a single-peaked domain on a tree.

An alternative is called the pairwise majority winner at a profile if it beats every other alternative according to pairwise majority comparison and a choice function is called the pairwise majority rule if it selects the pairwise majority winner at every preference profile. Condorcet (1785) argued that if such a majority winner exists at a profile, we should choose it on the basis of "straightforward reasoning." The analysis of the pairwise majority rule dates back to Borda (1784), Condorcet (1785), and Laplace (1820). Black (1948) shows that the pairwise majority rule exists on domains that are single-peaked on a line. Later, Demange (1982) generalizes this result by showing that the pairwise majority rule exists on a domain even if the domain is single-peaked on a tree. Hansen and Thisse (1981) consider the problem

[1] A connected graph is called a tree if it has no cycle.

of locating a public facility and show that the outcome of the pairwise majority rule on a single-peaked domain on a tree minimizes the total distance traversed by the users to go to the facility. They further prove that this property holds for a single-peaked domain only when the underlying graph is a tree. Moulin (1980) characterizes the pairwise majority rule on domains that are single-peaked on a line. Danilov (1994) shows that strategy-proof and tops-only SCFs on a single-peaked domain on a tree can be recursively decomposed into medians of constant and dictatorial rules.

Schummer and Vohra (2002) consider single-peaked domains on tree when preferences are Euclidean with respect to the graph distance and show that an SCF on such a domain is strategy-proof and unanimous if and only if it is an extended generalized median voter scheme. Nehring and Puppe (2007) introduce a class of generalized single-peaked domains based on an abstract betweenness property and show that an SCF is strategy-proof on a sufficiently rich domain of generalized single-peaked preferences if and only if it takes the form of voting by issues. They also provide a characterization of such domains that admit SCFs satisfying strategy-proofness, unanimity, neutrality, and non-dictatorship/anonymity. We provide a detailed discussion on the connection of our paper with these papers in Sect. 6.

May (1952) considers the problem of preference aggregation with exactly two alternatives and characterizes the pairwise majority aggregation rule in this setting by means of always decisiveness, equality, symmetry, and positive responsiveness. Later, Inada (1969) and Sen and Pattanaik (1969) provide necessary and sufficient conditions on a domain so that the pairwise majority aggregation rule is transitive.

It is worth mentioning that the tree-median rule coincides with the pairwise majority rule on domains that are single-peaked on a tree.[2] Thus, the main contributions of our paper can be considered as (i) a characterization of domains that are single-peaked on trees by means of choice functions satisfying natural conditions such as unanimity, anonymity, symmetry, and group strategy-proofness/strategy-proofness and (ii) a characterization of the pairwise majority rule on these domains as the only choice function satisfying the above-mentioned properties. Thus, in addition to the existing results where single-peakedness on trees is proved to be sufficient for the existence of the pairwise majority rule, we show that under some natural conditions, it is also necessary for the same.

Characterizing domains by means of the choice functions that they admit is considered as an important problem in the literature. Chatterji et al. (2016) characterize single-peaked domains on arbitrary trees by means of strategy-proof, unanimous, tops-only random social choice functions satisfying a compromise property, and Puppe (2018) shows that every minimally rich and connected Condorcet domain which contains at least one pair of completely reversed orders must be single-peaked.

The rest of the paper is organized as follows. Section 2 presents the notion of single-peaked domains on trees, and Sect. 3 introduces the notion of the tree-median rule. Main results of the paper are presented in Sect. 4. Section 5 shows how group

[2] Despite the fact that the tree-median rule is nothing but the pairwise majority rule, we use the former term as for the special case when the tree is a line, this rule is called the median rule in the literature.

strategy-proofness can be replaced with strategy-proofness in our main result. All the proofs, as well as the independence of the axioms used in our main result, are collected in Appendix.

2 Domains and their properties

Let A denote the set of *alternatives*, and let $N = \{1, \ldots, n\}$ denote the set of n *agents*, where n is at least 2. We denote by $\mathbb{L}(A)$ the set of all linear orders (reflexive, transitive, antisymmetric, and complete binary relations) on A. An element of $\mathbb{L}(A)$ is called a *preference*. Note that preferences are strict by definition. An admissible set of agents' preferences (or a domain) $\mathbb{D}$ is a subset of $\mathbb{L}(A)$. A *profile* is a collection of preferences, one for each agent. More formally, a profile p is an element of $\mathbb{D}^n$.

For ease of presentation, we do not use braces for singleton sets and use the following notations throughout the paper. Let R be a preference, and let a and b be two alternatives (not necessarily distinct) in A. To save parentheses, we write $ab \in R$ instead of $(a, b) \in R$, which has the usual interpretation that a is (weakly) preferred to b at R. When a and b are distinct, we write $R \equiv \cdots ab \cdots$ to mean that a is ranked just above b at R. In line with this, we write $R \equiv ab \cdots$ to mean that a is the top-ranked and b is the second-ranked alternative at R. Notations like $R \equiv \cdots a \cdots b \cdots$, $R \equiv a \cdots$, and $R \equiv \cdots a$ have self-explanatory interpretations.

The top-ranked alternative at a preference R is denoted by $\tau(R)$. The set of the top-ranked alternatives of the preferences in a domain $\mathbb{D}$ is denoted by $\tau(\mathbb{D})$, that is, $\tau(\mathbb{D}) = \{a \in A : \tau(R) = a \text{ for some } R \in \mathbb{D}\}$. We assume that $\tau(\mathbb{D})$ is a finite set of m alternatives.

Next, we introduce the notion of graphs. An (undirected) graph $G = (V(G), E(G))$ is a tuple where $V(G)$ is the set of vertices and $E(G) \subseteq \{\{a, b\} : a, b \in V(G)\}$ is the set of edges. A sequence of vertices $x^1, \ldots, x^k$ is called a path in G if $\{x^l, x^{l+1}\} \in E$ for all $1 \leq l < k$. A path $x^0, x^1, \ldots, x^k$ in G is called a *cycle* if $k \geq 3$, $x^0 = x^k$, and $x^s \neq x^t$ for all $0 \leq s < t \leq k$. A graph is called a *tree* if it has no cycles. For a tree and two vertices a and b, we denote by $\pi(a, b)$ (whenever the tree is clear from the context) the unique path between a and b.

Two alternatives a and b in A are called *top-connected* (in $\mathbb{D}$) if there are $R, R' \in \mathbb{D}$ such that $R \equiv ab \cdots$ and $R' \equiv ba \cdots$. We use the notation $a \leftrightsquigarrow b$ to mean that a and b are top-connected. The induced graph of a domain $\mathbb{D}$ is defined as the undirected graph $\mathcal{G}(\mathbb{D}) = (\tau(\mathbb{D}), E)$, where E is the set of edges consisting of all pairs of top-connected alternatives, that is, $E = \{\{a, b\} \subseteq \tau(\mathbb{D}) : a \leftrightsquigarrow b\}$. Two alternatives a and b are called *path-connected* if there is a path from a to b in $\mathcal{G}(\mathbb{D})$. A domain $\mathbb{D}$ is called path-connected if every two alternatives in $\tau(\mathbb{D})$ are path-connected (see Aswal et al. (2003)).

A subset S of N is called a coalition. The restriction of a profile p to a coalition S is denoted by $p|_S$. For a coalition S and preferences R and R' in $\mathbb{D}$, the N-tuple $((R)^S, (R')^{N \setminus S})$ denotes the profile p where $p(i) = R$ for all agents i in S and $p(i) = R'$ for all agents i in $N \setminus S$.

We introduce the notion of single-peaked domains on trees. A preference is single-peaked on a tree if it has the property that as one goes far away along any path from its top-ranked alternative, preference decreases.

Definition 2.1 Let T be a tree with $V(T) \subseteq A$. A domain $\mathbb{D}$ is called *single-peaked on T* if $\tau(\mathbb{D}) = V(T)$ and for all $R \in \mathbb{D}$ and all $a, b \in \tau(\mathbb{D})$, $a \in \pi(\tau(p(i)), b)$ implies $ab \in R$.

Note that for a domain $\mathbb{D}$ that is single-peaked on a tree, there is no restriction on the ordering of the alternatives outside $\tau(\mathbb{D})$. We present an example of a single-peaked domain on a tree.

Example 2.1 Let the set of alternatives be $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$. Consider the tree T in Fig. 1 with $V(T) = \{a_1, a_2, a_3, a_4, a_5\}$. In Table 1, we present a single-peaked domain on this tree.

3 Choice functions and their properties

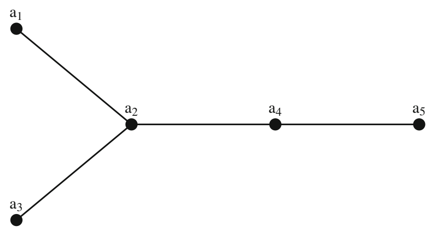
A *choice function* φ is a mapping from $\mathbb{D}^n$ to A. A choice function φ is *unanimous* if, whenever all the agents agree on their preferences, the top-ranked alternative of that common preference is chosen. More formally, $\varphi : \mathbb{D}^n \to A$ is unanimous if for all profiles $p \in \mathbb{D}^n$ such that $p(i) = R$ for all agents $i \in N$ and some $R \in \mathbb{D}$, we have $\varphi(p) = \tau(R)$. A choice function φ is called *anonymous* if it is symmetric in its arguments. In other words, anonymous choice functions disregard the identities of the agents. A choice function φ is *strategy-proof* if no agent can change its outcome in his/her favor by misreporting his/her sincere preference. More formally, $\varphi : \mathbb{D}^n \to A$ is strategy-proof if for all agents $i \in N$ and all profiles $p, q \in \mathbb{D}^n$ with $p|_{N \setminus i} = q|_{N \setminus i}$, we have $\varphi(p)\varphi(q) \in p(i)$. A choice function φ is *group strategy-proof* if for all non-empty coalitions S of N and all profiles $p, q \in \mathbb{D}^n$ with $p|_{N \setminus S} = q|_{N \setminus S}$, we have either $\varphi(p) = \varphi(q)$ or $\varphi(p)\varphi(q) \in p(i)$ for some $i \in S$.

Next, we introduce the notion of symmetry. Symmetry has some resemblance with neutrality; however, they are not the same.[3] Suppose that the agents are divided into two groups such that all agents in each group have the same preference. Suppose further that two alternatives a and b appear at the top two positions in each preference. Symmetry says that if the outcome of such a profile is a and the two groups interchange their preferences, then the outcome of the new profile will be b. In other words, symmetry ensures that if the roles of two alternatives are interchanged at certain type of profiles, the outcome is also interchanged accordingly. Note that symmetry is different from neutrality as it applies to a very specific class of profiles and only to the top-two ranked alternatives.

Definition 3.1 We say that a choice function φ satisfies *symmetry* if for all $R \equiv ab\cdots$ and $R' \equiv ba\cdots$, and all subsets S of N, we have

[3] Nehring and Puppe (2007) define a notion that is very similar to symmetry and call it neutrality. We use a different term to avoid confusion.

Fig. 1 Tree for Example 2.1



$$\varphi((R)^S, (R')^{N\setminus S}) = a \text{ if and only if } \varphi((R')^S, (R)^{N\setminus S}) = b.$$

3.1 Tree-median rule

The tree-median rule is an appropriate extension of the median rule defined in the context of single-peaked domains on lines. We first provide an verbal description of these rules. Suppose that the alternatives are named as $a_1, \ldots, a_m$ and that they are arranged on a line in the following order: $a_1 \prec \cdots \prec a_m$. Note that the median of a subset of alternatives B can be defined as the (unique) alternative a such that $|\{b \in B : b \prec a\}| < \frac{|B|}{2}$ and $|\{b \in B : b \succ a\}| < \frac{|B|}{2}$. For instance, if $B = \{a_1, a_3, a_4, a_9, a_{11}\}$, then a_4 is the unique alternative that satisfies the condition that $|\{b \in B : b \prec a_4\}| = |\{a_1, a_3\}| < 2.5$ and $|\{b \in B : b \succ a_4\}| = |\{a_9, a_{11}\}| < 2.5$. In other words, the number of alternatives which lie in any particular "direction" of the median must be less than the half of the cardinality of the set. Here, two alternatives are said to be in the same direction with respect to an alternative a if they lie in the same component of the (possibly disconnected) graph that is obtained by deleting the alternative a from the line. We implement this idea on a tree.

Consider a tree $T = (V, E)$. For a vertex a of T, we denote by T^{-a} the graph that is obtained by deleting the alternative a (and all the edges involving a) from T, that is, $T^{-a} = \{\hat{V}, \hat{E}\}$, where $\hat{V} = V \setminus a$ and $\{x, y\} \in \hat{E}$ if and only if $\{x, y\} \in E$ and $a \notin \{x, y\}$. Note that T^{-a} is a disconnected graph unless a is a terminal node in T.[4] A component $\mathcal{C}$ of T^{-a} is defined as a maximal set of vertices of T^{-a} that are connected via some path in T^{-a}. Below, we provide an example of a tree T and show the components of T^{-a} for some vertex a.

Example 3.1 Consider the tree T as given in Fig. 2. Consider the vertex a_6. The components of T^{-a_6} are shown in Fig. 3.

Now, we are ready to define the notion of the median with respect to a tree. Let $T = (V, E)$ be a tree. For a subset $\widehat{V}$ of V, define the median of $\widehat{V}$ (with respect to T) as the unique vertex $a \in V$ such that for each component $\mathcal{C}$ of T^{-a}, we have

$$|\widehat{V} \cap \mathcal{C}| < \frac{|\widehat{V}|}{2}.$$

Whenever the tree T is clear from the context, we denote the median of a set $\widehat{V} \subseteq V$

[4] A node is called terminal if it has degree 1.

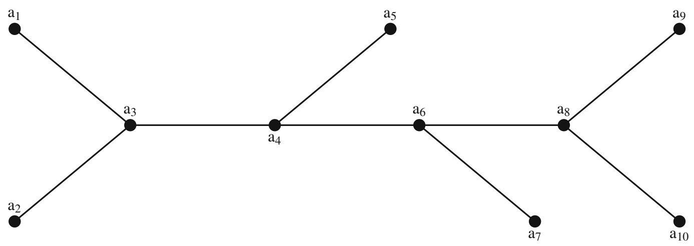


Fig. 2 Tree for Example 3.1 and Example 3.2

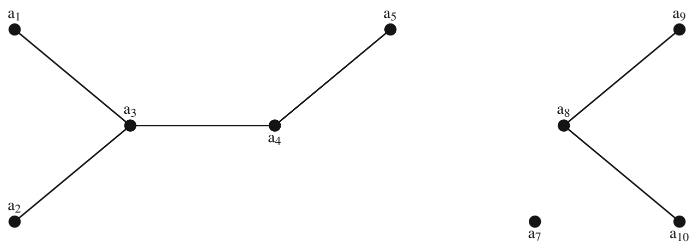


Fig. 3 Components of T^{-a_6}

with respect to T by median $(\widehat{V})$. The following example explains the idea of the median of a set. It should be clear from this example that the median of a set may not lie within the set.

Example 3.2 Consider the tree T with $V(T) = \{a_1, \ldots, a_{10}\}$ as given in Fig. 2. Consider the subset $\widehat{V} = \{a_1, a_4, a_7, a_8, a_9\}$ of V. We show that the median of $\widehat{V}$ is a_6. The components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of T^{-a_6} are shown in Fig. 3. Note that in each of these components, the number of elements from $\widehat{V}$ is less than the half of the cardinality of $\widehat{V}$. For instance, the elements of $\widehat{V}$ that are in Component $\mathcal{C}_1$ are a_1 and a_4. This proves that the median of $\widehat{V}$ is a_6. We proceed to show that a_6 is the unique vertex that satisfies this property. Note that since $\frac{n}{2} = 2.5$, a vertex v cannot be the median if a component in T^{-v} has more than two vertices. Consider the vertex a_4. Then, there is a component $C = \{a_6, a_7, a_8, a_9, a_{10}\}$ in T^{-a_4} that contains three elements a_7, a_8, a_9 from $\widehat{V}$. By using a similar logic, for any vertex v in $\{a_1, a_2, a_3, a_5\}$ there is a component in T^{-v} containing the vertices a_7, a_8, a_9, for any vertex v in $\{a_8, a_9, a_{10}\}$, there is a component in T^{-v} containing the vertices a_1, a_4, a_7, and for a_7, there is a component in T^{-a_7} containing the vertices a_1, a_4, a_8, a_9 from $\widehat{V}$. Since for each of these vertices, there is a component having more than two elements from $\widehat{V}$, none of them satisfies the requirement for being the median. This shows that a_6 is the unique median.

Now, we are ready to define the notion of the tree-median rule. It selects the median of the top-ranked alternatives at every profile.

Definition 3.2 A choice function $\varphi : \mathbb{D}^n \to A$ is called the tree-median rule with respect to a tree T with $V(T) = \tau(\mathbb{D})$ if for all $p \in \mathbb{D}^n$, $\varphi(p) =$ median $(\{\tau(p(i)) : i \in N\})$.

Remark 3.1 An alternative a is called *pairwise majority winner* at a profile if for all $b \neq a$, the number of agents who prefer a to b at that profile is more than $\frac{n}{2}$. It is worth noting that the outcome of a median rule at any profile is the pairwise majority winner (Condorcet winner) at that profile. To see this, suppose that the outcome of the tree-median rule is a at a profile p. Consider an alternative b other than a. Suppose b belongs to a component $\mathcal{C}$ of T^{-a}. By single-peakedness, every agent, whose top-ranked alternative is not in $\mathcal{C}$, will prefer a to b. By the definition of the tree-median rule, the number of agents in component $\mathcal{C}$ is strictly less than $\frac{n}{2}$. Therefore, the number of agents who prefer a to b must be more than $\frac{n}{2}$, implying that a beats b by pairwise majority comparison.

4 Results

Our first theorem characterizes the single-peaked domains on trees by means of choice functions that are unanimous, anonymous, symmetric, and group strategy-proof. It says that these domains are the only path-connected domains that admit such rules when the number of agents is odd.

Theorem 4.1 *Let $\mathbb{D}$ be a path-connected domain. Then, there exists a unanimous, anonymous, symmetric, and group strategy-proof choice function $\varphi : \mathbb{D}^n \to A$ if and only if $\mathbb{D}$ is single-peaked on a tree and n is odd.*

The proof of this theorem is relegated to Appendix A. In Sect. 4.1, we provide an idea of the proof of the only-if part of the theorem by considering the case of three alternatives.

Our next corollary says that if the number of agents is even, then there is no path-connected domain that admits a unanimous, anonymous, symmetric, and group strategy-proof rule. The intuition of this result is as follows. Since the number of agents is even, we can divide the agents into two groups N_1 and N_2 having equal size. Consider the profile where agents in N_1 have the same preference $ab\cdots$ and agents in N_2 have the same preference $ba\cdots$, for some $a, b \in A$. By unanimity and group strategy-proofness, the outcome at such a profile must be either a or b. Suppose that the outcome is a. Now, consider the profile where agents in N_1 have the same preference $ba\cdots$ and agents in N_2 have the same preference $ab\cdots$. By symmetry, the outcome at this profile must be b. However, this violates anonymity.

Corollary 4.1 *Let $\mathbb{D}$ be a path-connected domain, and let n be even. Then, there is no unanimous, anonymous, symmetric, and group strategy-proof choice function $\varphi : \mathbb{D}^n \to A$.*

Our next theorem characterizes the unanimous, anonymous, symmetric, and group strategy-proof rules on a single-peaked domain on a tree as the tree-median rules.

Theorem 4.2 *Let $\mathbb{D}$ be path-connected and single-peaked on a tree T and let n be odd. Then, a choice function $\varphi : \mathbb{D}^n \to A$ is unanimous, anonymous, symmetric, and group strategy-proof if and only if it is the tree-median rule with respect to T.*

The proof of this theorem is relegated to Appendix B. In Sect. 4.1, we provide an idea of the proof of the only-if part by considering the case of three alternatives.

4.1 An illustration of the proofs of Theorems 4.1 and 4.2

We illustrate the idea of the proof of the only-if parts of Theorem 4.1 and Theorem 4.2 by considering the case of three alternatives. Let $A = \{a, b, c\}$ be the set of three alternatives, and let $N = \{1, \ldots, n\}$ be the set of agents. Suppose $\mathbb{D}$ is a path-connected domain and let φ be a unanimous, anonymous, symmetric, and group strategy-proof choice function from $\mathbb{D}^n$ to A. We show that

1. n is odd,
2. $\mathbb{D}$ is a set of single-peaked preferences on a tree, and
3. φ chooses the median of the top-ranked alternatives at any profile in $\mathbb{D}^n$.

Because $\mathbb{D}$ is path-connected, we have, after a possible renaming of the alternatives, one of the following four cases

(i) $\mathbb{D} = \mathbb{L}(A)$

(ii) $\mathbb{D} = \mathbb{L}(A) \setminus \{acb\}$

(iii) $\mathbb{D} \subseteq \{abc, bac, bca, cba\}$ implying that $\mathbb{D}$ is single-peaked on a (sub)tree T_1 of the following tree

$$a \leftrightsquigarrow b \leftrightsquigarrow c.$$

(iv) $\mathbb{D} \subseteq \{abc, acb, cab, cba\}$ implying that $\mathbb{D}$ is single-peaked on a (sub)tree T_2 of the following tree[5]

$$a \leftrightsquigarrow c.$$

Consider a profile p and a coalition S such that $p(i) = xyz$ for all $i \in S$ and $p(i) = yxz$ for all $i \in N \setminus S$. By unanimity and group strategy-proofness, $\varphi(p) \neq z$, as otherwise the agents in S will manipulate by reporting their preferences as yxz. By anonymity and group strategy-proofness, the outcome of any profile $\hat{p}$ such that $\hat{p}(i) \in \{xyz, yxz\}$ for all $i \in N$ and $|\{i : \hat{p}(i) = xyz\}| \geq |S|$ is x. By symmetry, this means $\varphi(\hat{\hat{p}}) = y$ for any profile $\hat{\hat{p}}$ such that $\hat{\hat{p}}(i) \in \{xyz, yxz\}$ for all $i \in N$ and $|\{i : \hat{p}(i) = yxz\}| \geq |S|$. Therefore, it must be that $|S| > \frac{n}{2}$, as otherwise we can have a profile q such that both $|\{i : q(i) = xyz\}|$ and $|\{i : q(i) = yxz\}|$ are greater than or equal to $|S|$ and in view of the earlier observations, nothing can be defined as an outcome at q. Using similar logic, no outcome can be defined at a profile q such that

[5] Such a set of preferences is known as a single-dipped domain in the literature.

Table 1 The single-peaked domain with respect to the tree in Fig. 1

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}
a_1	a_1	a_2	a_2	a_2	a_2	a_2	a_3	a_3	a_4	a_4	a_4	a_4	a_4	a_5	a_5
a_2	a_2	a_1	a_6	a_3	a_4	a_4	a_7	a_2	a_2	a_2	a_6	a_5	a_5	a_4	a_6
a_6	a_4	a_4	a_1	a_7	a_5	a_7	a_2	a_6	a_7	a_5	a_2	a_6	a_2	a_6	a_4
a_3	a_7	a_7	a_4	a_1	a_6	a_5	a_6	a_4	a_6	a_3	a_3	a_2	a_6	a_2	a_2
a_4	a_3	a_6	a_3	a_4	a_3	a_1	a_4	a_1	a_1	a_6	a_7	a_7	a_1	a_3	a_1
a_5	a_6	a_5	a_5	a_5	a_7	a_3	a_5	a_5	a_3	a_1	a_5	a_1	a_3	a_7	a_7
a_7	a_5	a_3	a_7	a_6	a_1	a_6	a_1	a_7	a_5	a_7	a_1	a_3	a_7	a_1	a_3

Table 2 Primary structure of a unanimous, anonymous, symmetric, and group strategy-proof choice function

S	$N \setminus S$			
	zxy	xzy	xyz	yxz
zxy	z	z		
xzy	x	x		
xyz			x	x
yxz			y	y

$|\{i : q(i) = xyz\}| = |\{i : q(i) = yxz\}|$. This proves (1), that is, n is odd. This is formally proved in Lemma A.2 (see Appendix A).

Now, we proceed to prove (2). Consider a coalition S with $|S| > \frac{n}{2}$. We show that for any profile q such that $q(i) = q(j)$ for all $i, j \in S$ and $q(i) = q(j)$ for all $i, j \in N \setminus S$, the outcome is the top-ranked alternative of the agents in S. In Table 2, we present such profiles where agents' preferences lie in the set $\{zxy, xzy, xyz, yxz\}$. We also present the outcomes of φ at the profiles where it can be obtained by unanimity and Lemma A.2.

We proceed to show that the outcome at any profile in the table will be the top-ranked alternative of the agents in S. Since the outcome at the profile (xzy, zxy) is x, by group strategy-proofness, it must be x at (xyz, zxy).[6] In Table 3, we present the outcomes that can be obtained using similar logic.

Consider the profile (yxz, xzy). Since the outcome at (xyz, xzy) is x, by group strategy-proofness, the outcome at (yxz, xzy) must be x or y. Similarly, since the outcome at (yxz, xyz) is y, by group strategy-proofness, the outcome at (yxz, xzy) must be y. Moreover, since the outcome at (yxz, xzy) is y and y is the bottom-ranked alternative for the agents in $N \setminus S$, by group strategy-proofness, the outcome at (yxz, zxy) must be y. In Table 4, we present the outcomes that can be obtained using similar logic. Since S is arbitrary, Table 4 implies that the outcome of φ will be determined by the majority at any profile where the agents are partitioned into two groups such that agents in any group have the same preference.

[6] For ease of presentation, by (xzy, zxy) we denote the profile where the agents in S have the preference xzy and the agents in $N \setminus S$ have the preference zxy. We continue to use similar notations.

Table 3 Additional structure of a unanimous, anonymous, symmetric, and group strategy-proof choice function

S	$N \backslash S$			
	zxy	xzy	xyz	yxz
zxy	z	z		
xzy	x	x	x	x
xyz	x	x	x	x
yxz			y	y

Table 4 Final structure of a unanimous, anonymous, symmetric, and group strategy-proof choice function

S	$N \backslash S$			
	zxy	xzy	xyz	yxz
zxy	z	z	z	z
xzy	x	x	x	x
xyz	x	x	x	x
yxz	y	y	y	y

Since n is odd, there must be at least 3 agents. Therefore, the set of agents can be partitioned into non-empty sets S_1, S_2, S_3 such that $|S_i \cup S_j| > \frac{n}{2}$ for all $i \neq j$. Consider the profile v such that $v(i) = xyz$ for all $i \in S_1$, $v(i) = yzx$ for all $i \in S_2$, and $v(i) = zxy$ for all $i \in S_3$.[7] As $|S_1 \cup S_2| > \frac{n}{2}$, the outcome at the profile where agents in $S_1 \cup S_2$ have the preference yzx and the agents in S_3 have the preference zxy is y. Hence, by group strategy-proofness $\varphi(v) \neq z$. Using a similar logic, $|S_2 \cup S_3| > \frac{n}{2}$ implies $\varphi(v) \neq x$, and $|S_1 \cup S_3| > \frac{n}{2}$ implies $\varphi(v) \neq y$. So, no outcome can be defined at the profile v, and hence, a profile like v cannot lie in $\mathbb{D}^n$. Therefore, out of the four cases for $\mathbb{D}$ mentioned at the beginning, only Case (iii) and Case (iv) are possible. This proves that the domain $\mathbb{D}$ is a set of single-peaked preferences with respect to either the tree T_1 or the tree T_2. This completes the proof of (2).

We complete the sketch of the proof by showing (3). We deal with Case (iii) and Case (iv) separately.

Case (iii): Here, $\mathbb{D}$ is a subset of single-peaked preferences $\{abc, bac, bca, cba\}$ with respect to the alphabetical order $a \prec b \prec c$ and $\mathcal{G}(\mathbb{D})$ is a (sub)graph of

$$a \leftrightsquigarrow b \leftrightsquigarrow c.$$

Let p be a profile in $\mathbb{D}^n$. We prove that $\varphi(p)$ is the median of the top-ranked alternatives at p. We distinguish three cases.

Suppose $\varphi(p) = b$. Consider the profile q such that $q(i) = bca$ if $p(i) = abc$, and $q(i) = p(i)$ otherwise. By group strategy-proofness, $\varphi(q) = b$, as otherwise the agents i having preference bca at q will (group) manipulate at q by misreporting their preferences as $p(i)$. Next, consider the profile r such that $r(i) = bca$ if $q(i) = p(i) = bac$, and $r(i) = q(i) = cba$ otherwise. By group strategy-proofness, $\varphi(r) = b$. Since agents have one of the two preferences bca and cba at the profile

[7] Such a profile is called a Condorcet profile.

r, the outcome of φ at r will be the majority vote (winner) between b and c. As this outcome is b, it must be that majority of voters have the top-ranked alternative as b at the profile r. This implies that a majority of voters have top-ranked alternatives at p in the set $\{a, b\}$. Similarly, we can deduce that a majority of voters have top-ranked alternatives at p in the set $\{b, c\}$. Thus, it follows that at the profile p, there is a majority of voters having the top-ranked alternative in the set $\{a, b\}$ and a (possibly different) majority of voters having top-ranked alternatives in the set $\{b, c\}$, and hence, b is the median of the top-ranked alternatives at p.

Suppose $\varphi(p) = a$. Consider the profile v such that $v(i) = bac$ if $p(i) \neq abc$, and $v(i) = p(i) = abc$ otherwise. Since agents have one of the two preferences abc and bac at the profile v, the outcome of φ at v will be the majority vote between a and b. In particular, $\varphi(v) \in \{a, b\}$. Note that except for the preference abc, alternative b is strictly preferred to a at all other preferences in $\mathbb{D}$. So, group strategy-proofness implies that $\varphi(v) \neq b$, as otherwise the agents i having preference bac at v will manipulate at p via $v(i)$. So, $\varphi(v) = a$. Since the outcome of φ at v will be the majority vote between a and b, this means that there is a majority of voters having top-ranked alternative as a at p. So, a is the median of the top-ranked alternatives at p.

Suppose $\varphi(p) = c$. This case is similar to the latter case where $\varphi(p) = a$.

Case (iv): Here, we have $\tau(\mathbb{D}) = \{a, c\}$. We have already argued that the outcome will be determined by the majority at profiles where agents are partitioned into two groups with each group having the same preference. Since $\tau(\mathbb{D}) = \{a, c\}$, by group strategy-proofness, this implies that the outcome of φ will be the majority vote between a and c at any profile. This in particular means that φ chooses the median of the top-ranked alternatives at any profile.

So, φ is the median rule and the only-if parts of Theorem 4.1 and Theorem 4.2 are proved for the case of three alternatives.

5 Weakening group strategy-proofness to strategy-proofness

In this section, we show that group strategy-proofness cannot be replaced by strategy-proofness in Theorem 4.1 and, consequently, provide a version of Theorem 4.1 with strategy-proofness. The following example shows that Theorem 4.1 does not hold under strategy-proofness.

Example 5.1 Suppose that the set of alternatives is two-dimensional where each dimension or component has two elements: 0 and 1. More formally, the alternatives are $A = \{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Agents' preferences are such that if a is the top-ranked alternative in a preference and b differs from a in both components, then b will be the bottom-ranked alternative in that preference.[8] For instance, if (0, 1) is the top-ranked alternative in a preference, then (1, 0) will be the bottom-ranked alternative in that preference. Therefore, there will be two preferences with (0, 1) at the top for the two possible relative ordering of the remaining alternatives (0, 0) and (1, 1). The preferences are as follows:

[8] This is a special case of a more general condition known as separability in the literature.

Table 5 Domain for Example 5.1

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
(0, 0)	(0, 1)	(0, 1)	(1, 1)	(1, 1)	(1, 0)	(1, 0)	(0, 0)
(0, 1)	(0, 0)	(1, 1)	(0, 1)	(1, 0)	(1, 1)	(0, 0)	(1, 0)
(1, 0)	(1, 1)	(0, 0)	(1, 0)	(0, 1)	(0, 0)	(1, 1)	(0, 1)
(1, 1)	(1, 0)	(1, 0)	(0, 0)	(0, 0)	(0, 1)	(0, 1)	(1, 1)

(0, 1)(0, 0)(1, 1)(1, 0) and (0, 1)(1, 1)(0, 0)(1, 0). In Table 5, we present all (eight) preferences satisfying this property. Consider the domain with these preferences.

Suppose that there are three agents. We define an SCF called component-wise majority rule. For each component, it selects the element in that component that appears as the top-ranked element in that component for at least two agents. Note that the SCF depends only on the top-ranked alternatives in a profile. For an illustration of the rule, consider a profile with top-ranked alternatives as (1, 0), (0, 1), (1, 0). In the first component, element 1 appears at least two times as the top-ranked alternative, and hence, it is the outcome in that component. Similarly, 0 is the outcome in the second component. The final outcome of the rule is (1, 0) , which is obtained by combining the outcomes in each component.

It is shown in Barberà et al. (1991) (see Theorem 1) that the component-wise majority rule is strategy-proof. Unanimity and anonymity of the rule follow from the definition. For symmetry, consider a profile p where the agents in a group S, $\emptyset \neq S \neq N$, have the preference $R \equiv xy\cdots$ and others have the preferences $R' \equiv yx\cdots$ for some x and y in A. By the definition of the domain, x and y can differ only over one component. So, assume without loss of generality, $x = (0, 0)$ and $y = (0, 1)$, and suppose that the outcome of the component-wise majority rule at this profile is (0, 0). Since the outcome in the second component is 0, by the definition of the component-wise majority rule, it must be that S contains at least 2 agents. Now, suppose that the agents in S interchange their preference with those in $N \setminus S$. The outcome in the first component will still be 0 as it is the top-ranked element of each agent in that component. Moreover, since S contains at least 2 agents, the outcome in the second component will now become 1, and hence, the final outcome will be (0, 1). This shows that the component-wise majority rule satisfies symmetry.

Now, we argue that it is not group strategy-proof. Consider the profile of top-ranked alternatives (0, 0), (1, 1), (1, 0). Suppose that both agents 1 and 2 prefer (0, 1) to (1, 0). Note that this assumption is compatible with our domain restriction. The outcome of the component-wise majority rule at this profile is (1, 0). However, if agents 1 and 2 together misreport their preferences as one having the top-ranked alternative as (0, 1) , then the outcome of the component-wise majority rule will become (0, 1) , which is preferred to (1, 0) for both agents 1 and 2. Therefore, the component-wise majority rule is not group strategy-proof.

In what follows, we show that if we strengthen the notion of path-connectedness, then we can replace group strategy-proofness by strategy-proofness in Theorem 4.1.

Let a and b be two alternatives in $\tau(\mathbb{D})$. We say that a is *strongly top-connected to* b if there are R^a and R^b in $\mathbb{D}$ such that (i) $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$, and (ii) for all $x, y \notin \{a, b\}$, $xR^a y$ if and only if $xR^b y$. The notion of a strongly path-connected domain is defined in an obvious manner.

Our next theorem says that group strategy-proofness can be replaced by strategy-proofness if we strengthen path-connectedness by strongly path-connectedness.

Theorem 5.1 *Let $\mathbb{D}$ be a strongly path-connected domain. Then, there exists a unanimous, anonymous, symmetric, and strategy-proof choice function $\varphi : \mathbb{D}^n \to A$ if and only if $\mathbb{D}$ is single-peaked on a tree and n is odd.*

The proof of this theorem is relegated to Appendix C.

6 Relation to the literature

In this section, we discuss the connection of our results with some of the closely related papers.

6.1 Schummer and Vohra (2002)

Schummer and Vohra (2002) consider single-peaked domains on graphs (trees as a special case). Preferences are Euclidean with respect to the graph distance. They show that an SCF is strategy-proof and unanimous if and only if it is an extended generalized median voter scheme. Although tree-median rules are special cases of extended generalized median voter scheme, our result does not follow from their result because of the following reasons.

(i) In their model, for each alternative there is exactly one preference with it as the top-ranked alternative. Thus, SCFs on such a domain become tops-only vacuously. However, in our case, there can be more than one preference with the same top-ranked alternative, and hence, tops-onlyness is required to be proved additionally. Weymark (2011) shows that the maximal single-peaked domain on a line is tops-only, and recently, Achuthankutty and Roy (2018) generalize this result for arbitrary (that is, not necessary maximal) single-peaked domains on a line.[9] Chatterji and Sen (2011) provide a sufficient condition on a domain for it to be tops-only. None of these results applies to a path-connected single-peaked domain on a tree.

(ii) Schummer and Vohra (2002) use strategy-proofness, whereas we use group strategy-proofness. To the best of our knowledge, it is not known in the literature whether extended generalized median voter schemes are group strategy-proof or not on domains that are single-peaked on a tree. Barberà et al. (2010) provide a sufficient condition on a domain for the equivalence of group strategy-proofness and strategy-proofness; however, their result also does not apply to such domains.

[9] A domain is tops-only if every unanimous and strategy-proof SCF on it is tops-only.

6.2 Nehring and Puppe (2007)

Nehring and Puppe (2007) consider a class of single-peaked domains based on an abstract betweenness property. They have analyzed the structure of strategy-proof and unanimous SCFs on such domains. Furthermore, they provide a characterization of such domains that admit SCFs satisfying strategy-proof, unanimous, neutral, and non-dictatorial/anonymity. Two particular results of Nehring and Puppe (2007) are closely related to our work, which we explain below.

(i) Corollary 5 in Nehring and Puppe (2007) says that a strategy-proof, unanimous, neutral, and anonymous SCF exists on a "rich" single-peaked domain if and only if n is odd and the domains is a "median space." On the other hand, Theorem 4.1 of our paper says that a group strategy-proof, unanimous, anonymous, and symmetric SCF exists on a path-connected single-peaked domain if and only if n is odd and the domain is single-peaked on a tree. While neutrality and symmetry are similar in nature, the assumption of richness and the inclusion of median space make a significant difference between the two results as we explain below.
Richness: They assume the domains to be rich. In the context of domains that are single-peaked on a tree, this means the relative ordering of two alternatives that do not lie on the same path from the peak must be unrestricted. To see how strong this condition is, consider a single-peaked domain on a line. One implication of richness is that there must be preferences where the extreme left (or right) alternative is preferred to the "right-neighbor" (or the "left-neighbor") of the peak. For instance, if there are 100 alternatives $a_1, \ldots, a_{100}$ with the prior ordering $a_1 \prec \cdots \prec a_{100}$, then there must be a preference with a_2 at the top position where the "far away" alternative a_{100} is preferred to the neighboring one a_1. This is clearly a strong assumption for practical applications. Our notion of path-connectedness requires that for every two adjacent alternatives, say a_2 and a_3, there are two preferences where they swap their positions at the top two ranks; that is, preferences of the form $a_2 a_3 \cdots$ and $a_3 a_2 \cdots$ must be present. Thus, we do not require anything about the relative ordering of other alternatives.
Median space: A domain is a median space if the notion of median can be defined for any three alternatives in it; that is, for any three alternatives a, b, c, there is an alternative m called the "median" of a, b, c such that m lies between every pair of alternatives from a, b, c. Apart from domains that are single-peaked on a tree, there are several other domains that are median space (see Example 4 in Nehring and Puppe (2007)). Thus, domains that are single-peaked on a tree cannot be characterized by the properties used in Nehring and Puppe (2007)) and the use of group strategy-proofness does the job in our paper. As we have mentioned earlier, it is not yet known if group strategy-proofness and strategy-proofness are equivalent on domains that are single-peaked on a tree. Thus, (even the "if part" of) Theorem 4.1 of our paper does not follow from Corollary 5 of Nehring and Puppe (2007).

(ii) Theorem 4 of Nehring and Puppe (2007) says that an SCF on a rich median space is strategy-proof, unanimous, and neutral if and only if it is a particular type of voting by issues rules. Furthermore, if anonymity is imposed additionally, then these rules become tree median. Since the single-peaked domains on trees we consider do not satisfy richness, this result does not apply to these domains. Moreover, even if we additionally impose richness on such domains, since we work with group strategy-proofness, Theorem 4.1 of our paper does not follow from this result. The contribution of our result on these special class of rich domains is that it implies that strategy-proofness and group strategy-proofness are equivalent on those under unanimity, anonymity, and symmetry/neutrality. Such a result is not known in the literature, and we feel it is not straightforward either.

Appendix

Proof of Theorem 4.1

We introduce the following terminologies to facilitate the presentation of our proofs. For a coalition S, a preference R, and a profile q, we denote by $((R)^S, q|_{N\setminus S})$ the profile p where $p(i) = R$ for all $i \in S$ and $p(i) = q(i)$ for all $i \in N\setminus S$. We call such a profile S-*unanimous*. In a similar fashion, a profile of the form $((R)^S, (R')^{N\setminus S})$ is said to be $(S, N\setminus S)$-*unanimous*. Additionally, if $\tau(R) = a$ and $\tau(R') = b$, then such a profile is said to be (a, b)-$(S, N\setminus S)$-*unanimous*. Let V be a set of S-unanimous profiles in $\mathbb{D}^n$ for some coalition S. Given a choice function φ, we say that the coalition S is *decisive on* V (*for* φ), if $\varphi(R^S, p|_{N\setminus S}) = \tau(R)$ for all $(R^S, p|_{N\setminus S}) \in V$. The coalition S is said to be *decisive* if it is decisive on the set of all S -unanimous profiles in $\mathbb{D}^n$. For instance, N is decisive for a unanimous choice function φ.

We are now ready to present the proof of Theorem 4.1.

If part: Let T be a tree, and let $\mathbb{D}$ be a single-peaked domain on T. Suppose n is odd. Consider the median rule $\varphi : \mathbb{D}^n \to A$. By definition, φ satisfies unanimity, anonymity, and symmetry. In what follows, we show that it satisfies group strategy-proofness.

Consider a profile $p \in \mathbb{D}^n$. Suppose $\varphi(p) = a$. Assume for contradiction that some coalition S manipulates φ at the profile p. First note that by the definition of single-peaked domain on T, if the top-ranked alternatives of the agents in S at the profile p are in different components of T^{-a}, then there is no alternative b that is strictly preferred to a by each agent in S. So, since the agents in S manipulate, it must be that their top-ranked alternatives in p are in some component $\mathcal{C}$ of T^{-a}. By the definition of the median rule, the number of agents who have top-ranked alternatives in $\mathcal{C}$ is less than $\frac{n}{2}$. Therefore, no matter how the agents in S misreport their preferences, the outcome at the misreported profile cannot be an alternative of $\mathcal{C}$. This, in turn, means that the agents in S will not prefer the outcome at the misreported profile, a contradiction. This completes the proof of the if part of Theorem 4.1.

Only-if part: We prove the only-if part by means of the following lemmas. For all these lemmas, assume that $\mathbb{D}$ is a path-connected domain.

Lemma A.1 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous and group strategy-proof choice function, and let a coalition S be decisive on all $(S, N \backslash S)$-unanimous profiles. Then, S is decisive.*

Proof In order to prove that S is decisive, let $p \in \mathbb{D}^n$ be an S-unanimous profile such that $p(i) = R$ for all i in S, where $\tau(R) = a$. For an S-unanimous profile p, define $k(p) = |\{p(j) : j \in N\}|$ as the number of different preferences in p. We prove the lemma by using induction on k. Note that $k \geq 2$ by definition. Note that the base case where $k = 2$ follows from the definition of $(S, N \backslash S)$-unanimous profiles. Suppose S is decisive on all S-unanimous profiles p such that $k(p) \leq \bar{k}$, for some $\bar{k} \geq 2$. We show that S is decisive on all S-unanimous profiles p such that $k(p) \leq \bar{k} + 1$. Consider $p \in \mathbb{D}^n$ such that $k(p) = \bar{k} + 1$. Since $k(p) = \bar{k} + 1$, we can partition N as $T_1, \ldots, T_{k+1}$ such that for all $l \in \{1, \ldots, k + 1\}$, there exists $R^l \in \mathbb{D}$ such that $p(i) = R^l$ for all $i \in T_l$. Since p is an S-unanimous profile, assume without loss of generality $S \subseteq T_1$. Assume for contradiction, $\varphi(p) \neq a$. Suppose $\varphi(p) = b$ for some $b \in A \backslash \{a\}$. Consider $q \in \mathbb{D}^n$ such that $q(i) = R^l$ for all $i \in T_l$ and all $l \in \{1, \ldots, k\}$ and $q(i) = R^k$ for all $i \in T_{k+1}$. Since $k(q) = \bar{k}$ by construction, we have by our induction hypothesis that $\varphi(q) = a$. By means of group strategy-proofness for the agents in T_{k+1} at p via $q|_{T_{k+1}}$, we have

$$ba \in R^{k+1}. \tag{1}$$

Now, consider the preference $r \in \mathbb{D}^n$ such that $r(i) = R^l$ for all $i \in T_l$ and all $l \in \{1, \ldots, k - 2, k\}$ and $q(i) = R^{k+1}$ for all $i \in T_k$. Since $k(r) = \bar{k}$ by construction, we have by our induction hypothesis that $\varphi(r) = a$. By means of group strategy-proofness for the agents in T_k at r via $p|_{T_k}$, we have

$$ab \in R^{k+1}. \tag{2}$$

Combining (1) and (2), $\varphi(p) = a$. This completes the proof by induction. □

Lemma A.2 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function, and let $R^a \equiv ab \cdots$ and $R^b \equiv ba \cdots$ be two preferences in $\mathbb{D}$. Suppose a coalition S is such that $\varphi((R^a)^S, (R^b)^{N \backslash S}) = a$. Then, $|S| > \frac{n}{2}$.*

Proof Assume for contradiction $|S| \leq \frac{n}{2}$. By applying symmetry to $\varphi((R^a)^S, (R^b)^{N\setminus S}) = a$, we have

$$\varphi((R^b)^S, (R^a)^{N\setminus S}) = b. \tag{3}$$

Since $|S| \leq \frac{n}{2}$, there exists $T \subseteq N\setminus S$ such that $|S| = |T|$. We write $\varphi((R^a)^S, (R^b)^{N\setminus S}) = a$ as

$$\varphi((R^a)^S, (R^b)^T, (R^b)^{N\setminus(S\cup T)}) = a. \tag{4}$$

Now, applying anonymity to (4), since $|S| = |T|$,

$$\varphi((R^b)^S, (R^a)^T, (R^b)^{N\setminus(S\cup T)}) = a. \tag{5}$$

This, together with (3), implies that agents in $N\setminus(S\cup T)$ manipulate at $((R^b)^S, (R^a)^{N\setminus S})$ via $(R^b)^{N\setminus(S\cup T)}$, a contradiction. □

Lemma A.3 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous and group strategy-proof choice function, and let $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$ be two preferences in $\mathbb{D}$. Suppose a coalition S is such that $\varphi((R^a)^S, (R^b)^{N\setminus S}) = a$. Then, S is decisive on all (a, b)-$(S, N\setminus S)$-unanimous profiles.*

Proof Consider an (a, b) -$(S, N\setminus S)$-unanimous profile $p \in \mathbb{D}^n$. Assume for contradiction, $\varphi(p) \neq a$. Suppose $\varphi(p) = x$. Consider $q \in \mathbb{D}^n$ such that $q(i) = R^a$ for all $i \in S$ and $q|_{N\setminus S} = p|_{N\setminus S}$. We claim $\varphi(q) = b$. If $\varphi(q) = a$, then by means of unanimity agents in S manipulate at p via $q|_S$, a contradiction. If $\varphi(q) \notin \{a, b\}$, then agents in S manipulate at q via some preference where b is the top-ranked alternative for all agents in S. So, $\varphi(q) = b$. However, since $\varphi((R^a)^S, (R^b)^{N\setminus S}) = a$, this means agents in $N\setminus S$ manipulate at $((R^a)^S, (R^b)^{N\setminus S})$ via $q|_{N\setminus S}$, a contradiction. □

Remark A.1 It follows from Lemma A.2 and Lemma A.3 that there is a unanimous, anonymous, symmetric, and group strategy-proof choice function $\varphi : \mathbb{D}^n \to A$ only if n is odd. This completes the proof of Corollary 4.1.

Lemma A.4 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function, and let a and b be top-connected alternatives. Then, the following two are equivalent.*

(i) *S is decisive on all (a, b)-$(S, N\setminus S)$-unanimous profiles.*
(ii) $|S| > \frac{n}{2}$.

Proof Consider $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$. By group strategy-proofness and unanimity, $\varphi((R^a)^S, (R^b)^{N\setminus S}) \in \{a, b\}$. If (i) holds, then $\varphi((R^a)^S, (R^b)^{N\setminus S}) = a$, and by Lemma A.2, $|S| > \frac{n}{2}$.

Suppose (ii) holds. If $\varphi((R^a)^S, (R^b)^{N\setminus S}) = b$, then by Lemma A.2, we have $|N\setminus S| > \frac{n}{2}$, a contradiction to $|S| > \frac{n}{2}$. So, $\varphi((R^a)^S, (R^b)^{N\setminus S}) = a$. By Lemma A.2, this implies S is decisive on all (a, b) -$(S, N\setminus S)$-unanimous profiles. □

Lemma A.5 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous and group strategy-proof choice function, and let $x^1, \ldots, x^k$ be a path in $G(\mathbb{D})$ such that every three consecutive alternatives in the path are distinct. Suppose a coalition S is decisive on all (x^1, x^2)-$(S, N\setminus S)$-unanimous profiles. Then, S is decisive on all (x^s, x^t)-$(S, N\setminus S)$-unanimous profiles for all $1 \le s < t \le k$.*

Proof We prove this by using induction on the value of $t - s$ for (x^s, x^t)-$(S, N\setminus S)$-unanimous profiles. First, we prove that S is decisive on all (x^s, x^{s+1})-$(S, N\setminus S)$-unanimous profiles for all $0 \le s < k$. Consider $p \in \mathbb{D}^n$ such that $p(i) \equiv x^2x^3\cdots$ for all $i \in S$ and $p(i) \equiv x^3x^2\cdots$ for all $i \in N\setminus S$. We show that $\varphi(p) = x^2$. To ease the presentation of the proof, we use the notationR^{st} to denote a preference of the form $x^sx^t\cdots$. By our assumption, $\varphi((R^{12})^S, (R^{23})^{N\setminus S}) = x^1$. Consider a profile $((R^{12})^S (R^{32})^{N\setminus S}) \in \mathbb{D}^n$. By group strategy-proofness, $\varphi((R^{12})^S, (R^{32})^{N\setminus S}) \in \{x^1, x^2, x^3\}$. Since $\varphi((R^{12})^S, (R^{23})^{N\setminus S}) = x^1$, by using group strategy-proofness for the agents in $N\setminus S$, $\varphi((R^{12})^S, (R^{32})^{N\setminus S}) = x^1$. Consider a profile of the form $((R^{21})^S, (R^{32})^{N\setminus S})$. Since $\varphi((R^{12})^S, (R^{32})^{N\setminus S}) = x^1$, by using group strategy-proofness for agents in S, $\varphi((R^{21})^S, (R^{32})^{N\setminus S}) \in \{x^1, x^2\}$. If $\varphi((R^{21})^S, (R^{32})^{N\setminus S}) = x^1$, then agents in $N\setminus S$ manipulate at $((R^{21})^S, (R^{32})^{N\setminus S})$ via $(R^{21})^{N\setminus S}$. So, $\varphi((R^{21})^S, (R^{32})^{N\setminus S}) = x^2$. By group strategy-proofness, $\varphi((R^{23})^S, (R^{32})^{N\setminus S}) = x^2$ and by Lemma A.3, S is decisive on all (x^2, x^3)-$(S, N\setminus S)$-unanimous profiles. Continuing in this manner, it can be shown that S is decisive on all (x^s, x^{s+1})-$(S, N\setminus S)$-unanimous profiles for all $1 \le s < k$. Suppose S is decisive on all (x^s, x^t)-$(S, N\setminus S)$-unanimous profiles where $t - s \le l$ for some $l \le k - 1$. We show that S is decisive on all (x^s, x^t)-$(S, N\setminus S)$-unanimous profiles where $t - s = l + 1$. By our induction hypothesis, $\varphi((R^{(s+1)s})^S, (R^{t+1})^{N\setminus S}) = x^{s+1}$. By group strategy-proofness, this means $\varphi((R^{s(s+1)})^S, (R^{t+1})^{N\setminus S}) \in \{x^s, x^{s+1}\}$. Suppose $\varphi((R^{s(s+1)})^S, (R^{t+1})^{N\setminus S}) = x^{s+1}$. Then, by group strategy-proofness $\varphi((R^{s(s+1)})^S, (R^{(s+1)s})^{N\setminus S}) = x^{s+1}$, which contradicts our earlier step where we have shown that S is decisive on all (x^s, x^{s+1})-$(S, N\setminus S)$-unanimous profiles for all $0 \le s < k$. So, $\varphi((R^{s(s+1)})^S, (R^{t+1})^{N\setminus S}) = x^s$. By group strategy-proofness, this means $\varphi((R^s)^S, (R^{t+1})^{N\setminus S}) = x^s$ implying that S is decisive on all (x^s, x^t)-$(S, N\setminus S)$-unanimous profiles for all $0 \le s < t \le k$. This completes the proof of the lemma. □

Lemma A.6 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function, and let S be a coalition with $|S| > \frac{n}{2}$. Then, S is decisive.*

Proof By Lemma A.4, S is decisive on all (a, b) -$(S, N\setminus S)$-unanimous profiles where $a \leftrightsquigarrow b$. By Lemma A.5, S is decisive on all (x, y) -$(S, N\setminus S)$-unanimous profiles such that there exists a path in $G(\mathbb{D})$ connecting x and y. Since $G(\mathbb{D})$ is connected, this means S is decisive on all (x, y) -$(S, N\setminus S)$-unanimous profiles. Now, by Lemma A.1, we have that S is decisive on all profiles. □

The restriction of a preference $R \in \mathbb{L}(A)$ to a set $X \subseteq A$ is defined as $R|_X := \{xy : xy \in R \text{ and } xy \in X \times X\}$.

Lemma A.7 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function. Consider a path $x^1, \ldots, x^k$ in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$. Then, for all $R \in \mathbb{D}$, $\tau(R|_X) = x^k$ implies $R|_X \equiv x^k x^{k-1} \cdots x^1$, where $X = \{x^1, \ldots, x^k\}$.*

Proof Since n is odd, $n \geq 3$. Therefore, N can be partitioned into coalitions S, T and U such that $|S| = 1$, $|S \cup U| > \frac{n}{2}$, $|T \cup U| > \frac{n}{2}$ and $|S \cup T| > \frac{n}{2}$. Let a preference $R^1 \in \mathbb{D}$ be such that $\tau(R^1) = x^1$. Define the choice function $\psi : \mathbb{D}^{S \cup T} \to A$ such that for all $p \in \mathbb{D}^{S \cup T}$, $\psi(p) = \varphi(\tilde{p})$ where $\tilde{p}(i) = R^1$ for all $i \in U$ and $\tilde{p}(i) = p(i)$ for all $i \in S \cup T$. Since φ is group strategy-proof, ψ is group strategy-proof. Further, since $|S \cup T| > \frac{n}{2}$, by Lemma A.6, $S \cup T$ is decisive for φ. This together with the fact that φ is unanimous implies ψ is unanimous. Since $|S \cup U| > \frac{n}{2}$, by Lemma A.6, $\varphi(\tilde{p}) = x^1$ where $\tilde{p}(i) = R^1$ for all $i \in S \cup U$ and $\tau(\tilde{p}(i)) = x^2$ for all $i \in T$. This means $\psi(p) = x^1$, where p is a (x^1, x^2)- (S, T) -unanimous profile. Since ψ is unanimous and group strategy-proof, by Lemma A.5, we have for all $1 \leq s < t \leq k$ and all (x^s, x^t)- (S, T)-unanimous profiles q, $\psi(q) = x^s$. Using a similar logic, it follows that T is decisive on all (x^s, x^t)- (T, S)-unanimous profiles for all $1 \leq s < t \leq k$. Combining all these observations, we have

$$\varphi(q) = x^{\min\{s,t\}}. \tag{6}$$

Now, we are ready to complete the proof of the lemma. Assume for contradiction that there exists $R \in \mathbb{D}$ such that $\tau(R|_X) = x^k$ and $x^r x^s \in R$ for some $r < s$. Then, by (6), $\psi(p) = x^s$ where $p(i) = R$ for all $i \in S$ and $\tau(p(i)) = x^s$ for all $i \in T$. Consider $q \in \mathbb{D}^n$ such that $\tau(q(i)) = x^r$ for all $i \in S$ and $q|_T = p|_T$. By (6), $\psi(q) = x^r$, which means that the agents in S manipulate at p via $q|_S$ contradicting the group strategy-proofness (also, strategy-proofness) of ψ. □

Lemma A.8 *Let $\varphi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function. Then, $G(\mathbb{D})$ must be a tree, $\mathbb{D}$ must be single-peaked on $G(\mathbb{D})$, and n must be odd.*

Proof Assume for contradiction that there exists a cycle $x^1, \ldots, x^k, x^1$ in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$. Consider $R \in \mathbb{D}$ such that $\tau(R) = x^1$. Since $x^1, x^2, \ldots, x^k$ is a path in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$, by Lemma A.7, $x^2 x^3 \in R$. Again, since $x^1, x^k, x^{k-1}, \ldots, x^2$ is a path in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$, by Lemma A.7, $x^3 x^2 \in R$. However, this contradicts that R is a preference. So, $G(\mathbb{D})$ is a tree. Now, by means of Lemma A.7 it follows that $\mathbb{D}$ is single-peaked on $G(\mathbb{D})$.
Now, we show n is odd. Suppose not. Take $|S| = \frac{n}{2}$. Let $R^a \equiv ab \cdots$ and $R^b \equiv ba \cdots$. By unanimity and group strategy-proofness, $\varphi((R^a)^S, (R^b)^{N \setminus S}) \in \{a, b\}$. Assume without loss of generality, $\varphi((R^a)^S, (R^b)^{N \setminus S}) = a$. Since $|S| = \frac{n}{2}$, by anonymity of φ, this implies $\varphi((R^b)^S, (R^a)^{N \setminus S}) = a$. On the other hand, since $\varphi((R^a)^S, (R^b)^{N \setminus S}) = a$, by symmetry $\varphi((R^b)^S, (R^a)^{N \setminus S}) = b$, which is a contradiction. □

Proof of Theorem 4.2

Proof The proof of the if part of the theorem follows from the same of the if part of Theorem 4.1. We proceed to prove the only-if part.
Assume for contradiction that $\varphi(p) = x$ for some $p \in \mathbb{D}^n$ where x is such that there exists a component C in T^{-x} with $|\{i \in N : \tau(p(i)) \in C\}| \geq \frac{n}{2}$. Let $S = \{i \in N : \tau(p(i)) \in C\}$. Since n is odd, this means $|S| > \frac{n}{2}$. Consider $q \in \mathbb{D}^n$ such that $q(i) = p(i)$ for all $i \in S$ and $\tau(q(i)) = x$ for all $i \in N \backslash S$. By group strategy-proofness, $\varphi(q) = x$. Let $y \in C$ be the (unique) vertex in C such that $\{x, y\}$ is an edge in T. Consider $r \in \mathbb{D}^n$ such that $r(i) \equiv yx \cdots$ for all $i \in S$ and $r(i) = q(i)$ for all $i \in N \backslash S$. By unanimity and group strategy-proofness, $\varphi(r) \in \{x, y\}$. If $\varphi(r) = y$, then because preferences are single-peaked on T, agents in S manipulate at r via $q|_S$. So, $\varphi(r) = x$. However, since r is a (x, y) -$(S, N \backslash S)$-unanimous profile with $|S| > \frac{n}{2}$, this contradicts Lemma A.2. □

Proof of Theorem 5.1

The proof of Theorem 5.1 follows from following the steps in the proof of Theorem 4.1 with the following modifications.

Proof of Lemma A.5 Let $R, R' \in \mathbb{D}$ be such that $R \equiv x^1 x^2 \cdots$ and $R' \equiv x^3 x^2 \cdots$. Let S be a coalition and consider the profile p such that $p(i) = R$ for all $i \in S$ and $p(i) = R'$ for all $i \in N \backslash S$. In the proof of Lemma A.5, we use the fact that by unanimity and group strategy-proofness, $\varphi(p) \in \{x^1, x^2, x^3\}$. Clearly, this does not follow if we replace group strategy-proofness by strategy-proofness. However, since we additionally have the fact that the domain is strongly path-connected, this assertion follows. To see this, assume for contradiction that $\varphi(p) \notin \{x^1, x^2, x^3\}$. Consider the preference $\bar{R}$ such that $\bar{R} \equiv x^2 x^3 \cdots$, and for all $a, b \notin \{x^2, x^3\}$, $ab \in \bar{R}$ if and only if $ab \in R$. We can move the agents in $N \backslash S$ sequentially to $\bar{R}$, and each time, by strategy-proofness we can claim that the outcome will remain the same as $\varphi(p)$. Since $\varphi(p) \notin \{x^1, x^2, x^3\}$, this contradicts the assumption of the lemma that S is decisive on all (x^1, x^2)-$(S, N \backslash S)$-unanimous profiles. This completes the proof of Lemma A.5 for this case.
In every other place where group strategy-proofness is used, we can change the preferences of the agents in the corresponding group one by one (as discussed in the modified proof of Lemma A.5) and apply strategy-proofness at each step to obtain the desired conclusion. □

Independence of axioms in Theorem 4.1

In this section, we establish the independence of the conditions that we have used in Theorem 4.1 Furthermore, we show how to modify Theorem 4.1 if we replace group strategy-proofness by strategy-proofness.

In what follows, we introduce some special type of choice functions and discuss their properties. We will use these functions to establish the mentioned independence.

Let $a \in A$ be an alternative, and let $\mathbb{D}^{-a}$ be a domain such that $a \notin \tau(\mathbb{D}^{-a})$. A choice function $\varphi^a : (\mathbb{D}^{-a})^n \to A$ is called *constant at a* if $\varphi^a(p) = a$ for all profiles $p \in (\mathbb{D}^{-a})^n$. By definition, φ^a violates unanimity. Since the outcome of φ^a does not depend on the profiles, it satisfies anonymity, strategy-proofness, and group strategy-proofness. To apply symmetry, we need two preferences in the domain of the form $xy \cdots$ and $yx \cdots$ for some $x, y \in A$, and a profile where each agent has one of the two preferences such that the outcome at that profile is either x or y. Because a never appears at the top position in any preference in $\mathbb{D}^{-a}$, a cannot be one of x or y in the aforementioned preferences. Since both x and y are different from a, by definition the outcome of φ^a cannot be x or y. Thus, symmetry is vacuously satisfied by φ^a.

A choice function $\varphi_j^{dict} : \mathbb{L}(A)^n \to A$, where $j \in N$, is called *dictatorial* if $\varphi_j^{dict}(p) = \tau(p(j))$ for all $p \in \mathbb{L}(A)^n$. By definition, φ_j^{dict} satisfies unanimity, strategy-proofness, group strategy-proofness and violates anonymity. To see that φ_j^{dict} satisfies symmetry, consider a profile where a group $S, \emptyset \neq S \neq N$, of agents have a preference $P \equiv xy \cdots$ and other agents have the preference $P' \equiv yx \cdots$. Suppose that outcome of φ_j^{dict} at this profile is x. This means some agent in S is the dictator. Therefore, if agents in S and $N \backslash S$ interchange their preferences, then the top-ranked alternative of the dictator will be y, and consequently, the outcome will be y, ensuring symmetry.

A choice function $\varphi_a^{una} : (\mathbb{D}^{-a})^n \to A$, where $a \in A$, is called *unanimous with disagreement a*, if for all $p \in \mathbb{L}(A)^n$,

$$\begin{aligned} \varphi_a^{una}(p) &= b \quad \text{if} \quad \tau(p(i)) = b \text{ for all } i \in N \\ &= a \quad \text{otherwise.} \end{aligned}$$

The rule φ_a^{una} satisfies unanimity by definition. Anonymity of φ_a^{una} follows from the fact that if agents interchange their preferences, then a unanimous profile will remain unanimous and a non-unanimous profile will remain non-unanimous, and hence by definition, the outcome of φ_a^{una} will not change. Since a does not appear at the top position in any preference in $\mathbb{D}^{-a}$, as we have explained in the case of φ^a, symmetry holds vacuously for φ_a^{una}. To see that φ_a^{una} is manipulable, consider a profile where some alternative b is the top-ranked alternative of every agent except agent 1 and b is preferred to a for agent 1. By definition, the outcome of φ_a^{una} at this profile is a. However, if agent 1 misreports her preference as one with b at the top position, then the outcome will become b and agent 1 will be strictly better off. So, φ_a^{una} is not strategy-proof, and hence, it is not group strategy-proof either.

For the next choice function and its (restricted) domain, let the alternatives be numbered as $a_1, \ldots, a_m$. To ease our presentation, whenever we use minimum or maximum of a set of alternatives, we mean it with respect to the ordering $a_1 \prec \cdots \prec a_m$. A domain $\mathcal{S}$ is said to be semi-single-peaked domain if for all R in $\mathcal{S}$, $\tau(R) = a_k$ implies $R \equiv a_k \cdots a_{k-1} \cdots a_{k-2} \cdots a_2 \cdots a_1 \cdots$. Thus, each preference in a semi-single-peaked domain maintains single-peakedness *only* on the left side of the peak (that is, the top-ranked alternative); that is, as one moves away from the peak on the left side, preference declines. Note that there is no restriction on the relative ordering of two alternatives if at least one of them is on the right of the peak.

A choice function $\varphi^{low} : \mathcal{S}^n \to A$ is called *lowest peak* if for all $p \in \mathcal{S}^n$,

$$\varphi^{low}(p) = \min\{\tau(p(i)) : i \in N\}.$$

As the name suggests, φ^{low} selects the minimum peak (with respect to the ordering $\prec$) at every profile. Unanimity and anonymity of φ^{low} follow from the definition. In what follows, we argue that φ^{low} satisfies group strategy-proofness (and hence strategy-proofness).

Table 6 Independence of axioms in Theorem 4.1

	φ^a on $(\mathbb{D}^{-a})^n$	φ_j^{dict} on $\mathbb{L}(A)^n$	φ^{low} on $\mathcal{S}^n$	φ_a^{una} on $(\mathbb{D}^{-a})^n$
Unanimity	No	Yes	Yes	Yes
Anonymity	Yes	No	Yes	Yes
Symmetry	Yes	Yes	No	Yes
Group strategy-proofness	Yes	Yes	Yes	No
Strategy-proofness	Yes	Yes	Yes	No

Suppose a group of agents S manipulate φ^{low} at a profile p. Let $\min(p)$ be the minimum peak of p. Since $\varphi(p) = \min(p)$, the (sincere) peak of each agent in S must be strictly on the right of $\min(p)$. This in particular means that the peak of some agent outside S is $\min(p)$. Therefore, by the definition of φ, the only way the agents in S can change the outcome is to declare a peak which is on the (further) left of $\min(p)$. This will push the outcome to the left of $\min(p)$ as well. Since the sincere peaks of the agents in S are on the right of $\min(p)$ and the changed outcome is on the left of $\min(p)$, by the definition of semi-single-peakedness, the changed outcome will become even worse for them. So, no group of agents can manipulate φ^{low} at any profile.

Finally, we explain that φ^{low} does not satisfy symmetry. Consider two preferences $R \equiv a_k a_{k+1} \cdots$ and $R' \equiv a_{k+1} a_k \cdots$, and consider a profile p where each agent in a group S, $\emptyset \neq S \neq N$, has the preference R and each remaining agent has the preference R'. By the definition of φ^{low}, $\varphi^{low}(p) = a_k$. Now, consider the profile p' where each agent in S has the preference R' and each remaining agent has the preference R. In order to satisfy symmetry, the outcome at this profile must be x_{k+1}, but by the definition of φ^{low}, the outcome is x_k.

In Table 6, we present the conditions that are satisfied by each of the above-mentioned choice functions. Note that this table establishes the independence of the conditions that are used in Theorem 4.1.

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