Rationalizable strategies in games with incomplete preferences

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Abstract

This paper introduces a new solution concept for games with incomplete preferences. The concept is based on rationalizability and it is more general than the existing ones based on Nash equilibrium. In rationalizable strategies, we assume that the players choose nondominated strategies given their beliefs of what strategies the other players may choose. Our solution concept can also be used, e.g., in ordinal games where the standard notion of rationalizability cannot be applied. We show that the sets of rationalizable strategies are the maximal mutually nondominated sets. We also show that no new rationalizable strategies appear when the preferences are refined, i.e., when the information gets more precise. Moreover, noncooperative multicriteria games are suitable applications of incomplete preferences. We apply our framework to such games, where the outcomes are evaluated according to several criteria and the payoffs are vector valued. We use the sets of feasible weights to represent the relative importance of the criteria. We demonstrate the applicability of the new solution concept with an ordinal game and a bicriteria Cournot game.

Keywords Normal-form games · Incomplete preferences · Rationalizable strategies · Nondominated strategies · Multicriteria games

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1 Introduction

This paper examines games with incomplete preferences (Bade 2005). A player has incomplete preferences when he is unable to compare or is indecisive between some of the outcomes. This may be due to the fact that the outcomes are represented by multiple conflicting criteria, that the outcomes are simply uncertain, or that the player represents a group of individuals. Incomplete preferences have been mainly studied in the context of decision-making problems (Aumann 1962; Ok 2002; Heller 2012). In non-cooperative games, incomplete preferences have been studied in the special case of multicriteria games (Shapley 1959; Blackwell 1956; Corley 1985; Borm et al. 1988; de Marco and Morgan 2007; Marmol et al. 2017). Excluding the multicriteria games, only few papers have examined incomplete preferences in non-cooperative game models (Bade 2005; Park 2015; Shafer and Sonnenchein 1975). This paper extends the solution concept of rationalizability to the games with incomplete preferences.

Nash equilibrium is the main solution concept in game theory and it assumes that the players have correct beliefs about their opponents' strategies, i.e., they know the strategies that their opponents are going to choose. Rationalizability is a more general solution concept that allows the players to have erroneous but rational beliefs (Bernheim 1984; Pearce 1984), which means that the set of rationalizable strategies always contains the set of Nash equilibria. The standard rationalizability means that each player maximizes the expected utility given her probabilistic belief. However, the maximization of expected utility may not be possible under incomplete preferences or nonprobabilistic beliefs. This may, e.g., happen in ordinal games (Durieu et al. 2008) where the players only have a preference order but no numeric values are assigned to the outcomes. Thus, we need to generalize the definition of rationalizability.

We propose a notion of rationalizable strategies, where the players have non-probabilistic beliefs and they choose nondominated strategies given their beliefs. Nonprobabilistic beliefs mean that the players only reason about what pure strategies the opponents may choose but do not need to assign any probabilities to these strategies. Nondomination means that a player does not select a strategy if another strategy yields a better outcome with all combinations of the possible strategies of the other players given her belief. The probabilistic and nonprobabilistic beliefs have been discussed in Perea (2014). Chen et al. (2016) extend rationalizability beyond the probabilistic beliefs and expected utility maximization. The non-expected utility models have also been examined in Jungbauer and Ritzberger (2011) and Beauchêne (2016). Moreover, the closed under rational behavior (curb) sets (Basu and Weibull 1991) are closely related to the rationalizable strategies and our solution concept as well. Especially, the maximal tight curb sets are very close to our notion when the players' preferences are complete, except we use nondominated strategies whereas the curb sets are defined by the best-response correspondences.

To our knowledge, this paper is the first to study rationalizability in the games with incomplete preferences. Bade (2005) has shown that the Nash equilibria in games with incomplete preferences correspond to the union of Nash equilibria in all the completions of the game. More recently, Park (2015) has examined the existence of Nash equilibrium in potential games with incomplete preferences. Incomplete preferences have also been considered in nonmonetized games (Li 2013; Xie et al. 2013),

where the preference order is defined in a common outcome space for all players, whereas in our framework, the preference orders for each player are defined directly over the strategy combinations. The literature on nonmonetized games is also focused on generalizations of Nash equilibrium.

In this paper, we show that the sets of rationalizable strategies are the maximal mutually nondominated sets. The mutual nondominance is defined so that the sets of strategies are mutually nondominated if they contain no dominated strategies with respect to each other. We also show that providing more precise preference information does not enlarge the sets of rationalizable strategies. That is, if new preferences are added over some pairs of outcomes for which a player was previously indecisive and all the original preferences are maintained, then there will be no new rationalizable strategies.

We apply our framework to multicriteria games (Shapley 1959; Blackwell 1956; Corley 1985; Borm et al. 1988; de Marco and Morgan 2007; Marmol et al. 2017), which is a special class of games with incomplete preferences. In the multicriteria games, the outcomes are evaluated as vector-valued payoffs, where each component describes how good the outcome is with respect to that particular criterion. Since an outcome can be better in one criterion but worse in another criterion, it is natural that the players may have incomplete preferences over the outcomes in multicriteria games. The main solution concept in the existing literature on multicriteria games is the multicriteria extension of Nash equilibrium (e.g., Shapley 1959; Corley 1985; Borm et al. 1988; de Marco and Morgan 2007; Marmol et al. 2017). A combination of strategies is an equilibrium if the strategy of each player is nondominated when other players play the equilibrium strategies. However, this equilibrium concept does not take into account information about the relative importance of criteria.

In this paper, we use weights to represent the importance of the criteria and allow the weights to be set-valued. This idea has been proposed in the multicriteria/multiattribute decision analysis literature (e.g., White et al. 1982; Kirkwood and Sarin 1985; Hazen 1986; Weber 1987; Salo and Hämäläinen 1992, 2010). In the game context, Monroy et al. (2009) has considered the sets of feasible weights in a cooperative bargaining setting, while in the noncooperative setting using sets of feasible weights has been proposed only by us and independently by Marmol et al. (2017), who uses the Nash equilibrium as the solution concept. Our framework enables analyzing the impact of additional preference information about the relative importance of the criteria to the solutions of the multicriteria games. To our knowledge, this is the first paper to consider rationalizable strategies in the multicriteria games while representing incomplete preference information as sets of feasible weights.

The paper is structured as follows. Games with incomplete preferences as well as the rationality concept are defined in Sect. 2, and the rationalizable strategies are defined in Sect. 3. Sect. 4 provides properties of the sets of rationalizable strategies. First, the characterization in terms of mutual dominance is given in Sect. 4.1. Then, the relation between rationalizable strategies and the iterative elimination of dominated strategies is discussed in Sect. 4.2. The existence of the rationalizable strategies in the case of finite strategy sets and possible nonexistence in the infinite case are shown in Sect. 4.3. The result that adding preference information does not lead to any additional rationalizable strategies is shown in Sect. 4.4. Multicriteria games with incomplete

preference information are discussed in Sect. 5. Examples of a game with incomplete preferences with a finite number of strategies as well as of a multicriteria game with an infinite number of strategies are given in Sect. 6. Finally, concluding remarks are presented in Sect. 7.

2 Games with incomplete preferences

2.1 Elements of the game

Definition 1 (Bade 2005) A game with incomplete preferences consists of the following components:

- The set of players $I = \{1, \dots, n\}$.
- For each player $i \in I$, the set of strategies S_i .
- For each player $i \in I$, the preference relation: a transitive and reflexive binary relation $\succeq_i$ defined on $S = S_1 \times \dots \times S_n$.

The players are assumed to select their strategies simultaneously. The combination of the selected strategies then implies the outcome of the game and, thus, the outcome is formally defined as the combination of the selected strategies. Player $i \in I$ strictly prefers the outcome implied by strategies $(s_1, \dots, s_n) \in S$ to the outcome implied by strategies $(s'_1, \dots, s'_n) \in S$, denoted by $(s_1, \dots, s_n) \succ_i (s'_1, \dots, s'_n)$, if $(s_1, \dots, s_n) \succeq_i (s'_1, \dots, s'_n)$ and $(s'_1, \dots, s'_n) \not\succeq_i (s_1, \dots, s_n)$. If both $(s_1, \dots, s_n) \succeq_i (s'_1, \dots, s'_n)$ and $(s'_1, \dots, s'_n) \succeq_i (s_1, \dots, s_n)$, player i is indifferent between the outcomes. The relation $\succeq_i$ is allowed to be incomplete so that neither $(s_1, \dots, s_n) \succeq_i (s'_1, \dots, s'_n)$ nor $(s'_1, \dots, s'_n) \succeq_i (s_1, \dots, s_n)$ holds for some pair of outcomes. This means that the outcomes are incomparable for the player.

2.2 Rationality concept

We do not assume that the players possess probabilistic assessments about what strategies the opponents may select. Instead, we only assume that a player holds a nonprobabilistic belief of possible selections of strategies taken by the opponents. Without probabilities, the only assumption that can be made about the preferences over strategies is that a rational player does not select a strategy that leads to worse outcomes than some other strategy no matter which strategies the opponents select among those that are possible according to her belief. Thus, we take rationality to mean that the players select nondominated strategies with respect to their beliefs. The set of nondominated strategies for player i with belief $B_i \subseteq S_{-i}$ is¹ hereafter denoted by $\text{ND}(i, B_i)$ and defined as:

$$\text{ND}(i, B_i) = \{s_i \in S_i \mid \nexists s'_i \in S_i : \forall s_{-i} \in B_i : (s'_i, s_{-i}) \succ_i (s_i, s_{-i})\}. \quad (1)$$

¹ We use the notational convention that S_{-i} denotes the $(n - 1)$ -tuple of sets of strategies S_k for $k \neq i$. Expressions such as (s_i, s_{-i}) are implicitly assumed to refer to the tuple $(s_1, \dots, s_n)$ in the correct order.

It should also be noted that only the strict preferences $\succ$ are important for our rationality concept since it is based on domination.

A useful property of ND is that all strategies nondominated with respect to a belief remain nondominated if the belief is replaced with another belief containing additional possible strategies of the opponents. This is known as the monotonicity property and it is formalized in the following remark.

Remark 1 Let A and B be two beliefs of player i , and $A \subseteq B$. Then, $\text{ND}(i, A) \subseteq \text{ND}(i, B)$.

Our rationality concept formulated in this section can be seen as a relaxation of a similar rationality concept based on nonprobabilistic beliefs, viz. rationality* defined by Chen et al. (2007). Here, the difference is that the preferences can be incomplete and we use nondominated strategies.

3 Definition of rationalizable strategies

Bernheim (1984) motivates the concept of rationalizable strategies as the logical conclusion of assuming that the players view the selections of the opponents' strategies as uncertain events, that the players follow Savage's axioms of individual rationality, and that the latter as well as the game (i.e., the strategies and the payoffs) are common knowledge. In this paper, we assume the selection of nondominated strategies in the sense of Sect. 2.2 instead of Savage rationality. Furthermore, in our case, the game being common knowledge means that the strategies and the preference relations are common knowledge. The definition of rationalizable strategies used here is obtained by modifying the standard definition of Bernheim (1984) accordingly.

The following notation is taken from Bernheim (1984). Let Δ_I^i be the set of sequences $(i_1, \dots, i_m)$, $i_j \in I$ for $1 \leq j \leq m$, where $1 \leq m < \infty$, $i_1 \neq i$, and $i_j \neq i_{j+1}$ for all $1 \leq j \leq m$. Denote the last element of $\delta \in \Delta_I^i$ by $l(\delta)$, and a sequence formed by adding j to the end of δ by $\delta + (j)$. Similarly, a sequence formed by concatenating δ_1 and δ_2 is denoted by $\delta_1 + \delta_2$.

Definition 2 A mapping $\Theta : \Delta_I^i \rightarrow P(S_1) \cup \dots \cup P(S_n)$, where $P(S_k)$ denotes the power set of S_k , is called a system of beliefs for player i iff $\forall \delta \in \Delta_I^i : \Theta(\delta) \subseteq S_{l(\delta)}$.

The interpretation is that for $\delta = (i_1, \dots, i_m)$, $\Theta(\delta)$ is the set of strategies s for which player i believes that i_1 may believe that i_2 may believe that $\dots$ that i_{m-1} may believe that i_m may select s . Naturally, such strategies must belong to the set of strategies of player i_m , i.e., $\Theta(\delta)$ is a subset of $S_{l(\delta)}$. Bernheim (1984) requires $\Theta(\delta)$ to be a Borel subset. In our framework with nonprobabilistic beliefs, no such assumption is necessary and thus the systems of beliefs are allowed to contain any subsets of the strategy sets S_k .

The common knowledge of rationality implies that i believes that i_1 believes that $\dots$ i_{m-1} believes that i_m is rational. Hence, the strategies that i believes that i_1 believes that $\dots$ i_{m-1} believes that i_m may select must be nondominated with respect to the strategies that i believes that i_1 believes that $\dots$ i_{m-1} believes that i_m believes that

her opponents may select. This implies the following consistency condition for the systems of beliefs.

Definition 3 A system of beliefs of player i , denoted by Θ , is consistent iff $\forall \delta \in \Delta_I^i$

$$\Theta(\delta) \subseteq \text{ND}(l(\delta), \times_{j \neq l(\delta)} \Theta(\delta + (j))). \quad (2)$$

Bernheim (1984) requires that each strategy in $\Theta(\delta)$ is a best response to some probability distribution over the strategies of $l(\delta)$'s opponents, where only strategies that i believes that i_1 believes that i_2 believes that $\dots$ that $l(\delta)$ considers possible may have nonzero probability. Our definition captures the idea without probability distributions and with nondominance instead of best responses. Finally, the rationalizable strategies of player i are the strategies that are nondominated with respect to some consistent system of beliefs for player i , which leads to the following definition.

Definition 4 Strategy $s_i \in S_i$ is a rationalizable strategy iff a consistent system of beliefs Θ of player i exists such that

$$s_i \in \text{ND}(i, \times_{j \neq i} \Theta((j))). \quad (3)$$

The set of rationalizable strategies for player i is hereafter denoted by S_i^R . The following remark shows that the strategies contained in a consistent system of beliefs are rationalizable.

Remark 2 Let Θ be a consistent system of beliefs for player i and $\delta \in \Delta_I^i$. Then, $\Theta(\delta) \subseteq S_{l(\delta)}^R$.

Proof If $\delta' \in \Delta_I^{l(\delta)}$, then $\delta + \delta' \in \Delta_I^i$. Therefore, we may define a system of beliefs Θ' for player $l(\delta)$ as follows:

$$\Theta'(\delta') = \Theta(\delta + \delta'). \quad (4)$$

Consistency of Θ then implies consistency of Θ' as well as that the condition of Eq. (3) is fulfilled for any $s_j \in \Theta(\delta)$. Thus, any such s_j is rationalizable. $\square$

4 Properties of rationalizable strategies

In this section, we first characterize the sets of rationalizable strategies as the largest mutually nondominated sets of players' strategies. Then, we show that the iterative elimination of dominated strategies never removes rationalizable strategies. Furthermore, in the case of finite strategy sets, the iterative elimination converges exactly to the rationalizable strategies, which in turn implies the existence of rationalizable strategies in the finite case.

These results are close in spirit to those of Chen et al. (2007) who showed that their iterative elimination concept, IESDS*, converges to the largest stable set with respect to dominance, which in turn is the implication of common knowledge of rationality*

as defined in their paper. We give independently developed proofs of our results since our rationality concept differs from rationality* in that we allow the preferences to be incomplete. Our solution concept is also related to the maximal tight curb sets (Basu and Weibull 1991; Jungbauer and Ritzberger 2011) except that we allow incomplete preferences and use nondominated strategies rather than best response.

Note that while IESDS* uses uncountably infinite number of iterations, we focus only on countable iteration, since our main motivation for studying the iterative elimination is to show the existence of rationalizable strategies in the finite case, and to use it as a practical algorithm for actually finding the rationalizable strategies in games.

In Sect. 4.4, we formulate and prove the result that adding preference information to the preference relations does not lead to any new rationalizable strategies. This result is novel and has no correspondence in the rationality* framework of Chen et al. (2007) since they do not consider incomplete preferences.

4.1 Characterization

The sets of rationalizable strategies are characterized here in terms of mutually nondominated subsets. We define mutually nondominated subsets as subsets consisting of strategies that are nondominated with respect to belief that the opponents select from the same mutually nondominated sets. This concept is an adaptation of the best response property used by Bernheim (1984) to the rationality concept used in this paper. In the following, we show (1) that the sets of rationalizable strategies must be mutually nondominated, (2) that any mutually nondominated sets are subsets of the rationalizable strategies, and (3) that the union of all mutually nondominated sets is mutually nondominated. This implies that the set of rationalizable strategies is the union of all mutually nondominated sets. In other words, the rationalizable strategies comprise the maximal mutually nondominated set.

Definition 5 An n -tuple of sets of strategies for each player $(S'_1, \dots, S'_n)$ is mutually nondominated, denoted by $(S'_1, \dots, S'_n) \in \text{MND}$, if

$$\forall i \in I : S'_i \subseteq \text{ND}(i, S'_{-i}). \quad (5)$$

Lemma 1 *The sets of rationalizable strategies are mutually nondominated, i.e., $S^R = (S_1^R, \dots, S_n^R) \in \text{MND}$.*

Proof All strategies contained in consistent systems of beliefs must be rationalizable (Remark 2). On the other hand, a rationalizable strategy must be nondominated with respect to strategies contained in a consistent system of beliefs. Therefore, a rationalizable strategy is nondominated with respect to a subset of the rationalizable strategies. Based on Remark 1, this implies that a rationalizable strategy is nondominated with respect to the rationalizable strategies. Hence, the sets of rationalizable strategies are mutually nondominated. $\square$

Lemma 2 *All strategies contained in an n -tuple of mutually nondominated sets of strategies are rationalizable, i.e.,*

$$\forall (S'_1, \dots, S'_n) \in \text{MND} : S'_1 \subseteq S_1^R, \dots, S'_n \subseteq S_n^R.$$

Proof Define the following systems of beliefs for all players: $\forall \delta \in \Delta_I^i : \Theta(\delta) = S'_{I(\delta)}$. Mutual nondominance implies consistency of Θ and that Θ rationalizes all strategies in $(S'_1, \dots, S'_n)$. $\square$

Definition 6 We denote by $\bigcup \text{MND}$ the ordered n -tuple $(\bigcup \text{MND}_1, \bigcup \text{MND}_2, \dots, \bigcup \text{MND}_n)$ such that for all $i \in I$, $\bigcup \text{MND}_i = \bigcup \{S'_i \mid \exists S' = (S'_1, \dots, S'_n) \in \text{MND}\}$.

Note that this is an abuse of notation since $\bigcup \text{MND}$ is not technically the union of MND but the tuple of the unions of the components of the members of MND. See the example after Theorem 1.

Lemma 3 $\bigcup \text{MND} \in \text{MND}$.

Proof For any strategy s_i in $\bigcup \text{MND}$, there exists a $S' \in \text{MND}$ such that $s_i \in S'_i$. By the definition of MND, this implies

$$s_i \in \text{ND}(i, S'_{-i}). \quad (6)$$

Furthermore, ND has the property that all nondominated strategies remain nondominated if the belief set is replaced with its superset. Therefore,

$$s_i \in \text{ND}\left(i, \bigcup \text{MND}_{-i}\right), \quad (7)$$

and thus $\bigcup \text{MND}$ satisfies Definition 5. $\square$

Theorem 1 $S^R = \bigcup \text{MND}$.

Proof Lemma 3 states that $\bigcup \text{MND} \in \text{MND}$. Hence, Lemma 2 implies that $\bigcup \text{MND} \subseteq S^R$. On the other hand, Lemma 1 states that $S^R \in \text{MND}$. Hence, $S^R \subseteq \bigcup \text{MND}$. Therefore, $S^R = \bigcup \text{MND}$. $\square$

For example, let the rationalizable strategies be T and B for player 1 and L for player 2. Then, we have $\text{MND} = (\{T, B\}, \{L\})$, $(\{T\}, \{L\})$, $(\{B\}, \{L\})$, i.e., MND consists of three tuples. We get that $\bigcup \text{MND} = (\{T, B\}, \{L\})$ since $\bigcup \text{MND}_1 = \{T, B\} \cup \{T\} \cup \{B\} = \{T, B\}$. We also have $\bigcup \text{MND}_2 = (\{T, B\}, \{L\}) \in \text{MND}$.

4.2 Iterative elimination of dominated strategies

Next, we show that the iterative elimination of dominated strategies never removes rationalizable strategies. If the strategy sets are finite, the iterative elimination of dominated strategies converges to the rationalizable strategies. However, in the case of infinite strategy sets, nonrationalizable strategies may survive the iterative elimination.

Definition 7 We define the sets of strategies surviving k steps of the iterative elimination recursively as follows. $\forall i \in I : S_i^0 = S_i$ and

$$\forall i \in I, k \in \mathbb{N} : S_i^{k+1} = \text{ND}(i, S_{-i}^k). \quad (8)$$

Note that clearly $\forall i, k : S_i^{k+1} \subseteq S_i^k$. Then, the strategies surviving the iterative elimination of dominated strategies are

$$\forall i : S_i^\infty = \bigcap_{k \in \mathbb{N}} S_i^k. \quad (9)$$

Theorem 2 *All rationalizable strategies survive the iterative elimination of dominated strategies, i.e., $S_i^R \subseteq S_i^\infty$.*

Proof We show by induction that the set of rationalizable strategies of player i is a subset of the strategies surviving k steps of the iterative elimination for all k . Initially, $S_i^0 = S_i$ and thus $S_i^R \subseteq S_i^0$. Assume that $\forall i \in I : S_i^R \subseteq S_i^k$. The sets of rationalizable strategies are mutually nondominated and thus also nondominated with respect to S_{-i}^k . Hence $\forall i \in I : S_i^R \subseteq S_i^{k+1}$. $\square$

Theorem 3 *If the strategy sets S_i are finite, $\exists K : k > K \Rightarrow \forall i S_i^k = S_i^R$.*

Proof The strategy sets S_i are finite and thus strategies can be removed from them for only a finite number of times. Therefore, $\exists K : \forall k > K : \forall i S_i^k = S_i^{k+1}$. Assume $k > K$. Then, because $\forall i S_i^k = S_i^{k+1}$, the sets S_i^k are mutually nondominated and, therefore, by Lemma 2, $\forall i S_i^k \subseteq S_i^R$. On the other hand, according to Theorem 2, $\forall i S_i^R \subseteq S_i^\infty$. Therefore, $\forall i, \forall k > K : S_i^k = S_i^R$. $\square$

Remark 3 There exists a game with incomplete preferences for which $S_i^R \neq S_i^\infty$.

Proof Consider the following game with two players where the players pick numbers from the set of natural numbers extended with a “small infinity” (∞') and a “big infinity” (∞). The player who selects the largest number wins. A player always prefers a win to a tie and a tie to a loss. Furthermore, between two losing outcomes, the players prefer ones where they select larger numbers. However, the players have no preference between two winning outcomes. Formally, the sets of strategies are $S_1 = S_2 = \{0, 1, 2, \dots, \infty', \infty\}$ and the preference relation of player 1, $\succ_1$, is defined so that $(n_1, n_2) \succ_1 (m_1, m_2)$ if and only if

- $m_1 < m_2$ and $n_1 \geq n_2$, or
- $m_1 = m_2$ and $n_1 > n_2$, or
- $m_1 < m_2$ and $n_1 > m_1$,

where the relations $>$ and $<$ are extended so that $\forall k \in \mathbb{N} : \infty, \infty' > k$ and $\infty > \infty'$. The iterative elimination removes at each step the smallest number from the remaining strategy sets of both players, and thus

$$\forall k \in \mathbb{N} : S_1^k = S_2^k = \{k, k+1, \dots, \infty', \infty\}, \quad (10)$$

and, therefore, $S_1^\infty = S_2^\infty = \{\infty', \infty\}$. However, these sets are not mutually nondominated as only ∞ is nondominated with respect to the belief that the opponent selects from $\{\infty', \infty\}$.

On the other hand, if one considers a transfinite number of elimination rounds as in, e.g., Chen et al. (2007); Lipman (1994), the elimination would converge to the rationalizable strategies $(\{\infty\}, \{\infty\})$. $\square$

4.3 Existence

Here, we show in Theorem 4 that when the strategy sets are finite, the sets of rationalizable strategies are nonempty. However, in the case of infinite strategy sets, the existence of rationalizable strategies is not guaranteed, which is illustrated in Remark 4.

Theorem 4 *If the strategy sets are finite, the sets of rationalizable strategies are nonempty.*

Proof Theorem 3 implies that the iterative elimination of dominated strategies converges to the set of rationalizable strategies. When the strategy sets are finite, a step of the iterative elimination of dominated strategies will never remove all strategies. Hence, the sets of rationalizable strategies are nonempty. $\square$

Remark 4 When the strategy sets are allowed to be infinite, the sets of rationalizable strategies may be empty.

Proof Consider the game discussed in Remark 3 with the strategies ∞ and ∞' removed. For this game,

$$\forall k : S_1^k = S_2^k = \{k, k+1, \dots\}, \quad (11)$$

and thus $S_1^\infty = S_2^\infty = \emptyset$. Then, by applying Theorem 2 one can conclude that $S_1^R = S_2^R = \emptyset$. $\square$

4.4 Effect of additional preference information

Next, we consider the effect of taking into account additional information about the preferences of the players. That is, how the sets of rationalizable strategies change when we extend the preference relations. We denote the dependence of the rationalizable strategies and mutually nondominated sets on the preference relations by $S^R(>_1, \dots, >_n)$, $\text{MND}(>_1, \dots, >_n)$.

Definition 8 The preference relation $>'$ extends the preference relation $>$ if $(s_1, \dots, s_n) >_i (s'_1, \dots, s'_n) \rightarrow (s_1, \dots, s_n) >'_i (s'_1, \dots, s'_n)$ for any $i \in I$ and $(s_1, \dots, s_n), (s'_1, \dots, s'_n) \in S_1 \times \dots \times S_n$.

Remark 5 The concept of extending can be defined for the weak preferences $\geq$ by requiring both $(s_1, \dots, s_n) >_i (s'_1, \dots, s'_n) \Rightarrow (s_1, \dots, s_n) >'_i (s'_1, \dots, s'_n)$ and $(s_1, \dots, s_n) \geq_i (s'_1, \dots, s'_n) \Rightarrow (s_1, \dots, s_n) \geq'_i (s'_1, \dots, s'_n)$. This is interpreted as adding strict preferences and indifferences between pairs of outcomes that were incomparable. This corresponds to the definition of completion by Bade (2005), except that we do not require the extended preference relations to be complete. However, since our solution concept is based on strict preferences only, the possible addition of indifferences is irrelevant, and thus for simplicity we formulate the result in terms of Definition 8.

Theorem 5 *If $>'$ extends $>$, then $S^R(>'_1, \dots, >'_n) \subseteq S^R(>_1, \dots, >_n)$.*

Proof According to Lemma 1, $S^R(\succ'_1, \dots, \succ'_n) \subseteq \text{MND}(\succ'_1, \dots, \succ'_n)$, i.e.,

$$\begin{aligned} \forall i \in I : \forall s_i \in S_i^R(\succ'_1, \dots, \succ'_n) : \nexists s'_i \in S_i : \\ \forall s_{-i} \in S_{-i}^R(\succ'_1, \dots, \succ'_n) : (s'_i, s_{-i}) \succ'_i (s_i, s_{-i}). \end{aligned} \quad (12)$$

Then, the assumption $s \succ_i s' \rightarrow s \succ'_i s'$ implies that the nonexistence in Eq. (12) extends to $\succ$ so that

$$\begin{aligned} \forall i \in I : \forall s_i \in S_i^R(\succ'_1, \dots, \succ'_n) : \nexists s'_i \in S_i : \\ \forall s_{-i} \in S_{-i}^R(\succ'_1, \dots, \succ'_n) : (s'_i, s_{-i}) \succ_i (s_i, s_{-i}). \end{aligned} \quad (13)$$

Thus, $S^R(\succ'_1, \dots, \succ'_n) \in \text{MND}(\succ_1, \dots, \succ_n)$. Lemma 2 then implies $S^R(\succ'_1, \dots, \succ'_n) \subseteq S^R(\succ_1, \dots, \succ_n)$. $\square$

The above result can be interpreted as follows. The new preference relations $\succ'$ are more accurate than $\succ$. Here, more accurate is understood to mean that while all preference statements composing the original incomplete preference information are correct, the more accurate information contains additional preferences between outcomes that were incomparable according to the original information. Theorem 5 then shows that incomplete preference information will not cause the exclusion of strategies that might be selected with more accurate information about the preferences. On the other hand, more accurate preference information may lead to ruling out more strategies.

It should be noted that Theorem 5 holds also when we allow adding both indifference and strict preferences into the preference relations $\succeq$ (see Remark 5).

5 Multicriteria games

Multicriteria games (Shapley 1959; Blackwell 1956; Corley 1985; Borm et al. 1988; Ghose and Prasad 1989; Zhao 1991; de Marco and Morgan 2007; Monroy et al. 2009) are games where players evaluate outcomes according to several criteria and thus the outcomes correspond to vector-valued payoffs. If one outcome is better than another with respect to one criterion but worse with respect to another criterion, a player may not be able to state her preference between these outcomes. Therefore, the following multicriteria game is naturally a game with incomplete preferences.

Definition 9 A multicriteria game is a game with incomplete preferences (c.f. Definition 1) defined so that

- For each player $i \in I$, we have a vector-valued payoff function $f^i : S \rightarrow \mathbb{R}^m$.
- The preferences of the players are defined so that $s \succeq_i s'$ iff

$$\forall k \in \{1, \dots, m\} : f_k^i(s) \geq f_k^i(s'). \quad (14)$$

Remark 6 In a multicriteria game according to Definition 9, the strict preferences are such that $s \succ_i s'$ iff

$$\exists k \in \{1, \dots, m\} : f_k^i(s) > f_k^i(s'), \quad (15)$$

$$\forall k \in \{1, \dots, m\} : f_k^i(s) \geq f_k^i(s'). \quad (16)$$

Multicriteria games have been analyzed in the literature mainly from the point of view of equilibrium strategies (e.g., Shapley 1959; Corley 1985; Borm et al. 1988; de Marco and Morgan 2007). Ghose and Prasad (1989) considered additionally solutions based on the so-called security strategies, and Zhao (1991) discussed cooperative solutions. Since the multicriteria games are a special case of games with incomplete preferences, rationalizable strategies and the rationality concept elaborated in this paper can be applied to the multicriteria games as well.

5.1 Incomplete preference information

In the existing literature (e.g., Corley 1985; Borm et al. 1988), information about the relative importance of criteria has been taken into account by introducing weight vectors so that each component of the payoff vectors is weighted according to its relative importance. A multicriteria game is then turned into a scalar game via multiplying the payoff vectors by these weight vectors. However, defining the weights would require complete information about the preferences of the players. Monroy et al. (2009) mentioned the possibility of using incomplete information about the weights, in the form of inequality constraints, to obtain a unique bargaining solution. Preference programming (Salo and Hämäläinen 2010) is a similar idea in the multicriteria decision analysis literature. In preference programming, incomplete preference information of a decision maker is represented by a set of feasible weights. In this paper, we apply the concept of preference programming into multicriteria games as follows. The preferences of player i are described by a set of feasible weights $W_i \subseteq W^0$, where $W^0 = \{w \in \mathbb{R}^m : w_k \geq 0, \sum w_k = 1\}$. Here, the k th component of a weight vector w describes the relative importance of the k th criterion. The player prefers an outcome to another if it is better with at least some weight vector in the set of feasible weights and at least as good with all weights in the set of feasible weights. This leads to the following game:

Definition 10 A multicriteria game with incomplete preference information is a game with incomplete preferences (c.f. Definition 1) defined so that for each player $i \in I$:

- A vector valued payoff function: $f^i : S \rightarrow \mathbb{R}^m$ is given.
- The set of feasible weights $W_i \subseteq W^0 = \{w \in \mathbb{R}^m : w_k \geq 0, \sum w_k = 1\}$ is given.
- The preference relation $\succeq_i$ is defined so that $s \succeq_i s'$ iff

$$\forall w \in W_i : w^\top f^i(s) \geq w^\top f^i(s'). \quad (17)$$

Note that $w^\top f^i$ is linear in w and thus W_i can be replaced with the set of the extreme points of W_i in Eq. (17) if W_i are polyhedral sets.

Note that the relation defined by Eq. (17) is reflexive and transitive and thus Definition 10 indeed defines a game with incomplete preferences following Definition 1. Furthermore, the multicriteria game of Definition 9 is a special case of Definition 10 where the sets of feasible weights are equal to the set of all possible weights W^0 .

Remark 7 In a multicriteria game with incomplete preference information according to Definition 10, the strict preference $s \succ_i s'$ is equivalent to

$$\exists w \in W_i : w^\top f^i(s) > w^\top f^i(s'), \quad (18)$$

$$\forall w \in W_i : w^\top f^i(s) \geq w^\top f^i(s'). \quad (19)$$

In preference programming, additional preference information is treated by constraining the set of feasible weights, i.e., replacing W_i with $W'_i \subseteq W_i$. A known result (see, e.g., Liesiö et al. 2007) is that limiting the set of the feasible weights extends the preference relation in the sense of Definition 8 under a certain technical condition shown in Lemma 4 below.

Lemma 4 Assume that a new multicriteria game with incomplete preference information is formed from an original multicriteria game with incomplete preference information so that the original weight sets W_i are replaced with weight sets W'_i so that $W'_i \subseteq W_i$ and $\text{int}(W_i) \cap W'_i \neq \emptyset$. Denote by $\succ_i$ the preference relations of the original game and by $\succ'_i$ the preference relations of the new game. Then, $\succ'_i$ extends $\succ_i$ (cf. Definition 8). That is, for any pair of outcomes such that player i prefers $(s_1, \dots, s_n)$ over $(s'_1, \dots, s'_n)$ in the original game, she has the same preference in the new game.

Then, Theorem 5 can be applied to multicriteria games with incomplete preference information as follows.

Theorem 6 If a new multicriteria game with incomplete preference information is formed from an original multicriteria game with incomplete preference information so that the original weight sets W_i are replaced with weight sets W'_i so that $W'_i \subseteq W_i$ and $\text{int}(W_i) \cap W'_i \neq \emptyset$, the sets of rationalizable strategies of the new game are subsets of the sets of rationalizable strategies of the original game.

Proof The result is directly implied by Lemma 4 and Theorem 5. $\square$

6 Examples

In this section, we present two examples. The first one is an ordinal game with a finite number of strategies that is solved by the iterative elimination of dominated strategies. The second example deals with a multicriteria game containing an infinite number of strategies that is solved by deriving equations for the maximal mutually nondominated sets. The effect of adding preference information by limiting the set of feasible weights is also illustrated.

Fig. 1 Preferences in the example of Sect. 6.1. The graph on the left represents the preferences of player 1 and the graph on the right the preferences of player 2. An arrow from outcome B to outcome A refers to $A \succ B$, i.e., the player wants to move from B to A

6.1 Game with finite strategy sets

Consider a game with two players having strategy sets $S_1 = \{T, M, B\}$, $S_2 = \{L, C, R\}$ and the preference relations

- $\succ_1$: $(T, L) \succ_1 (M, L) \succ_1 (B, L)$, $(B, C) \succ_1 (M, C) \succ_1 (T, C)$,
- $\succ_2$: $(T, C) \succ_2 (T, R)$, $(M, C) \succ_2 (M, R)$, $(B, C) \succ_2 (B, R)$.

These relations are illustrated in Fig. 1. The strategy sets are finite and thus the rationalizable strategies can be found by the iterative elimination of dominated strategies, as shown in Sect. 4.2. For player 2, strategy C yields a preferred outcome to strategy R no matter which strategy player 1 selects. For player 1, no strategies are dominated. Thus, $S_1^1 = \{T, M, B\}$, $S_2^1 = \{L, C\}$. Both remaining strategies of player 2 are nondominated with respect to S_1^1 as player 2 has no preferences defined for the relevant outcomes. For player 1, all strategies are nondominated as the preference order of the outcomes is reversed when the strategy of player 2 is switched. Thus, as all remaining strategies of both players are nondominated, the rationalizable strategies of the game are $S_1^R = \{T, M, B\}$ and $S_2^R = \{L, C\}$. Note that strategy M of player 1 is not nondominated with respect to any singleton belief among the rationalizable strategies of player 2 since it is dominated by strategy T if player 2 selects strategy L and by strategy B if player 2 selects strategy C . However, strategy M of player 1 is rationalized by the belief that player 2 may select either strategy L or strategy C .

Now, let us incorporate additional preference information to the game. Define $\succ'_2$ to consist of $\succ_2$ and the additional preferences $(T, L) \succ'_2 (T, C)$, $(T, L) \succ'_2 (T, R)$, $(M, L) \succ'_2 (M, C)$, $(M, L) \succ'_2 (M, R)$, $(B, L) \succ'_2 (B, C)$, and $(B, L) \succ'_2 (B, R)$. The new preference relations are illustrated in Fig. 2. Now, for player 2, strategy C is dominated by strategy L and the only nondominated strategy is L . Therefore, $S_2^R = \{L\}$. When player 2 is known to select strategy L , the only nondominated response of player 1 is strategy T . Thus, the rationalizable strategies of the new game are $S_1^R = \{T\}$ and $S_2^R = \{L\}$. Note that the new rationalizable strategies are subsets of the rationalizable strategies of the original game, which is implied by Theorem 5.

Fig. 2 Preferences in the example of Sect. 6.1 with additional preference information. The graph on the left represents the preferences of player 1 and the graph on the right the preferences of player 2. An arrow from outcome B to outcome A refers to $A > B$

6.2 Multicriteria game with infinite strategy sets

A bicriteria Cournot game is discussed by Bade (2005) from the point of view of equilibrium strategies. In this game setting, there are n players representing firms that select produced quantities simultaneously; see also Marmol et al. (2017). Denote the quantity selected by player i by s_i . Assume that the market clearing price is $2 - \sum s_k$ and that the unit cost of production is 1. The profit for player i is $s_i (1 - \sum_{k \neq i} s_k) - s_i^2$. Besides profits, the firms desire to maximize the sales as long as the profits are nonnegative, i.e., when $1 - \sum s_k \geq 0$. When the firm cannot make profit, i.e., $1 - \sum_{k \neq i} s_k \leq 0$, then we set the utility function to take the constant value of $1 - \sum_{k \neq i} s_k$. This makes the utility function concave and continuous in the firm's own action s_i . This can be expressed as a bicriteria game where the strategy sets are $\forall i \in I = \{1, \dots, n\} : S_i = [0, \infty)$, and the payoff vectors are

$$\forall i \in I : f_i(s_1, \dots, s_n) = \left(s_i (1 - \sum_{k \neq i} s_k) - s_i^2, \min(s_i, 1 - \sum_{k \neq i} s_k) \right). \quad (20)$$

6.2.1 Two players

Here, we obtain the rationalizable strategies of the bicriteria Cournot game with 2 players by deriving the maximal mutually nondominated sets of the game. We consider the game first as a multicriteria game in the sense of Definition 9. Then, we consider additional incomplete preference information in the sense of Definition 10.

If the opponent selects strategy s_j , the payoff vector of player i is

$$f_i(s_i, s_j) = (s_i (1 - s_j) - s_i^2, \min(s_i, 1 - s_j)). \quad (21)$$

For player i , any strategy $s_i > 1$ is dominated by $s_i = 1$ as the profits will be less than and the sales at most equal to what is obtained by selecting $s_i = 1$, no matter which strategy the opponent selects. Hence, no strategies $s_i > 1$ belong to any mutually nondominated sets. When player i believes the opponent may select $s_j \in [0, 1]$,

$1 - s_j$ is a nondominated strategy since the sales criterion is lower with any strategy $s_i < 1 - s_j$ and the profit criterion is lower with any strategy $s_i > 1 - s_j$. This implies that $S'_1 = [0, 1]$ and $S'_2 = [0, 1]$ are mutually nondominated. As no strategies $s_i > 1$ belong to any mutually nondominated sets, it has been shown that $S^R_1 = [0, 1]$ and $S^R_2 = [0, 1]$.

Now, let us add preference information in the sense of Definition 10. Assume that one unit of profits is known to be at least as valuable as α units of sales for both firms. Thus, the sets of feasible weights are $W_1 = W_2 = \{w \in W^0 \mid w_2 \leq \frac{1}{1+\alpha}\}$, where $W^0 = \{(w_1, w_2) \mid w_1, w_2 \geq 0, w_1 + w_2 = 1\}$. The extreme points of these sets are $(1, 0)$ and $(\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha})$. Thus, the payoffs at the extreme points are

$$(1, 0)^T f_i(s_i, s_j) = s_i(1 - s_j) - s_i^2, \quad (22)$$

$$\left(\frac{\alpha}{1+\alpha}, \frac{1}{1+\alpha}\right)^T f_i(s_i, s_j) = \frac{\alpha(s_i(1 - s_j) - s_i^2) + \min(s_i, 1 - s_j)}{1+\alpha}. \quad (23)$$

When s_j is fixed, the value of Eq. (22) as a function of s_i is increasing when $s_i \in [0, \frac{1-s_j}{2}]$ and decreasing when $s_i > \frac{1-s_j}{2}$. The value of Eq. (23) as a function of s_i is increasing when $s_i \in \left[0, \min\left(\frac{1+\frac{1}{\alpha}-s_j}{2}, 1 - s_j\right)\right]$. Next, we argue that if the infimum of S'_j is $s_{\min}$ and the supremum of S'_j is $s_{\max}$, the set of nondominated strategies with respect to S'_j is

$$ND(i, S'_j) = \left[\max\left(0, \frac{1-s_{\max}}{2}\right), \max\left(0, \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2}, 1 - s_{\min}\right)\right) \right]. \quad (24)$$

First, any $s_i < \max(0, \frac{1-s_{\max}}{2})$ is dominated by $\max(0, \frac{1-s_{\max}}{2})$ and any $s_i > \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2}, 1 - s_{\min}\right)$ is dominated by $\max\left(0, \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2}, 1 - s_{\min}\right)\right)$. Thus, any strategy outside the interval is dominated. On the other hand, for any s_i in the interval, all $s'_i < s_i$ lead to a lower value of Eq. (23) with some $s_j \in S'_j$ and all $s'_i > s_i$ result in a lower value of Eq. (22) with some $s_j \in S'_j$. Therefore, the nondominated strategies indeed consist of the interval.

For symmetry reasons, $S^R_1 = S^R_2$ and thus it suffices to search for the maximal symmetric mutually nondominated sets. Eq. (24) implies that the maximal symmetric mutually nondominated sets are intervals $[s_{\min}, s_{\max}]$ that satisfy

$$\begin{cases} s_{\min} = \max\left(0, \frac{1-s_{\max}}{2}\right), \\ s_{\max} = \min\left(\frac{1+\frac{1}{\alpha}-s_{\min}}{2}, 1 - s_{\min}\right), \end{cases} \quad (25)$$

whose solution is

$$\begin{cases} s_{\min} = \max\left(\frac{1}{3} - \frac{1}{3\alpha}, 0\right), \\ s_{\max} = \min\left(\frac{1}{3} + \frac{2}{3\alpha}, 1\right). \end{cases} \quad (26)$$

To summarize, with no information about the relative importance of the criteria, all strategies between $[0, 1]$ are rationalizable. For example, $s_1 = 1, s_2 = 1$ is not an equilibrium (Bade 2005), but in the absence of any equilibrium selection mechanism, it is possible that both players select strategy 1 unaware that also the opponent will select strategy 1. When the game is considered with the additional preference information that the players consider one unit of profits at least as valuable as α units of sales, if $\alpha \leq 1$, the rationalizable strategies given by Eq. (26) are $S_1^R = S_2^R = [0, 1]$, i.e., the preference information does not change the rationalizable strategies. However, when $\alpha > 1$, the rationalizable strategies given by Eq. (26) are $S_1^R = S_2^R = \left[\frac{1}{3} - \frac{1}{3\alpha}, \frac{1}{3} + \frac{2}{3\alpha}\right]$. Increasing the value of the bound α corresponds to adding preference information to the game and as we have shown in Sect. 4.4, Theorem 5, no new rationalizable strategies appear. When $\alpha \rightarrow \infty$, i.e., profits become more important, the rationalizable strategies approach $S_1^R = S_2^R = \left\{\frac{1}{3}\right\}$, i.e., the equilibrium (Bade 2005) and the rationalizable strategies (Bernheim 1984) of the single-criterion profit-maximizing Cournot game.

6.2.2 Multiple players

With $n > 2$, the analysis of the bicriteria Cournot game is similar to the case with two players. When the opponents are believed to select strategies from $[s_{\min}, s_{\max}]$, the total quantity produced by the opponents varies in $[(n-1)s_{\min}, (n-1)s_{\max}]$. Hence, the rationalizable strategies are obtained by solving the equations

$$\begin{cases} s_{\min} = \max\left(0, \frac{1-(n-1)s_{\max}}{2}\right), \\ s_{\max} = \min\left(\frac{1+\frac{1}{\alpha}-(n-1)s_{\min}}{2}, 1-(n-1)s_{\min}\right). \end{cases} \quad (27)$$

With some values of n and α , these equations have multiple solutions. Since the rationalizable strategies are the largest mutually nondominated sets, the solution corresponding to the rationalizable strategies is the one that produces an interval that contains the intervals produced by possible other solutions. That is,

$$\begin{cases} s_{\min} = 0, \\ s_{\max} = \min\left(\frac{1+\frac{1}{\alpha}}{2}, 1\right). \end{cases} \quad (28)$$

When $\alpha \leq 1$, all strategies in $[0, 1]$ are rationalizable as in the two-player case, whereas with $\alpha > 1$, the rationalizable strategies are $\left[0, \frac{1+\frac{1}{\alpha}}{2}\right]$. When $\alpha \rightarrow \infty$, the rationalizable strategies approach the rationalizable strategies of the single-criterion profit-maximizing Cournot game, i.e., $\left[0, \frac{1}{2}\right]$ (Bernheim 1984).

7 Conclusions

In this paper, we considered normal-form games with incomplete preferences (Bade 2005). We proposed a new and more general solution concept for these games that is based on rationalizable strategies (Bernheim 1984; Pearce 1984). This is an alternative solution concept to the standard notion of Nash equilibrium where the players possess correct beliefs about their opponents' choices. In rationalizable strategies, the players choose reasonable strategies but may hold incorrect beliefs; see the motivation in Perea (2014).

We revised the standard notion of rationalizable strategies that uses probabilistic beliefs. Instead, we assume nonprobabilistic beliefs where the players only specify the strategies that they consider possible but do not assign probabilities to these strategies; see Perea (2014) for the earlier use of such models. Moreover, the players select nondominated strategies given these beliefs. Our framework can be used in games where the expected utility maximization is not possible and the information that is needed to define the probabilities and utilities does not exist.

We showed that no new rationalizable strategies appear in a game with incomplete preferences when preference information is added in the sense of adding new preferences over pairs of outcomes into the preference relations. Another interpretation of this result is that no rationalizable strategies disappear when preferences are relaxed in the sense of removing preferences over pairs of outcomes from the preference relations. Thus, the game can be analyzed using only such preference information that one is definitely willing to assume, with confidence that no strategies are wrongly ruled out.

We considered multicriteria games as a special case of games with incomplete preferences and introduced a way of adding preference information to the multicriteria games by modeling incomplete preferences with sets of feasible weights of the criteria, following the treatment in the literature on multicriteria/multiattribute decision analysis (e.g., Salo and Hämäläinen 2010). The idea of using sets of feasible weights in noncooperative multicriteria games has recently been proposed also by Marmol et al. (2017) who considered equilibrium solutions. Besides multicriteria games, the game and solution concept developed in this paper could be applied to, for example, interval games (Levin 1999), where the payoffs are not known exactly, but only as intervals of possible values.

We showed that the sets of rationalizable strategies as defined in this paper are nonempty in the case of finite strategy sets. In the infinite case, nonemptiness is not guaranteed. However, the nonexistence of rationalizable strategies might be due to unreasonable structure of the preference relations. A possible topic for future research would be to define intuitively reasonable conditions on the preference relations that guarantee the existence of rationalizable strategies. Another topic for future research is the extension to probabilistic beliefs, mixed strategies and/or extensive-form games. This requires defining the preferences over lotteries and introducing appropriate restrictions on belief updating.

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