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## AXIOMATIZATIONS OF A CLASS OF EQUAL SURPLUS SHARING SOLUTIONS FOR TU-GAMES

**ABSTRACT.** A situation, in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game. A (point-valued) solution for TU-games assigns a payoff distribution to every TU-game. In this article we discuss a class of equal surplus sharing solutions consisting of all convex combinations of the CIS-value, the ENSC-value and the equal division solution. We provide several characterizations of this class of solutions on variable and fixed player set. Specifications of several properties characterize specific solutions in this class.

**KEY WORDS:** TU-game, equal surplus sharing, CIS-value, ENSC-value, equal division solution, reduced game consistency

**JEL CLASSIFICATION:** C71

### 1. INTRODUCTION

A situation, in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utility*, or simply a TU-game, being a pair  $(N, v)$ , where  $N \subset \mathbf{N}$  is a finite set of players with  $|N| \geq 2$ , and  $v: 2^N \rightarrow \mathbb{R}$  is a characteristic function on  $N$  such that  $v(\emptyset) = 0$ . For any coalition  $S \subseteq N$ ,  $v(S)$  is called the *worth* of coalition  $S$ . This is what the members of coalition  $S$  can obtain by agreeing to cooperate. We denote the class of all TU-games by  $\mathcal{G}$ .

A *payoff vector* of game  $(N, v)$  is an  $|N|$ -dimensional real vector  $x \in \mathbb{R}^n$ ,  $n = |N|$ , which represents a distribution of the payoffs that can be earned by cooperation over the individual players. A (point-valued) *solution* for TU-games is a function  $\psi$ , which assigns a payoff vector  $\psi(N, v)$  to every TU-game

$(N, v)$ . If a solution assigns to every game a payoff vector that exactly distributes the worth of the ‘grand coalition’  $N$ , then the solution is called *efficient*.<sup>1</sup>

In this article, we discuss a class of solutions for TU-games that all have some egalitarian flavour in the sense that they assign to every player some initial payoff and distribute the remainder of the worth  $v(N)$  of the grand coalition  $N$  equally among all players. Examples of such solutions are the *Centre-of-gravity of the Imputation-Set value*, shortly denoted by *CIS-value* (see Driessen and Funaki, 1991), *Egalitarian Non-Separable Contribution value*, shortly denoted by *ENSC-value* and the *equal division solution*. The CIS-value assigns to every player its individual worth, and distributes the remainder of  $v(N)$  equally among all players. The ENSC-value assigns to every game  $(N, v)$  the CIS-value of its dual game. The equal division solution just distributes  $v(N)$  equally among all players. In this article, we consider the class of solutions that consists of the above mentioned solutions and their convex combinations. We begin by defining this class for two-player games. For this class of games, our solutions are defined by a generalized *standardness for two-player games*. The usual standardness for two-player games states that in a two player game every player earns its own worth plus half of what remains of the worth of the two-player (‘grand’) coalition (see, e.g. Hart and Mas-Colell, 1988, 1989). We discuss a weaker standardness, stating that both players in a two-player game get a (uniform) share of their individual worth, and the remainder of the worth of the two-player coalition is shared equally among the two players. Besides the usual standardness, this also contains egalitarian standardness for two-player games stating that in two player games both players earn the same payoff. We also discuss properties that characterize this standardness.

After defining the class of solutions for two-player games, we extend the definition to  $n$ -player games by applying some *reduced game consistency*. In the reduced game that is played after one player has left the game, coalitions of remaining players either have the participation of the leaving player or

not. In the first case, the worth of the coalition in the reduced game is what it earned in the original game with the cooperation of the leaving player, but has to subtract the payoff, with which the leaving player leaves the game. When the leaving player does not cooperate then the coalition of remaining players just earns its worth in the original game. For every coalition of players in the reduced game, we thus have to specify whether the leaving player does cooperate or not. Besides cooperation to the grand coalition, we require that in the reduced game, the leaving player cooperates with all coalitions that consist of all, but one player, and does not cooperate with all one-player coalitions. For intermediate coalitions, both situations are possible. This reduced game is well defined as long as the original game has at least four players. In case of a three-player game, the above definition is inconsistent for one-player coalitions. Therefore, we assume for those coalitions, a probability distribution with respect to the cooperation of the leaving player.

We show that convex combinations of the equal division solution, the CIS-value and the ENSC-value are characterized by a particular standardness and reduced game consistency. The weight put on the equal division solution is determined by the share that the players get from their individual worths in the standardness axiom. The distribution of weight between the CIS- and ENSC-value is determined by the probability of cooperation of the leaving player in reduced games.

We also provide an axiomatization of this class of solutions on fixed player sets using the well-known axioms of efficiency, linearity and local monotonicity, and adding two new axioms with respect to dictator and veto players. The *dictator property* states that (i) the difference between the payoffs of a dictator and another (null) player in that dictator game cannot be more than the worth of the 'grand coalition' (which by efficiency equals the total worth to be distributed), and (ii) in any other monotone game, the difference between the payoffs of two players is never more than that described in (i). The *veto equal loss property* states that making one of the players in a zero-normalized game a veto player yields the same change

in payoff for the other players. Adding a weak null player out property stating that deleting a null player from a game changes the payoffs of all other players by the same amount, yields an axiomatization on the class of all TU-games (with variable player set).

Finally, we show how strengthening some axioms yield characterizations of some specific solutions in this class.

The article is organized as follows. Section 2 discusses some preliminaries on TU-games and solutions. In Section 3, we define our class of solutions and state some properties and relations between solutions in this class. In Section 4, we characterize these solutions for two-player games. In Section 5, we extend this definition to  $n$ -player games using a reduced game consistency. In Section 6, we give an axiomatic characterization on fixed player sets and show how this characterization holds on variable player set by adding a weak null player out property. In Section 7, we give axiomatic characterizations of some specific solutions in this class. Finally, Section 8, contains some concluding remarks.

## 2. PRELIMINARIES

A (point-valued) solution  $\psi$  on  $\mathcal{G}$  assigns a payoff vector  $\psi(N, v) \in \mathbb{R}^n$  to every TU-game  $(N, v) \in \mathcal{G}$ . Examples of solutions are the CIS-value, the ENSC-value and the equal division solution. The *CIS-value* (see Driessen and Funaki, 1991) assigns to every player its individual worth, and distributes the remainder of the worth of the grand coalition  $N$  equally among all players, i.e.

$$CIS_i(N, v) = v(\{i\}) + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} v(\{j\}) \right) \quad \text{for all } i \in N.$$

The *dual game*  $(N, v^*)$  of game  $(N, v)$  is the game that assigns to each coalition  $S \subseteq N$  the worth that is lost by the grand coalition  $N$  if coalition  $S$  leaves  $N$ , i.e.

$$v^*(S) = v(N) - v(N \setminus S) \quad \text{for all } S \subseteq N.$$

The *ENSC-value* assigns to every game  $(N, v)$  the CIS-value of its dual game, i.e.

$$\begin{aligned} ENSC_i(N, v) &= CIS_i(N, v^*) = v^*({i}) \\ &+ \frac{1}{|N|} \left( v^*(N) - \sum_{j \in N} v^*({j}) \right) \\ &= v(N) - v(N \setminus {i}) \\ &+ \frac{1}{|N|} \left( v(N) - \sum_{j \in N} (v(N) - v(N \setminus {j})) \right) \\ &= -v(N \setminus {i}) + \frac{1}{|N|} \left( v(N) + \sum_{j \in N} v(N \setminus {j}) \right) \end{aligned}$$

for all  $i \in N$ .

Thus, the ENSC-value assigns to every player in a game its marginal contribution to the ‘grand coalition’ and distributes the (positive or negative) remainder equally among the players. Using these two solutions, we can define a class of solutions, by taking any convex combination of the two, i.e. for  $\beta \in [0, 1]$  we define

$$ENCIS^\beta(N, v) = \beta CIS(N, v) + (1 - \beta) ENSC(N, v). \tag{1}$$

The solutions discussed above have some egalitarian flavour, in the sense that they equally split a surplus that is left after all players receive some individual payoff. Ignoring these individual payoffs, we obtain the *equal division solution* given by

$$ED_i(N, v) = \frac{1}{|N|} v(N) \text{ for all } i \in N.$$

Next, we state some well-known properties of solutions for TU-games. Players  $i, j \in N$  are *symmetric* in game  $(N, v)$  if  $v(S \cup {i}) = v(S \cup {j})$  for all  $S \subseteq N \setminus {i, j}$ . Player  $i \in N$  is a *null player* in game  $(N, v)$  if  $v(S \cup {i}) = v(S)$  for all  $S \subseteq N \setminus {i}$ . For game  $(N, v) \in \mathcal{G}$  and permutation  $\pi : N \rightarrow N$ , the permuted game  $(N, \pi v)$  is defined by  $\pi v(S) = v(\cup_{i \in S} {\pi(i)})$  for all  $S \subseteq N$ . Finally, for  $(N, v), (N, w) \in \mathcal{G}$  and  $a, b \in \mathbb{R}$ , the game

$(N, av + bw) \in \mathcal{G}$  is defined by  $(av + bw)(S) = av(S) + bw(S)$  for all  $S \subseteq N$ . Solution  $\psi$

- is *efficient* if  $\sum_{i \in N} \psi_i(N, v) = v(N)$  for all  $(N, v) \in \mathcal{G}$ ;
- is *symmetric* if  $\psi_i(N, v) = \psi_j(N, v)$  whenever  $i$  and  $j$  are symmetric players in  $(N, v) \in \mathcal{G}$ ;
- is *anonymous* if for every permutation  $\pi : N \rightarrow N$ , it holds that  $\psi_i(N, v) = \psi_{\pi(i)}(N, \pi v)$  for every  $(N, v) \in \mathcal{G}$ ;
- is *linear* if  $\psi(N, av + bw) = a\psi(N, v) + b\psi(N, w)$  for all  $(N, v), (N, w) \in \mathcal{G}$  and  $a, b \in \mathbb{R}$ ;
- satisfies the *null player property* if  $\psi_i(N, v) = 0$  whenever  $i$  is a null player in  $(N, v) \in \mathcal{G}$ ;
- satisfies *local monotonicity* if  $\psi_i(N, v) \geq \psi_j(N, v)$  whenever  $v(S \cup \{i\}) \geq v(S \cup \{j\})$  for all  $S \subseteq N \setminus \{i, j\}$ ;
- is *self-dual* if  $\psi(N, v) = \psi(N, v^*)$  for all  $(N, v) \in \mathcal{G}$ ;
- is *covariant* if  $\psi_i(N, w) = k\psi_i(N, v) + p_i$  for every  $(N, v) \in \mathcal{G}$ ,  $k \in \mathbb{R}_+$  and  $p \in \mathbb{R}^n$ , where  $w$  is given by  $w(S) = kv(S) + \sum_{j \in S} p_j$  for all  $S \subseteq N$ ;
- satisfies *individual rationality* if  $\psi_i(N, v) \geq v(\{i\})$  for all  $i \in N$  and  $(N, v) \in \mathcal{G}$  satisfying  $\sum_{i \in N} v(\{i\}) \leq v(N)$ ;<sup>2</sup>
- satisfies *dual individual rationality* if  $\psi_i(N, v) \geq v^*(\{i\}) = v(N) - v(N \setminus \{i\})$  for all  $i \in N$  and  $(N, v) \in \mathcal{G}$  satisfying  $\sum_{i \in N} v^*(\{i\}) \leq v(N)$ ;
- is *non-negative* if  $\psi_i(N, v) \geq 0$  for all  $i \in N$  and  $(N, v) \in \mathcal{G}$  satisfying  $v(S) \geq 0$  for all  $S \subseteq N$ .

All solutions  $ENCIS^\beta$ , as defined in (1), are covariant. The only self-dual solution in this class is the ‘average’ of the CIS- and ENSC-value obtained by taking  $\beta = \frac{1}{2}$ . The equal division solution is self-dual but not covariant.

We conclude this section by mentioning two important classes of games. Consider a player set  $N$ . The *unanimity game* of coalition  $T \subseteq N$ ,  $T \neq \emptyset$ , is the game  $(N, u_T)$  given by  $u_T(S) = 1$  if  $T \subseteq S$ , and  $u_T(S) = 0$  otherwise. The *standard game* of coalition  $T \subseteq N$ ,  $T \neq \emptyset$ , is the game  $(N, b_T)$  given by  $b_T(S) = 1$  if  $T = S$ , and  $b_T(S) = 0$  otherwise. It is well known that every characteristic function  $v$  can be written as  $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$  with  $\Delta_v(T) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S)$  being

the Harsanyi dividends (see Harsanyi, 1959), and also as  $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} v(T) b_T$ .

### 3. A CLASS OF EQUAL SURPLUS SHARING SOLUTIONS

In this article, we discuss the class of solutions that consists of all convex combinations of the equal division solution, the CIS-value and the ENSC-value, i.e. for  $\alpha, \beta \in [0, 1]$ , we consider solutions  $\varphi^{\alpha, \beta}$  given by

$$\varphi^{\alpha, \beta}(N, v) = \alpha ENCIS^\beta(N, v) + (1 - \alpha) ED(N, v), \tag{2}$$

where  $ENCIS^\beta$  is given by (1). We denote the class of all solutions that are obtained in this way by  $\Phi := \{\varphi^{\alpha, \beta} \mid \alpha, \beta \in [0, 1]\}$ . Clearly, the extreme solutions in this class are the CIS-value, which is obtained by taking  $\alpha = \beta = 1$  (i.e.  $CIS(N, v) = \varphi^{1, 1}(N, v)$ ), the ENSC-value, which is obtained by taking  $\alpha = 1, \beta = 0$  (i.e.  $ENSC(N, v) = \varphi^{1, 0}(N, v)$ ) and the equal division solution, which is obtained by taking  $\alpha = 0$  (i.e.  $ED(N, v) = \varphi^{0, \beta}(N, v), \beta \in [0, 1]$ ). We thus can write  $\varphi^{\alpha, \beta}$  as

$$\begin{aligned} \varphi^{\alpha, \beta}(N, v) &= \alpha \varphi^{1, \beta}(N, v) + (1 - \alpha) \varphi^{0, 1}(N, v) \\ &= \alpha \beta \varphi^{1, 1}(N, v) + \alpha(1 - \beta) \varphi^{1, 0}(N, v) \\ &\quad + (1 - \alpha) \varphi^{0, 1}(N, v) \end{aligned}$$

for  $\alpha, \beta \in [0, 1]$ . Without proof we state two more relations.<sup>3</sup>

**PROPOSITION 3.1.** *For every  $\alpha, \beta \in [0, 1]$  and  $(N, v) \in \mathcal{G}$ , it holds that*

1.  $\varphi^{\alpha, \beta}(N, v) + \varphi^{\alpha, 1 - \beta}(N, v) = \varphi^{\alpha, 1}(N, v) + \varphi^{\alpha, 0}(N, v)$ ;
2.  $\varphi^{\alpha, \beta}(N, v) + \varphi^{\gamma, \beta}(N, v) = \varphi^{\alpha + \gamma, \beta}(N, v) + \varphi^{0, \beta}(N, v)$  for all  $\gamma \in [0, 1]$  such that  $\alpha + \gamma \in [0, 1]$ .

Next we provide an expression of the solutions  $\varphi^{\alpha, \beta}$  showing that they have some egalitarian flavour in the sense that they give each player  $i$  in a game  $(N, v)$  some value  $\lambda_i^{\alpha, \beta}(N, v)$ , and the remainder of  $v(N)$  is equally split among all players.

**PROPOSITION 3.2.** *For every  $(N, v) \in \mathcal{G}$  and  $\alpha, \beta \in [0, 1]$  it holds that*

$$\varphi_i^{\alpha, \beta}(N, v) = \lambda_i^{\alpha, \beta}(N, v) + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \lambda_j^{\alpha, \beta}(N, v) \right), \quad (3)$$

where  $\lambda_i^{\alpha, \beta}(N, v) = \alpha(\beta v(\{i\}) - (1 - \beta)v(N \setminus \{i\}))$  for  $i \in N$ .

*Proof.* For  $(N, v) \in \mathcal{G}$  and  $\alpha, \beta \in [0, 1]$  we have,

$$\begin{aligned} \varphi_i^{\alpha, \beta}(N, v) &= \alpha ENCIS^\beta(N, v) + (1 - \alpha)ED(N, v) \\ &= \alpha(\beta v(\{i\}) - (1 - \beta)v(N \setminus \{i\})) \\ &\quad + \frac{\alpha}{|N|} \left( v(N) - \sum_{j \in N} (\beta v(\{j\}) - (1 - \beta)v(N \setminus \{j\})) \right) \\ &\quad + \frac{1 - \alpha}{|N|} v(N) \\ &= \alpha(\beta v(\{i\}) - (1 - \beta)v(N \setminus \{i\})) \\ &\quad + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \alpha(\beta v(\{j\}) - (1 - \beta)v(N \setminus \{j\})) \right) \\ &= \lambda_i^{\alpha, \beta}(N, v) + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \lambda_j^{\alpha, \beta}(N, v) \right). \end{aligned}$$

□

Although not all solutions in  $\Phi$  are self-dual, the class  $\Phi$  itself is closed under duality. More specific, we state the following.

**PROPOSITION 3.3.** *For every  $\alpha, \beta \in [0, 1]$  and  $(N, v) \in \mathcal{G}$  it holds that  $\varphi^{\alpha, \beta}(N, v^*) = \varphi^{\alpha, 1-\beta}(N, v)$ .*



*Proof.* Let  $\alpha, \beta \in [0, 1]$  and  $(N, v) \in \mathcal{G}$ . Then

$$\begin{aligned}
 \varphi^{\alpha, \beta}(N, v^*) &= \lambda_i^{\alpha, \beta}(N, v^*) + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \lambda_j^{\alpha, \beta}(N, v^*) \right) \\
 &= \alpha (\beta v^*({i}) - (1 - \beta)v^*(N \setminus {i})) \\
 &\quad + \frac{1}{|N|} \left( v(N) - \alpha \sum_{j \in N} (\beta v^*({j}) - (1 - \beta)v^*(N \setminus {j})) \right) \\
 &= \alpha (\beta v(N) - \beta v(N \setminus {i}) - (1 - \beta)v(N) \\
 &\quad + (1 - \beta)v({i})) \\
 &\quad + \frac{1}{|N|} \left( v(N) - \alpha \sum_{j \in N} (\beta v(N) - \beta v(N \setminus {j}) - (1 - \beta)v(N) + (1 - \beta)v({j})) \right) \\
 &= v(N) \left( \alpha \beta - \alpha(1 - \beta) \right. \\
 &\quad \left. + \frac{1}{|N|} - \frac{|N|\alpha\beta}{|N|} + \frac{|N|\alpha(1 - \beta)}{|N|} \right) \\
 &\quad + \alpha \left( (1 - \beta)v({i}) - \beta v(N \setminus {i}) \right. \\
 &\quad \left. + \frac{1}{|N|} \sum_{j \in N} (\beta v(N \setminus {j}) - (1 - \beta)v({j})) \right) \\
 &= \frac{1}{|N|} v(N) + \alpha((1 - \beta)v({i}) - \beta v(N \setminus {i}))
 \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{|N|} \left( \sum_{j \in N} \alpha((1-\beta)v(\{j\}) - \beta v(N \setminus \{j\})) \right) \\
& = \varphi^{\alpha, 1-\beta}(N, v). \quad \square
\end{aligned}$$

#### 4. TWO-PLAYER GAMES

On the class of two-player games, the CIS- and ENSC-value coincide. Thus, on this class we consider convex combinations of the CIS-value and the equal division solution. It is well known that on the class of two-player games, the CIS-value satisfies *standardness for two-player games* as considered in, e.g. Hart and Mas-Colell (1988, 1989):  $\psi_i(N, v) = \frac{1}{2}v(\{i\}) - \frac{1}{2}v(\{j\}) + \frac{1}{2}v(N) = v(\{i\}) + \frac{1}{2}(v(N) - v(\{i\}) - v(\{j\}))$  with  $N = \{i, j\}$ . On the other hand, the equal division solution satisfies *egalitarian standardness for two-player games*:  $\psi_i(N, v) = \frac{1}{2}v(N)$  for  $i \in N$ . We denote the class of two-player games by  $\mathcal{G}^2 = \{(N, v) \in \mathcal{G} \mid |N| = 2\}$ . On this class,  $\Phi$  consists of all solutions for two-player games that assign to both players the same share (between zero and one) in their individual worth, and distributes the remainder of the worth of the two-player coalition equally among the two players.

**DEFINITION 4.1.** *Let  $\alpha \in [0, 1]$ . A solution  $\psi$  satisfies  $\alpha$ -standardness for two-player games if for every  $(N, v) \in \mathcal{G}$  with  $N = \{i, j\}$ ,  $i \neq j$ , it holds that*

$$\begin{aligned}
\psi_i(N, v) &= \alpha v(\{i\}) + \frac{1}{2}(v(N) - \alpha(v(\{i\}) + v(\{j\}))) \\
&\text{for } N = \{i, j\}.
\end{aligned}$$

*A solution  $\psi$  satisfies weak standardness for two-player games if there exists an  $\alpha \in [0, 1]$  such that  $\psi$  satisfies  $\alpha$ -standardness for two-player games.*

Clearly, standardness for two-player games coincides with  $\alpha = 1$ , and egalitarian standardness coincides with  $\alpha = 0$ . Weak

standardness is equivalent to requiring efficiency, symmetry and linearity on  $\mathcal{G}^2$ .

**PROPOSITION 4.2.** *A solution  $\psi$  on  $\mathcal{G}^2$  satisfies weak standardness for two-player games if and only if it is efficient, symmetric and linear.*

*Proof.* The ‘only if’ part is straightforward. To prove the ‘if’ part, suppose that  $\psi$  is efficient, symmetric and linear on  $\mathcal{G}^2$ . We show that  $\psi$  must satisfy weak standardness in four steps.

*Step 1.* Linearity of  $\psi$  implies that there exist  $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ , such that for every two-player game  $(N, v) \in \mathcal{G}^2$  with  $N = \{i, j\}$ , we have

$$\psi_i(N, v) = \alpha_i v(\{i\}) + \beta_i v(\{j\}) + \gamma_i v(N) + \delta_i \quad (4)$$

(see Weber, 1988, Theorem 1).

*Step 2.* Suppose that  $\psi$  is linear and symmetric, and let  $i, j \in N$  be symmetric players in  $(N, v)$ . By symmetry of  $\psi$ , it then must hold that  $\psi_i(N, v) = \psi_j(N, v)$ , which with (4) is equivalent to  $\alpha_i v(\{i\}) + \beta_i v(\{j\}) + \gamma_i v(N) + \delta_i = \alpha_j v(\{j\}) + \beta_j v(\{i\}) + \gamma_j v(N) + \delta_j$ . By symmetry of  $i, j \in N$  in game  $(N, v)$ , we have  $v(\{i\}) = v(\{j\})$ , and thus  $(\alpha_i + \beta_i)v(\{i\}) + \gamma_i v(N) + \delta_i = (\alpha_j + \beta_j)v(\{i\}) + \gamma_j v(N) + \delta_j$ . Since this must hold for all  $(N, v) \in \mathcal{G}^2$  with  $v(\{i\}) = v(\{j\})$ , the parameters in (4) must satisfy

$$\alpha_i + \beta_i = \alpha_j + \beta_j, \quad \gamma_i = \gamma_j \quad \text{and} \quad \delta_i = \delta_j. \quad (5)$$

*Step 3.* Suppose that  $\psi$  is linear and efficient. Efficiency of  $\psi$  implies that  $\psi_i(N, v) + \psi_j(N, v) = \alpha_i v(\{i\}) + \beta_i v(\{j\}) + \gamma_i v(N) + \delta_i + \alpha_j v(\{j\}) + \beta_j v(\{i\}) + \gamma_j v(N) + \delta_j = (\alpha_i + \beta_j)v(\{i\}) + (\alpha_j + \beta_i)v(\{j\}) + (\gamma_i + \gamma_j)v(N) + \delta_i + \delta_j = v(N)$ . Since this must hold for all  $(N, v) \in \mathcal{G}^2$  the parameters in (4) must satisfy

$$\alpha_i + \beta_j = \alpha_j + \beta_i = 0, \quad \gamma_i + \gamma_j = 1 \quad \text{and} \quad \delta_i + \delta_j = 0. \quad (6)$$

*Step 4.* Combining steps 1, 2 and 3, suppose that  $\psi$  is linear, symmetric and efficient. (5) and (6) imply that  $\alpha_i - \alpha_j = \beta_i - \beta_j = \alpha_j - \alpha_i$ , which implies that  $\alpha_i = \alpha_j$ . Then also  $\beta_i = \beta_j$ . With (6), it also follows that  $\alpha_i = \alpha_j = -\beta_i = -\beta_j$ . Then it follows straightforward that  $\gamma_i = \gamma_j = \frac{1}{2}$  and  $\delta_i = \delta_j = 0$ . However, this means that  $\psi$  satisfies weak standardness. In particular, it satisfies  $\alpha$ -standardness with  $\frac{\alpha}{2} = \alpha_i = \alpha_j = -\beta_i = -\beta_j$ .  $\square$

Note that equality of  $\alpha_i$  and  $\alpha_j$  is not implied by linearity and symmetry, but with efficiency it is<sup>4</sup>. The same can be said for  $\beta_i = \beta_j$ . In the next section, we extend the class of weak standard solutions on  $\mathcal{G}^2$  to  $n$ -player games using a reduced game consistency.

## 5. AN EXTENSION TO $n$ -PLAYER GAMES USING A REDUCED GAME CONSISTENCY

Take a game  $(N, v) \in \mathcal{G}$ , a payoff vector  $x \in \mathbb{R}^n$  and a player  $j \in N$ . The player set of a *reduced game* is obtained by removing player  $j$  from the original player set  $N$ . The worths of the coalitions in this reduced game reflect what these coalitions can earn if player  $j$  has left the game with its payoff  $x_j$ . The worth of the coalition  $N \setminus \{j\}$  (the ‘grand coalition’) in the reduced game is equal to the worth of  $N$  minus the payoff  $x_j$  assigned to player  $j$ . Clearly, this is what is left to be allocated to the players in  $N \setminus \{j\}$  after removing player  $j$  from the game with payoff  $x_j$ . For the other coalitions  $S \subset N \setminus \{j\}$ , we assume that they have the participation of the leaving player  $j$  (but must pay  $x_j$  to  $j$ ) or not. In case  $j$  cooperates with  $S \subset N \setminus \{j\}$ , the worth of  $S$  in the reduced game thus equals  $v(S \cup \{j\}) - x_j$ , while in case  $j$  does not cooperate, it equals  $v(S)$ . Although we do not want to specify which coalitions have the participation of  $j$ , we require that besides the ‘grand coalition’ also its largest proper subcoalitions have the participation of  $j$ , while the smallest coalitions, i.e. the singletons, do not have  $j$ ’s participation. These reduced games are

well defined as long as  $|N| \geq 4$ . In case  $|N| = 3$ , the reduced games are two-player games, and thus the one-player coalitions are the smallest and largest proper subcoalitions of the ‘grand coalition’. Therefore, in these games, it is not clear whether these coalitions have player  $j$ ’s participation or not, and we assume that with probability  $\beta \in [0, 1]$  player  $j$  does not cooperate with the one-player coalitions, and with probability  $(1 - \beta)$  does cooperate. This yields the following reduced games.

**DEFINITION 5.1.** *Given a game  $(N, v) \in \mathcal{G}$  with  $|N| \geq 4$ , a player  $j \in N$ , and a payoff vector  $x \in \mathbb{R}^n$ , a reduced game with respect to  $j$  and  $x$  is a game  $(N \setminus \{j\}, v^x)$  that satisfies*

$$v^x(S) = \begin{cases} v(N) - x_j & \text{if } S = N \setminus \{j\} \\ v(S \cup \{j\}) - x_j & \text{if } S \subset N \setminus \{j\} \text{ with } |S| = |N| - 2 \\ v(S) & \text{if } S \subset N \setminus \{j\} \text{ with } |S| = 1 \\ 0 & \text{if } S = \emptyset. \end{cases}$$

*If  $|N| = 3$ , then for  $\beta \in [0, 1]$ , the  $\beta$ -reduced game with respect to  $j \in N$  and  $x \in \mathbb{R}^n$  is uniquely determined as the game  $(N \setminus \{j\}, v^{x,\beta})$  given by*

$$v^{x,\beta}(S) = \begin{cases} v(N) - x_j & \text{if } S = N \setminus \{j\} \\ \beta v(S) + (1 - \beta)(v(S \cup \{j\}) - x_j) & \text{if } S \subset N \setminus \{j\} \text{ with } |S| = 1 \\ 0 & \text{if } S = \emptyset. \end{cases}$$

These reduced games are also considered in Funaki (1994) and contain that of Funaki and Yamato (2001).<sup>5</sup> With a slight abuse of notation, we denote in the remainder of the article the characteristic function  $v^{x,\beta}$  just by  $v^x$ , also when  $|N| = 3$ . We are ready to give a definition of the consistency property of a solution associated with a reduced game.

**DEFINITION 5.2.** *Let  $\psi$  be a solution on  $\mathcal{G}$ , and  $\beta \in [0, 1]$ . Solution  $\psi$  satisfies  $\beta$ -consistency if and only if for every  $(N, v) \in \mathcal{G}$  with  $|N| \geq 3$ ,  $j \in N$ , and  $x = \psi(N, v)$ , it holds that  $\psi_i(N \setminus \{j\}, v^x) = \psi_i(N, v)$  for  $i \in N \setminus \{j\}$ .*

Consistency implies that given a game  $(N, v)$ , if  $x$  is a solution payoff vector for  $(N, v)$ , then for every player  $j \in N$ , the payoff vector  $x_{N \setminus \{j\}}$  with payoffs for the players in  $N \setminus \{j\}$ , must be a solution payoff vector of the reduced game  $(N \setminus \{j\}, v^x)$ . It is a kind of internal consistency requirement to guarantee that players respect the recommendations made by the solution.

**PROPOSITION 5.3.** *Take  $\beta \in [0, 1]$ . For every  $\alpha \in [0, 1]$ , the solution  $\varphi^{\alpha, \beta}$  satisfies  $\beta$ -consistency on the class of all games  $\mathcal{G}$ .*

*Proof.* Take any  $\beta \in [0, 1]$  and fix  $\alpha \in [0, 1]$ . Take any  $(N, v) \in \mathcal{G}$ . First, suppose that  $|N| \geq 4$ . For  $x = \varphi^{\alpha, \beta}(N, v)$ , we have

$$\begin{aligned} \lambda_i^{\alpha, \beta}(N \setminus \{j\}, v^x) &= \alpha(\beta v^x(\{i\}) - (1 - \beta)v^x(N \setminus \{i, j\})) \\ &= \alpha(\beta v(\{i\}) - (1 - \beta)(v(N \setminus \{i\}) - x_j)) \\ &= \lambda_i^{\alpha, \beta}(N, v) + (1 - \beta)x_j \end{aligned} \tag{7}$$

since in that case,  $|\{i\}| = 1$  and  $|N \setminus \{i, j\}| = |N| - 2$ . Denote  $\lambda_i = \lambda_i^{\alpha, \beta}(N, v)$  and  $\lambda_i^x = \lambda_i^{\alpha, \beta}(N \setminus \{j\}, v^x)$  with  $x = \varphi^{\alpha, \beta}(N, v)$ ,  $i \in N$ . Then with (7), it follows that

$$\begin{aligned} \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^x) &= \lambda_i^x + \frac{1}{|N| - 1} \left( v^x(N \setminus \{j\}) - \sum_{k \in N \setminus \{j\}} \lambda_k^x \right) \\ &= \lambda_i + (1 - \beta)x_j + \frac{1}{|N| - 1} \\ &\quad \times \left( v(N) - x_j - \sum_{k \in N \setminus \{j\}} (\lambda_k + (1 - \beta)x_j) \right) \\ &= \lambda_i + (1 - \beta)x_j + \frac{1}{|N| - 1} \\ &\quad \times \left( v(N) - \lambda_j - \frac{1}{|N|} \left( v(N) - \sum_{k \in N} \lambda_k \right) \right. \\ &\quad \left. - \sum_{k \in N \setminus \{j\}} \lambda_k \right) - (1 - \beta)x_j \end{aligned}$$

$$\begin{aligned}
&= \lambda_i + \frac{1}{|N|-1} \left( \frac{|N|-1}{|N|} \left( v(N) - \sum_{k \in N} \lambda_k \right) \right) \\
&= \varphi_i^{\alpha, \beta}(N, v).
\end{aligned}$$

Second, consider the case  $|N|=3$ . Let  $x = \varphi^{\alpha, \beta}(N, v)$ ,  $\bar{\beta} = 1 - \beta$  and  $N = \{i, j, k\}$ . (Note that  $|N|=3$  implies that  $i, j$  and  $k$  are distinct players.) First, we remark that

$$\begin{aligned}
\varphi_i^{\alpha, \beta}(N, v) &= \alpha(\beta v(\{i\}) - \bar{\beta} v(N \setminus \{i\})) \\
&\quad + \frac{1}{3} \left( v(N) - \sum_{l \in N} \alpha(\beta v(\{l\}) - \bar{\beta} v(N \setminus \{l\})) \right) \\
&= \alpha(\beta v(\{i\}) - \bar{\beta} v(\{j, k\})) \\
&\quad + \frac{1}{3} (v(N) - \alpha\beta v(\{i\}) - \alpha\beta v(\{j\}) - \alpha\beta v(\{k\}) \\
&\quad + \alpha\bar{\beta} v(\{j, k\}) + \alpha\bar{\beta} v(\{k, i\}) + \alpha\bar{\beta} v(\{i, j\})).
\end{aligned}$$

From the definitions of the solution and the reduced game, we have for  $x = \varphi^{\alpha, \beta}(N, v)$  that

$$\begin{aligned}
\varphi_i^{\alpha, \beta}(N \setminus \{k\}, v^x) &= \varphi_i^{\alpha, \beta}(\{i, j\}, v^x) = \frac{\alpha}{2} (v^x(\{i\}) - v^x(\{j\})) \\
&\quad + \frac{1}{2} v^x(\{i, j\}) \\
&= \frac{\alpha}{2} ((\beta v(\{i\}) + \bar{\beta} v(\{i, k\}) - \bar{\beta} x_k) - (\beta v(\{j\}) \\
&\quad + \bar{\beta} v(\{j, k\}) - \bar{\beta} x_k)) \\
&\quad + \frac{1}{2} [v(N) - \alpha(\beta v(\{k\}) - \bar{\beta} v(\{i, j\})) \\
&\quad - \frac{1}{3} (v(N) - \alpha\beta v(\{i\}) - \alpha\beta v(\{j\}) - \alpha\beta v(\{k\}) \\
&\quad + \alpha\bar{\beta} v(\{j, k\}) + \alpha\bar{\beta} v(\{k, i\}) + \alpha\bar{\beta} v(\{i, j\}))] \\
&= (\alpha\beta v(\{i\}) - \alpha\bar{\beta} v(\{j, k\})) + \frac{1}{2} (-\alpha\beta v(\{i\}) \\
&\quad + \alpha\bar{\beta} v(\{i, k\}) - \alpha\beta v(\{j\}) + \alpha\bar{\beta} v(\{j, k\})) \\
&\quad + \frac{1}{3} (v(N) - \alpha\beta v(\{k\}) + \alpha\bar{\beta} v(\{i, j\}))
\end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{6}(-\alpha\beta v(\{i\}) - \alpha\beta v(\{j\}) + \alpha\bar{\beta}v(\{j, k\}) \\
 & \quad + \alpha\bar{\beta}v(\{k, i\})) \\
 & = \alpha(\beta v(\{i\}) - \bar{\beta}v(\{j, k\})) \\
 & \quad + \frac{1}{3}(v(N) - \alpha\beta v(\{k\}) + \alpha\bar{\beta}v(\{i, j\})) - \alpha\beta v(\{i\}) \\
 & \quad - \alpha\beta v(\{j\}) + \alpha\bar{\beta}v(\{j, k\}) + \alpha\bar{\beta}v(\{k, i\})) \\
 & = \varphi_i^{\alpha, \beta}(N, v).
 \end{aligned}$$

This completes the proof. □

In the previous section, we saw that  $\varphi^{\alpha, \beta}$  satisfies  $\alpha$ -standardness for two-player games. Next, we characterize those solutions.

**THEOREM 5.4.** *Take any  $\alpha, \beta \in [0, 1]$ . A solution  $\psi$  satisfies efficiency,  $\alpha$ -standardness for two-player games and  $\beta$ -consistency on the class of all games  $\mathcal{G}$ , if and only if  $\psi = \varphi^{\alpha, \beta}$ .*

*Proof.*  $\varphi^{\alpha, \beta}$  satisfying efficiency and  $\alpha$ -standardness for two-player games is straightforward.  $\varphi^{\alpha, \beta}$  satisfying  $\beta$ -consistency follows from Proposition 5.3. Here we prove the ‘only if’ part. Take  $\alpha, \beta \in [0, 1]$ , and let  $\psi$  be a solution, which satisfies efficiency,  $\alpha$ -standardness for two-player games and  $\beta$ -consistency. If  $|N| = 2$ , then  $\psi(N, v) = \varphi^{\alpha, \beta}(N, v)$  follows from  $\alpha$ -standardness for two-player games.

In the following, we denote  $x = \psi(N, v)$  and  $y = \varphi^{\alpha, \beta}(N, v)$ .

If  $|N| = 3$ , let  $N = \{i, j, k\}$ . By  $\alpha$ -standardness,  $\beta$ -consistency and the induction hypothesis, we have

$$\begin{aligned}
 x_i - y_i & = \psi_i(N \setminus \{j\}, v^x) - \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^y) \\
 & = \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^x) - \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^y) \\
 & = \frac{\alpha}{2}(v^x(\{i\}) - v^x(\{k\})) + \frac{1}{2}v^x(\{i, k\}) - \frac{\alpha}{2}(v^y(\{i\}) \\
 & \quad - v^y(\{k\})) - \frac{1}{2}v^y(\{i, k\})
 \end{aligned}$$



$$\begin{aligned}
 &= \frac{\alpha}{2} [\beta v(\{i\}) + (1 - \beta)(v(\{i, j\}) - x_j) - \beta v(\{k\}) \\
 &\quad - (1 - \beta)(v(\{k, j\}) - x_j) \\
 &\quad - \beta v(\{i\}) - (1 - \beta)(v(\{i, j\}) - y_j) + \beta v(\{k\}) \\
 &\quad + (1 - \beta)(v(\{k, j\}) - y_j)] \\
 &\quad + \frac{1}{2}(v(N) - x_j - v(N) + y_j) \\
 &= -\frac{1}{2}(x_j - y_j)
 \end{aligned}$$

Similarly, we find that  $x_j - y_j = \varphi_j^{\alpha, \beta}(N \setminus \{i\}, v^x) - \varphi_j^{\alpha, \beta}(N \setminus \{i\}, v^y) = -\frac{1}{2}(x_i - y_i)$ . So,  $x_i - y_i = \frac{1}{4}(x_i - y_i)$ , and thus  $x_i - y_i = 0$ . Similarly, it follows that  $x_j - y_j = x_k - y_k = 0$ . This shows that  $\psi(N, v) = \varphi^{\alpha, \beta}(N, v)$ .

Proceeding by induction, for  $|N| \geq 4$ , suppose that  $\psi(N', v') = \varphi^{\alpha, \beta}(N', v')$  whenever  $|N'| = |N| - 1$ . We will show that  $\psi(N, v) = \varphi^{\alpha, \beta}(N, v)$ .

Take any  $i, j \in N$  such that  $i \neq j$ . Again, let  $x = \psi(N, v)$  and  $y = \varphi^{\alpha, \beta}(N, v)$ . For the two reduced games  $(N \setminus \{j\}, v^x)$  and  $(N \setminus \{j\}, v^y)$ , by the induction hypothesis, we have

$$\begin{aligned}
 x_i - y_i &= \psi_i(N \setminus \{j\}, v^x) - \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^y) \\
 &= \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^x) - \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^y)
 \end{aligned} \tag{8}$$

By (7) and the definitions of  $\varphi^{\alpha, \beta}$  and the reduced game, we have

$$\begin{aligned}
 &\varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^x) - \varphi_i^{\alpha, \beta}(N \setminus \{j\}, v^y) \\
 &= \lambda_i^{\alpha, \beta}(N \setminus \{j\}, v^x) - \lambda_i^{\alpha, \beta}(N \setminus \{j\}, v^y) \\
 &\quad + \frac{1}{|N| - 1} [v^x(N \setminus \{j\}) - v^y(N \setminus \{j\})] \\
 &\quad - \sum_{k \in N \setminus \{j\}} (\lambda_k^{\alpha, \beta}(N \setminus \{j\}, v^x) - \lambda_k^{\alpha, \beta}(N \setminus \{j\}, v^y)) \\
 &= (1 - \beta)(x_j - y_j) \\
 &\quad + \frac{1}{|N| - 1} \left( -x_j + y_j - \sum_{k \in N \setminus \{j\}} (1 - \beta)(x_k - y_k) \right)
 \end{aligned}$$

$$\begin{aligned}
&= (1 - \beta)(x_j - y_j) + \frac{1}{|N| - 1} \left( -(x_j - y_j) - (1 - \beta)(v(N) \right. \\
&\quad \left. - x_j - v(N) + y_j) \right) \\
&= (1 - \beta)(x_j - y_j) + \frac{1}{|N| - 1} \left( -(x_j - y_j) - (1 - \beta)(-x_j + y_j) \right) \\
&= \left( 1 - \frac{|N|\beta}{|N| - 1} \right) (x_j - y_j).
\end{aligned}$$

With (8), this implies that  $x_i - y_i = (1 - \frac{|N|\beta}{|N|-1})(x_j - y_j)$  for all  $i \in N \setminus \{j\}$ . This implies that  $x_i - y_i = x_l - y_l$  for all  $i, l \in N \setminus \{j\}$ . Take any  $k \in N \setminus \{i, j\}$ . Since  $x_j - y_j = (1 - \frac{|N|\beta}{|N|-1})(x_k - y_k)$  for any  $j \in N \setminus \{k\}$ , it holds that  $x_i - y_i = x_j - y_j$ . This implies that  $x_i - y_i = x_j - y_j = x_l - y_l$  for all  $l \in N \setminus \{i, j\}$ . Then efficiency<sup>6</sup> implies that  $\sum_{i \in N} (x_i - y_i) = v(N) - v(N) = 0$ , and thus  $x_i - y_i = 0$  for all  $i \in N$ . This shows that  $\psi(N, v) = \varphi^{\alpha, \beta}(N, v)$ .

This completes the proof.  $\square$

Similar to the definition of weak standardness for two-player games, we can say that a solution satisfies *weak consistency* if and only if there exists a  $\beta \in [0, 1]$  such that it satisfies  $\beta$ -consistency.

From the previous section, we know that for two-player games a solution satisfying weak standardness is equivalent to the solution being efficient, symmetric and linear. From Theorem 5.4, it follows that on the class of all TU-games, the class of solutions  $\Phi$  is characterized by efficiency, weak standardness and weak consistency.

**COROLLARY 5.5.** *A solution  $\psi$  satisfies efficiency, weak standardness for two-player games and weak consistency on the class of all games  $\mathcal{G}$ , if and only if  $\psi \in \Phi$ .*

The  $\alpha$ -standardness for two-player games and  $\beta$ -consistency also make clear how the solutions  $\varphi^{\alpha, \beta}$  depend on these two parameters. Note that the parameter  $\beta$  does not appear in the standardness property, while the parameter  $\alpha$  does not appear in the consistency property. The parameter  $\alpha \in [0, 1]$  is the share that the players get from their individual worths in

the standardness axiom and determines the weight put on the equal division solution. The parameter  $\beta \in [0, 1]$  determines the probability about cooperation of the leaving player  $j$  in the coalitions in the reduced game, and determines the distribution of weight between the CIS- and ENSC-values.

6. AXIOMATIZATIONS USING DICTATOR AND VETO PLAYER PROPERTIES

In the previous section, we characterized the class of solutions  $\Phi$ , and every solution in this class, on a variable player set. Next, we provide a characterization of the class  $\Phi$  on a fixed player set using the well-known axioms of efficiency, linearity, local monotonicity and two new axioms that we introduce below. Moreover, adding a weak null player out property yields an axiomatization for variable player set. The first of the two new axioms concerns a dictator property. We say that player  $i \in N$  is a *dictator* in a monotone game  $(N, v)$  if all coalitions containing  $i$  earn the same worth  $c > 0$ , and all other coalitions earn zero. We denote this dictator game by  $d_i^c$ , i.e.  $d_i^c(S) = c$  if  $i \in S$ , and  $d_i^c(S) = 0$  otherwise.<sup>7</sup> Note that in the dictator game  $d_i^c$ , all players in  $N \setminus \{i\}$  are null players. The dictator property now states that (i) the difference between the payoffs of a dictator and another (null) player in that dictator game cannot be more than the worth of the ‘grand coalition’ (which by efficiency equals the total worth to be distributed), and (ii) in any other monotone game, the difference between the payoffs of two players is never more than that described in (i). A TU-game  $(N, v)$  is monotone if  $v(S) \leq v(T)$  whenever  $S \subseteq T \subseteq N$ .

**AXIOM 6.1.** (*Dictator property*) *A solution  $\psi$  satisfies the dictator property if*

- (i)  $\psi_i(N, d_i^c) - \psi_j(N, d_i^c) \leq c$  for all  $j \in N \setminus \{i\}$ , and
- (ii)  $\psi_i(N, v) - \psi_j(N, v) \leq \psi_i(N, d_i^{v(N)}) - \psi_j(N, d_i^{v(N)})$  for all  $i, j \in N$ , and monotone games  $(N, v)$ .

Note that this dictator property implies the following *difference property* which states that the maximal difference between the payoffs of two players in a monotone game cannot be more than the worth of the ‘grand coalition’. (The straightforward proof is omitted.)

**PROPOSITION 6.2.** *If solution  $\psi$  satisfies the dictator property then  $\psi_i(N, v) - \psi_j(N, v) \leq v(N)$  for all  $i, j \in N$ , and monotone game  $(N, v)$ .*

To introduce the veto equal loss property, suppose that player  $h \in N$  becomes a veto player in game  $(N, v)$ , i.e. instead of characteristic function  $v$ , we consider the characteristic function  $v^h$  given by

$$v^h(S) = \begin{cases} v(S) & \text{if } h \in S \\ 0 & \text{otherwise.} \end{cases}$$

Next, we require that this yields the same change in payoff for the other players when  $v$  is zero-normalized, i.e.  $v(\{i\}) = 0$  for all  $i \in N$ .

**AXIOM 6.3. (Veto equal loss property)** *A solution  $\psi$  satisfies the veto equal loss property if  $\psi_i(N, v) - \psi_i(N, v^h) = \psi_j(N, v) - \psi_j(N, v^h)$  for every  $h \in N$ ,  $i, j \in N \setminus \{h\}$  and zero-normalized  $(N, v) \in \mathcal{G}$ .*

Note that  $(N, v)$  being zero-normalized implies that  $(N, v^h)$  is zero-normalized. The above mentioned five axioms characterize the class  $\Phi$  on a fixed player set  $N$ . In proving this, we use the following expressions for unanimity games. For  $T \subseteq N$ ,  $|T| = 1$ , it holds that

$$CIS_i(N, u_T) = ENSC_i(N, u_T) = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases}$$

and

$$ED_i(N, u_T) = \frac{1}{|N|} \quad \text{for all } i \in N.$$

For  $T \subseteq N$ ,  $|T| \geq 2$ , it holds that

$$CIS_i(N, u_T) = ED_i(N, u_T) = \frac{1}{|N|} \quad \text{for all } i \in N,$$

and

$$ENSC_i(N, u_T) = \begin{cases} \frac{(|N|-|T|+1)}{|N|} & \text{if } i \in T \\ \frac{(1-|T|)}{|N|} & \text{otherwise.} \end{cases}$$

Note that for  $(N, u_N)$ , all payoffs are equal to  $\frac{1}{|N|}$  in all three solutions. As a consequence, we can express every solution in  $\Phi$  for unanimity games  $(N, u_T)$  with  $|T|=1$  as

$$\varphi_i^{\alpha, \beta}(N, u_T) = \begin{cases} \frac{(1-\alpha)}{|N|} + \alpha & \text{if } i \in T \\ \frac{(1-\alpha)}{|N|} & \text{otherwise} \end{cases} \quad (9)$$

and for  $|T| \geq 2$ ,

$$\varphi_i^{\alpha, \beta}(N, u_T) = \begin{cases} \frac{1}{|N|}((1-\alpha) + \alpha\beta + \alpha(1-\beta)(|N|-|T|+1)) & \text{if } i \in T \\ \frac{1}{|N|}((1-\alpha) + \alpha\beta + \alpha(1-\beta)(1-|T|)) & \text{otherwise.} \end{cases} \quad (10)$$

So, for  $i \in T$ ,  $j \in N \setminus T$  we have  $\varphi_i^{\alpha, \beta}(N, u_T) = \varphi_j^{\alpha, \beta}(N, u_T) + \alpha$  if  $|T|=1$ , and  $\varphi_i^{\alpha, \beta}(N, u_T) = \varphi_j^{\alpha, \beta}(N, u_T) + \alpha(1-\beta)$  if  $|T| \geq 2$ . We use these expressions in proving the following characterization on a fixed player set. We denote by  $\mathcal{G}^N$ , the collection of all TU-games on player set  $N \subset \mathbb{N}$ . A solution on  $\mathcal{G}^N$  is a function  $\psi$ , which assigns a payoff vector  $\psi(N, v)$  to every TU-game  $(N, v) \in \mathcal{G}^N$ .<sup>8</sup> We denote by  $\Phi^N$  the set of all solutions on  $\mathcal{G}^N$  obtained by (2) with  $\alpha, \beta \in [0, 1]$ .

**THEOREM 6.4.** *A solution  $\psi$  on  $\mathcal{G}^N$  belongs to  $\Phi^N$  if and only if it satisfies efficiency, linearity, local monotonicity, the dictator property and the veto equal loss property.*

*Proof.* It is straightforward to verify that any  $\varphi^{\alpha, \beta}$ ,  $\alpha, \beta \in [0, 1]$ , satisfies efficiency, linearity and local monotonicity on  $\mathcal{G}^N$ .

Next, we show the dictator property. (i) Considering the dictator game  $(N, d_i^c)$ , for  $j \in N \setminus \{i\}$ , it follows with (9) that  $\varphi_i^{\alpha,\beta}(N, d_i^c) - \varphi_j^{\alpha,\beta}(N, d_i^c) = c\alpha \leq c$  since  $\alpha \leq 1$ . (ii) For unanimity games  $(N, u_{\{j\}})$  (i.e. dictator games  $(N, d_j^1)$ ), it holds that  $\varphi_i^{\alpha,\beta}(N, u_{\{j\}}) - \varphi_j^{\alpha,\beta}(N, u_{\{j\}}) = -\alpha < \alpha = \varphi_i^{\alpha,\beta}(N, d_i^1) - \varphi_j^{\alpha,\beta}(N, d_i^1)$ . Considering any unanimity game  $(N, u_T)$  with  $|T| \geq 2$ , it follows with (10) for  $i, j \in T$  or  $i, j \in N \setminus T$  that  $\varphi_i^{\alpha,\beta}(N, u_T) = \varphi_j^{\alpha,\beta}(N, u_T)$ , and for  $i \in T$  and  $j \in N \setminus T$  that  $\varphi_i^{\alpha,\beta}(N, u_T) - \varphi_j^{\alpha,\beta}(N, u_T) = \alpha(1 - \beta) \leq \alpha$  since  $\beta \in [0, 1]$ . The dictator property then follows with linearity and the fact that  $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$  for all  $(N, v) \in \mathcal{G}^N$ .

To show the veto equal loss property, consider the unanimity game  $(N, u_T)$ ,  $|T| \geq 2$ , and player  $h \in N$ . (Note that we do not have to consider cases with  $|T| = 1$  since the veto equal loss property only concerns zero-normalized games.) If  $h \in T$  then  $(u_T)^h = u_T$ , and thus payoffs are not changing when  $h$  becomes a veto player. If  $h \in N \setminus T$  then  $(u_T)^h = u_{T \cup \{h\}}$ . Obviously,  $\varphi_i^{\alpha,\beta}(N, u_T) - \varphi_i^{\alpha,\beta}(N, (u_T)^h)$  is equal for all  $i \in T$ , and  $\varphi_j^{\alpha,\beta}(N, u_T) - \varphi_j^{\alpha,\beta}(N, (u_T)^h)$  is equal for all  $j \in N \setminus (T \cup \{h\})$ . Take an  $i \in T$  and  $j \in N \setminus (T \cup \{h\})$ . Then it follows with (10) that  $\varphi_i^{\alpha,\beta}(N, u_T) - \varphi_i^{\alpha,\beta}(N, (u_T)^h) = \frac{1}{|N|} \alpha(1 - \beta)(|N| - |T| + 1 - |N| + |T|) = \frac{1}{|N|} \alpha(1 - \beta) = \varphi_j^{\alpha,\beta}(N, u_T) - \varphi_j^{\alpha,\beta}(N, (u_T)^h)$ . The veto equal loss property then follows by linearity of  $\varphi^{\alpha,\beta}$  and the fact that  $v^h = \sum_{\substack{T \subseteq N \\ |T| \geq 2}} \Delta_v(T) (u_T)^h$  when  $v$  is zero-normalized.

Now, suppose that solution  $\psi$  satisfies the five properties on  $\mathcal{G}^N$ . Let  $(N, v) \in \mathcal{G}^N$ . If  $|N| = 1$ , then  $\psi(N, v) \in \Phi^N$  by efficiency. If  $|N| = 2$ , then  $\psi \in \Phi^N$  follows from efficiency, local monotonicity, linearity and Proposition 4.2. (Note that local monotonicity implies symmetry.)

Next, suppose that  $|N| \geq 3$ . If  $(N, v)$  is a null game given by  $v(S) = 0$  for all  $S \subseteq N$ , then efficiency and local monotonicity imply that  $\psi_i(N, v) = 0$  for all  $i \in N$ .

Next, we consider unanimity games  $(N, u_T) \in \mathcal{G}^N$ ,  $T \subset N$ ,  $|T| = 1$ . Local monotonicity implies that there is a  $c^* \in \mathbb{R}$  such that  $\psi_j(N, u_T) = c^*$  for all  $j \in N \setminus T$ . For  $i \in T$ , local monotonicity implies that  $\psi_i(N, u_T) \geq c^*$ . By efficiency,

$\psi_i(N, u_T) = 1 - (|N| - 1)c^*$ , and thus  $1 - (|N| - 1)c^* \geq c^*$ , which is equivalent to  $c^* \leq \frac{1}{|N|}$ . The dictator property implies that  $\psi_i(N, v) - c^* \leq 1$ , which with efficiency yields  $c^* \geq 1 - (|N| - 1)c^* - 1$ , which is equivalent to  $c^* \geq 0$ . Thus, there is an  $\alpha_T \in [0, 1]$  such that  $\psi(N, u_T) = \alpha_T CIS(N, u_T) + (1 - \alpha_T)ED(N, u_T)$ . (Note that in this case  $CIS(N, u_T) = ENSC(N, u_T)$ .)

Next, we show that  $\alpha_T = \alpha_{T'}$  for all  $T, T' \subset N$  with  $|T| = |T'| = 1$ . Take  $i, j \in N$ ,  $i \neq j$ , and  $h \in N \setminus \{i, j\}$ . Since  $(u_{\{i\}})^j = u_{\{i,j\}}$ , the veto equal loss property implies that  $\psi_h(N, u_{\{i,j\}}) - \psi_h(N, u_{\{i\}}) = \psi_i(N, u_{\{i,j\}}) - \psi_i(N, u_{\{i\}})$  and  $\psi_h(N, u_{\{i,j\}}) - \psi_h(N, u_{\{j\}}) = \psi_j(N, u_{\{i,j\}}) - \psi_j(N, u_{\{j\}})$ , and thus  $\psi_i(N, u_{\{i,j\}}) - \psi_i(N, u_{\{i\}}) + \psi_h(N, u_{\{i\}}) = \psi_h(N, u_{\{i,j\}}) = \psi_j(N, u_{\{i,j\}}) - \psi_j(N, u_{\{j\}}) + \psi_h(N, u_{\{j\}})$ . Local monotonicity implies that  $\psi_i(N, u_{\{i,j\}}) = \psi_j(N, u_{\{i,j\}})$ , and thus  $-\psi_i(N, u_{\{i\}}) + \psi_h(N, u_{\{i\}}) = -\psi_j(N, u_{\{j\}}) + \psi_h(N, u_{\{j\}})$ , meaning that

$$\begin{aligned} & -\alpha_{\{i\}}CIS_i(N, u_{\{i\}}) - (1 - \alpha_{\{i\}})ED_i(N, u_{\{i\}}) + \alpha_{\{i\}}CIS_h(N, u_{\{i\}}) \\ & \quad + (1 - \alpha_{\{i\}})ED_h(N, u_{\{i\}}) \\ & = -\alpha_{\{j\}}CIS_j(N, u_{\{j\}}) - (1 - \alpha_{\{j\}})ED_j(N, u_{\{j\}}) \\ & \quad + \alpha_{\{j\}}CIS_h(N, u_{\{j\}}) + (1 - \alpha_{\{j\}})ED_h(N, u_{\{j\}}) \end{aligned}$$

which is equivalent to

$$\alpha_{\{i\}} + 2\frac{(1 - \alpha_{\{i\}})}{|N|} = \alpha_{\{j\}} + 2\frac{(1 - \alpha_{\{j\}})}{|N|},$$

which is equivalent to

$$(|N| - 2)\alpha_{\{i\}} + 2 = (|N| - 2)\alpha_{\{j\}} + 2, \quad \text{which is equivalent to } \alpha_{\{i\}} = \alpha_{\{j\}}.$$

So, there is an  $\alpha \in [0, 1]$  such that  $\alpha_T = \alpha$  for all  $T \subset N$  with  $|T| = 1$ .

Next, consider unanimity games  $(N, u_T) \in \mathcal{G}^N$ ,  $T \subset N$ ,  $2 \leq |T| \leq |N| - 1$ . Local monotonicity again implies that there is a  $c^* \in \mathbb{R}$  such that  $\psi_j(N, u_T) = c^*$  for all  $j \in N \setminus T$ . Moreover, local monotonicity implies that there is a  $c^{**} \geq c^*$  such that  $\psi_i(N, u_T) = c^{**}$  for all  $i \in T$ . Efficiency implies that  $c^{**} = \frac{1 - (|N| - |T|)c^*}{|T|}$ . Thus  $c^{**} \geq c^*$  if and only if  $\frac{1 - (|N| - |T|)c^*}{|T|} \geq c^*$  if and only if  $c^* \leq \frac{1}{|N|}$ . By Proposition 6.2, it follows that  $c^* \geq c^{**} - 1$ ,

and thus  $c^* \geq \frac{1-(|N|-|T|)c^*}{|T|} - 1$ , which is equivalent to  $c^* \geq \frac{1-|T|}{|N|}$ . From this, it follows that there is a  $\delta_T \in [0, 1]$  such that  $\psi(N, u_T) = \delta_T CIS(N, u_T) + (1 - \delta_T) ENSC(N, u_T)$ . (Note that in this case  $CIS(N, u_T) = ED(N, u_T)$ .)

Next, we show that there is a  $\delta \in [0, 1]$  such that  $\delta_T = \delta$  for all  $T \subset N$  with  $2 \leq |T| \leq |N| - 1$ . We distinguish the case  $|N| = 3$  from the case  $|N| \geq 4$ .

For  $|N| = 3$ , we only have to show that  $\delta_T = \delta_{T'}$  for  $T, T' \subset N$  with  $|T| = |T'| = 2$ . Suppose that  $N = \{i, j, h\}$  and consider the game  $v = u_{\{i,j\}} + u_{\{i,h\}}$ . By linearity, we have that  $\psi_j(N, v) = \psi_j(N, u_{\{i,j\}}) + \psi_j(N, u_{\{i,h\}})$  and  $\psi_h(N, v) = \psi_h(N, u_{\{i,j\}}) + \psi_h(N, u_{\{i,h\}})$ . So

$$\begin{aligned} \psi_j(N, u_{\{i,j\}}) + \psi_j(N, u_{\{i,h\}}) &= \psi_j(N, v) \\ &= \psi_h(N, v) = \psi_h(N, u_{\{i,j\}}) + \psi_h(N, u_{\{i,h\}}), \end{aligned} \tag{11}$$

where the second equality follows from local monotonicity. Since  $\psi(N, u_T) = \delta_T CIS(N, u_T) + (1 - \delta_T) ENSC(N, u_T)$  as shown above, (11) yields that

$$\begin{aligned} &\left(\frac{\delta_{\{i,j\}}}{3} + \frac{2(1 - \delta_{\{i,j\}})}{3}\right) + \left(\frac{\delta_{\{i,h\}}}{3} - \frac{1 - \delta_{\{i,h\}}}{3}\right) \\ &= \left(\frac{\delta_{\{i,j\}}}{3} - \frac{1 - \delta_{\{i,j\}}}{3}\right) + \left(\frac{\delta_{\{i,h\}}}{3} + \frac{2(1 - \delta_{\{i,h\}})}{3}\right), \end{aligned}$$

showing that  $\delta_{\{i,j\}} = \delta_{\{i,h\}}$ . Similarly, it can be shown for all  $T, T' \subset N$  with  $|T| = |T'| = 2$  that  $\delta_T = \delta_{T'}$  for  $|N| = 3$ .

For  $|N| \geq 4$ , consider a  $T \subset N$  with  $|T| \in \{2, \dots, |N| - 2\}$ , and take some  $h \in N \setminus T$ . Since  $(u_T)^h = u_{T \cup \{h\}}$ , the veto equal loss property implies for  $i \in T, j \in N \setminus (T \cup \{h\})$  that

$$\psi_i(N, u_T) - \psi_i(N, u_{T \cup \{h\}}) = \psi_j(N, u_T) - \psi_j(N, u_{T \cup \{h\}})$$

which, as shown above, is equivalent to

$$\begin{aligned} &\delta_T CIS_i(N, u_T) + (1 - \delta_T) ENSC_i(N, u_T) - \delta_{T \cup \{h\}} CIS_i(N, u_{T \cup \{h\}}) \\ &\quad - (1 - \delta_{T \cup \{h\}}) ENSC_i(N, u_{T \cup \{h\}}) \\ &= \delta_T CIS_j(N, u_T) + (1 - \delta_T) ENSC_j(N, u_T) \\ &\quad - \delta_{T \cup \{h\}} CIS_j(N, u_{T \cup \{h\}}) - (1 - \delta_{T \cup \{h\}}) ENSC_j(N, u_{T \cup \{h\}}), \end{aligned}$$



which, by definition of the CIS-value, is equivalent to

$$\begin{aligned} & (1 - \delta_T)(ENSC_i(N, u_T) - ENSC_j(N, u_T)) \\ & = (1 - \delta_{T \cup \{h\}})(ENSC_i(N, u_{T \cup \{h\}}) - ENSC_j(N, u_{T \cup \{h\}})), \end{aligned}$$

which, by definition of the ENSC-value, is equivalent to

$$\begin{aligned} & (1 - \delta_T) \left( \frac{|N| - |T| + 1}{|N|} - \frac{1 - |T|}{|N|} \right) \\ & = (1 - \delta_{T \cup \{h\}}) \left( \frac{|N| - |T| - 1 + 1}{|N|} - \frac{1 - |T| - 1}{|N|} \right), \end{aligned}$$

which is equivalent to

$$\delta_T = \delta_{T \cup \{h\}}.$$

But then there is a  $\delta \in [0, 1]$  such that  $\delta_T = \delta$  for all  $T \subset N$  with  $|T| \in \{2, \dots, |N| - 1\}$ .

For the unanimity game  $(N, u_N) \in \mathcal{G}^N$ , efficiency and local monotonicity imply that  $\psi_i(N, u_N) = ED(N, u_N) = CIS(N, u_N) = ENSC(N, u_N) = \frac{1}{|N|}$  for all  $i \in N$ .

If  $\alpha > 0$ , then defining  $\beta = \frac{\delta - (1 - \alpha)}{\alpha}$ , we obtain from above that  $\psi(N, u_T) = \varphi^{\alpha, \beta}(N, u_T)$  for all  $T \subseteq N$ ,  $T \neq \emptyset$ . We are left to show that  $\beta \in [0, 1]$ . Obviously,  $\delta - (1 - \alpha) \leq \alpha$  since  $\delta \in [0, 1]$ , and thus  $\beta \leq 1$ . Considering  $(N, u_T)$  with  $|T| = 2$ , take  $i \in T$  and  $j \in N \setminus T$ . The dictator property then yields that  $\psi_i(N, u_T) - \psi_j(N, u_T) = \varphi_i^{\alpha, \beta}(N, u_T) - \varphi_j^{\alpha, \beta}(N, u_T) = \alpha(1 - \beta) \leq \psi_i(N, u_{\{i\}}) - \psi_j(N, u_{\{i\}}) = \varphi_i^{\alpha, \beta}(N, u_{\{i\}}) - \varphi_j^{\alpha, \beta}(N, u_{\{i\}}) = \alpha$ . So,  $1 - \beta \leq 1$ , and thus  $\beta \geq 0$ . So, for  $\alpha > 0$ , we have shown that  $\beta \in [0, 1]$ . For  $\alpha = 0$ , we can just take  $\beta = \delta \in [0, 1]$ . So, we have determined that  $\psi(N, u_T) = \varphi^{\alpha, \beta}(N, u_T)$  for  $\alpha, \beta \in [0, 1]$  and all  $T \subseteq N$ .

Finally, linearity and the fact that  $v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \Delta_v(T) u_T$  implies that  $\psi(N, v) = \varphi^{\alpha, \beta}(N, v)$  for all  $(N, v) \in \mathcal{G}^N$ . □

Next, we provide an axiomatization on the class of all TU-games  $\mathcal{G}$ . In order to do that, we introduce the following axiom that relates payoffs in games with different sets of players and can be seen as a weak consistency axiom. It states

that deleting a null player from a game changes the payoffs of all other players by the same amount. For  $(N, v) \in \mathcal{G}$  and  $T \subset N$ , the restricted game  $(T, v_T)$  is given by  $v_T(S) = v(S)$  for all  $S \subseteq T$ .

**AXIOM 6.5.** (*Weak null player out property*) *A solution  $\psi$  satisfies the weak null player out property if  $\psi_i(N \setminus \{h\}, v_{N \setminus \{h\}}) - \psi_i(N, v) = \psi_j(N \setminus \{h\}, v_{N \setminus \{h\}}) - \psi_j(N, v)$  for every  $i, j \in N$  and  $h \in N \setminus \{i, j\}$  being a null player in  $(N, v)$ .*

This is a weaker version of the null player out property as considered in Derks and Haller (1999), which states that deleting a null player does not change the payoffs of the other players.

**THEOREM 6.6.** *A solution  $\psi$  on  $\mathcal{G}$  belongs to  $\Phi$  if and only if it satisfies efficiency, linearity, local monotonicity, the dictator property, the veto equal loss property and the weak null player out property.*

*Proof.* Similar to the proof of Theorem 6.4 it follows that any  $\varphi^{\alpha, \beta}$ ,  $\alpha, \beta \in [0, 1]$ , satisfies efficiency, linearity, local monotonicity, the dictator property and the veto equal loss property on  $\mathcal{G}$ . To show the weak null player out property, consider unanimity games  $(N, u_T)$ ,  $h, j \in N \setminus T$ ,  $h \neq j$ , and  $i \in T$ . If  $|T| = 1$  then  $\varphi_i^{\alpha, \beta}(N \setminus \{h\}, u_T) - \varphi_i^{\alpha, \beta}(N, u_T) = \varphi_j^{\alpha, \beta}(N \setminus \{h\}, u_T) + \alpha - \varphi_j^{\alpha, \beta}(N, u_T) - \alpha = \varphi_j^{\alpha, \beta}(N \setminus \{h\}, u_T) - \varphi_j^{\alpha, \beta}(N, u_T)$ . If  $|T| \geq 2$  then  $\varphi_i^{\alpha, \beta}(N \setminus \{h\}, u_T) - \varphi_i^{\alpha, \beta}(N, u_T) = \varphi_j^{\alpha, \beta}(N \setminus \{h\}, u_T) + \alpha(1 - \beta) - \varphi_j^{\alpha, \beta}(N, u_T) - \alpha(1 - \beta) = \varphi_j^{\alpha, \beta}(N \setminus \{h\}, u_T) - \varphi_j^{\alpha, \beta}(N, u_T)$ . The weak null player out property then follows by linearity of  $\varphi^{\alpha, \beta}$ .

To show uniqueness, suppose that solution  $\psi$  satisfies the six properties on  $\mathcal{G}$ . From Theorem 6.4, it follows that for every  $N \subset \mathbb{N}$ , there exist numbers  $\alpha^{|N|}, \beta^{|N|} \in [0, 1]$  such that  $\psi(N, v) = \varphi^{\alpha^{|N|}, \beta^{|N|}}(N, v)$  for all  $(N, v) \in \mathcal{G}$ . Since efficiency determines the solution for one-player games, we are left to show that there exist  $\alpha, \beta \in [0, 1]$  such that  $\alpha^n = \alpha$  and  $\beta^n = \beta$  for all  $n \in \{2, \dots\}$ . First consider games  $(N, u_T)$ ,  $(N \cup \{h\}, u_T)$  for  $h \in \mathbb{N} \setminus N$  and  $|T| = 1$ . For  $i \in T$ ,  $j \in N \setminus T$ , the weak null

player out property implies that  $\psi_j(N, u_T) - \psi_j(N \cup \{h\}, u_T) = \psi_i(N, u_T) - \psi_i(N \cup \{h\}, u_T)$ , so  $\frac{1-\alpha^{|N|}}{|N|} - \frac{1-\alpha^{|N|+1}}{|N|} = \frac{1-\alpha^{|N|}}{|N|} + \alpha^{|N|} - \frac{1-\alpha^{|N|+1}}{|N|} - \alpha^{|N|+1}$ , yielding  $\alpha^{|N|} = \alpha^{|N|+1}$ . So, we conclude that there exists an  $\alpha \in [0, 1]$  such that  $\alpha^n = \alpha$  for all  $n \in \{2, \dots\}$ .

Next, consider games  $(N, u_T)$ ,  $(N \cup \{h\}, u_T)$  for  $h \in \mathbb{N} \setminus N$  and  $2 \leq |T| \leq |N| - 2$ . For  $i \in T$ ,  $j \in N \setminus T$ , the weak null player out property again implies that  $\psi_j(N, u_T) - \psi_j(N \cup \{h\}, u_T) = \psi_i(N, u_T) - \psi_i(N \cup \{h\}, u_T)$ . Using  $\alpha^n = \alpha$  for all  $n \in \mathbb{N}$ , we then find that  $\frac{1}{|N|}((1-\alpha) + \alpha\beta^{|N|} + \alpha(1-\beta^{|N|})(1-|T|)) - \frac{1}{|N|+1}((1-\alpha) + \alpha\beta^{|N|+1} + \alpha(1-\beta^{|N|+1})(1-|T|)) = \frac{1}{|N|}((1-\alpha) + \alpha\beta^{|N|} + \alpha(1-\beta^{|N|})(|N|-|T|+1)) - \frac{1}{|N|+1}((1-\alpha) + \alpha\beta^{|N|+1} + \alpha(1-\beta^{|N|+1})(|N|-|T|+1))$ , and thus  $\beta^{|N|} = \beta^{|N|+1}$ . So, we conclude that there exists a  $\beta \in [0, 1]$  such that  $\beta^n = \beta$  for all  $n \in \{2, \dots\}$ .  $\square$

Note that, similar to Corollary 5.5, we characterized the class of solutions  $\Phi$  using axioms that do not depend on the parameters  $\alpha$  and  $\beta$ . We conclude this section by showing logical independence of the six axioms of Theorem 6.6.

- (1) Solution  $\psi$  given by  $\psi_i(N, v) = 0$  for all  $i \in N$  and  $(N, v) \in \mathcal{G}$  satisfies the axioms of Theorem 6.6 except efficiency.
- (2) Solution  $\psi$  given by  $\psi(N, v) = CIS(N, v)$  if  $v(N) \leq 10$ , and  $\psi(N, v) = ENSC(N, v)$  if  $v(N) > 10$  satisfies the axioms of Theorem 6.6 except linearity.
- (3) Define  $\overline{ENCIS}^\beta(N, v) = \sum_{T \subseteq N} \Delta_v(T) \overline{ENCIS}^\beta(N, u_T)$ , where  $\overline{ENCIS}^\beta(N, u_T) = ENCIS^\beta(N, u_T)$  if  $|T| = 1$ , and  $\overline{ENCIS}^\beta(N, u_T) = ENCIS^\beta(N, u_{N \setminus T})$  if  $|T| \geq 2$ . Then the solution  $\overline{\varphi}^{\alpha, \beta}$  given by  $\overline{\varphi}^{\alpha, \beta}(N, v) = \alpha \overline{ENCIS}^\beta(N, v) + (1-\alpha)ED(N, v)$  for all  $(N, v) \in \mathcal{G}$ , satisfies the axioms of Theorem 6.6, except local monotonicity.
- (4) Solution  $\psi = \varphi^{\alpha, \beta}$  with  $\alpha > 1$  satisfies the axioms of Theorem 6.6 except the dictator property.
- (5) The Shapley value (Shapley, 1953) satisfies the axioms of Theorem 6.6 except the veto equal loss property.
- (6) Consider numbers  $\alpha^n \in [0, 1]$ ,  $n \in \mathbb{N}$  such that there exists  $n \in \mathbb{N}$  with  $\alpha^n \neq \alpha^{n+1}$ . Then solution  $\psi(N, v) = \varphi^{\alpha^{|N|}, \beta^{|N|}}(N, v)$

for all  $(N, v) \in \mathcal{G}$  satisfies the axioms of Theorem 6.6 except the weak null player out property.

## 7. SPECIFIC SOLUTIONS IN $\Phi$

Taking specific values for  $\alpha$ , respectively,  $\beta$  in the standardness, respectively, reduced game consistency, yields characterizations of specific solutions in  $\Phi$ . The CIS- and ENSC-value and all their convex combinations are obtained with  $\alpha = 1$ , and thus satisfy standardness for two-player games. The corresponding characterizations follow directly from Theorem 5.4, and therefore, are stated without further proof.

**COROLLARY 7.1.** *A solution  $\psi$  satisfies*

1. *efficiency, standardness for two-player games and  $\beta$ -consistency,  $\beta \in [0, 1]$ , on  $\mathcal{G}$  if and only if  $\psi = \text{ENCIS}^\beta = \varphi^{1,\beta}$ .*
2. *efficiency, standardness for two-player games and 1-consistency on  $\mathcal{G}$  if and only if  $\psi = \text{CIS} = \varphi^{1,1}$ .*
3. *efficiency, standardness for two-player games and 0-consistency on  $\mathcal{G}$  if and only if  $\psi = \text{ENSC} = \varphi^{1,0}$ .*
4. *efficiency, egalitarian standardness for two-player games and  $\beta$ -consistency,  $\beta \in [0, 1]$ , on  $\mathcal{G}$  if and only if  $\psi = \text{ED}$ .*

Note that the last statement in this corollary implies that the equal division solution is axiomatized by egalitarian standardness for two-player games and any consistency, as discussed in the previous section.

In the previous section, we characterized the class of solutions  $\Phi$ . Adding additional properties yields characterizations of specific solutions in  $\Phi$  (where some of the other properties might be deleted). The subclass of self-dual solutions in  $\Phi$  is characterized as follows.

**PROPOSITION 7.2.** *Consider solution  $\varphi^{\alpha,\beta} \in \Phi$ . Then  $\varphi^{\alpha,\beta}$  is self-dual if and only if  $\alpha = 0$  or  $\beta = \frac{1}{2}$ .*

*Proof.* From Proposition 3.3, it is straightforward to show that these solutions are self-dual. To show that these are the

only self-dual solutions in the class  $\Phi$ , suppose that  $\varphi^{\alpha,\beta}(N, v) = \varphi^{\alpha,\beta}(N, v^*)$  for all  $(N, v) \in \mathcal{G}$ . Then with Proposition 3.3, we have

$$\begin{aligned} \varphi^{\alpha,\beta}(N, v) &= \varphi^{\alpha,\beta}(N, v^*) \\ \Leftrightarrow \varphi^{\alpha,\beta}(N, v) &= \varphi^{\alpha,1-\beta}(N, v) \\ \Leftrightarrow \lambda_i^{\alpha,\beta}(N, v) + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \lambda_j^{\alpha,\beta}(N, v) \right) \\ &= \lambda_i^{\alpha,1-\beta}(N, v) + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \lambda_j^{\alpha,1-\beta}(N, v) \right) \\ \Leftrightarrow \alpha(\beta v(\{i\}) - (1 - \beta)v(N \setminus \{i\})) \\ &\quad + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \alpha(\beta v(\{j\}) - (1 - \beta)v(N \setminus \{j\})) \right) \\ &= \alpha((1 - \beta)v(\{i\}) - \beta v(N \setminus \{i\})) \\ &\quad + \frac{1}{|N|} \left( v(N) - \sum_{j \in N} \alpha((1 - \beta)v(\{j\}) - \beta v(N \setminus \{j\})) \right) \end{aligned}$$

Since this must hold for all  $(N, v) \in \mathcal{G}$ , we have  $\alpha\beta = \alpha(1 - \beta)$ , which is equivalent to  $\alpha(2\beta - 1) = 0$ . So,  $\alpha = 0$  or  $\beta = \frac{1}{2}$ .  $\square$

So, the self-dual solutions in  $\Phi$  are the equal division solution, the average of the CIS- and ENSC-value, and all convex combinations of these two solutions. The only covariant solutions in  $\Phi$  are the ones with  $\alpha = 1$ . These are the CIS-value, the ENSC-value and their convex combinations. In the following, we use the expression

$$\begin{aligned} \varphi_i^{\alpha,\beta}(N, v) &= \alpha\beta v(\{i\}) - \alpha(1 - \beta)v(N \setminus \{i\}) + \frac{1}{|N|}v(N) \\ &\quad - \frac{1}{|N|}\alpha\beta \sum_{j \in N} v(\{j\}) + \frac{1}{|N|}\alpha(1 - \beta) \\ &\quad \times \sum_{j \in N} v(N \setminus \{j\}) \end{aligned} \tag{12}$$

**PROPOSITION 7.3.** *Consider solution  $\varphi^{\alpha,\beta} \in \Phi$ . Then  $\varphi^{\alpha,\beta}$  is covariant if and only if  $\alpha = 1$ .*

*Proof.* Let  $(N, v), (N, w) \in \mathcal{G}$  be such that there exist  $k \in \mathbb{R}_+$  and  $p \in \mathbb{R}^n$  with  $w(S) = kv(S) + \sum_{j \in S} p_j$  for all  $S \subseteq N$ . Denoting  $p_S = \sum_{j \in S} p_j$  for all  $S \subseteq N$ , with (12) it then follows that

$$\begin{aligned} \varphi_i^{\alpha,\beta}(N, w) &= \alpha\beta w(\{i\}) - \alpha(1 - \beta)w(N \setminus \{i\}) + \frac{1}{|N|}w(N) \\ &\quad - \frac{1}{|N|}\alpha\beta \sum_{j \in N} w(\{j\}) + \frac{1}{|N|}\alpha(1 - \beta) \sum_{j \in N} w(N \setminus \{j\}) \\ &= \alpha\beta(kv(\{i\}) + p_i) - \alpha(1 - \beta)(kv(N \setminus \{i\}) + p_{N \setminus \{i\}}) \\ &\quad + \frac{1}{|N|}(kv(N) + p_N) - \frac{1}{|N|}\alpha\beta \sum_{j \in N} (kv(\{j\}) + p_j) \\ &\quad + \frac{1}{|N|}\alpha(1 - \beta) \sum_{j \in N} (kv(N \setminus \{j\}) + p_{N \setminus \{j\}}) \\ &= \alpha\beta kv(\{i\}) - \alpha(1 - \beta)kv(N \setminus \{i\}) + \frac{1}{|N|}kv(N) \\ &\quad - \frac{1}{|N|}\alpha\beta \sum_{j \in N} kv(\{j\}) + \frac{1}{|N|}\alpha(1 - \beta) \sum_{j \in N} kv(N \setminus \{j\}) \\ &\quad + \alpha\beta p_i - \alpha(1 - \beta)p_{N \setminus \{i\}} + \frac{1}{|N|}p_N - \frac{1}{|N|}\alpha\beta p_N \\ &\quad + \frac{1}{|N|}\alpha(1 - \beta)(|N| - 1)p_N \\ &= k\varphi_i^{\alpha,\beta}(N, v) + \left( \frac{1 + (|N| - 1)\alpha}{|N|} \right) p_i + \frac{1 - \alpha}{|N|} p_{N \setminus \{i\}}. \end{aligned}$$

But then,  $\varphi_i^{\alpha,\beta}(N, w) = k\varphi_i^{\alpha,\beta}(N, v) + p_i$  for all  $k \in \mathbb{R}_+$ ,  $p \in \mathbb{R}^N$  if and only if  $\alpha = 1$ .  $\square$

As a corollary from Propositions 7.2 and 7.3, we immediately get a characterization of the average of the CIS and ENSC-value.

**COROLLARY 7.4.** *Consider solution  $\varphi^{\alpha,\beta} \in \Phi$ . Then  $\varphi^{\alpha,\beta}$  is self-dual and covariant if and only if it is the average of the CIS- and ENSC-value  $\varphi^{1,\frac{1}{2}}$ .*

Recall from the previous section that  $\psi \in \Phi$  is equivalent to saying that  $\psi$  satisfies efficiency, linearity, local monotonicity, the dictator property, the veto equal loss property and the weak null player out property.

The equal division solution satisfies self-duality, but is not covariant. Instead, it is characterized as the unique solution in  $\Phi$  that is non-negative.<sup>9</sup> We even can weaken the conditions, under which the solution belongs to  $\Phi$  (as given in Theorem 6.4) by requiring it to be efficient, symmetric and linear.

**PROPOSITION 7.5.** *Solution  $\psi$  satisfies efficiency, symmetry, linearity and non-negativity if and only if it is the equal division solution.*

*Proof.* Obviously, the equal division solution satisfies efficiency, symmetry, linearity and non-negativity. Next, suppose that solution  $\psi$  satisfies these four properties. Consider the standard game  $(N, b_T)$  for  $T \subset N$ ,  $T \neq N$ . Efficiency of  $\psi$  implies that  $\sum_{i \in N} \psi_i(N, b_T) = 0$ . Non-negativity of  $\psi$  implies that  $\psi_i(N, b_T) \geq 0$  for all  $i \in N$ . Thus,  $\psi_i(N, b_T) = 0 = ED_i(N, b_T)$  for all  $i \in N$ .

For the standard game  $(N, b_N)$ , symmetry implies that all  $\psi_i(N, b_N)$ ,  $i \in N$ , are equal. With efficiency, it then follows that  $\psi_i(N, b_N) = \frac{1}{|N|} = ED_i(N, b_N)$  for all  $i \in N$ .

So,  $\psi$  is equal to the equal division solution on the class of standard games. Since every characteristic function  $v$  can be written as a linear combination of standard games by  $v = \sum_{\substack{T \subset N \\ T \neq \emptyset}} v(T)b_T$ , linearity then implies that  $\psi$  is equal to the equal division solution on the class of all games. □

Note that from Propositions 7.2 and 7.5, we also derive that the CIS- and ENSC-values are not self-dual (in fact, they are each others dual), nor non-negative. They are

characterized by individual rationality properties. The CIS-value is the unique solution in  $\Phi$  that satisfies individual rationality. Again, the conditions, under which a solution belongs to  $\Phi$  can be weakened.

**PROPOSITION 7.6.** *Solution  $\psi$  satisfies efficiency, symmetry, linearity and individual rationality if and only if it is the CIS-value  $\varphi^{1,1}$ .*

*Proof.* It is known that the CIS-value satisfies efficiency, symmetry, linearity and individual rationality. Next, suppose that solution  $\psi$  satisfies these four properties. Consider the unanimity game  $(N, u_T)$  for  $T \subset N$ , with  $|T| = 1$ . Individual rationality of  $\psi$  implies that  $\psi_i(N, u_T) \geq 0$  for all  $i \in N \setminus T$ , and  $\psi_i(N, u_T) \geq 1$  for  $i \in T$ . Efficiency of  $\psi$  implies that  $\sum_{i \in N} \psi_i(N, u_T) = 1$ , and thus the above given inequalities are equalities, i.e.  $\psi_i(N, u_T) = 0 = CIS_i(N, u_T)$  for all  $i \in N \setminus T$ , and  $\psi_i(N, u_T) = 1 = CIS_i(N, u_T)$  for  $i \in T$ .

Next, consider the standard game  $(N, b_T)$  with  $2 \leq |T| \leq |N| - 1$ . Then individual rationality of  $\psi$  implies that  $\psi_i(N, b_T) \geq 0$  for all  $i \in N$ . Since efficiency of  $\psi$  implies that  $\sum_{i \in N} \psi_i(N, b_T) = 0$ , it follows that  $\psi_i(N, b_T) = 0 = CIS_i(N, b_T)$  for all  $i \in N$ .

Finally, consider the standard game  $(N, b_N)$ . Then symmetry and efficiency imply that  $\psi_i(N, b_N) = \frac{1}{|N|} = CIS_i(N, b_N)$  for all  $i \in N$ .

Since every characteristic function  $v$  can be written as a linear combination of standard- and one-player unanimity games by  $v = \sum_{i \in N} v(\{i\})u_{\{i\}} + \sum_{\substack{T \subset N \\ |T| \geq 2}} (v(T) - \sum_{i \in T} v(\{i\}))b_T$ , linearity implies that  $\psi$  is equal to the CIS-value on the class of all games.  $\square$

The ENSC-value is the unique solution in  $\Phi$  satisfying dual individual rationality. (The proof goes along similar lines as the proof of Proposition 7.6, and is, therefore, omitted.)

**PROPOSITION 7.7.** *Solution  $\psi$  satisfies efficiency, symmetry, linearity and dual individual rationality if and only if it is the ENSC-value  $\varphi^{1,0}$ .*



## 8. CONCLUDING REMARKS

In this article, we characterized the class of solutions that are obtained as convex combinations of the equal division solution, the CIS-value and the ENSC-value using a general  $\alpha$ -standardness for two-player games and a  $\beta$ -consistency,  $\alpha, \beta \in [0, 1]$ . We restricted both parameters  $\alpha$  and  $\beta$  to be between zero and one. This is obvious for  $\beta$ , which is interpreted as a probability of cooperation of the leaving player in the reduced games. However, we might consider  $\alpha$  to be any real number. Repeating the analysis done in this article, we would obtain similar results for the class of all affine combinations of the equal division solution and any convex combination of the CIS-value and the ENSC-value.

An  $n$ -player TU-game is described by the  $2^n - 1$  worths of non-empty coalitions. However, the solutions studied in this article only depend on the worths of the singletons, coalitions of size  $n - 1$  and the 'grand coalition'. This is a consequence of the egalitarian approach. In the literature, various classes of games that are fully determined by the worths of these special coalitions can be found. For example, in auction games (see Graham et al., 1990) and bankruptcy games (see O'Neill, 1982; Aumann and Maschler, 1985), the game is fully described by the worth of the 'grand coalition'  $N$  and the worths of all coalitions of size  $n - 1$ , while an airport game (see Littlechild and Owen, 1973) is fully described by the  $n$  worths of the singletons. Although the solutions studied in this article are especially useful for these classes of games, we support these solutions also in other applications of games, even if the game depends on the worths of more coalitions. Consider, for example, a firm which organization is managed by a group of managers. In wage negotiations between a manager and the firm, usually the wage of the manager is determined by the managers reservation wage (i.e. its singleton worth), the contribution of the manager to the firm (determined by its singleton worth in the dual game) and the worth of the fully employed firm.

The solutions  $\varphi^{\alpha,\beta} \in \Phi$  also can be characterized as *compromise values*. The literature on compromise values starts with the introduction of the  $\tau$ -value in Tijs (1981). This is an efficient solution for a special class of TU-games, which assigns to every so-called quasi-balanced TU-game the unique efficient payoff vector on the line segment between some lower- and upper bound. This set of quasi-balanced games are exactly those, for which the upper bound is above the lower bound and an efficient payoff vector on the line segment between these bounds exists. Taking as lower bound for player  $i \in N$  in game  $(N, v)$  the value  $m_i^{\alpha,\beta}(N, v) = \lambda_i^{\alpha,\beta}(N, v)$ , and as upper bound  $M_i^{\alpha,\beta}(N, v) = v(N) - \sum_{j \in N \setminus \{i\}} m_j^{\alpha,\beta}(N, v) = v(N) - \sum_{j \in N \setminus \{i\}} \lambda_j^{\alpha,\beta}(N, v)$ , we can adapt the class of quasi-balanced games accordingly. On this class, the solution  $\varphi^{\alpha,\beta}$  is the solution that assigns to every game the unique efficient payoff vector between the lower and upper bounds. The characterization of the  $\tau$ -value by efficiency, the *minimal right property* and the *restricted proportionality property*, as given in Tijs (1987), can be reformulated by adapting the lower and upper bounds to give characterizations of  $\varphi^{\alpha,\beta}$  on the adapted class of games.<sup>10</sup> Although the  $\tau$ -value is not a solution in  $\Phi$ , for *convex* games,<sup>11</sup> it always yields a convex combination of the CIS- and ENSC-value. However, for different games, the weight assigned to these two solutions is different. In the future, we will study equal surplus sharing solutions, which allow the parameter  $\beta(v)$  to depend on the game.

We once more remark that besides  $\beta$ , also the parameter  $\alpha$  in the solutions  $\varphi^{\alpha,\beta}$  is fixed and the same for every game. We obtain a larger class of solutions if we also allow this parameter to depend on the game. Allowing the value of  $\alpha$  to depend on the game we could take  $\alpha(v) = \frac{v(N)}{v(\{i\})+v(\{j\})}$  whenever  $v(\{i\}) + v(\{j\}) \neq 0$ . This yields *proportional standardness for two-player games* as satisfied by, e.g. the Proper Shapley value as introduced in Vorob'ev and Liapunov (1998): for  $N = \{i, j\}$ ,  $i \neq j$ , we have  $\psi_i(N, v) = \frac{v(N)}{2(v(\{i\})+v(\{j\}))} (v(\{i\}) - v(\{j\})) + \frac{1}{2}v(N) = \frac{v(\{i\})-v(\{j\})+v(\{i\})+v(\{j\})}{2(v(\{i\})+v(\{j\}))} v(N) = \frac{v(\{i\})}{v(\{i\})+v(\{j\})} v(N)$ . In

the future, we study the generalized class of solutions with parameters  $\alpha$  and  $\beta$  depending on the game, and in that way obtain new characterizations of (solutions related to) the  $\tau$ -value and the Proper Shapley value.

Given our class of solutions  $\Phi$ , we can also define a new set-valued solution for TU-games by assigning to every game the union of all payoff vectors assigned by any solution in  $\Phi$  to that game, i.e. we can consider the set-valued solution  $\Psi$  given by  $\Psi(N, v) = \{\varphi^{\alpha, \beta}(N, v) \mid \alpha, \beta \in [0, 1]\}$ . Finding axiomatic characterizations for this set-valued solution is a plan for future research.

Further, the solutions considered in Joosten (1996), van den Brink et al. (2007) and Ju et al., (2007) who, respectively, consider all convex combinations of the equal division solution and the Shapley value, and all convex combinations of the CIS-value and the Shapley value, can be generalized by considering all convex combinations of any solution  $\varphi^{\alpha, \beta} \in \Phi$  and the Shapley value. Then also other properties that the solutions in  $\Phi$  have in common with the Shapley value, such as Chun (1989)'s fair ranking and van den Brink (2001)'s fairness, will be useful. Finally, we mention future research on implementation of the solutions in  $\Phi$ .

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## NOTES

1. Efficient solutions are often called *values*.
2. TU-games satisfying this property are called *weakly essential*.
3. The straightforward proof can be obtained from the authors on request.
4. The stronger anonymity property does imply equality of  $\alpha_i$  and  $\alpha_j$  with linearity. In fact, it can be shown that a solution  $\psi$  satisfies (4) and is anonymous if and only if  $\alpha_i = \alpha_j$ ,  $\beta_i = \beta_j$ ,  $\gamma_i = \gamma_j$  and  $\delta_i = \delta_j$ . The 'if' part is straightforward. To show the 'only if' part, consider the permutation  $\pi : N \rightarrow N$  given by  $\pi(i) = j$  and  $\pi(j) = i$ . Then

$\pi v(\{i\}) = v(\{j\})$ ,  $\pi v(\{j\}) = v(\{i\})$  and  $\pi v(N) = v(N)$ . By anonymity of  $\psi$  it then must hold that  $\psi_i(N, v) = \psi_{\pi(i)}(N, \pi v) = \psi_j(N, \pi v)$ , which is equivalent to  $\alpha_i v(\{i\}) + \beta_i v(\{j\}) + \gamma_i v(N) + \delta_i = \alpha_j \pi v(\{j\}) + \beta_j \pi v(\{i\}) + \gamma_j \pi v(N) + \delta_j = \alpha_j v(\{i\}) + \beta_j v(\{j\}) + \gamma_j v(N) + \delta_j$ . The statement then follows since this must hold for all  $(N, v) \in \mathcal{G}^2$ .

5. Here we only consider the class  $\mathcal{G}$  of all TU-games. If one considers subclasses  $\mathcal{C} \subset \mathcal{G}$ , then in the definition of consistency one should additionally require that the reduced game  $(N \setminus \{j\}, v^x)$  in this definition also belongs to  $\mathcal{C}$ .
6. In case  $\beta \in (0, 1]$  we can even do without efficiency since  $x_i - y_i = 0$  for all  $i \in N$  follows from  $x_i - y_i = (1 - \frac{|N|\beta}{|N|-1})(x_j - y_j)$  for all  $i, j \in N$ ,  $i \neq j$ .
7. Note that  $d_i^c$  equals the scaled unanimity game  $cu_{\{i\}}$ .
8. Note that we can easily adapt the axioms by requiring them on  $\mathcal{G}^N$  instead of  $\mathcal{G}$ .
9. Alternative characterizations of the equal division solution and CIS-value are given in van den Brink (2007).
10. In the proof given in Tijs (1987), use of the specific formula's of the lower- and upper-bounds is made only twice. This is to show that for every  $v \in \mathcal{G}$ , the rescaled game, in which we subtract the minimal rights from the worths of coalitions is the nullvector (which is satisfied for all bounds satisfying covariance), and to show that the upper bound satisfies covariance, see also van den Brink (1994).
11. A TU-game  $(N, v)$  is *convex* if  $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$  for all  $S, T \subseteq N$ .

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