CORRECTION

Correction to: The number of different true permutation polynomial based interleavers under Zhao and Fan sufficient conditions

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In the original version of this article [1] unfortunately, there are mistakes in some formulas for determining the number of true different cubic, fourth degree, and fifth degree permutation polynomial based interleavers under Zhao and Fan sufficient conditions. The reason for these mistakes was considering all null polynomials. However under these sufficient conditions, the null polynomials have to be restricted in some cases. These particular cases are indicated, and the corresponding formulas are corrected.

In the original version, we have determined the number of true different cubic, fourth degree, and fifth degree permutation polynomial (PP) based interleavers under Zhao and Fan (ZF) sufficient conditions given in [2]. To determine the number of true different PPs of a certain degree n , we took into account the equivalence conditions for PPs considering null polynomials (NPs) of degree up to n . In [1], we have not considered the fact that by adding a NP to a PP fulfilling ZF sufficient conditions does not always result a PP that also fulfills ZF sufficient conditions. This omission generated a mistake in some formulas for the number of true different PPs of degree 3 to 5 and for the lengths leading this number to be equal to zero. In this paper, we will correct these formulas.

1 Null polynomials under Zhao and Fan sufficient conditions

Let there be a PP of degree n modulo a positive integer N defined as:

$$\pi(x) = q_0 + q_1 \cdot x + q_2 \cdot x^2 + \cdots + q_n \cdot x^n \pmod{N},$$

$$x = \overline{0, N-1}, \quad (1)$$

and let the prime factorization of N be

$$N = \prod_{\substack{p \in \mathcal{P}, \\ p|N}} p^{n_{N,p}}, \quad (2)$$

where $\mathcal{P}$ is the set of prime numbers, the notation $p \mid N$ means that p divides N and $n_{N,p} \geq 1$ for a finite number of prime numbers.

ZF sufficient conditions on the coefficients of the polynomial in (1) so that it is PP modulo N are given in Table 1, which is similar to Theorem 1 in [2].

Let there be

$$z(x) = \sum_{k=1}^n z_k \cdot x^k \pmod{N} \quad (3)$$

a NP of degree n modulo N , i.e., $z(x) = 0, \forall x = \overline{0, N-1}$.

We define a *NP valid under ZF sufficient conditions* (denoted by ZF NP) a NP $z(x)$ fulfilling the condition that if $\pi(x)$ is PP under ZF sufficient conditions then $\pi(x) + z(x)$ is also a PP under ZF sufficient conditions. From Table 1, we can determine the conditions on the coefficients of a NP valid under ZF sufficient conditions. Since ZF sufficient conditions are also necessary for the prime $p = 2$, we have to take into account only the conditions for primes $p > 2$ (i.e., the last row in Table 1). This means:

$$z_1 \neq (-q_1) \pmod{p}, z_2 = z_3 = \cdots = z_n = 0 \pmod{p},$$

$$\forall p \mid N, p > 2. \quad (4)$$

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Table 1 Zhao and Fan sufficient conditions for coefficients $q_1, q_2, \dots, q_n$ so that $\pi(x)$ in (1) is a PP

1.a	$p = 2$	$n_{N,2} = 1$	$(q_1 + q_2 + \dots + q_n) \not\equiv 0 \pmod{2}$
1.b		$n_{N,2} > 1$	$q_1 \not\equiv 0, q_2 + q_4 + q_6 + \dots \equiv 0 \pmod{2}$ and $q_3 + q_5 + q_7 + \dots \equiv 0 \pmod{2}$
2	$p > 2$	$n_{N,p} \geq 1$	$q_1 \not\equiv 0, q_2 = q_3 = \dots = q_n \equiv 0 \pmod{p}$

We know that the general form of a NP of degree n is [3]:

$$z(x) = \sum_{k=1}^n \left\{ \frac{N}{\gcd(k!, N)} \cdot \tau_k \cdot \prod_{m=0}^{k-1} (x - m) \right\} \pmod{N},$$

where $0 \leq \tau_k \leq \gcd(k!, N) - 1$

(5)

In (5), gcd stands for the greatest common divisor. The expanded form of (5) for $n = 5$ was given in [1] as

$$\begin{aligned} z_{n=5}(x) = & \frac{N}{\gcd(5!, N)} \cdot \tau_5 \cdot x^5 \\ & + \left(\frac{N}{\gcd(4!, N)} \cdot \tau_4 - \frac{N}{\gcd(5!, N)} \cdot 10 \cdot \tau_5 \right) \cdot x^4 \\ & + \left(\frac{N}{\gcd(3!, N)} \cdot \tau_3 - \frac{N}{\gcd(4!, N)} \cdot 6 \cdot \tau_4 \right. \\ & \quad \left. + \frac{N}{\gcd(5!, N)} \cdot 35 \cdot \tau_5 \right) \cdot x^3 \\ & + \left(\frac{N}{\gcd(2!, N)} \cdot \tau_2 - \frac{N}{\gcd(3!, N)} \cdot 3 \cdot \tau_3 \right. \\ & \quad \left. + \frac{N}{\gcd(4!, N)} \cdot 11 \cdot \tau_4 \right. \\ & \quad \left. - \frac{N}{\gcd(5!, N)} \cdot 50 \cdot \tau_5 \right) \\ & \cdot x^2 + \left(\frac{N}{\gcd(3!, N)} \cdot 2 \cdot \tau_3 - \frac{N}{\gcd(2!, N)} \cdot \tau_2 \right. \\ & \quad \left. - \frac{N}{\gcd(4!, N)} \cdot 6 \cdot \tau_4 \right. \\ & \quad \left. + \frac{N}{\gcd(5!, N)} \cdot 24 \cdot \tau_5 \right) \cdot x, \end{aligned}$$

(6)

with $0 \leq \tau_2 \leq \gcd(2!, N) - 1$,
 $0 \leq \tau_3 \leq \gcd(3!, N) - 1$,
 $0 \leq \tau_4 \leq \gcd(4!, N) - 1$ and
 $0 \leq \tau_5 \leq \gcd(5!, N) - 1$.

For $\tau_5 = 0$ in (6), we obtain the expanded form of a fourth degree NP $z_{n=4}(x)$, and for $\tau_5 = \tau_4 = 0$ in (6), we obtain the expanded form of a third degree NP $z_{n=3}(x)$ (a cubic NP).

We have to impose additional constraints for $z_{n=5}(x)$ to be a valid ZF NP only if $3 \mid N$ and/or $5 \mid N$ because for every prime $p > 5$, with $p \mid N$, the five coefficients of $z_{n=5}(x)$ will be divisible by p , and thus, they fulfill conditions (4).

We denote by g_k the value $\gcd(k!, N)$. Let the prime factorization of g_k , for $k = 3, 4, 5$, be of the form

$$g_k = \gcd(k!, N) = 2^{n_{g_k,2}} \cdot 3^{n_{g_k,3}} \cdot 5^{n_{g_k,5}}, \text{ where}$$

$$0 \leq n_{g_k,2} \leq 3, 0 \leq n_{g_k,3} \leq 1, \text{ and } 0 \leq n_{g_k,5} \leq 1. \quad (7)$$

Obviously, for $k = 3$ or $k = 4$, we have $n_{g_k,5} = 0$ in (7).

We write the prime factorization of N as

$$N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot 5^{n_{N,5}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}}, \text{ with}$$

$$n_{N,p} \geq 0 \text{ for } p = 2, 3, 5, \text{ and } n_{N,p_j} \geq 1 \quad \forall j = \overline{4, s}. \quad (8)$$

We note that if $n_{N,3} = 0$, then $3 \nmid N$ and we have not to impose additional constraints for prime $p = 3$ for $z_{n=k}(x)$, $k = 3, 4, 5$, to be a valid ZF NP. Also, if $n_{N,5} = 0$, then $5 \nmid N$ and we have not to impose additional constraints for prime $p = 5$ for $z_{n=5}(x)$.

If $n_{N,3} > 1$, we have $3 \mid \frac{N}{\gcd(k!, N)}$, $\forall k = 3, 4, 5$ and from (6) it results that 3 divides all coefficients of $z_k(x)$, thus fulfilling the conditions for prime $p = 3$ for a valid ZF NP. If $n_{N,5} > 1$, we have $5 \mid \frac{N}{\gcd(5!, N)}$ and from (6) it results that 5 divides all coefficients of $z_k(x)$, thus fulfilling the conditions for prime $p = 5$ for a valid ZF NP.

From those above it results that we have to impose additional constraints for $z_{n=5}(x)$ to be a valid ZF NP only if $n_{N,3} = 1$ and/or $n_{N,5} = 1$. We analyze these cases below.

As it results from (5), the coefficient of the maximum degree term of a NP, i.e., z_n , is any multiple of $\frac{N}{\gcd(n!, N)}$. Thus, as it is shown in Sect. 3 from [1], any PP modulo N is equivalent to a PP having the coefficient of the k degree term $q_k < \frac{N}{\gcd(k!, N)}$, $\forall k = \overline{2, n}$. However, for a PP under ZF sufficient conditions this condition is not always valid. We detail below these conditions for a NP of degree three, four, and five.

If $n_{N,3} = 1$, then $3 \nmid \frac{N}{\gcd(k!, N)}$, $\forall k = 3, 4, 5$. In this case for $z_{n=k}(x)$, $\forall k = 3, 4, 5$, to be a valid ZF NP, we have to impose the condition that $3 \mid \tau_k$. Similarly, if $n_{N,5} = 1$, then $5 \nmid \frac{N}{\gcd(5!, N)}$. In this case for $z_{n=5}(x)$ to be a valid ZF NP we have to impose the condition that $5 \mid \tau_5$.

Thus, when $n_{N,3} = 1$ any PP modulo N under ZF sufficient conditions is equivalent to a PP under ZF sufficient conditions having the coefficient of the third degree term $q_3 < 3 \cdot \frac{N}{\gcd(3!, N)}$.

For similar reasons when $n_{N,3} = 1$, any PP modulo N under ZF sufficient conditions is equivalent to a PP under ZF sufficient conditions having the coefficient of the fourth degree term $q_4 < 3 \cdot \frac{N}{\gcd(4!, N)}$.

When $n_{N,3} = 1$ and $n_{N,5} = 0$, any PP modulo N under ZF sufficient conditions is equivalent to a PP under ZF sufficient conditions having the coefficient of the fifth degree term $q_5 < 3 \cdot \frac{N}{\gcd(5!, N)}$.

When $n_{N,3} = 0$ and $n_{N,5} = 1$, any PP modulo N under ZF sufficient conditions is equivalent to a PP under ZF sufficient conditions having the coefficient of the fifth degree term $q_5 < 5 \cdot \frac{N}{\gcd(5!, N)}$.

When $n_{N,3} = 1$ and $n_{N,5} = 1$, any PP modulo N under ZF sufficient conditions is equivalent to a PP under ZF sufficient conditions having the coefficient of the fifth degree term $q_5 < 15 \cdot \frac{N}{\gcd(5!, N)}$.

We have not to impose additional constraints for coefficients z_1 and z_2 of the NP $z_{n=5}(x)$ because when $3 \mid \tau_k \cdot \frac{N}{\gcd(k!, N)}$, $\forall k = 3, 4, 5$ and $5 \mid \tau_5 \cdot \frac{N}{\gcd(5!, N)}$, then $p \mid z_1$ and $p \mid z_2$, for $p = 3$ or 5 . Thus, $z_2 = z_1 = 0 \pmod{p}$, for $p = 3$ or 5 , and the conditions from (4) for primes 3 and 5 are fulfilled.

The analysis from Sect. 4 in [1] regarding the three types of prime factors from Table 1 remains valid. For primes 3 and 5, we can summarize the formula for the number of coefficients as follows.

For $n = 3$ and $n_{N,3} \geq 1$, there are $3^{n_{N,3}-1}$ coefficients $q_{2,j}$ and $3^{\max\{0; n_{N,3}-2\}}$ coefficients $q_{3,j}$, out of which one is zero.

For $n = 4$ and $n_{N,3} \geq 1$, there are $3^{n_{N,3}-1}$ coefficients $q_{2,j}$ and $3^{\max\{0; n_{N,3}-2\}}$ coefficients $q_{3,j}$ and $q_{4,j}$, respectively, out of which one is zero.

For $n = 5$, if $p_j = 3$ (i.e., $n_{N,3} \geq 1$) there are $3^{n_{N,3}-1}$ coefficients $q_{2,j}$ and $3^{\max\{0; n_{N,3}-2\}}$ coefficients $q_{3,j}$, $q_{4,j}$, and $q_{5,j}$, respectively, out of which one is zero.

If $p_j = 5$ (i.e., $n_{N,5} \geq 1$), there are $5^{n_{N,5}-1}$ coefficients $q_{2,j}$, $q_{3,j}$, and $q_{4,j}$, respectively, and $5^{\max\{0; n_{N,5}-2\}}$ coefficients $q_{5,j}$, out of which one is zero.

2 Corrections for formulas in Sects. 4.1, 4.2, and 4.3 from [1] considering null polynomials under Zhao and Fan sufficient conditions

In the following, we correct the formulas in Sects. 4.1, 4.2, and 4.3 from [1] in cases when the power of 3 and/or 5 in the factorization of N is equal to one. As in some cases the formulas become simpler, we restate and proof the complete theorems.

2.1 Corrections for formulas in Sect. 4.1 from [1]

In Sect. 4.1, the corrected equivalence conditions for ZF CPPs are:

- $q_2 < N/2$ and $q_3 < N/2$, when $2 \mid N$ and $3 \nmid N$, or when $6 \mid N$ and $9 \nmid N$;
- $q_3 < N/3$, when $2 \nmid N$ and $9 \mid N$;
- $q_2 < N/2$ and $q_3 < N/6$, when $18 \mid N$.

The formulas from Theorems 4.1, 4.2, and 4.3 remain valid. The formulas from Theorems 4.4, 4.5, and 4.6 remain valid only when $n_{N,3} > 1$. We correct these formulas for the case when $n_{N,3} = 1$, restating and proving the complete theorems.

Theorem 2.1 (Updated Theorem 4.4 from [1]) *If $2 \nmid N$ and $3 \mid N$, i.e., $N = 3^{n_{N,3}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1$, $\forall j = 2, s$, the number of ZF CPPs will be equal to:*

$$C_{N, CPPs, ZF} = 2 \cdot 3^{2 \cdot (n_{N,3}-1)} \cdot \prod_{j=2}^s \left(p_j^{2 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (9)$$

Proof See Appendix A.1. $\square$

From (9), we see that the number of ZF CPPs is equal to 0, if N is three or nine times a product of prime numbers greater than three, each of them to the power of 1.

Theorem 2.2 (Updated Theorem 4.5 from [1]) *If $2 \mid N$, $4 \nmid N$ and $3 \mid N$, i.e., $N = 2 \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1$, $\forall j = 3, s$, the number of ZF CPPs will be equal to:*

$$C_{N, CPPs, ZF} = 2 \cdot 3^{2 \cdot (n_{N,3}-1)} \cdot \prod_{j=3}^s \left(p_j^{2 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (10)$$

Proof See Appendix A.2. $\square$

From (10), we see that the number of ZF CPPs is equal to 0 if N is 6 or 18 times a product of prime numbers greater than three, each of them to the power of 1.

Theorem 2.3 (Updated Theorem 4.6 from [1]) *If $4 \mid N$ and $3 \mid N$, i.e., $N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,2} > 1$, $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1$, $\forall j = 3, s$, the number of ZF CPPs will be equal to:*

$$\begin{aligned}
C_{N, CPPs, ZF} &= 2^{2 \cdot (n_{N,2}-1)} \cdot 3^{2 \cdot (n_{N,3}-1)} \\
&\cdot \prod_{j=3}^s \left(p_j^{2 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
&\cdot \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned} \quad (11)$$

Proof See Appendix A.3. $\square$

From (11), we see that the number of ZF CPPs is equal to 0, if N is 12 or 36 times a product of prime numbers greater than three, each of them to the power of 1.

The conclusions from the end of Sect. 4.1 in [1] have to be corrected as follows.

Under ZF sufficient conditions, the number of true different CPPs is 0, when the interleaver length is

$$\begin{aligned}
N &= 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j, \quad n_{N,2} = \overline{0, 2}, \\
n_{N,3} &= \overline{0, 2}, \quad p_j > 3, \quad \forall j = \overline{3, s}
\end{aligned} \quad (12)$$

Because all the lengths from the LTE standard [4] are multiples of 8, from (12) it results that for all these lengths there exist CPPs under ZF sufficient conditions.

The remark that the LTE standard contains 58 of all 188 lengths for which ZF sufficient conditions of CPPs become also necessary remains valid. However, for all lengths for which in their prime factorization there exists the prime 3 at power of 1, but no primes p of the form $(p = 2 \pmod{3})$ at power of 1, the number of all true different CPPs is equal to the number of true different CPPs under ZF sufficient conditions. There exist 25 such lengths among the LTE lengths (namely 48, 96, 168, 192, 312, 336, 384, 456, 624, 672, 768, 912, 1248, 1344, 1536, 1824, 2496, 2688, 3072, 3648, 4800, 4992, 5376, 5952, and 6144). Thus for 83 lengths from the LTE standard, the number of all true different CPPs is equal to the number of true different CPPs under ZF sufficient conditions. For another 105 lengths, the ratio (in percentage) between the number of true different CPPs under ZF sufficient conditions and the number of all true different CPPs is less than 50%. This is shown in Fig. 1 (the corrected Fig. 1 from [1]).

The example for Theorem 2.3 (i.e., updated Theorem 4.6 from [1]) has to be corrected as in Appendix D.1.

2.2 Corrections for formulas in Sect. 4.2 from [1]

In Sect. 4.2, the corrected equivalence conditions for 4-PPs under ZF sufficient conditions are written as

Fig. 1 The ratio (in percentage) between the number of true different CPPs under Zhao and Fan sufficient conditions and the number of all true different CPPs, for all lengths of LTE standard

- $q_2 < N/2$, $q_3 < N/2$ and $q_4 < N/2$, when $2 \mid N$, $3 \nmid N$ and $4 \nmid N$, or when $4 \nmid N$, $6 \mid N$, and $9 \nmid N$;
- $q_2 < N/2$, $q_3 < N/2$ and $q_4 < N/4$, when $3 \nmid N$, $4 \mid N$ and $8 \nmid N$, or when $8 \nmid N$, $9 \nmid N$, and $12 \mid N$;
- $q_2 < N/2$, $q_3 < N/2$ and $q_4 < N/8$, when $3 \nmid N$ and $8 \mid N$, or when $9 \nmid N$ and $24 \mid N$;
- $q_3 < N/3$ and $q_4 < N/3$, when $2 \nmid N$ and $9 \mid N$;
- $q_2 < N/2$, $q_3 < N/6$ and $q_4 < N/6$, when $4 \nmid N$ and $18 \mid N$;
- $q_2 < N/2$, $q_3 < N/6$ and $q_4 < N/12$, when $8 \nmid N$ and $36 \mid N$;
- $q_2 < N/2$, $q_3 < N/6$ and $q_4 < N/24$, when $72 \mid N$.

The formulas from Theorems 4.7, 4.8, 4.10, and 4.12 remain valid. The formulas from Theorems 4.9, 4.11, 4.13, and 4.14 remain valid only when $n_{N,3} > 1$. We correct these formulas for the case when $n_{N,3} = 1$, restating and proving the complete theorems.

Theorem 2.4 (Updated Theorem 4.9 from [1]) *If $2 \nmid N$ and $3 \mid N$, i.e., $N = 3^{n_{N,3}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{2, s}$, the number of ZF 4-PPs will be equal to:*

$$\begin{aligned}
C_{N, 4-PPs, ZF} &= 2 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \\
&\cdot \prod_{j=2}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned} \quad (13)$$

Proof See Appendix B.1. $\square$

From (13), we see that the number of ZF 4-PPs is equal to 0 if N is three or nine times a product of prime numbers greater than three, each of them to the power of 1.

Theorem 2.5 (Updated Theorem 4.11 from [1]) *If $6 \mid N$ and $4 \nmid N$, i.e., $N = 2 \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1, \forall j = \overline{3, s}$, the number of ZF 4-PPs will be equal to:*

$$C_{N,4-PPs,ZF} = 2 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (14)$$

Proof See Appendix B.2. $\square$

From (14), we see that the number of ZF 4-PPs is equal to 0 if N is 6 or 18 times a product of prime numbers greater than three, each of them to the power of 1.

Theorem 2.6 (Updated Theorem 4.13 from [1]) *If $12 \mid N$ and $8 \nmid N$, i.e., $N = 4 \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1, \forall j = \overline{3, s}$, the number of ZF 4-PPs will be equal to:*

$$C_{N,4-PPs,ZF} = 4 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (15)$$

Proof See Appendix B.3. $\square$

From (15), we see that the number of ZF 4-PPs is equal to 0 if N is 12 or 36 times a product of prime numbers, greater than three, each of them to the power of 1.

Theorem 2.7 (Updated Theorem 4.14 from [1]) *If $24 \mid N$, i.e., $N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,2} \geq 3$, $n_{N,3} \geq 1$, $p_j > 3$ and $n_{N,p_j} \geq 1, \forall j = \overline{3, s}$, the number of ZF 4-PPs will be equal to:*

Fig. 2 The ratio (in percentage) between the number of true different 4-PPs under Zhao and Fan sufficient conditions and the number of all true different 4-PPs, for all lengths used in the LTE standard

$$C_{N,4-PPs,ZF} = 2^{3 \cdot n_{N,2} - 4} \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(2^{n_{N,2}-3} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (16)$$

Proof See Appendix B.4. $\square$

From (16), we see that the number of ZF 4-PPs is equal to 0 if N is 24 or 72 times a product of prime numbers greater than three, each of them to the power of 1.

The conclusions from the end of Sect. 4.2 have to be corrected as follows.

Under ZF sufficient conditions, the number of true different 4-PPs is 0, when the interleaver length is of the form

$$N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j, n_{N,2} = \overline{0, 3}, n_{N,3} = \overline{0, 2}, p_j > 3, \forall j = \overline{3, s}. \quad (17)$$

The LTE standard [4] contains 27 lengths (of form (17) with $n_{N,2} = 3$) for which there are no 4-PPs under ZF sufficient conditions, namely the lengths 40, 56, 72, 88, 104, 120, 136, 152, 168, 184, 232, 248, 264, 280, 296, 312, 328, 344, 360, 376, 408, 424, 440, 456, 472, 488, and 504.

We compared the total number of true different 4-PPs, obtained with the algorithm from [5], for all lengths from the LTE standard with the number of true different 4-PPs under ZF sufficient conditions. Figure 2 shows the ratio (in percentage) between the number of true different 4-PPs under

ZF sufficient conditions and the number of all true different 4-PPs for all the 188 lengths of the LTE standard. We note that for 23 from the 188 lengths, the number of all true different 4-PPs is zero (i.e., there are no true 4-PPs for these lengths), and thus, the percentage is 100%. For 84 from the 188 lengths (from which 23 lengths are those mentioned above), the ZF sufficient conditions are also necessary (i.e., this percentage is 100%). For other 4 lengths, this percentage is 0% (i.e., there are no true 4-PPs under ZF sufficient conditions, but there are true different 4-PPs, for these lengths, namely 56, 168, 280, and 504), for 12 lengths the percentage is equal to 50%, and for the other 88 lengths, the percentage is less than 20%.

In the example for Theorem 2.7 (i.e., updated Theorem 4.14 from [1]), the number of ZF 4-PPs computed according to updated formula (16) will be equal to:

$$C_{1080,4-PPs,ZF} = 2^{3 \cdot 3 - 4} \cdot 3^{2 \cdot (3-1) + \max\{0; 3-2\}} \cdot 5^{3 \cdot (1-1)} \cdot (5-1) \\ \cdot \left(2^{3-3} \cdot 3^{\max\{0; 3-2\}} \cdot 5^{1-1} - 1 \right) \\ = 32 \cdot 243 \cdot 1 \cdot 4 \cdot (3-1) = 62,208,$$

which is the same as that obtained by means of the old formula from [1], because in this case $n_{N,3} > 1$.

2.3 Corrections for formulas in Sect. 4.3 from [1]

In Sect. 4.3, the corrected equivalence conditions for 5-PPs under ZF sufficient conditions are written as

- $q_2 < N/2, q_3 < N/2, q_4 < N/2$, and $q_5 < N/2$ when $2 \mid N, 3 \nmid N, 4 \nmid N$, and $5 \nmid N$, or when $4 \nmid N, 5 \nmid N, 6 \mid N$, and $9 \nmid N$, or when $3 \nmid N, 4 \nmid N, 10 \mid N$, and $25 \nmid N$, or when $4 \nmid N, 9 \nmid N, 25 \nmid N$, and $30 \mid N$;
- $q_2 < N/2, q_3 < N/2, q_4 < N/4$, and $q_5 < N/4$ when $3 \nmid N, 4 \mid N, 5 \nmid N$, and $8 \nmid N$, or when $5 \nmid N, 8 \nmid N, 9 \nmid N$, and $12 \mid N$, or when $3 \nmid N, 8 \nmid N, 20 \mid N$, and $25 \nmid N$, or when $8 \nmid N, 9 \nmid N, 25 \nmid N$, and $60 \mid N$;
- $q_2 < N/2, q_3 < N/2, q_4 < N/8$, and $q_5 < N/8$ when $3 \nmid N, 5 \nmid N$, and $8 \mid N$, or when $5 \nmid N, 9 \nmid N$, and $24 \mid N$, or when $3 \nmid N, 25 \nmid N$, and $40 \mid N$, or when $9 \nmid N, 25 \nmid N$, and $120 \mid N$;
- $q_3 < N/3, q_4 < N/3$, and $q_5 < N/3$ when $2 \nmid N, 5 \nmid N$, and $9 \mid N$, or when $2 \nmid N, 25 \nmid N$, and $45 \mid N$;
- $q_2 < N/2, q_3 < N/6, q_4 < N/6$, and $q_5 < N/6$ when $4 \nmid N, 5 \nmid N$, and $18 \mid N$, or when $4 \nmid N, 25 \nmid N$, and $90 \mid N$;
- $q_5 < N/5$ when $2 \nmid N, 3 \nmid N$, and $25 \mid N$, or when $2 \nmid N, 9 \nmid N$, and $75 \mid N$;
- $q_2 < N/2, q_3 < N/6, q_4 < N/12$, and $q_5 < N/12$ when $5 \nmid N, 8 \nmid N$, and $36 \mid N$, or when $8 \nmid N, 25 \nmid N$, and $180 \mid N$;

- $q_2 < N/2, q_3 < N/2, q_4 < N/2$, and $q_5 < N/10$ when $3 \nmid N, 4 \nmid N$, and $50 \mid N$, or when $4 \nmid N, 9 \nmid N$, and $150 \mid N$;
- $q_2 < N/2, q_3 < N/6, q_4 < N/24$, and $q_5 < N/24$ when $5 \nmid N$ and $72 \mid N$, or when $25 \nmid N$ and $360 \mid N$;
- $q_2 < N/2, q_3 < N/2, q_4 < N/4$, and $q_5 < N/20$ when $3 \nmid N, 8 \nmid N$, and $100 \mid N$, or when $8 \nmid N, 9 \nmid N$, and $300 \mid N$;
- $q_2 < N/2, q_3 < N/2, q_4 < N/8$, and $q_5 < N/40$ when $3 \nmid N$ and $200 \mid N$, or when $9 \nmid N$ and $600 \mid N$;
- $q_3 < N/3, q_4 < N/3$, and $q_5 < N/15$ when $2 \nmid N$ and $225 \mid N$;
- $q_2 < N/2, q_3 < N/6, q_4 < N/6$, and $q_5 < N/30$ when $4 \nmid N$ and $450 \mid N$;
- $q_2 < N/2, q_3 < N/6, q_4 < N/12$, and $q_5 < N/60$ when $8 \nmid N$ and $900 \mid N$;
- $q_2 < N/2, q_3 < N/6, q_4 < N/24$, and $q_5 < N/120$ when $1800 \mid N$.

The formulas from Theorems 4.15, 4.16, 4.18, and 4.21 remain valid. The formulas from Theorems 4.17, 4.20, 4.23, and 4.26 remain valid only when $n_{N,3} > 1$. The formulas from Theorems 4.19, 4.22, 4.25, and 4.28 remain valid only when $n_{N,5} > 1$. The formulas from Theorems 4.24, 4.27, 4.29, and 4.30 remain valid only when $n_{N,3} > 1$ and $n_{N,5} > 1$. We correct these formulas for the cases when $n_{N,3} = 1$ and/or $n_{N,5} > 1$, restating and proving the complete theorems.

Theorem 2.8 (Updated Theorem 4.17 from [1]) *If $2 \nmid N, 3 \mid N$ and $5 \nmid N$, i.e., $N = 3^{n_{N,3}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1, p_j > 5$ and $n_{N,p_j} \geq 1, \forall j = \overline{2, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 2 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \\ \cdot \prod_{j=2}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (18)$$

Proof See Appendix C.1. □

From (18), we see that the number of ZF 5-PPs is equal to 0 if N is three or nine times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.9 (Updated Theorem 4.19 from [1]) *If $2 \nmid N, 3 \nmid N$ and $5 \mid N$, i.e., $N = 5^{n_{N,5}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}}$, with*

$n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{2, s}$, the number of ZF 5-PPs will be equal to:

$$C_{N,5-PPs,ZF} = 4 \cdot 5^{4 \cdot (n_{N,5}-1)} \cdot \prod_{j=2}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (19)$$

Proof See Appendix C.2. $\square$

From (19), we see that the number of ZF 5-PPs is equal to 0 if N is 5 or 25 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.10 (Updated Theorem 4.20 from [1]) *If $4 \nmid N$, $5 \nmid N$ and $6 \mid N$, i.e., $N = 2 \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 2 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (20)$$

Proof See Appendix C.3. $\square$

From (20), we see that the number of ZF 5-PPs is equal to 0 if N is 6 or 18 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.11 (Updated Theorem 4.22 from [1]) *If $2 \mid N$, $3 \nmid N$, $4 \nmid N$ and $5 \mid N$, i.e., $N = 2 \cdot 5^{n_{N,5}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 4 \cdot 5^{4 \cdot (n_{N,5}-1)} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (21)$$

Proof See Appendix C.4. $\square$

From (21), we see that the number of ZF 5-PPs is equal to 0 if N is 10 or 50 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.12 (Updated Theorem 4.23 from [1]) *If $5 \nmid N$, $8 \nmid N$ and $12 \mid N$, i.e., $N = 4 \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 4 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (22)$$

Proof See Appendix C.5. $\square$

From (22), we see that the number of ZF 5-PPs is equal to 0 if N is 12 or 36 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.13 (Updated Theorem 4.24 from [1]) *If $2 \nmid N$, $3 \mid N$ and $5 \mid N$, i.e., $N = 3^{n_{N,3}} \cdot 5^{n_{N,5}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 8 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \cdot 5^{4 \cdot (n_{N,5}-1)} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (23)$$

Proof See Appendix C.6. $\square$

From (23), we see that the number of ZF 5-PPs is 0 if N is 15, 45, 75, or 225 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.14 (Updated Theorem 4.25 from [1]) *If $3 \nmid N$, $4 \mid N$, $5 \mid N$ and $8 \nmid N$, i.e., $N = 4 \cdot 5^{n_{N,5}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 8 \cdot 5^{4 \cdot (n_{N,5}-1)} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (24)$$

Proof See Appendix C.7. $\square$

From (24), we see that the number of ZF 5-PPs is equal to 0 if N is 20 or 100 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.15 (Updated Theorem 4.26 from [1]) *If $8 \mid N$, $3 \mid N$ and $5 \nmid N$, i.e., $N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,2} \geq 3$, $n_{N,3} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 2^{4 \cdot n_{N,2} - 7} \cdot 3^{2 \cdot (n_{N,3} - 1) + 2 \cdot \max\{0; n_{N,3} - 2\}} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j} - 1)} \cdot (p_j - 1) \right) \cdot \left(2^{n_{N,2} - 3} \cdot 3^{\max\{0; n_{N,3} - 2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j} - 1} - 1 \right) \quad (25)$$

Proof See Appendix C.8. $\square$

From (25), we see that the number of ZF 5-PPs is equal to 0 if N is 24 or 72 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.16 (Updated Theorem 4.27 from [1]) *If $2 \mid N$, $3 \mid N$ and $4 \nmid N$ and $5 \mid N$, i.e., $N = 2 \cdot 3^{n_{N,3}} \cdot 5^{n_{N,5}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{4, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 8 \cdot 3^{2 \cdot (n_{N,3} - 1) + 2 \cdot \max\{0; n_{N,3} - 2\}} \cdot 5^{4 \cdot (n_{N,5} - 1)} \cdot \left(\prod_{j=4}^s p_j^{4 \cdot (n_{N,p_j} - 1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3} - 2\}} \cdot 5^{\max\{0; n_{N,5} - 2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j} - 1} - 1 \right) \quad (26)$$

Proof See Appendix C.9. $\square$

From (26), we see that the number of ZF 5-PPs is 0 if N is 30, 90, 150, or 450 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.17 (Updated Theorem 4.28 from [1]) *If $3 \nmid N$, $5 \mid N$ and $8 \mid N$, i.e., $N = 2^{n_{N,2}} \cdot 5^{n_{N,5}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}}$, with $n_{N,2} \geq 3$, $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{3, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 2^{4 \cdot n_{N,2} - 6} \cdot 5^{4 \cdot (n_{N,5} - 1)} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j} - 1)} \cdot (p_j - 1) \right) \cdot \left(2^{n_{N,2} - 3} \cdot 5^{\max\{0; n_{N,5} - 2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j} - 1} - 1 \right) \quad (27)$$

Proof See Appendix C.10. $\square$

From (27), we see that the number of ZF 5-PPs is equal to 0 if N is 40 or 200 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.18 (Updated Theorem 4.29 from [1]) *If $3 \mid N$, $4 \mid N$, $5 \mid N$ and $8 \nmid N$, i.e., $N = 4 \cdot 3^{n_{N,3}} \cdot 5^{n_{N,5}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}}$, with $n_{N,3} \geq 1$, $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{4, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 16 \cdot 3^{2 \cdot (n_{N,3} - 1) + 2 \cdot \max\{0; n_{N,3} - 2\}} \cdot 5^{4 \cdot (n_{N,5} - 1)} \cdot \prod_{j=4}^s \left(p_j^{4 \cdot (n_{N,p_j} - 1)} \cdot (p_j - 1) \right) \cdot \left(3^{\max\{0; n_{N,3} - 2\}} \cdot 5^{\max\{0; n_{N,5} - 2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j} - 1} - 1 \right) \quad (28)$$

Proof See Appendix C.11. $\square$

From (28), we see that the number of ZF 5-PPs is equal to 0 if N is 60, 180, 300, or 900 times a product of prime numbers greater than five, each of them to the power of 1.

Theorem 2.19 (Updated Theorem 4.30 from [1]) *If $3 \mid N$, $5 \mid N$ and $8 \mid N$, i.e., $N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot 5^{n_{N,5}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}}$, with $n_{N,2} \geq 3$, $n_{N,3} \geq 1$, $n_{N,5} \geq 1$, $p_j > 5$ and $n_{N,p_j} \geq 1$, $\forall j = \overline{4, s}$, the number of ZF 5-PPs will be equal to:*

$$C_{N,5-PPs,ZF} = 2^{4 \cdot n_{N,2} - 5} \cdot 3^{2 \cdot (n_{N,3} - 1) + 2 \cdot \max\{0; n_{N,3} - 2\}} \cdot 5^{4 \cdot (n_{N,5} - 1)} \cdot \prod_{j=4}^s \left(p_j^{4 \cdot (n_{N,p_j} - 1)} \cdot (p_j - 1) \right) \cdot \left(2^{n_{N,2} - 3} \cdot 3^{\max\{0; n_{N,3} - 2\}} \cdot 5^{\max\{0; n_{N,5} - 2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j} - 1} - 1 \right) \quad (29)$$

Proof See Appendix C.12. $\square$

From (29), we see that the number of ZF 5-PPs is equal to 0 if N is 120, 360, 600, or 1800 times a product of prime numbers greater than five, each of them to the power of 1.

The conclusions from the end of Sect. 4.3 have to be corrected as follows.

Fig. 3 The ratio (in percentage) between the number of true different 5-PPs under Zhao and Fan sufficient conditions and the number of all true different 5-PPs, for all lengths used in the LTE standard

Under ZF sufficient conditions, the number of true different 5-PPs is 0, when the interleaver length is of the form

$$N = 2^{n_{N,2}} \cdot 3^{n_{N,3}} \cdot 5^{n_{N,5}} \cdot \prod_{j=4}^s p_j, \quad n_{N,2} = \overline{0, 3}, \quad n_{N,3} = \overline{0, 2}, \quad (30)$$

$$n_{N,5} = \overline{0, 2}, \quad p_j > 5, \quad \forall j = \overline{4, s}.$$

The LTE standard [4] contains 28 lengths (of form (30) with $n_{N,2} = 3$) for which there are no 5-PPs under ZF sufficient conditions, namely the lengths 40, 56, 72, 88, 104, 120, 136, 152, 168, 184, 200, 232, 248, 264, 280, 296, 312, 328, 344, 360, 376, 408, 424, 440, 456, 472, 488, and 504.

We compared the total number of true different 5-PPs, obtained with the algorithm from [5], for all lengths from the LTE standard with the number of true different 5-PPs under ZF sufficient conditions. Figure 3 shows the ratio (in percentage) between the number of true different 5-PPs under ZF sufficient conditions and the number of all true different 5-PPs for all the 188 lengths of the LTE standard. We note that for 8 from the 188 lengths, the number of all true different 5-PPs is zero (i.e., there are no true 5-PPs for these lengths, namely 40, 88, 120, 248, 264, 328, 440, and 488), and thus, the percentage is 100%. For 31 from the 188 lengths (from which 8 lengths are those mentioned above), the ZF sufficient conditions are also necessary (i.e., this percentage is 100%). For other 20 lengths, this percentage is 0% (i.e., there are no true 5-PPs under ZF sufficient conditions, but there are true different 5-PPs, for these lengths) and for the other 137 lengths, the percentage is less than 20%.

In the first example for Theorem 2.19 (i.e., updated Theorem 4.30 from [1]), the number of ZF 5-PPs computed according to updated formula (29) will be equal to:

$$\begin{aligned} C_{25200,5-PPs,ZF} &= 2^{4 \cdot 4 - 5} \cdot 3^{2 \cdot (2-1) + 2 \cdot \max\{0; 2-2\}} \cdot 5^{4 \cdot (2-1)} \cdot 7^{4 \cdot (1-1)} \cdot (7-1) \\ &\quad \cdot (2^{4-3} \cdot 3^{\max\{0; 2-2\}} \cdot 5^{\max\{0; 2-2\}} \cdot 7^{1-1} - 1) \\ &= (2048 \cdot 9 \cdot 625 \cdot 1 \cdot 6) \cdot (2 \cdot 1 \cdot 1 \cdot 1 - 1) = 6912000, \end{aligned}$$

which is the same as that obtained by means of the old formula from [1], because in this case $n_{N,3} > 1$ and $n_{N,5} > 1$.

The second example for Theorem 2.19 has to be corrected as in Appendix D.2.

Appendix A

A.1 Proof [Theorem 2.1 (Updated Theorem 4.4 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 is equal to $3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_3 is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, out of which one value is zero. By removing the value $q_3 = 0$, the number of ZF CPPs will be equal to:

$$\begin{aligned} C_{N,CPPs,ZF} &= \left(2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ &\quad \cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\ &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (31) \\ &= 2 \cdot 3^{2 \cdot (n_{N,3}-1)} \cdot \prod_{j=2}^s \left(p_j^{2 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \end{aligned}$$

A.2 Proof [Theorem 2.2 (Updated Theorem 4.5 from [1])]

The proof for Theorem 2.2 is similar to that of Theorem 2.1, by replacing N by $N/2$. Therefore, the number of ZF CPPs will be equal to:

$$\begin{aligned} C_{N,CPPs,ZF} &= \left(2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ &\quad \cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
& = 2 \cdot 3^{2 \cdot (n_{N,3}-1)} \cdot \prod_{j=3}^s \left(p_j^{2 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned} \quad (32)$$

A.3 Proof [Theorem 2.3 (Updated Theorem 4.6 from [1])]

The number of possible coefficients q_1 is equal to $2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 is equal to $2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_3 is equal to $2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, out of which one is zero. By removing the value $q_3 = 0$, the number of ZF CPPs will be equal to:

$$\begin{aligned}
C_{N, CPPs, ZF} &= \left(2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
&\cdot \left(2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
&\cdot \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
&= 2^{2 \cdot (n_{N,2}-1)} \cdot 3^{2 \cdot (n_{N,3}-1)} \cdot \prod_{j=3}^s \left(p_j^{2 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
&\cdot \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned} \quad (33)$$

Appendix B

B.1 Proof [Theorem 2.4 (Updated Theorem 4.9 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 is equal to $3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_3 and q_4 , respectively, is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, out of which one value is zero. By removing the value $q_4 = 0$, the number of ZF 4-PPs will be equal to:

$$\begin{aligned}
C_{N, 4-PPs, ZF} &= 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
&\cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
&= 2 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \\
&\cdot \prod_{j=2}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned} \quad (34)$$

B.2 Proof [Theorem 2.5 (Updated Theorem 4.11 from [1])]

The proof for Theorem 2.5 is similar to that of Theorem 2.4, by replacing N by $N/2$. Thus, the number of ZF 4-PPs will be equal to:

$$\begin{aligned}
C_{N, 4-PPs, ZF} &= 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
&\cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
&= 2 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \\
&\cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
&\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned} \quad (35)$$

B.3 Proof [Theorem 2.6 (Updated Theorem 4.13 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coef-

ficients q_2 is equal to $3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, the number of coefficients q_3 is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_4 is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, out of which one is zero. Therefore, the number of ZF 4-PPs will be equal to:

$$\begin{aligned} C_{N,4-PPs,ZF} &= 4 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ &\quad \cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \quad (36) \\ &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\ &= 4 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \\ &\quad \cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \end{aligned}$$

B.4 Proof [Theorem 2.7 (Updated Theorem 4.14 from [1])]

The number of possible coefficients q_1 is equal to $2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_3 is equal to $2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, the number of even or odd coefficients q_2 is equal to $2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, and the number of even or odd coefficients q_4 is equal to $2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ if $n_{N,2} \geq 4$, out of which one is zero. If $n_{N,2} = 3$, there are $2 \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_3 , $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_4 (all even), and $2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_2 (all even). Therefore, if $n_{N,2} \geq 4$, the number of ZF 4-PPs will be equal to:

$$\begin{aligned} C_{N,4-PPs,ZF} &= 2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \end{aligned}$$

$$\begin{aligned} &\cdot \left(2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ &\cdot \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ &\cdot \left(2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\ &+ 2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ &\cdot \left(2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \quad (37) \\ &\cdot \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ &\cdot \left(2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ &= 2^{3 \cdot n_{N,2}-4} \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \\ &\cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ &\cdot \left(2^{n_{N,2}-3} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \end{aligned}$$

We mention that, if $n_{N,2} = 3$, the number of ZF 4-PPs will be equal to:

$$\begin{aligned} C_{N,4-PPs,ZF} &= 2^{3-1} \cdot 2 \cdot 3^{n_{N,3}-1} \\ &\cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ &\cdot \left(2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ &\cdot \left(2 \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \quad (38) \\ &\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\ &= 32 \cdot 3^{2 \cdot (n_{N,3}-1) + \max\{0; n_{N,3}-2\}} \\ &\cdot \prod_{j=3}^s \left(p_j^{3 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ &\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \end{aligned}$$

so that relation (37) is also true.

Appendix C

C.1 Proof [Theorem 2.8 (Updated Theorem 4.17 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 is equal to $3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_3 , q_4 and q_5 , respectively, is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, out of which one value is zero. By removing the value $q_5 = 0$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 2 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \\
 &\quad \cdot \prod_{j=2}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned} \quad (39)$$

C.2 Proof [Theorem 2.9 (Updated Theorem 4.19 from [1])]

The number of possible coefficients q_1 is equal to $4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 , q_3 , and q_4 , respectively, is equal to $5^{n_{N,5}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_5 is equal to $5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1}$, out of which one value is zero. By removing the value $q_5 = 0$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=2}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 4 \cdot 5^{4 \cdot (n_{N,5}-1)} \\
 &\quad \cdot \prod_{j=2}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=2}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned} \quad (40)$$

C.3 Proof [Theorem 2.10 (Updated Theorem 4.20 from [1])]

The proof for Theorem 2.10 is similar to that of Theorem 2.8, by replacing N by $N/2$. Therefore, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 2 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \\
 &\quad \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned} \quad (41)$$

C.4 Proof [Theorem 2.11 (Updated Theorem 4.22 from [1])]

The proof for Theorem 2.11 is similar to that of Theorem 2.9, by replacing N by $N/2$. Therefore, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 4 \cdot 5^{4 \cdot (n_{N,5}-1)} \\
 &\quad \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned} \quad (42)$$

C.5 Proof [Theorem 2.12 (Updated Theorem 4.23 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 is equal to $3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, the number of coefficients q_3 , q_4 and q_5 , respectively, is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, out of which one is zero. Therefore, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 4 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right)
 \end{aligned}$$

$$\begin{aligned}
 &\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 4 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \\
 &\quad \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned} \quad (43)$$

C.6 Proof [Theorem 2.13 (Updated Theorem 4.24 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 4 \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 is equal to $3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, the number of coefficients q_3 and q_4 , respectively, is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$. The number of coefficients q_5 is equal to $3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, out of which one is zero. By removing the value $q_5 = 0$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 8 \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 8 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \cdot 5^{4 \cdot (n_{N,5}-1)}
 \end{aligned} \quad (44)$$

$$\cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)$$

C.7 Proof [Theorem 2.14 (Updated Theorem 4.25 from [1])]

The number of possible coefficients q_1 is equal to $2 \cdot 4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of coefficients q_2 , q_3 and q_4 , respectively, is equal to $5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, and the number of coefficients q_5 is equal to $5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, out of which one is zero. Therefore, the number of ZF 5-PPs will be equal to:

$$C_{N,5-PPs,ZF} = 8 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ \cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\ = 8 \cdot 5^{4 \cdot (n_{N,5}-1)} \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ \cdot \left(5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (45)$$

C.8 Proof [Theorem 2.15 (Updated Theorem 4.26 from [1])]

The number of possible coefficients q_1 is equal to $2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of even or odd coefficients q_2 is equal to $2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, the number of even or odd coefficients q_3 is equal to $2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, and the number of even or odd coefficients q_4 and q_5 , respec-

tively, is equal to $2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ if $n_{N,2} \geq 4$, out of which one is zero. If $n_{N,2} = 3$, there are $2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_2 , all of them even, $2 \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_3 , all of them even, and $3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_4 and q_5 , respectively, all of them even. Therefore, if $n_{N,2} \geq 4$, the number of ZF 5-PPs will be equal to:

$$C_{N,5-PPs,ZF} \\ = 2 \cdot 2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \\ \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ \cdot \left(2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ \cdot \left(2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ \cdot \left\{ \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \right. \\ \cdot \left(2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\ \left. + \left(2^{n_{N,2}-2} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \right. \\ \cdot \left. \left(2^{n_{N,2}-4} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \right\} \\ = 2^{4 \cdot n_{N,2}-7} \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \\ \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\ \cdot \left(2^{n_{N,2}-3} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (46)$$

If $n_{N,2} = 3$, the number of ZF 5-PPs will be equal to:

$$C_{N,5-PPs,ZF} = 4 \cdot 2 \cdot 3^{n_{N,3}-1} \\ \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\ \cdot \left(2 \cdot 3^{n_{N,3}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\ \cdot \left(2 \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \quad (47)$$

$$\begin{aligned}
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
& = 32 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \\
& \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right),
\end{aligned}$$

so that relation (46) is also true.

C.9 Proof [Theorem 2.16 (Updated Theorem 4.27 from [1])]

The proof for Theorem 2.16 is similar to that of Theorem 2.13, by replacing N by $N/2$. Thus, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
C_{N,5-PPs,ZF} & = 8 \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
& \cdot \left(3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \quad (48) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
& = 8 \cdot 3^{2 \cdot (n_{N,3}-1) + 2 \cdot \max\{0; n_{N,3}-2\}} \cdot 5^{4 \cdot (n_{N,5}-1)} \\
& \cdot \left(\prod_{j=4}^s p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
& \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned}$$

C.10 Proof [Theorem 2.17 (Updated Theorem 4.28 from [1])]

The number of possible coefficients q_1 is equal to $2^{n_{N,2}-1} \cdot 4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of even or odd coefficients q_2 and q_3 , respectively, is equal to $2^{n_{N,2}-2} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, the number of even or odd coefficients q_4 is equal to $2^{n_{N,2}-4} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$, and the number of even or odd coefficients q_5 is equal to $2^{n_{N,2}-4} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ if $n_{N,2} \geq 4$, out of which one value is zero. If $n_{N,2} = 3$, there are $2 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_2 and q_3 , respectively, all of them even, $5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_4 , all of them even, and $5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1}$ coefficients q_5 , all of them even, out of which one is zero. Therefore, if $n_{N,2} \geq 4$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
C_{N,5-PPs,ZF} & = 2 \cdot 2^{n_{N,2}-1} \cdot 4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
& \cdot \left(2^{n_{N,2}-2} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
& \cdot \left(2^{n_{N,2}-4} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
& \cdot \left\{ \left(2^{n_{N,2}-2} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \right. \\
& \cdot \left(2^{n_{N,2}-4} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (49) \\
& \left. + \left(2^{n_{N,2}-2} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \right. \\
& \cdot \left. \left(2^{n_{N,2}-4} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \right\} \\
& = 2^{4 \cdot n_{N,2}-6} \cdot 5^{4 \cdot (n_{N,5}-1)} \\
& \cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
& \cdot \left(2^{n_{N,2}-3} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right)
\end{aligned}$$

If $n_{N,2} = 3$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 4 \cdot 4 \cdot 5^{n_{N,5}-1} \\
 &\cdot \prod_{j=3}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\cdot \left(2 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(2 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(5^{n_{N,5}-1} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 64 \cdot 5^{4 \cdot (n_{N,1}-1)} \\
 &\cdot \prod_{j=3}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\cdot \left(5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=3}^s p_j^{n_{N,p_j}-1} - 1 \right), \quad (50)
 \end{aligned}$$

so that relation (49) is also true.

C.11 Proof [Theorem 2.18 (Updated Theorem 4.29 from [1])]

The proof for Theorem 2.18 is similar to that of Theorem 2.13, by replacing N by $N/4$ and multiplying the number of coefficients q_1 by 2. The number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 16 \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\cdot \left(3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(3^{\max\{0;n_{N,3}-2\}} \cdot 5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right) \quad (51)
 \end{aligned}$$

$$\begin{aligned}
 &= 16 \cdot 3^{2 \cdot (n_{N,3}-1)+2 \cdot \max\{0;n_{N,3}-2\}} \cdot 5^{4 \cdot (n_{N,5}-1)} \\
 &\cdot \prod_{j=4}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\cdot \left(3^{\max\{0;n_{N,3}-2\}} \cdot 5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned}$$

C.12 Proof [Theorem 2.19 (Updated Theorem 4.30 from [1])]

The number of possible coefficients q_1 is equal to $2^{n_{N,2}-1} \cdot 2 \cdot 3^{n_{N,3}-1} \cdot 4 \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right)$, the number of even or odd coefficients q_2 is equal to $2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1}$, and the number of even or odd coefficients q_3 is equal to $2^{n_{N,2}-2} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1}$. The number of even or odd coefficients q_4 is equal to $2^{n_{N,2}-4} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1}$ if $n_{N,2} \geq 4$. If $n_{N,2} = 3$, the number of coefficients q_4 is equal to $3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1}$, all of them even. The number of even or odd coefficients q_5 is equal to $2^{n_{N,2}-4} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1}$ if $n_{N,2} \geq 4$. If $n_{N,2} = 3$, the number of coefficients q_5 is equal to $3^{\max\{0;n_{N,3}-2\}} \cdot 5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1}$, all of them even, out of which one is zero. By removing the value $q_5 = 0$, if $n_{N,2} \geq 4$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 2 \cdot 8 \cdot 2^{n_{N,2}-1} \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \\
 &\cdot \prod_{j=4}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\cdot \left(2^{n_{N,2}-2} \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left(2^{n_{N,2}-4} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\cdot \left\{ \left(2^{n_{N,2}-2} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \right. \\
 &\cdot \left(2^{n_{N,2}-4} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &\left. + \left(2^{n_{N,2}-2} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \right. \\
 &\cdot \left. \left(2^{n_{N,2}-4} \cdot 3^{\max\{0;n_{N,3}-2\}} \cdot 5^{\max\{0;n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \right\} \quad (52)
 \end{aligned}$$

$$\begin{aligned}
 &= 2^{4 \cdot n_{N,2}-5} \cdot 3^{2 \cdot (n_{N,3}-1)+2 \cdot \max\{0; n_{N,3}-2\}} \cdot 5^{4 \cdot (n_{N,5}-1)} \\
 &\quad \cdot \prod_{j=4}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(2^{n_{N,2}-3} \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right)
 \end{aligned}$$

If $n_{N,2} = 3$, the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{N,5-PPs,ZF} &= 8 \cdot 4 \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s \left(p_j^{n_{N,p_j}-1} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(2 \cdot 3^{n_{N,3}-1} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(2 \cdot 3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{n_{N,5}-1} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} \right) \quad (53) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right) \\
 &= 128 \cdot 3^{2 \cdot (n_{N,3}-1)+2 \cdot \max\{0; n_{N,3}-2\}} \cdot 5^{4 \cdot (n_{N,5}-1)} \\
 &\quad \cdot \prod_{j=4}^s \left(p_j^{4 \cdot (n_{N,p_j}-1)} \cdot (p_j - 1) \right) \\
 &\quad \cdot \left(3^{\max\{0; n_{N,3}-2\}} \cdot 5^{\max\{0; n_{N,5}-2\}} \cdot \prod_{j=4}^s p_j^{n_{N,p_j}-1} - 1 \right),
 \end{aligned}$$

so that relation (52) is also true.

Appendix D

D.1 Example for Theorem 2.3 (Updated example for Theorem 4.6 from [1])

Let the interleaver length be $N = 2100 = 2^2 \cdot 3 \cdot 5^2 \cdot 7$. According to (11), the number of ZF CPPs will be equal to:

$$\begin{aligned}
 C_{2100, CPPs, ZF} &= 2^{2 \cdot (2-1)} \cdot 3^{2 \cdot (1-1)} \cdot 5^{2 \cdot (2-1)} \cdot (5 - 1) \\
 &\quad \cdot 7^{2 \cdot (1-1)} \cdot (7 - 1) \\
 &\quad \cdot \left(2^{2-2} \cdot 3^{\max\{0; 1-2\}} \cdot 5^{2-1} \cdot 7^{1-1} - 1 \right) \\
 &= 4 \cdot 1 \cdot 25 \cdot 4 \cdot 1 \cdot 6 \cdot (1 \cdot 3^0 \cdot 5 \cdot 1 - 1) \\
 &= 2400 \cdot (5 - 1) = 9600.
 \end{aligned}$$

Further, we detail the coefficients of the 9600 true different ZF CPPs. As $6 \mid N$ and $9 \nmid N$, it is required that $q_3 < 3 \cdot N/6 = N/2 = 1050$ and $q_2 < N/2 = 1050$. The coefficients q_2 and q_3 have to be multiples of $2 \cdot 3 \cdot 5 \cdot 7 = 210$, and coefficient q_1 has to be relatively prime with 2100. Therefore, coefficient q_3 can take 4 values: 210, 420, 630 and 840, q_2 can take 5 values: 0, 210, 420, 630 and 840, and coefficient q_1 can take $\Phi(2100) = 2100 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} = 480$ values. The total number of ZF CPPs is equal to $4 \cdot 5 \cdot 480 = 9600$, just the number found by the formula we determined. We note that the number of all true different CPP is also 9600. However, the equivalence conditions for CPPs, in this case, are: $q_3 < N/6 = 350$ and $q_2 < N/2 = 1050$. We note that every ZF CPP having coefficient $q_3 > 350$ (i.e., 420, 630, or 840) has an equivalent CPP having coefficient $q_3 < 350$ and coefficient $q_2 < 1050$. For example, ZF CPP $\pi_{CPP,ZF}(x) = 1 \cdot x + 0 \cdot x^2 + 420 \cdot x^3 \pmod{2100}$ is equivalent to CPP $\pi_{CPP}(x) = 351 \cdot x + 0 \cdot x^2 + 70 \cdot x^3 \pmod{2100}$, since $\pi_{CPP}(x) = \pi_{CPP,ZF}(x) + z(x) \pmod{2100}$ only for NP $z(x) = 350 \cdot x + 0 \cdot x^2 + 1750 \cdot x^3 \pmod{2100}$. This NP is not a ZF NP since $3 \nmid 1750$.

D.2 Example 2 for Theorem 2.19 (Updated example 2 for Theorem 4.30 from [1])

Let the interleaver length be $N = 2400 = 2^5 \cdot 3 \cdot 5^2$. According to (29), the number of ZF 5-PPs will be equal to:

$$\begin{aligned}
 C_{2400,5-PPs,ZF} &= 2^{4 \cdot 5-5} \cdot 3^{2 \cdot (1-1)+2 \cdot \max\{0; 1-2\}} \cdot 5^{4 \cdot (2-1)} \\
 &\quad \cdot \left(2^{5-3} \cdot 3^{\max\{0; 1-2\}} \cdot 5^{\max\{0; 2-2\}} - 1 \right) \\
 &= 32768 \cdot 1 \cdot 625 \cdot (4 \cdot 1 \cdot 1 - 1) \\
 &= 61440000.
 \end{aligned}$$

In this case, since $9 \nmid N$ and $600 \mid N$, it is required that $q_5 < N/40 = 60$, $q_4 < N/8 = 300$, $q_3 < N/2 = 1200$ and $q_2 < N/2 = 1200$. Coefficients q_5 , q_4 , q_3 , and q_2 have to be multiples of $3 \cdot 5 = 15$. The other conditions coefficients q_5 , q_4 , q_3 and q_2 have to meet are the same as those for $N = 25200$ (in Example 1 for Theorem 4.30 from [1]), and coefficient q_1 has to be relatively prime with 2400. Therefore, the pair of coefficients (q_5, q_3) can take the values (15, 15), (15, 45), (15, 75), ..., (15, 1185), (45, 15), (45, 45), (45, 75), ..., (45, 1185) ($2 \cdot 40 = 80$ pairs with both coefficients odd) and (30, 0), (30, 30), (30, 60), ... (30, 1170) (40 pairs with both coefficients even), i.e., a total of 120 pairs. The pair of coefficients (q_4, q_2) can take the values (15, 15), (15, 45), (15, 75), ..., (15, 1185), (45, 15), (45, 45), (45, 75), ..., (45, 1185), (75, 15), (75, 45), (75, 75), ..., (75, 1185), ..., (285, 15), (285, 45), (285, 75), ..., (285, 1185) ($10 \cdot 40 = 400$ pairs with both coefficients odd) and (0, 0), (0, 30), (0, 60),

$\dots, (0, 1170), (30, 0), (30, 30), (30, 60), \dots, (30, 1170), (60, 0), (60, 30), (60, 60), \dots, (60, 1170), (90, 0), (90, 30), (90, 60), \dots, (90, 1170), \dots, (270, 0), (270, 30), (270, 60), \dots, (270, 1170)$ ($10 \cdot 40 = 400$ pairs with both coefficients even), i.e., a total of 800 pairs. Coefficient q_1 can take $\Phi(2400) = 2400 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 640$ values. The total number of ZF 5-PPs is equal to $120 \cdot 800 \cdot 640 = 61440000$, just the number we found by the formula we determined.

We note that the equivalence conditions for 5-PPs in this case are: $q_5 < N/120 = 20$, $q_4 < N/24 = 100$, $q_3 < N/6 = 400$ and $q_2 < N/2 = 1200$. We note that every ZF 5-PP having coefficient $q_5 > 20$ (i.e., 30 or 45) and/or coefficient $q_4 > 100$ and/or coefficient $q_3 > 400$ has an equivalent 5-PP having coefficients $q_5 < 20$, $q_4 < 100$, $q_3 < 400$, and $q_2 < 1200$. For example, ZF 5-PP $\pi_{5-PP, ZF}(x) = 1 \cdot x + 0 \cdot x^2 + 1185 \cdot x^3 + 270 \cdot x^4 + 45 \cdot x^5 \pmod{2400}$ is equivalent to 5-PP $\pi_{5-PP}(x) = 1041 \cdot x + 200 \cdot x^2 + 185 \cdot x^3 + 70 \cdot x^4 + 5 \cdot x^5 \pmod{2400}$ since $\pi_{5-PP}(x) = \pi_{5-PP, ZF}(x) + z(x) \pmod{2400}$ only for NP $z(x) = 1040 \cdot$

$x + 200 \cdot x^2 + 1400 \cdot x^3 + 2200 \cdot x^4 + 2360 \cdot x^5 \pmod{2400}$. This NP is not a ZF NP since $3 \nmid 2360$, $3 \nmid 2200$, $3 \nmid 1400$, and $3 \nmid 200$.

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