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ORIGINAL RESEARCH

# On metaphors of mathematics: Between Blumenberg's nonconceptuality and Grothendieck's waves

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## Abstract

How can metaphors account for the formation of mathematical concepts, for changes in mathematical practices, or for the handling of mathematical problems? Following Hans Blumenberg's thought, this paper aims to unfold a possible answer to these questions by viewing the metaphorical frameworks accompanying these changes as essential for an understanding of how changes in mathematical practices have been accounted for. I will focus especially on cases in which these changes were caused by encounters with a mathematical object which did not yet have a well-defined concept, but also show that such indeterminacy remains with the mathematical concept even after it is considered 'well-defined'. As the paper will show, this 'forefield' [*Vorfeld*] of the concept is addressed by Blumenberg's account of metaphorology on the one hand, and accompanied by his later account of nonconceptuality [*Unbegrifflichkeit*] on the other hand. While Blumenberg himself did not develop a full-fledged philosophy of mathematics or of mathematical practices, I aim to show that one can nevertheless extract from his writings a unique position concerning the role metaphors play in mathematics. To this end, Blumenberg's account of nautical and oceanic metaphors and Alexandre Grothendieck's philosophy of mathematical practice provide fruitful starting points.

**Keywords** Hans Blumenberg · Nonconceptuality · Metaphorology · Nautical metaphors · Oceanic metaphors · Alexandre Grothendieck

In his notes on the formation of concepts and on the possibility of nonconceptuality written during the 1970s, Hans Blumenberg (1920–1996), one of the best-known German philosophers of the 20th century, states that "the desire for linguistic definiteness

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[unambiguity] […] will, as a utopia, no more disappear than Esperanto has." This longing is "bound to the ideal of mathematics to cover, with the use of a constructive set of instruments, all noncontradictory possibilities."[1] (2020, p. 282) Isolated from its context and without any connection to Blumenberg's writings on metaphors on the one hand, and on nonconceptuality on the other hand, this statement– that mathematical language ideally does not contain (or tolerate) any ambiguity– may be considered to represent a rather simplistic view within the philosophy of mathematics. But can one extract from Blumenberg's view of nonconceptuality another position toward both mathematical concepts and their emergence?

Considering 'nonconceptuality' as a designation for a field of study which deals with encounters with not yet conceptualized events or objects, or even with objects which do not have a place in a given conceptual system, two questions arise: first, how mathematicians deal and have dealt with these kinds of encounters; second, whether such encounters are also a part of the concept itself. To answer these questions, however, one should note that there are two ways of viewing these encounters: the first concerns how nonconceptuality arises in mathematics itself, even though the 'ideal' of mathematics is to consider this discipline as being based on nonambiguous definitions, laws of deduction, and theorems, while the second relates to how mathematicians, philosophers, and historians (of mathematics) have discussed unsolved problems, unclear mathematical concepts or configurations, or possibly ambiguous definitions. These two views certainly overlap and relate to each other, but they are not the same: the first deals with (un)ambiguity within mathematics, while the second deals with how such encounters (with what is considered ambiguous or undefined) are discussed. Hence, to follow Corry (1989, 2004), the first view deals with the body of mathematics (or of a mathematical configuration),[2] whereas the second is concerned with its image.[3] While this paper will deal mostly with the second view, Corry stresses that the two views cannot be treated separately– that is, encounters with 'inexistent,' impossible, or not well-defined mathematical concepts are accompanied by a certain image of the corresponding mathematical configuration, which specifies how those concepts are considered; moreover, discussions of such images usually point toward the emergence of nonconceptuality in the body of the mathematical

[1] Blumenberg (2007, p. 51): "Die Sehnsucht nach sprachlicher Eindeutigkeit […] wird als Utopie so wenig verschwinden wie die des Esperanto. […] Sie ist […] an das Ideal der Mathematik gebunden, durch ein konstruktives Instrumentarium alle widerspruchsfreien Möglichkeiten […] abzudecken […]."

[2] A mathematical configuration, to follow Moritz Epple's discussion of epistemic configurations (2004), is an array of mathematical objects researched by a group of mathematicians in a certain specific temporal and geographical setting, as well as of techniques developed to study those objects.

[3] Here, I employ the distinction introduced by Corry between the body and the image of mathematical knowledge, who follows (Elkana, 1981). Statements included in the body of knowledge are about the subject matter of the discipline involved, where these may be theories, conjectures, methods, problems, proofs, etc. Statements belonging to the image of knowledge function as "guiding principles or selectors" (Corry, 1989, p. 411), and answer questions about the discipline as such. These questions may be about authority, the correct and valid methods and proofs that can be used, methodology, and what, how, and with whom one should investigate. Though this distinction is essential for analytical purposes, Corry stresses not only that the "body and the image of mathematics appear as organically interconnected domains in the actual history of the discipline," but also that one should analyze "the subsequent transformations in both the body and the images of mathematics." (Corry, 2004, p. 5).

configuration itself. In this sense, the paper joins and is motivated by the recent studies in the philosophy of mathematics respectively of mathematical practices, which deviate from this ideal of mathematics and stress, as will be elaborated in Sect. 1.3, the semantic indeterminacy, vagueness and 'open texture' of mathematical concepts, following later Wittgenstein, Lakatos, and Waismann.

How do metaphors enter the picture? One of the most famous examples of a reaction to such an encounter with a not yet conceptualized object is the story told concerning the alleged rejection by the Pythagoreans of irrational numbers,[4] metaphorized by drowning in a sea (see Sect. 2.2). Another example would be the slow acceptance of imaginary numbers– first by treating them as 'mental torture,' as Gerolamo Cardano called them (Corry, 2015, p. 144), but finally by accepting them as legitimate, and assigning them a symbol of their own: $i = \sqrt{-1}$. These episodes are well known and well researched. Blumenberg's theory of metaphors and his view of nonconceptuality allow to examine these and similar events not from a (conceptual) historical point of view but from the point of view of the metaphorical frameworks accompanying these concepts and the histories of the acceptance of these concepts. This approach, as will be shown in Sect. 1.3, is opposed to the one developed by George Lakoff and Rafael E. Núñez, since it does not aim to uncover "where mathematics"– or any other science– "comes from," to cite the title of Lakoff's and Núñez's book.

This paper will therefore deal with how encounters with and discoveries of mathematically impossible or still undefinable objects as well as their conceptualization may be viewed from the perspective of the metaphors, parables, and fables that frame such encounters and concepts, whereby these issues will be considered in relation to metaphorology, Blumenberg's theory of metaphors. After reviewing Blumenberg's work on metaphors and his complicated relation to mathematics in Sect. 1.1, I focus in Sect. 1.2 on how his approach to nonconceptuality deals with mathematics and mathematical concepts. Section 1.3 examines other approaches in philosophy of mathematics, which may be related to Blumenberg's approach: its differences from Lakoff's and Núñez's appraoch, and its possible affinities with Friedrich Waismann's views on the 'open texture' of concepts and with later Wittgenstein's philosophy of mathematics. Section 2 discusses one specific set of metaphors: the complex of nautical and oceanic metaphors and images, such as the shipwreck, the sea, and the coast, as well as certain 'demonic' events, such as plagues, floods, and tides, which are sometimes associated with such metaphors. The question that stands at the center of this section concerns the philosophical insights that Blumenberg's approach to nautical and oceanic metaphors may uncover. The last section, Sect. 3, goes on to examine a specific case study: it looks at Alexandre Grothendieck's philosophy of mathematical practices and discovery through the lens of Blumenberg's understanding of nautical-oceanic metaphors. Grothendieck suggests considering these practices as a never-ending series of waves crashing on the shore. An examination of this metaphor hence offers a unique way to reflect on and frame– both philosophically and metaphorically– acts of practicing mathematics, which is further reflected in the concluding Sect. 4.

[4] As will become clear below, I am not claiming that the Pythagoreans indeed rejected irrational numbers.

# 1 Mathematics between Blumenberg's metaphorology and nonconceptuality: how to account for mathematical concept formation?

Hans Blumenberg is known for his work in the field of the history and philosophy of ideas, which focuses on the role played by metaphors in the formation of concepts and of images of knowledge.[5] One of his early starting points is his 1960 text *Paradigms for a Metaphorology* [*Paradigmen zu einer Metaphorologie*], which starts with a critique of René Descartes, in which Blumenberg rejects the Cartesian project of developing a language which consists only of clear and distinct concepts,[6] and in which all figurative elements are eliminated. This rejection leads Blumenberg to an inquiry into the various metaphors to be found in the history of European thought.[7] This section will briefly review Blumenberg's metaphorology, mainly examining in Sect. 1.1 his relationship to mathematics and mathematical concepts. Section 1.2 goes on to examine Blumenberg's later views on nonconceptuality, which point toward a revised understanding of the formation of mathematical concepts. Section 1.3 examines Blumenberg's reflections on mathematics in the wider context of philosophy of mathematics in the 20th and the 21st centuries.

## 1.1 Blumenberg's metaphorology and mathematical concepts

One of Blumenberg's main claims in *Paradigms for a Metaphorology* is not only that not every element of language can be reduced to a concept, but that there are also linguistic elements which are necessary for human thought and irreducible to concepts. These elements are later termed 'absolute metaphors,' on which I will elaborate below. The investigation of these fundamental metaphors is therefore a part of the investigation of the history of ideas. Blumenberg's project, called *metaphorology*, represents an alternative to the history of concepts not so much because it deals primarily with metaphors, whereas the history of concepts deals primarily with concepts, but because it suggests that concepts themselves operate on a metaphorical foundation.[8] This mode of operation is not to be understood as implying that concepts are a crystallization of metaphors,[9] but rather that there are fundamental metaphors which guide the emergence and production of concepts.[10] It is therefore

[5] Blumenberg is also known for his other works, for example, on the history of the early modern period and on the readability of the world (see: Blumenberg, 1986; 1987).

[6] The reference to the Cartesian 'clear' and 'distinct' concepts is explicitly expressed in: (Blumenberg, 2020, p. 261).

[7] The secondary literature on Blumenberg is vast; for recent overviews of his works, see for example: (Zill, 2020; Wetz, 2014). See also, among others, Haverkamp's discussions on Blumenberg's metaphorology in: (Haverkamp, 2012, 2018). On Blumenberg's own usage of metaphors, see: (Gehring, 2022).

[8] On the development of Blumenberg's metaphorology and how it was reshaped by him over the decades, see: (Zill, 2019).

[9] Cf. Haverkamp (2012, p. 42), who notes that it is essential "to realize that the project of metaphorology was not primarily modeled upon the […] concept-focused history of ideas."

[10] (Blumenberg, 2010, p. 5): "[M]etaphorology seeks to burrow down to the substructure of thought, the underground, the nutrient solution of systematic crystallizations; but it also aims to show with what

essential to remember that Blumenberg stresses that the (historical) research of concepts and the (historical) research of metaphors, along with their respective philosophical frameworks, are interwoven. The necessity of this interwovenness for our thinking is termed by Blumenberg as a "sad necessity" ["traurige Notwendigkeit"] (Blumenberg, 2020, p. 294), when he cites the opening passages of a letter Georg Jonathan von Holland sent to Johann Heinrich Lambert in 1765: "I think that we owe a large part of our knowledge [*Erkenntnis*], and an even larger part of our errors, to the development of metaphors."[11] (Ibid.) Blumenberg then adds: "Here, everything is said with the fewest possible words: the sad necessity of a makeshift solution leads to the ambiguous excess of a guidance for knowledge on the one hand,"– that is, the guideline of the formation of concepts, "and to a relegation of deception on the other"[12]– implying that perhaps other metaphors will be needed to overcome such future misapprehensions (ibid.). I will return to the 1765 letter below, as it also shows Blumenberg's complicated relation to mathematics.

To be more explicit, in *Paradigms for a Metaphorology*, Blumenberg does not formulate a complete theory of metaphor,[13] but rather discusses numerous examples of metaphors (of light, the machine, the organism, or the circle), and from this discussion emerges a multifaceted definition of the absolute metaphor, whereby, with each example, Blumenberg illuminates only certain aspects.[14] For Blumenberg, what needs to be addressed is "the fundamental question of the conditions under which metaphors can claim legitimacy in philosophical language. Metaphors can first of all be *leftover elements*, rudiments on the path *from mythos to logos* […]." (2010, p. 3) But in opposition to the Cartesian understanding of the metaphor, which ultimately dismisses metaphor as a dispensable tool, Blumenberg presents absolute metaphors, which "can also […] be *foundational elements* of philosophical language, 'translations' [*'Übertragungen'*] that resist being converted back into authenticity and logicality." (Ibid.) These metaphors "prove resistant to terminological claims and cannot be dissolved into conceptuality." (Ibid., p. 5) Absolute metaphors are not to be taken as a substrate which can be transformed into concepts, but function "as a catalytic sphere from which the universe of concepts continually renews itself, with-

'courage' the mind preempts itself in its images, and how its history is projected in the courage of its conjectures."

[11] (Blumenberg, 2007, p. 90): "Ich denke, daß wir der Entwickelung von Metaphern einen großen Teil unserer Erkenntnis und einen noch größeren unserer Irrtümer zu danken haben."

[12] (Blumenberg, 2007, p. 90): "Hier ist auf engstem Raum alles gesagt: die traurige Notwendigkeit eines Notbehelfs führt in den doppeldeutigen Überschuss eines Leitfadens der Erkenntnis einerseits, einer Verweisung der Irreführung andererseits."

[13] Here, I follow Haverkamp (2012, p. 40): "It is remarkable that *Metaphorology* does not contain even the slightest hint of a definition of the term metaphor itself, and retrospectively it can only be doubly striking that Blumenberg makes no attempt […] to deduce a definition of metaphor in terms of its conceptual history, almost as if metaphor– possibly it alone– had no history: as if it excelled and exceeded all history in the usual sense."

[14] As is clear from the title of the essay (*Paradigms for a Metaphorology*), Blumenberg offers "paradigms," but certainly not a closed, finite list of absolute metaphors. Cf. Haverkamp (2012, p. 45), who stresses that these paradigms are not to be thought as if they would result in concepts: "Blumenberg's paradigms serve as the epistemo-pragmatic parameters of metaphorological range: parameters that form their comprehension by means of their reach, rather than finding it in fixed, conceptually preformatted concepts."

out thereby converting and exhausting this founding reserve." (Ibid., p. 4) Moreover, these metaphors do not have to be expressed explicitly as metaphors (though they can be), but can remain latent, in the background, when other expressions and modes of articulation draw on the semantic field of these (absolute) metaphors without the metaphors themselves being explicitly stated as such.

Absolute metaphors are therefore metaphors which, by definition, cannot be reduced to concepts or to dead metaphors (such as in the expression 'footnote'). They rather unfold a dynamic conceptual space which allows the emergence of specific concepts.[15] Moreover, these concepts themselves, as is already clear from Blumenberg's reference to Holland's letter to Lambert, do not necessarily have to follow the Cartesian "ideal of clarity", but rather have a certain relation to the dynamicity, to "the elasticity of the latitude" of the absolute metaphor (Blumenberg, 2020, p. 262). In this sense, as will be explicated later, also the concept itself, and not just the dynamic space enabling its emergence, "must possess enough indeterminacy [*Unbestimmtheit*]" (ibid.).[16]

As noted, Blumenberg's rejection of the Cartesian project consists in a rejection of a conception of a language which is cleansed and devoid of any residue of metaphorical language, and this rejection may also be considered as implicitly articulating a suspicion against mathematics' claim to be the (only) adequate language for all domains of the natural sciences, that is, the only language which allows one to 'read the book of nature.'[17] If mathematical concepts are (or should be) understood as well defined and unambiguous, then this understanding of mathematical language follows the Cartesian project. Mathematics can be thought of as providing this security to language, since by means of mathematics– at least according to the analytical conception of mathematics– everything can be reduced in theory to logical terms and relations, following what Blumenberg calls the "Cartesian teleology of logicization" (2010, p. 3);[18] such a reduction does not allow an 'elastic space' of interpretation of concepts. That this conception of security in mathematics is already expressed by Descartes himself can be seen in his *Discours de la méthode*, published in 1637: "I was especially delighted with the mathematics, on account of the certitude and evidence of their reasonings; but I had not as yet a precise knowledge of their true use; and thinking that they but contributed to the advancement of the mechanical arts, I was astonished that foundations, so strong and solid, should have had no loftier superstructure reared on them. On the other hand, I compared the disquisitions of the ancient moralists to very towering and magnificent palaces with no better foundation than sand and mud." (1951, p. 6)

[15] See for example Chapter I in Blumenberg's *Paradigms for a Metaphorology*: "Metaphorics of the 'Mighty' Truth," which deals with how the metaphor of light points to various concepts of 'truth,' adding that the "metaphorics of light cannot be translated back into concepts" (Blumenberg, 2010, p. 7).

[16] See also: (Blumenberg, 2007, p. 32)– the question "what is this?" and its possible answers already imply such an "indeterminacy."

[17] This is a reference to a passage in Galileo Galilei's *Il Saggiatore* which claims that the universe is written in the language of mathematics. On Blumenberg's discussions on the history of the image of the readability of the world and especially on this passage in Galileo, see: (Blumenberg, 1986).

[18] That this teleology still prevails in the 21st century can be seen with the attempts to formalize mathematical proofs with the development of computational proof assistants (see e.g. the works of Jeremy Avigad).

In this passage Descartes not only stresses the security mathematics provides, but he also employs architecture as an image of knowledge. That is, Descartes interlaces claims about the body of mathematics and the image of mathematics. He sees himself as an architect whose role is to rebuild secure and safe foundations for science.[19] How geometry is constructed is an important issue for Descartes. After being asked to set out his arguments in a geometrical fashion, he presents in an appendix to the sixth meditation of his *Meditations on First Philosophy* a 'geometrical' exposition of some of his central lines of argument, organized as definitions, postulates, axioms or common notions, and propositions (Descartes, 1904, pp. 160–170).

However, that Descartes uses architectural images and metaphors to present his ideas,[20] is not noted by Blumenberg. When, in his book *The Legitimacy of the Modern Age*, Blumenberg notes that Descartes sees logic and mathematics as that which guarantees certainty,[21] he considers Descartes' theory without the accompanying metaphorical images, and sees the Cartesian project only through the lens of the explicit aim of eliminating metaphors or of reducing them to logic and mathematics. Geometry plays a double role for Descartes– as a guiding architectural image and as a domain of unshakable knowledge, though one which may be expanded and revised (as seen in Descartes' *La Géométrie*).[22] This double role of geometry enables Descartes to employ this metaphor both as rhetoric and as that which prompts the emergence of new knowledge; but this double role is not mentioned in the short reference to Descartes in the introduction to Blumenberg's 1960 text *Paradigms for a Metaphorology*.

Here, it is already instructive to approach the question of how Blumenberg considers mathematics and its history. Blumenberg's relation to these subjects is rather difficult to reconstruct, since a detailed discussion on the history or philosophy of mathematics is not to be found in his (published) writings.[23] A possible reason for this may be that for Blumenberg, as noted above, mathematics might be considered

[19] See: (Descartes, 1904, p. 536): "Testatus sum ubique in meis scriptis, me Architectos in eo imitari." Cf. also the first meditation in Descartes' *Meditations on First Philosophy* (1641), where he notes that, due to his doubts, he needed to demolish everything completely and start again from the foundations in order to establish something in the sciences that would be stable.

[20] See also: (Purdy, 2011, pp. 87–93).

[21] Explicitly, Blumenberg notes the following: "The guarantee that Descartes will seek to found on the most perfect being, which he gains through his proof of God's existence, relates, however, not only to the reality of the physical objects that present themselves through our clear and distinct ideas but also […] to the propositions of logic and mathematics." (1983, pp. 197–198).

[22] To emphasize: Descartes does not employ the architectural metaphor inside mathematics but when describing mathematics' relation to other domains (e.g. the mechanical arts). For examples of this type of usage of the architectural metaphor in mathematics at the end of the 19th century, see: (Schlimm, 2016).

[23] The question of how to transfer Blumenberg's insights on the history of science and technique to 21st -century research within the history of science, as well as the question of how to consider Blumenberg's own analysis critically, is dealt with, in among other places, the volume *Hans Blumenberg beobachtet: Wissenschaft, Technik und Philosophie*, edited by Cornelius Borck (see: Borck, 2013). Borck not only argues that the "world of Blumenberg is in the past," and hence that one obtains a critical distance from his claims, but also– in view of the domination of "partial and micro histories" (among other contemporary methods) in today's history of science– asks "what potential do his reflections unfold if Blumenberg's broad conceptual landscapes today all the more sharply raise the question of the persuasiveness of such analyses?" (Borck, 2013, p. 19).

as fulfilling (or at least attempting to fulfill) the Cartesian project. Nevertheless, one may still extract from Blumenberg's writings a rather convoluted approach regarding mathematics: Blumenberg is aware of its role for the development of modern science, but he ignores its many metaphorical frameworks, except for specific cases. This focus on specific cases can be seen in several of Blumenberg's writings in which he discusses the ways in which metaphors are tolerated within mathematics; this can be seen, for example, in *Paradigms for a Metaphorology*. Two chapters of this book deal directly with mathematical concepts or with metaphors coming from mathematics: Chapter VIII ("Terminologization of a Metaphor: From 'Verisimilitude' to 'Probability'"[24]) discusses the "transitions from metaphors to concepts" (2010, p. 81), explicitly analyzing how the mathematical concept of probability emerged. Chapter X ("Geometric Symbolism and Metaphorics") considers the transition from geometrical symbols to metaphors through a discussion of the circle and the sphere.

Blumenberg's emphasis on the role of mathematics in the early modern period is noted when he underlines that the mathematization of nature was a criterion for this period's efforts to find "a set of instruments for man that would be usable in any possible world" (Blumenberg, 1983, p. 164).[25] But as he notes, while this set of instruments was thought in this period to reach the "naked truth,"[26] it was exactly this nakedness that was later criticized by Husserl as deceptive, being only a "well-fitting garb of ideas."[27] This transformation of the metaphor of clothes and clothing, of covering, uncovering, and revealing shows that the metaphors used to illustrate the role and importance of mathematics change over time. This again might be one of the few examples of Blumenberg which discusses the changing metaphors within the history of mathematics.[28]

Yet, as already noted in relation to his discussion on Descartes, the ways in which mathematical concepts function and operate, and especially how this mode of operation and emergence is framed metaphorically, is rarely discussed explicitly by Blu-

[24] Note that the title in German is "Terminologisierung einer Metapher: Wahrscheinlichkeit." Here, the translator explicitly notes that he modified the title to account for the various meanings of *Wahrscheinlichkeit* in German (Blumenberg, 2010, p. 81).

[25] Cf. also Blumenberg's view on the role of mathematics in early modern astronomy in: (Blumenberg, 1987).

[26] Chapter IV of *Paradigms for a Metaphorology* is titled "Metaphorics of the 'Naked' Truth" (Blumenberg, 2010, pp. 40–51). Blumenberg's analysis of the metaphorics of the naked truth is presented in (Blumenberg, 2019). See also: (Blumenberg, 1981, p. 30), which notes that for Husserl, starting in the early modern period, the "modern consciousness" believed that the "exact sciences could discover and represent the 'intrinsically true world' hidden behind appearances with the help of mathematics." ["[…] die exakte Wissenschaft könnte mit Hilfe der Mathematik die hinter den Erscheinungen gleichsam versteckte 'an sich wahre Welt' entdecken und darstellen."].

[27] The full quotation is as follows: "It is exemplary that the clothing metaphor recurs precisely where the early and high modern age thought to have reached through to the bare core of Being in itself, in the mathematical knowledge of nature: what looked like nakedness turns out to be a 'well-fitting garb of ideas' whose measurements are taken in the geometrical and natural-scientific mathematization of the lifeworld–so Edmund Husserl in his interpretation of Galileo." (Blumenberg, 2010, p. 48) Cf. also (Blumenberg, 1981, p. 30).

[28] Moreover, with respect to Blumenberg's discussion on Husserl presented above, it seems that Blumenberg adopts Husserl's critique of mathematics and its formalization without critically examining Husserl's own historical presentation. See: (Blumenberg, 1981, p. 31).

menberg; the episodes from the history of mathematics which he does discuss (as examined above) are brought up less to explicate the formation of mathematical concepts than to elucidate other domains, such as Blumenberg's claims about the history of metaphors. The choice to ignore the role of metaphors in the formation of mathematical concepts can be noted in two examples from Blumenberg's own writings.

The first can be found in the abovementioned letter from Holland to Lambert, where, according to Blumenberg, "everything is said with the fewest possible words […] [concerning the] sad necessity" of metaphors (Blumenberg, 2020, p. 294). But if one continues reading the letter, Holland discusses misapprehensions and errors which mainly arise in *mathematics*. In fact, the first topic Holland presents in this letter is a discussion on possible differences between the "symbolic Nothing [*symbolische[ ] Nichts*]" and the "conceivable Nothing [*gedenkbare[ ] Nichts*]" (Lambert, 1782, p. 40), a subject which one would have thought might be relevant to Blumenberg's views of the concept. The first example Holland provides is the square root of (-1)– that is, √-1– as a case in which the two types of Nothing coincide (ibid., p. 41). Other examples, such as the differential or limits of fractions (when both denominator and numerator converge to zero) are also discussed.[29] These examples of concepts whose mathematical status was at that time ambiguous and unclear, though they are presented immediately after Holland's statement on the necessity of metaphors, are, however, not mentioned by Blumenberg.

The second example of Blumenberg's failure to consider this issue is the absence in his writings of any discussion of the crises of mathematics at the end of the 19th century and the beginning of the 20th century (for example, the *Grundlagenkrise* or the crisis of *Anschauung*), even though he does underline that the "tendency of formalization" starting in the early modern period, prompted by the usage of "mathematical means of representation," leads first to an "arithmetization of geometry," then to its "algebraization," and eventually to a "purely empty set theory ['*Mannigfaltigkeitslehre*']" (Blumenberg, 1981, pp. 30–31). While Blumenberg is aware of the *Grundlagenkrise*, implicitly referring to it when citing Hilbert's phrase that "from the paradise [of set theory], that Cantor created for us, no one shall be able to expel us" (Hilbert, 1926, p. 170; see: Blumenberg, 1989, p. 789),[30] the absence of a more detailed discussion is somewhat surprising, since in several of his writings Blumenberg sets out to delineate a number of encounters with nonconceptuality, encounters which are also present in mathematics and may be detected clearly in the abovementioned crises of mathematics. At the same time, Blumenberg does explicate the relation between nonconceptuality and mathematics, which makes it somewhat surprising that he hardly deals with how mathematics and its concepts operate. Here, I would like to return to Blumenberg's philosophical conception of mathematical encounters with the nonconceptualized, as presented at the beginning of the paper, in

[29] This correspondence is also situated within the framework of Lambert's theory of metaphors; while outside the scope of this paper, I refer to (Müller, 2011), who discusses this theory, its relations to mathematics, and its relation to Blumenberg's reflections. Müller notes that Lambert's philosophy of science shows that a philosophical and scientific revaluation of metaphor may even grow out of a theory adhering to the mathematical ideal of the modern natural sciences (ibid., p. 49).

[30] Blumenberg mentions Hilbert's call in the framework of his discussion on Wittgenstein's philosphy (see: Blumenberg, 1989, pp. 752–792); see also Section 1.3.

order to show how, despite the above, it is still possible to account for the formation of mathematical concepts using Blumenberg's metaphorology.

### 1.2 Blumenberg's nonconceptuality and mathematical concepts

A discussion on nonconceptuality [*Unbegrifflichkeit*] is already found in Blumenberg's writings in 1960, when he describes the relationship between absolute metaphor and nonconceptuality in the following way: "the function of 'absolute metaphor' [is that it] springs into a nonconceptualizable, conceptually unfillable gap and lacuna […]." (2010, p. 122) In 1979, in "Prospect for a Theory of Nonconceptuality," in which Blumenberg reframes his conception of metaphorology, he notes that "metaphorology's function has not changed, but its referent has, primarily in that metaphorics is now understood as merely a limited special case of nonconceptuality." (1997, p. 81) That is, as Paul Fleming stresses, nonconceptuality also includes "myth, gloss, example, anecdote, etc." as other forms which may "spring" into this "unfillable gap." (Fleming, 2012, p. 25) Moreover, in 2007, the notes Blumenberg had made on the concept of nonconceptuality during the 1970s were published,[31] notes in which he stresses the operativeness of nonconceptuality. These notes begin by remarking that "concepts developed from the *actio per distans*, from action across spatial and temporal distance."[32] (Blumenberg, 2020, p. 261), and that the concept is "an action in […] absence" (ibid., p. 263). This is because the object, which we should touch, see, or sense, is missing– and the concept comes as a replacement and a representation of the lack of tangibility caused due to the distance from the object. Hence, Blumenberg designates the concept as "a trap" (ibid., p. 260). And, yet again, Blumenberg expresses his critique of the Cartesian conception of the concept, according to which concepts should be clear and distinct (ibid., p. 261). The interest here, to emphasize, does not lie in concepts; as with *metaphorology*, the goal is not necessarily to inquire into how concepts are formed. Nevertheless, as we will see later, Blumenberg does delineate a possibility to think on an essentially indeterminate conceptual system. Returning to nonconceptuality, it "is less concerned with the 'what' or even the 'how' of thinking (i.e., its relation to concept formation, though this is certainly also the case)." (Fleming, 2012, p. 25) Here, Fleming notes that "what a theory of nonconceptuality attempts to outline" is a "horizon of thinking, where knowledge is tied to disappointment"[33] (ibid.), that is, where thinking encounters boundaries. This gives rise to the question of where one may hope that such a disappointment would not be

[31] The notes were published under the title "Theory of nonconceptuality" [*Theorie der Unbegrifflichkeit*].

[32] (Blumenberg, 2007, p. 11): "Der Begriff ist aus der *actio per distans*, aus dem Handeln auf räumliche und zeitliche Entfernung entstanden."

[33] This disappointment is what leads Blumenberg (1997, p. 89) to underline that one needs a different form of language to account for the "ineffable" [*Unsagbaren*], following Wittgenstein's dictum "What we cannot speak about we must pass over in silence." Cf. (Ifergan, 2020, p. 134): "For Blumenberg, Wittgenstein's dictum is not an appeal for total silence, but an acknowledgment that a different mode of expression, a different form of language, is necessary. Blumenberg invokes Wittgenstein in an attempt to implicitly undermine the […] insistence that there is no recourse but silence. […] [M]etaphorology, absolute metaphor, and the theory of non-conceptuality can be read as ways of transgressing the boundaries that otherwise compel us to remain silent." I will return to later Wittgenstein's philosophy in Sect. 1.3.

encountered; and a possible answer given by Blumenberg refers to two domains: a legal one and a mathematical one.

By also explicating the relations between metaphor and concept, Blumenberg clarifies to some extent his position toward mathematics. A metaphor appears when "the determination of the context is weak enough. In a legal text […] the metaphor becomes impossible."[34] (Blumenberg, 2007, p. 61) According to Blumenberg, if a system (such as a system of laws) is understood as a system of unambiguous, precisely determined concepts, then mathematics would also have to be imagined as a field in which metaphors are impossible.[35] Blumenberg explicitly addresses this idea with the statement quoted in the introduction to this paper, according to which the longing for linguistic definiteness is "bound to the ideal of mathematics." (2020, p. 282) It seems, moreover, that Blumenberg here is following Kant, since, a few passages before the above statement, he quotes a footnote from Kant's *Critique of Pure Reason*: "In mathematics definitions belong *ad esse*, in philosophy *ad melius esse*. Attaining them is fine, but often very difficult. Jurists are still searching for a definition of their concept of right."[36] (Ibid., p. 281) What Blumenberg does not cite, but he may very well have agreed with it, is the statement by Kant which comes almost immediately after the statement which is cited by Blumenberg: "Mathematical definitions can never err. For since [in mathematics] the concept is first given through the definition, it contains just that which the definition would think through it." (Kant, 1998, A731/B759)[37]

The possibility of never being mistaken corresponds to the longing for linguistic unambiguity, which is in turn bound to the "ideal of mathematics." Accordingly, one may interpret Blumenberg's position as claiming that the semantic clarity of mathematical definitions vouches for the reliability of the mathematical procedure.[38] It is essential to note here that he underlines that this is an "ideal," which may imply that even mathematics and mathematical concepts never reach this state, and hence this never-ending process may emphasize the historicity of mathematics. I will elaborate on this position in Sect. 2.1, since Blumenberg himself stresses that even in mathematics, there might be essential incompleteness, in the sense that it can (or should) be again and again reconstructed. This ideal but unreachable state may be seen in

[34] "[…] die Determination des Kontextes schwach genug ist. In einem Gesetzestext […] wird die Metapher unmöglich."

[35] See also: (Blumenberg, 2010, p. 87): "If science were ever to be completed, there would be no more perplexity […], all rhetoric would also be superfluous and ineffective." ["Wo die Wissenschaft jemals vollendet wäre, gäbe es keine Verlegenheit mehr […], es wäre auch jede Rhetorik überflüssig und wirkungslos."].

[36] The quotation is taken from: (Kant, 1998, A731/B759, footnote).

[37] This is not to imply that Blumenberg does not criticize Kant's *Critique of Pure Reason*. Blumenberg describes Kant's project of the "critique of pure reason according to the constitutive restriction of its representational extension" (Blumenberg, 2007, p. 93), whereby one of the central questions raised by Kant's critique is the one concerning the possibility of mathematical concepts and their construction, or the uniqueness of mathematical reasoning. Blumenberg claims that "a critique of pure rationality […] which presents itself in metaphorics" is also needed (ibid.), which stresses again that the metaphors and the forefield of the concept should also be taken into account, and not just the construction of concepts.

[38] Note that exactly this position has been criticized in recent years within the philosophy of mathematics, as I will show below. See also: (De Toffoli, 2021).

the following statement by Blumenberg on the relation between concept and nonconceptuality: “for the benefit of concepts, there has to be a preliminary field [fore-field; *Vorfeld*] of nonconceivability [incomprehensibility; *Unbegreiflichkeit*], even if, under the criteria of the possible perfected concept, one were inclined to cross this field disparagingly and let it be altogether forgotten in the state of perfection.”[39] (Blumenberg, 2020, p. 281) This statement may open a way of thinking about mathematical nonconceptuality, or at least about the ‘forefield’ of incomprehensibility and nonconceivability, before but also during the emergence and formation of mathematical concepts. First, Blumenberg’s sarcastic tone concerning the disparaging crossing of the field implies that such a state of completion is not really possible. Second, immediately after the passage cited above, Blumenberg warns us not to consider “nonconceptuality in the service of concepts” [“Die Unbegrifflichkeit im Dienst des Begriffs”] as a “mere philosophical auxiliary discipline [*Hilfswissenschaft*],” (ibid.) but rather to note that the work in this forefield “of concepts does not arrive at its goal.”[40] (Ibid., p. 282) It thus becomes clear that between the ‘solid’ concept and its forefield of incomprehensibility there is a tension (and certainly not a smooth transition), since it is impossible to know when one has ‘successfully’ defined the concept (being the “goal” mentioned above) without any exceptions or ‘holes’. This is since not only are both entities (the forefield and the full-fledged concept) are interwoven, but also a transition from one to the other is practically impossible; nonconceptuality should not be seen in the service of the production of well-defined concepts.[41] It is hence no wonder that Blumenberg compares the concept to a “fishing net,” as it represents the tolerance between the accuracy and imprecision, between “exactness and inexactness of its reference object that can only be created by way of concepts.”[42] (Ibid., p. 263) That is, “[i]n principle, a concept must be definable” (ibid., p. 271), but in practice it also lets the absence on which it is founded be present as such. Or, to formulate it more concretely, the holes in the ‘fishing net’ are also present when the concept is defined. This is how nonconceptuality is present, a presence which– to return to the beginning of this subsection– can be accounted for, among other things, by metaphors.

If we return to Blumenberg’s statement about the forefield of incomprehensibility, and how one would like to leave it in order to define, at least in theory, clear and

[39] (Blumenberg, 2007, p. 51): “[…] zugunsten des Begriffs muß es ein Vorfeld der Unbegreiflichkeit geben, auch wenn man geneigt sein sollte, unter den Kriterien des möglichen vollendeten Begriffs dieses Vorfeld geringschätzig zu durchqueren und im Zustand der Vollendung ganz und gar vergessen zu machen.”

[40] “[…] die Arbeit im Vorfeld des Begriffs nicht zu ihrem Ziel gelangt.”

[41] The relations between *Unbegreiflichkeit* and *Unbegrifflichkeit*, which certainly demand a further investigation, are also underlined by Haverkamp (2018, p. 90). Discussing Georges Canguilhem’s notion of the concept of life, Haverkamp cites Canguilhem, who claims: “C’est le concept d’un être sans concept.” Haverkamp remarks: “Although not conceptual, it is not incomprehensible. On the contrary, nothing else happens in the process of science than to generate a concept of this […] nonconceptuality […].” [“Wiewohl unbegrifflich, ist es doch nicht unbegreiflich. Im Gegenteil, nichts anderes passiert im Prozeß der Wissenschaften, als einen Begriff von dieser […] Unbegrifflichkeit zu erwirtschaften […].”].

[42] (Blumenberg, 2007, p. 14): “[…] die Toleranz zwischen Genauigkeit und Ungenauigkeit des Bezugsobjektes, das nur durch den Begriff hergestellt werden kann.”

distinct concepts,[43] one may claim that the history of mathematics can show almost the opposite of this 'ideal' state: that some mathematical concepts were or will be ambiguous and unclear, and that their emergence, conception, and coining were (or will be) accompanied by metaphors and fables. Before showing how discussions on mathematical concepts and their emergence have been and are accompanied and framed by such metaphors, I would like to take a step back and examine Blumenberg's reflections on metaphorology in the wider context of the 20th and 21st century philosophy of mathematics.

### 1.3 Metaphors, nonconceptuality and indeterminacy of mathematics in context

First, it is important to note that the immediate context of Blumenberg's 1960 paper on metaphorology is to be found on the backdrop of the emerging field of *Begriffsgeschichte* (back then developed by Erich Rothacker, Otto Brunner and Reinhart Koselleck) during these years in Germany. Invited to write to the journal *Archiv für Begriffsgeschichte*, with the intention that he would contribute to the field of *Begriffsgeschichte*, Blumenberg explicitly goes against any possibility of writing such a history of concepts, at least if one does not consider the essential role of metaphors. Moreover, as can be noted above, Blumenberg's critique on the 20th century analytic philosophy is clear, if one considers it as a logical analysis of concepts and their usages. Indeed, in the wider context of philosophy of mathematics and of the sciences of the first half of the 20th century, Blumenberg's metaphorology criticizes analytic philosophy and logical positivism; he underlines the limits of the philosophy of Karl Popper (and also of Alfred Ayer), stressing that metaphors "do not admit of verification, and that the alternative already decided in them […] theoretically undecidable. Metaphors are unable to satisfy the requirement that truth, by definition, be the result of a methodologically secure procedure of verification." (2010, pp. 13–14)[44]

Moving forward to the late 20th century and the first decades of the 21st century, Blumenberg's research on metaphors and their historical frameworks can be considered as opposed to the metaphor theory developed by George Lakoff and Mark Johnson in the 1980s, and developed further in the context of mathematics by Lakoff and Rafael E. Núñez (2000) in their book *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being*. According to Lakoff and Núñez's approach (which is informed mainly by cognitive science), human conceptual thinking, and hence also mathematics, are framed by metaphors drawn from our embodied experience, whereas metaphors are defined as a universal cognitive mechanism and as an "inference-preserving cross-domain mapping" (Lakoff & Núñez, 2000, p. 6). While this approach has considerable explanatory power, its very strength can also be a weakness, namely the temptation to reduce all metaphors to the body or to 'embodiment.' This is clearly expressed in the preface of the book: "Abstract human ideas

[43] Though, as noted, leaving it behind is practically impossible.

[44] Blumenberg mentions Ayer explicitly in (2010, p. 14). However, this is not to imply that analytical philosophers did not employ themselves metaphors: for example, Frege, in his *Begriffsschrift* from 1879, notes that his concept writing is like a "microscope" (Frege, 1879, p. v). On Frege's usage of the tree metaphor, see (Schlimm, 2016).

make use of precisely formulatable cognitive mechanisms such as conceptual metaphors that import modes of reasoning from sensory-motor experience. It is *always* an empirical question just what human ideas are like, mathematical or not." (ibid., p. xii) While Lakoff and Núñez stress here that they are discussing "conceptual metaphors," by emphasizing that the question concerning the nature of "human ideas" (and hence mathematical concepts) is "*always* an empirical" one, they note that the sensory-motor framework is the only framework they take into consideration. Admittedly, the reference of numerous metaphors to corporal, physical, or embodied (in the world) experience is an aspect which is not taken into account by Blumenberg.[45] However, despite Johnson and Lakoff's metaphor theory and despite Lakoff and Núñez's contribution to the understanding of mathematical metaphors and concepts, one of the main critiques regarding Lakoff and Núñez's work concerns their ignorance not only of the history of mathematics in particular and the history of ideas in general, but also of the particular way metaphors emerge in mathematics– that is, the way metaphors only acquire their meaning in the specific historical and social context within which and for which they were employed as a metaphor.[46] The critique of Lassègue (2003, pp. 228-9) is essential to recall: by stressing the universal character of cognitive mechanisms, Lakoff and Núñez carry out a flattening of the various meanings of mathematical statements by always presupposing the univocity of such meaning. In addition, viewing the metaphor as a correspondence (a "mapping") between two domains, this conception only considers the metaphor as a connection between two already existing domains, domains which moreover must be assumed to exist innately– but such correspondence of domains does not account for the appearance of new concepts.

While Lakoff and Núñez's approach to mathematical metaphors is highly influential, they do not claim that mathematical concepts may have a dynamic or indeterminate character, a claim which may be deduced from Blumenberg's position. However, such a claim, it should be emphasized, is not a revolutionary one. To give one example, as already noted, the debates on the status of imaginary numbers– that is, whether they should only be considered as a fictitious auxiliary machinery, mere symbols, or whether they should be accepted as numbers, and not understood as 'torture'– are well researched. Moreover, that mathematical concepts are dynamic and subject to change, and hence already have a kernel of indeterminacy, has been noted not only by Lakatos (1976) but also by the later Wittgenstein and by Waismann. As Pérez-Escobar (2022, p. 171) shows, "Wittgenstein's late philosophy of mathematics moves the emphasis away from the foundations of mathematics […] and closer to mathematical practices and ways of living", when Wittgenstein discusses 'bending' of mathematical rules, which may very well point to their indeterminacy (cf. also: Scheppers, 2023). Interestingly enough, Blumenberg knew Wittgenstein's works on such subjects, as he discusses in (Blumenberg, 1989 p. 757ff) Wittgenstein's reflec-

[45] In this sense, one may very well argue that also Lakoff and Núñez reject the Cartesian project, though they do so for another reason: they do not wish to cancel, doubt, or reduce the bodily experience.

[46] For such critiques see for example: (Gold, 2001; Langendoen, 2002; Davis, 2005). Note also that it seems that Lakoff and Núñez claim, somewhat ahistorically, that there is a *finite* number of metaphors on which modern mathematics is based: "The modern notion of mathematical rigor and the Foundations of Mathematics movement both rest on a *sizable* collection of crucial conceptual metaphors." (Lakoff & Núñez, 2000, p. xiv; my italics).

tions on 'mathematical' flies and the discovery of surprising solutions to mathematical problems.[47] Moreover, recent works, such as those by Tanswell (2018) or Zayton (2022), on whether mathematical concepts can be 'open textured,' a notion taken from Waismann (1968),[48] or whether they can display semantic indeterminacy or be essentially vague,[49] indicate that this debate certainly should not be philosophically limited to specific case studies such as imaginary numbers or the foundations of set theory, but should encapsulate the entirety of mathematics. In this sense, one may argue that when committing to the thesis that a mathematical concept can be open textured,[50] one vouches also for the possible existence of a forefield of that concept or, in Waismann's words, for an "essential incompleteness" of the concept (Waismann, 1968, p. 121).

With these philosophical discussions on the one hand, and with Blumenberg's own remarks on mathematics on the other hand, the possibility of bringing mathematics and nonconceptuality together may be detected. Hence, following this conception, one can consider mathematics as an elastic, dynamic space that is transformed again and again with every encounter with 'inexistent' objects,[51] which is to say with objects for which a concept is yet to be coined– and this encounter can be unfolded with glosses, anecdotes, and fables, or metaphorically, as underlined above, and for example, with nautical and oceanic metaphors.

## 2 Nautical metaphors, oceanic metaphors, and histories of mathematics

Before examining these encounters and their accompanying metaphors in more detail, one question should be answered in advance concerning the focus of this paper: why concentrate on nautical and oceanic metaphors, appearing, for example, in the form of the sea and shipwreck when examining the formation of mathematical concepts and theories?

[47] A detailed discussion on Blumenberg's reading of Wittgenstein is outside the scope of this paper; see however: Friedman (forthcoming).

[48] Tanswell presents two ways to characterize the open texture of concepts: first, "that the concepts we deploy and use to understand the world around us are not delimited in all possible ways" (Tanswell, 2018, p. 3) and, second, a "concept or term displays open texture iff there are cases for which a competent, rational agent may acceptably assert either that the concept applies or that it disapplies" (ibid., p. 5).

[49] See the discussion on vagueness in: (Shapiro, 2006).

[50] See: (Tanswell, 2018, p. 29): "Mathematical concepts are open textured, […] with the upshot that the eternal and definite fixity of mathematical concepts is only a limited phenomenon and often illusory." Moreover, note that even new concepts, coined to solve the vagueness of the former system, may also be 'open textured'.

[51] The notion of 'inexistent' mathematical objects is taken from the work of the French philosopher Alain Badiou. When, in his book *Theory of the Subject*, Badiou discusses mathematical objects which were considered impossible or inexistent, such as the square root of 2 or the square root of (-1) (Badiou, 2009, pp. 201–207), he points out, among other things, the implications of considering these objects as inexistent: "Pythagorean mathematics posits that the countable or denumerable is made up of whole numbers, or of relations among whole numbers. […] [T]he whole field of Pythagorean mathematics is prescribed by this latent decision […]. You thus obtain a constitutive remainder of the field in which the mathematical knowledge of the era operates. This remainder is the undenumerable, posited as inexistent […]." (Ibid., p. 202).

It is clear that other metaphors have been used to describe how mathematics develops– the organic metaphor and the architectural metaphor are well-known examples; see: (Schlimm, 2016; Friedman, 2020, 2022).[52] Blumenberg himself notes the metaphor of clothing, covering, and uncovering with respect to mathematics, as was mentioned above. But the nautical and oceanic metaphors play a special role in Blumenberg's examination of the history of metaphors, as can be seen especially in his 1979 book *Shipwreck with Spectator*; reflections on the components of these metaphors are also to be found in his posthumous publication *Quellen, Ströme, Eisberge* (2012), which contains essays written in 1980 on the various metaphors of fluidity and of water.[53] To emphasize: while nautical metaphors point more to the relation of man to the sea, having to do with navigation, boats and their construction, navies, and steering in the sea, oceanic (or marine) metaphors deal with the sea itself, having to do with water and currents, and not necessarily with attempts at controlling them. Thus, *Shipwreck with Spectator* can be regarded as unfolding mainly an array of nautical metaphors, while *Quellen, Ströme, Eisberge* unfolds mainly an array of oceanic or marine metaphors.[54] But this separation is not a strict one, as components of one array can certainly appear in another. In *Shipwreck with Spectator*, the metaphor of the sea and the shipwreck functions for Blumenberg, as the subtitle of this book suggests, as a "paradigm of a metaphor for existence." Blumenberg notes: "Humans live their lives and build their institutions on dry land. Nevertheless, they seek to grasp the movement of their existence above all through a metaphorics of the perilous sea voyage." (1997, p. 7) In this sense, one can consider that Blumenberg viewed transgressions and crises not pejoratively or as something to be appeased, but as an essential part of human existence; this becomes apparent via the nautical metaphors used also in mathematical discourse.

To explicate: Several components of the shipwreck metaphor– the meeting of sea and shore, the stormy sea, the unpredictability of the waves, or the ever-shifting shoreline– refer to an encounter with a frontier or crisis, and in this they may correspond to the way the transformation of mathematics has been perceived, either by the mathematicians contributing to it or by the philosophers and historians describing it. This is not to suggest that the changes undergone by mathematics could not have

[52] Other historians and philosophers have also noticed the use of metaphors in mathematical discourse. See for example: (Mehrtens, 1990, p. 509): "They [metaphors] create and structure old and new perspectives; they fix or move positions; they confirm or alter identities of mathematicians, of fields of work and of over-arching social orders." In his discussion, Mehrtens follows the theory of Max Black presented in his *Models and Metaphors*, published in 1962. Mehrtens focuses on metaphors such as 'intuition' and 'system,' though one can certainly argue that those terms were not necessarily (only) metaphors. To give another example, Hesseling (2003, pp. 302–311) deals with the metaphors of 'crisis' and 'revolution' in the mathematical discourse of the early decades of the 20th century.

[53] Moreover, in these essays Blumenberg stresses that "it is not the function of metaphor as such that is to be eliminated and replaced by more appropriate conceptual performances; rather, confusions are to be corrected on the metaphorical level itself" ["[…] nicht die Funktion der Metapher als solche ausgeschaltet und durch passendere begriffliche Leistungen ersetzt werden soll, sondern auf dem metaphorischen Niveau selbst Verwechslungen berichtigt werden."] Here, Blumenberg emphasizes that the coinage of terminology and (re)metaphorization are two poles between which metaphors can move back and forth (Blumenberg, 2012, p. 167).

[54] (Timm, 1999) discusses these metaphors in the larger context of Blumenberg's works.

been approached with other metaphors, as we saw above with Descartes' architectural metaphor. During the 20th century, mathematics and the changes mathematics underwent were also metaphorized, for example, with the story of the Tower of Babel (see: Friedman, 2021).[55] As will be shown, however, the numerous nautical and oceanic metaphors that frame the history of mathematics suggest that these are background metaphors to account for such frontier experiences. Moreover, with the sea, a physical place is indicated which has always set limits to the human capacity for sovereignty and action– and it is precisely to the transgression of these limits which some mathematicians refer. That is, the usage of this metaphor to account for and unfold a certain image of mathematics implies an encounter with not yet well-defined mathematical concepts. The following two subsection will elaborate on those metaphors: Sect. 2.1 deals with Blumenberg's reflections on mathematics in *Shipwreck with Spectator*, whereas Sect. 2.2 examines other nautical metaphors which account for mathematical crises or concepts.

### 2.1 Blumenberg on geometrical salvation and mathematical shipwrecks

Blumenberg's approach to geometry and to mathematical concepts, as presented in *Shipwreck with Spectator*, consists of two somewhat opposed positions. To begin with the first position, the second chapter of this book deals with geometry and, more precisely, with geometrical figures, through a discussion of the philosopher Aristippus (Blumenberg, 1997, p. 12). Blumenberg cites the following passage in Book VI of Vitruvius's *De architectura*: "It is related of the Socratic philosopher Aristippus that, being shipwrecked and cast ashore on the coast of the Rhodians, he observed geometrical figures drawn thereon, and cried out to his companions: 'Let us be of good cheer, for I see the traces of man.'" (Vitruvius, 1914, p. 167) Blumenberg notes that, "even in the hopeless situation of being shipwrecked on a foreign shore, a philosophically trained person still knows what to do, when he recognizes civilized reason in geometrical diagrams." (1997, p. 12) These geometrical drawings are associated by Aristippus with a sign of human habitation, civilization, and safety. Geometry itself becomes a safe land. This story is recounted again and again over the centuries as a metaphor of safety, as can be seen in Blumenberg's own discussion on the 16th century astronomer Joachim Rheticus and the 18th century mathematician Abraham Gotthelf Kästner: both turn the shore in Vitruvius's historical description into an image of knowledge (ibid., pp. 13–14). As Blumenberg is aware, however, it would be wrong to automatically associate the shore with an image of safety, as this would imply that the shore metaphor does not change and does not have a history.

What arises from Blumenberg's reading of certain chapters of the history of this metaphor is an image of the human as seeking guidance and protection, whereas the sea is depicted as a place of constant change, or even, in certain epochs, as "the place where evil appears" (ibid., p. 8). But Blumenberg also points out that this metaphor is a story of a search for safe foundations. While one of the characteristics of this metaphor is the demonization of the sea as a lawless place without order, another is its designation of a limit to human activity. This leads me to Blumenberg's sec-

[55] On the metaphor of the Tower of Babel in German thought, see: (Purdy, 2011).

ond approach regarding mathematics and especially mathematical concepts, which is also presented in *Shipwreck with Spectator*, and which underlines how even ‘well-defined’ mathematical concepts themselves may have a kernel of indeterminacy, and in this sense such a stable conceptual system may not be so stable after all. The last chapter of *Shipwreck with Spectator* deals with various critiques expressed towards logical positivism and whether “the foundation of scientific language is possible” (ibid., p. 76). Blumenberg notes that the critiques of Otto Neurath, and afterwards of Paul Lorenzen, explicitly use ship metaphors: Neurath (1932, p. 206) affirms that “[w]e are like sailors who have to rebuild their ship on the high seas” with no place to dock, hence, so Blumenberg, if conceptual “imprecision is diminished in one place, it may reappear in a stronger form elsewhere.” (Blumenberg, 1997, p. 77) Already this remark underlines the indeterminacy of any conceptual (and hence mathematical) system. Blumenberg then notes that Lorenzen in 1965 takes an “extreme variant of the [ship] metaphor”, by supplementing it with a prehistory– that our ancestors had an initial point of beginning.[56] Blumenberg suggests at the end of the book that even a scientific language that presents itself as the “philosophical zero point”– and an example for that would be the mathematical set theory, which was considered as the foundation of mathematics– cannot be the promised initial point: it “contains material other than what has already been used. Where can it come from […] Perhaps from earlier shipwrecks?” (ibid., p. 79) Hence, not only Blumenberg doubts the safety that such “philosophical zero point” would gaurantee, but he also doubts the safety that any conceptual construction may ever provide. Following this double approach, I aim in the following to show that several other episodes in the history of mathematics display as well ambivalent and ambiguous positions toward the sea and shipwreck metaphor.

## 2.2 Nautical metaphors of geometrical disasters?

To begin with a first example, and to see how the metaphor of the sea framed encounters with ‘impossible’ mathematical objects, one can recall the following story. In a legend told by Pappus of Alexandria, Hippasus, who published his discovery of irrationality (or incommensurability), was drowned in the sea precisely because of this discovery:

> [in] the sect […] of Pythagoras […] a saying became current in it, namely, that he who first disclosed the knowledge of surds or irrationals and spread it abroad among the common herd, perished by drowning: […] it is better to conceal (or veil) every surd, or irrational, or inconceivable in the universe, […] [since] the soul, which by error or heedlessness discovers or reveals anything of this nature which is in it or in this world, wanders [thereafter] hither and thither on the sea of non-identity […] immersed in the stream of the coming-to-be and the passing away where there is no standard of measurement. (Pappus, 1930, p. 64).

[56] That is, that our ancestors were “using scraps of wood floating around-they somehow initially put together a raft” (Blumenberg, 1997, p. 77).

This legend should not be read as a historical account and it should certainly not imply that the Pythagoreans rejected irrational numbers or incommensurable quantities.[57] Following this legend, Pappus employs several nautical and oceanic expressions and metaphors: the drowning of man, the "sea of non-identity," the "stream of the coming-to-be." If one considers these expressions not as mere decorative elements, then Hippasus drowned because he made a controversial claim regarding the body of the mathematical configuration during this epoch: he showed that what exists in geometry cannot exist in the alleged Pythagorean arithmetic– for example, the irrational diagonal of a square. It is essential to emphasize that while one may interpret the above quotation as referring to the philosophical tradition of Heraclitus,[58] and thus as advocating an ever-present change, this story appears also in more recent accounts.[59] That is, the fable of drowning in the sea is employed again and again– without even attempting to offer another metaphorical framework– in order to frame a meeting with an object which allegedly had no place in the Pythagorean system.

Another example of how mathematical problems are accounted for with nautical metaphors is the problem of the doubling of the cube. In contrast to Pappus's description, the following parable, told by Plutarch, presents a different type of description, one in which the problem is posed by an oracle to plague-stricken Delians. In Plutarch's account, Simmias of Thebes, who was traveling with Plato, recounts a meeting with the inhabitants of the island of Delos on the coast of Caria, when Simmias and Plato were returning home from Egypt:

> [...] as we were sailing from Egypt, about [the shores of][60] Caria some Delians met us, who desired Plato [...] to solve [a problem which] an odd oracle lately delivered [...] [namely, the doubling of the cube]. [...] They, not comprehending the meaning of the words, after many ridiculous endeavors [...] made application to Plato to clear the difficulty. He [...] said that the God was merry upon the Greeks, who despised learning; [...] it required skill to find the true proportion by which alone a body of a cubic figure can be doubled, all its dimensions being equally increased. (Plutarch, 1878, vol. 2, p. 385)

It is easy to see that the problem of the doubling of the cube is equivalent to constructing two segments whose ratio is $\sqrt[3]{2}$, or just constructing a segment of length $\sqrt[3]{2}$ (given a segment whose length is 1). It is also well known that, if one constructs segments only with straightedge and compass, then constructing a segment whose length is $\sqrt[3]{2}$ is impossible. The impossibility of such a construction was nevertheless only proven during the first half of the 19th century (Wantzel, 1837).[61]

[57] See for example the work of Fowler (1999), who advocates the position that the Pythagoreans very much dealt with such quantities.

[58] Recall that the story of the death of Hippasus originates from Iamblichus.

[59] To give only two examples, see: (von Fritz, 1945, p. 244; Richeson, 2019, p. 62). The story is recounted in numerous accounts of Hippasus's discovery of incommensurability.

[60] The German translation of Plutarch's text states explicitly that the meeting took place on the shores of Caria; see: (Plutarch, 1797, p. 397).

[61] However, this result was overlooked for decades. See: (Lützen, 2009).

If we return to Plutarch's account, then one can find there almost all the ingredients of the shipwreck metaphor (voyage across the sea, island, danger in the form of a plague, coast), but they serve a different function than in Pappus's parable. While one may argue that the elements of the parable do not seem to directly relate to the mathematical problem itself, the problem is nevertheless framed via a metaphorical framework of danger and with nautical metaphors. In this sense, and following Blumenberg, who notes that metaphors do not have to be explicitly expressed as such, I claim that the result of not solving the mathematical problem is metaphorized by a plague which can be only stopped by a passenger of a nautical voyage. In this case, no one drowns in the sea, but it is clear that the story encapsulates a meeting with an object (a doubled cube) whose constructability– at least with straightedge and compass– is not yet known. Moreover, the metaphors employed also unfold a certain image of mathematics as a threatening, ever-changing realm; this image, in its turn, points to the abovementioned meeting. Here, however, one should stress that the construction of a segment of length $^3\sqrt{2}$ was a problem which was solved numerous times in antiquity, though not with a straightedge and a compass.[62] Hence, while the problem was not really unsolvable, despite what Plutarch states above,[63] the encounter with a certain impossibility is nevertheless framed via nautical metaphors.

The above mathematical episodes are not the only ones in which the metaphor of the sea and the shore plays an essential role; there are other episodes in the history of mathematics where the metaphor of nautical voyages and their possible hazardous consequences come into expression. In his book *Geometrical Landscapes*, Amir Alexander notes that "some early modern mathematicians adopted the imagery of geographical discovery and made it their own." (2002, p. 2) Alexander deals with several mathematicians, mainly from the 16th and 17th centuries, showing that "the imagery of a mathematics of adventure and exploration went hand in hand with the emergence of infinitesimal methods." (Ibid., p. 200) While some of the examples discussed by Alexander deal mainly with the exploration of land, other examples underline the nautical imagery used by a number of mathematicians. These include Bonaventura Cavalieri, who at the end of the 1630s described Galileo Galilei as the one who dared to "steer the immensity of the sea, and plunge into the ocean […], [and who] managed easily to navigate the immense ocean of indivisibles […] and a thousand other hard and distant things which could shipwreck anyone" (ibid., p. 184); Evangelista Torricelli, who calls Cavalieri's method of indivisibles an "immense ocean," noting that he himself prefers to stay near to the safe shore (ibid., p. 179); Thomas Harriot, for whom the mathematical continuum resembles "the coasts of undiscovered land" (ibid., p. 168); or John Davies, who in 1614 praises the mathematician Edward Wright, noting that as a geometer he resembles a navigator on the high seas, and hence equating the practices of the one with the practices of the other (ibid., p. 199).

As is clear from the above list of examples from Alexander's book, the nautical voyage in the sea of mathematics is much more than an imagery of adventure

[62] For a survey of the various solutions to this problem during antiquity, see: (Heath, 1921, pp. 244–270).

[63] Compare also Plato's critique regarding the various devices and techniques (other than compass and straightedge) used to double the cube in: (Plutarch, 1961, vol. 9, pp. 121–123).

and exploration; rather, it points toward the conception of the nonconceptualized as potentially a “place where evil appears,” to quote Blumenberg; the encounter with the nonconceptualized within mathematics itself– which occurs in opposition to the ‘ideal’ of mathematics– cannot be fully integrated or transferred into a concept. But as we saw with Blumenberg’s own position, also ‘well-defined’ concepts may contain ‘holes’ and their system may need to be reconstructed. This encounter with indeterminacy is hence at times considered dangerous, and at times metaphorized.

## 3 On Grothendieck’s mathematical and oceanic practices

In the examples discussed so far, one may note that each such encounter with the sea is unique, and thus each of the components of the metaphor illustrating it functions differently. To the dangerous, perplexing, or diverting images that serve to convey the mathematical encounter with what is not yet conceptualized, another metaphorical horizon can be added. Above, it was noted that Plutarch’s nautical metaphor implicitly frames the discussion on the ‘correct’ or ‘right’ practices to be employed when solving geometrical problems (i.e. with straightedge and compass). Hence the meeting on the shore framed the geometer’s uncertainty with respect to the instruments to be used. Nevertheless, the metaphors framing the discussion on the practices were more implicit, and thus remained in the background. How such oceanic-nautical metaphors may account explicitly for how one should practice mathematics is presented in the work of Alexandre Grothendieck (1928–2014), one of the pioneers of modern algebraic geometry.[64] To stress: these metaphors are not just attempts to present a claim *about* mathematical practices (i.e. as being basically meta-mathematical descriptions) but also– at least according to Grothendieck– a description of those practices, that is, how one can practice mathematics and how vagueness and ambiguity may arise in mathematics itself.

In his autobiographical manuscript *Récoltes et semailles*, Grothendieck describes two methodological approaches to mathematical problems. The first is to approach a problem as if it were a goal, using all tools to solve it, calling this method the one “of the chisel and the hammer” (1985–1987, p. 552). The second approach takes us to the image of the sea– or, more precisely, to what Grothendieck describes as the “rising sea” [“la mer qui monte”]. In Grothendieck’s words, “the sea advances imperceptibly and without noise, nothing seems to break, nothing moves, the water is so far away that it can hardly be heard […]. However, it ends up surrounding the reluctant substance, which gradually becomes a peninsula, then an island, then an islet, which ends up being submerged in its turn, as if it had finally dissolved in the ocean stretching as far as the eye can see […].”[65] (Ibid.) For Grothendieck, one has to let a problem– which he metaphorizes as a nut– be submerged and dissolved by a

[64] For an overview of Grothendieck’s mathematical work, see: (Zalamea, 2012, pp. 133–172; Zalamea, 2019).

[65] “La mer s’avance insensiblement et sans bruit, rien ne semble se casser rien ne bouge l’eau est si loin on l’entend à peine […]. Pourtant elle finit par entourer la substance rétive, celle-ci peu à peu devient une presqu’île, puis une île, puis un îlot, qui finit par être submergé à son tour, comme s’il s’était finalement dissous dans l’océan s’étendant à perte de vue […].”

vast theory, one that goes well beyond the results originally to be established (ibid., p. 555). In this way, not only does the nut become so soft that it opens by itself, but this approach also reshapes the entire mathematical landscape– one discovers "'new' worlds" (ibid., p. 554). This is not to suggest that Grothendieck considers that every mathematician employs (or should employ) this method. Indeed, he stresses that the mathematician Jean-Pierre Serre mainly uses the first method: "Serre's mathematical work, his approach to mathematics, […] to a difficulty would rather be that of the chisel and the hammer, very rarely that of the sea which rises and submerges, or that of the water which soaks and dissolves."[66] (Ibid., p. 558) Moreover, Grothendieck explicitly notes that he uses this method to solve several mathematical problems, emphasizing that this description is not merely metaphorical but also very much practical. Among these problems he lists the Hirzebruch–Riemann–Roch theorem for any characteristic or the structure of the algebraic fundamental group of an algebraic curve over an algebraic closed field of any characteristic (ibid., p. 554, footnote ***).[67]

What is unique in the use of this metaphor is that the sea, and not the shore, lets a solution arise, a solution which emerges almost unexpectedly, as if by itself. The method of submersion, absorption, and dissolution– in fact, of letting the problem drown in the sea of mathematics– is presented as a legitimate way to practice mathematics. One may very well claim that the metaphors presented here are oceanic, but they are also implicitly nautical, as they deal with how the mathematician should steer his way in the sea of mathematics. While Grothendieck's metaphor of the rising sea deals with the encounter between unsolved problems and a possibly already existing sea of theory, in Grothendieck's work there is yet another way in which the sea metaphor is discussed: in relation to the wave. In a subchapter titled "The arrow and the wave" ["La flèche et la vague"], Grothendieck again describes the two methods of mathematical research he favors. The first, once again, is characterized by solving a specific problem, reaching a goal, described as "this impatience to have reached the end of a task, this impulse toward such and such a 'point' […], this attraction of the 'goal' on me which throws me forward, like an arrow rushing toward its target."[68] (Ibid, p. 594) But the mathematical work itself, of practically dealing with mathematical problems, can also be described otherwise:

> There is no longer an arrow, rushing toward a target, but a wave that stretches out far and wide, moving who knows where, wherever the moving force that animates it takes it– a wave followed by another wave, followed by yet another […]. In each moment there is a progression, one cannot say toward what, there

[66] "[…] le travail mathématique de Serre, son approche de la mathématique, […] ][s]on approche d'une difficulté serait plutôt celle du burin et du marteau, bien rarement celle de la mer qui monte et submerge, ou celle de l'eau qui imbibe et dissout."

[67] On Grothendieck's treatment of the Weil conjectures, in which he employs the "rising sea" approach, see: (McLarty, 2007).

[68] "Cette impatience d'être arrivé au bout d'une tâche, cet élan vers tel 'point' […], cette attirance du 'but' sur moi qui me projette en avant, comme une flèche fonçant sur sa cible."

> is a 'work' accomplished in a movement which ignores the effort– and there is no goal. The very idea of a 'goal' here seems strangely absurd […].[69] (Ibid.)

Here, the image of the waves complements and expands the metaphor of the rising sea. What is unique in the wave metaphor, however, is the explicit emphasis on the countless meetings between the sea and shore, that is, the changes undergone by mathematical practice itself. For Grothendieck, just as the sea always meets the shore in new and unique ways, the changes occurring within mathematics and affecting its boundaries also takes place in a unique way. If one takes into account both metaphors as comprising one image of mathematical practice, then on the shore of the sinking island of mathematical problems, which borders in a non-predetermined way with the sea of mathematics, the waves of the rising sea are crashing one after the other. According to Grothendieck's philosophy of mathematical practice, not only is there no goal, as an attempt to construct a stable mathematical theory, which may be metaphorized, for example, with a 'solid' nautical metaphor (e.g. constructing a ship or controlling the sea) or with an architectonical metaphor (e.g. with a stable building), but also the result of practicing mathematics– described as the repeated crashing of the waves– is never known in advance, implying an unusual interlacement of nautical and oceanic metaphors. This, if we take notice of Blumenberg's views, encapsulates his statement, that the goal of reaching a well-defined concept while working in its forefield is doomed to fail. This is reflected in both Blumenberg's and Grothendieck's conception of conceptual or mathematical practice; both note that the "goal" of concept formation is not what one should strive for: while Blumenberg explicitly stresses that a 'successful' definition of a concept cannot be obtained, in Grothendieck's work, in which concepts are dynamic and may change, concept formation can be viewed as a byproduct. In this sense, even a 'successful' concept formation may result in concepts which are open-textured or essentially incomplete. Hence, with Grothendieck, one observes another use of the nautical-oceanic metaphor in relation to mathematics: there is no longer a possibly demonic sea, nor is there an island which guarantees certainty; there is rather an immense sea into which the islands of the problems should– or must– sink.

## 4 Conclusion: towards metaphorology of mathematics

If one views Grothendieck's philosophy of mathematical practices through the lens of Blumenberg's metaphorology and nonconceptuality, then one may conclude that the nautical-oceanic metaphors and their components are mostly suitable to account for the change and unpredictability of mathematical practices and of mathematics itself. This is seen not only with Grothendieck's own examples, but also with the changing practices and concepts employed in the other examples presented above:

[69] "Il n'y a plus de flèche, se hâtant vers une cible, mais une vague qui s'étend très loin et qui s'avance on ne sait où, là où la force mouvante qui l'anime la porte– une vague suivie par une autre vague, suivie par une autre encore […]. En chaque moment il y a une progression, on ne saurait dire vers quoi, il y a un 'travail' accompli dans une mouvance qui ignore l'effort– et il n'y a pas de but. L'idée même d'un 'but' ici paraît étrangement saugrenue […]".

the "emergence of infinitesimal methods," (Alexander, 2002, p. 200) the uncertainty with respect to the use of geometrical instruments (besides compass and straightedge), the encounter with incommensurability, or the changing conceptual framework itself. Hence, nautical and oceanic metaphors– as well as other metaphors– are not a decoration, a byproduct, or a redundant appendix to the (historical) account, and the arguments presented here give reason to believe that metaphors play a fundamental role also in other examples and domains. This set of metaphors in its turn points toward an encounter with a mathematical nonconceptuality, and calls for a consideration of Blumenberg's metaphorology with the latest discussions on open texture and the philosophy of mathematical practices.

To summarize: As seen in this paper, Blumenberg considered metaphors and metaphorology as a way to account for the emergence of concepts, later expanding his reflections to the field of nonconceptuality, hence reflecting also how mathematical concepts themselves may be elastic or indeterminable. While he did not explicitly deal with an extensive historical research of mathematical objects, concepts, or practices, I would like to suggest that his methodology can assist in understanding how these are conceptualized and used. Concepts in general, and hence also mathematical concepts, always have a forefield of what has not yet been or cannot be conceptualized, and this forefield is still a part of the concept, even when the latter is considered as well-defined. The (always occurring) encounter with this forefield is accounted for with metaphors. Hence, nautical and oceanic metaphors (and obviously not only these metaphors) should be taken into consideration within the philosophical and historical investigation, as they point toward an encounter with indeterminacy, and with nonconceptuality; this encounter, as both Grothendieck and Blumenberg stress, does not and cannot end or reach a definite conclusion. These metaphors therefore expand Blumenberg's discussion toward a consideration also of scientific and mathematical practices– as elaborated in the case of Grothendieck.

These considerations also point to two open questions, with which I would like to conclude this paper. First, how are the various metaphors interwoven with one another? The most explicit example examined here is the interlacement of nautical and oceanic metaphors. As we have seen, in order to account for certain images of mathematics, other images besides nautical ones are employed, including the clothing metaphors and the architectonic or organic images mentioned above. Taking into account the history of these metaphors, how are these histories interlaced with the history of the nautical metaphor, when discussing mathematical concepts, practices, and their history? And how is this interlacement reflected in the philosophy of mathematical practices?

While the first question aims at a broader inspection of other metaphors, the second returns to the sea and to the nautical and oceanic metaphors. How is the history of these metaphors and their components to be seen in relation to the history of the ocean and its exploration on the one hand, and to possible attempts at its mathematization[70] on the other hand? Against this background, Mentz (2015, 2020) proposes

[70] A variety of subjects may be included under this mathematization, ranging from Laplace's tidal equations in 1776 to the development of the Navier–Stokes equations during the first half of the 19th century to the mathematics and calculations behind the construction of ships and navigation at sea.

a retelling of human history and culture from an oceanic point of view. The task of examining nautical metaphors historically does not consist only in bringing Blumenberg into the fold, but also in more historical approaches stemming from the *Blue Humanities*. As Mentz and Blumenberg remind us, shipwreck, crashing waves, and unknown shores are not just metaphors but have their own materiality, with their own past, present, and future. Accordingly, this paper aims to point not only toward a possible future historical-metaphorical research on the nautical and oceanic metaphors found in mathematics, but also toward the philosophical implications of this research concerning how mathematics develops and is practiced.

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