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ORIGINAL RESEARCH

Paul Cohen's philosophy of mathematics and its reflection in his mathematical practice

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Abstract
This paper studies Paul Cohen's philosophy of mathematics and mathematical practice as expressed in his writing on set-theoretic consistency proofs using his method of forcing. Since Cohen did not consider himself a philosopher and was somewhat reluctant about philosophy, the analysis uses semiotic and literary textual methodologies rather than mainstream philosophical ones. Specifically, I follow some ideas of Lévi-Strauss's structural semiotics and some literary narratological methodologies. I show how Cohen's reflections and rhetoric attempt to bridge what he experiences as an uncomfortable tension between reality and the formal by means of his notion of intuition.

Keywords Structural semiotics · Lévi-Strauss · Narratology · Set theory · Consistency proofs · Forcing · Paul Cohen

1 Introduction

In this paper, I will discuss Paul Cohen's philosophy of mathematics and his mathematical practice based on his discussions of independence results in set theory (primarily the independence of the Continuum Hypothesis from the standard axioms of set theory, Cohen, 1963, 1964, 1965, 1966, 1971, 2002, 2005, 2011).[1] I will organize the discussion around two perspectives. The first perspective is the philosophy of mathematical practice, namely, a philosophical analysis that focuses on how mathematics is done, rather than rationally reconstructed. By "how mathematics is done" I mean how mathematicians think, experiment, discuss and write, all while engaging with real or imagined interlocutors (for various earlier and later formulations of the philosophy of mathematical practice see Buldt et al., 2008; Ferreirós, 2015; Kerkhove & Bendegem,

[1] From this point on references to Cohen's papers will appear without the label "Cohen".

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2007; Mancosu, 2011; Wagner, 2017). The second point of view is structural semiotics. I will take my inspiration from some of the ideas of Lévi-Strauss about how people think with signs.

Discussions of Cohen's philosophy of mathematics are not common in the literature. He did write several texts that explicitly engage with philosophical issues (1966, 1971, 2002, 2005, 2011), but for him philosophizing was "a rather strange and uncomfortable position". He was "struck at once by the futility of trying to state opinions which will be universally or even widely accepted, and also by the inconsistencies and difficulties of [his] own point of view" (1971, p. 9). Elsewhere, he wrote more bluntly: "I had never been attracted to philosophy" (2011, p. 441), whose nuances he "never found interesting" (2002, p. 1083).

When Cohen looks at philosophy from the point of view of its value for mathematics, rather than his own personal attitude, he is no less damning. Concerning syntactical, proof-theoretical approaches to mathematics, he wrote that they "seemed to be too close to philosophical discussions, which I felt would not ultimately be fruitful" (2011, p. 437). He also believed that "the more philosophical orientation of logicians of the time [early twentieth century], even the great Hilbert, distorted their view of the field and its results" (2005, p. 2411). Even when he starts with stating that "it would be truly sad if this wave of success [of independence proofs] should succeed in totally dismissing all philosophical concern over CH [the Continuum Hypothesis] and similar questions as inconsequential", he concludes with a condemnation: "of course, good mathematics is beautiful, while most philosophical discussion is barren and certainly not beautiful" (1971, p. 12).

Given such reluctance, it is not surprising to find little engagement with Cohen's views in contemporary professional philosophy. Two exceptions are the last section of Giaquinto's (1983), which points out the tensions in Cohen's philosophical view, and Leven (2019), who does not so much deal with Cohen's views directly, but emphasizes, as I will do below, the importance of notions of "intuition" in the reception of his results.

Given Cohen's attitude to philosophy, I believe it would be unfair and counterproductive to try to reconstruct his views with the tools of the mainstream philosophical discourse that he rejected. Instead, I followed a more anthropological approach to human thinking, namely that of Lévi-Strauss's structuralist semiotics. For Lévi-Strauss, all human thinking, from the myths of those societies marked by his contemporary colonial discourse as "primitive" to the achievements of modern science, is deployed within a universal structural mechanism. I do not wish to promote such a universalistic and monopolistic structuration of thought, but I often find Lévi-Strauss's approach useful. Lévi-Strauss readily accepts that human thought is full of unstable binary oppositions that are hard-pressed to find universal consistent deployment (e.g. the binary opposition life : death, considering all the biological, ecological, ethical and religious conundrums in trying to decide where life ends and death begins). Instead of a philosophical analysis or re-engineering of such terms into a system of more rigorous, consistent terms, the structuralist methodology follows how humans dynamically rearrange them in a growing system of global conceptual relations, which may reach some sort of practical saturation, rather than consistency.

The first aspect of the structuralist point of view that I follow here is the claim that signs obtain their meanings not by reference to objects, but by their relations of opposition and association with respect to other signs.[2] This means that a given term changes its meaning as it enters new relations with other terms. The fact that it is the same term, however, which enters different relations and thus acquires different, sometimes conflicting, meanings, is crucial for the system to hold together as a unified conceptual system. According to this approach, it would be wrong to segregate, for example, some of the technical and philosophical uses of Cohen's use of terms such as "real", "formal" and "true" (I also couldn't salvage much of interest by attempting such segregation). Indeed, the use of the same term keeps bringing together the supposedly segregated fields and blurs their boundaries. According to the structuralist approach, such segregation would constrain our understanding of the term, rather than improve it.

Of course, I don't expect everyone (not even myself) to simply abandon all segragatory conceptual analysis, and I don't know whether Cohen would have appreciated this approach any more than he did mainstream philosophy. However, given Cohen's embracing of his own contradictions and his rejection of mainstream philosophy, I believe that a methodological approach that considers inconsistencies necessary for productive thought, rather than nonsense that should be banished from thought, would be more fitting for an analysis of his work (for different approaches, kindred in spirit, see Byers, 2007; Grosholz, 2007; Fisch 2017, chs. 5-7).

According to Lévi-Strauss, when humans encounter irreconcilable oppositions in their symbolic understanding of the world, they react by deploying mechanisms that he associates with "savage-" or "wild thought" ("*sauvage*" in the original French, which is an ironic term, as it relates to the structural ground of all human thought, rather than the thinking of some kind of "savages"). The first mechanism is mediation: articulating symbolic positions that mediate between the irreconcilable extremes (e.g. between the irreconcilable opposition of life and death, there's hunting, which is killing to stay alive, a notion that helps, along with further intermediaries, integrate the opposite poles into a single, more continuous system).

The second, and more subtle thought mechanism, is analogy. The thinker deploys an analogy between the irreconcilable opposites and other oppositions, which have already been "tamed" and are considered more acceptable and reconcilable. For example, by forming an analogy between *life : death* and *wakefulness : sleep*, one may feel that the insurmountable transition between the poles of the former pair is perhaps not so insurmountable, as the transition between wakefulness and sleep is a daily occurrence.[3]

Section 2 of this paper will take the first step of a structuralist analysis: articulating concepts (here: the real and the formal) by their oppositions and associations, rather than their supposed referents, which Cohen, not being a philosopher, never tries to define. This will be based on a review of all appearances of some terms in Cohen's mathematical and philosophical works referenced in this paper (while these works

[2] This does not mean that meaning has nothing to do with the outside world, since this very articulation of signs and their relations is the result of an (underdetermined) encounter of the symbolic matter (in our case, language) with the world (or with its impressions on human perception)—but we needn't get into the details of this process here.

[3] Self contained and accessible examples of the application of these analytic approaches to the analysis of myth are available in Lévi-Strauss (1955, 1966).

span several decades, I do not think that their differences concerning what is of interest to us here justify setting them apart). Then, in Sect. 3, I will explore mediating positions (truth and intuition), which help reconcile the previously encountered opposites. Section 4 will explore an understanding of these oppositions by analogy—I will show how an analogy between the opposition *real : formal* and internal set theoretical oppositions helps render the former less perplexing.

Altogether, the above sections will study Cohen's conceptual structure by considering him as a "savage" or "wild" thinker, in the terminology of Lévi-Strauss. Finally, in Sect. 5, I will try to see how this conceptual structure is expressed in Cohen's mathematical practice, namely in his construction of so called "non-constructible" sets,[4] which is the cornerstone of his proofs of consistency of the continuum hypothesis and other consistency results. In this last section the approach will be more classically literary-theoretic.

In terms of prerequisites, this paper assumes knowledge of the concepts involved in introductions to model theory and formal systems. One doesn't need to be acquainted with Cohen's own work (which is somewhat different than the later Boolean algebraic approach to forcing), because the tools that I mention should be well known from other discussions in model theory. Readers without such experience will hopefully be able to "black box" such issues, except, perhaps, in the somewhat more technical Sect. 5.

2 The real and the formal

2.1 Cohen's "reality" is in tension with the formal and higher infinities, whereas Realism as a philosophy extrapolates an unfounded reality

Let's begin with the notion of the real. At the level of specific instances, Cohen sets the real in contrast to the formal. This contrast spans from small details to entire theories. For example, he says of a certain symbol that "in reality, we have treated [it] as a variable, even though we called it a constant [in terms of the relevant formal system]" (1966, 10).[5] Similarly, when performing the Löwenheim-Skolem construction of countable models of axiomatic theories, the "real" sets and "real" membership relation are not expected to be identified with those of the formally derived model, and in the context of primitive recursive functions, the "real" functions are not objects of any formal system (1996, pp. 17, 26–27; the scare-quotes around "real" are in the original text). In general, given this tension, one might be tempted to view the Zermelo-Fraenkel formal system (henceforth: ZF) as "a highly successful shell which has nothing to do with "real" sets but at best describes some type of mental process used in describing the real objects such as integers" (1966, p. 150).

But reality is not characterized only in opposition to the formal. A sense of reality (or lack thereof) is also correlated with the finiteness (or infinity) of the objects at hand. Realists "must flinch when contemplating cardinals of a sufficiently inaccessible

4 The apparent contradiction is due to the technical meaning of "non-constructible" as opposed to an informal use of the term "construction".

5 The context is the inference rule that derives $\forall x\, A(x)$ from $A(c)$, where c is subject to the stated ambiguity.

type" (1971, p. 11), "more iterations [of the power set axiom] diminish our sense of the reality of the objects involved", and "the reality of the [ever larger inaccessible] cardinals … becomes more and more dubious" (2005, p. 2416). Proper classes, as in Gödel–Bernays set theory, "are primarily fictions introduced so as to simplify the presentation of the axioms" (1965, p. 53). This is why, I suppose, some of the objects mentioned in the previous paragraph are considered "real" rather than simply real.

From a more general point of view, Cohen states that set theory is grounded in extrapolation from the finite, but the extrapolation itself (that is, applying what is extrapolated beyond the finite) "has no basis in reality" (2005, p. 2416). Ultimately, "The only reality we truly comprehend is that of our own experience" (2002, p. 1099), which is presumably mostly finite. However, Cohen's use of "reality" is not strictly confined to the finite, and not restricted by limitations imposed on cardinalities in formal systems. For example, he states that that "there is no reason to believe that in the real world [a certain construction] cannot be done countably many times", even though in ZF it can only be performed finitely many times (1966, p. 79).

These notions of reality (with or without scare quotes), associated with the finite or lower infinity and contrasted with the formal, are linked to the philosophy that Cohen marks as *Realism*, a term he prefers to *Platonism* (1971, p. 11). The realist, according to Cohen, believes that every mathematical statement is either true or false. They would claim that "all questions such as the Continuum Hypothesis are either true or false in the real world despite their independence from various axiom systems" (1971, p. 11). In fact, instead of allowing independence results to challenge the realist's faith "the greater facility for handling [independence] questions [due to Cohen's work] has given people greater motivation to believe that set theory refers to "real" mathematical objects" (1971, p. 12).

We see, therefore, that the realists' notion of reality is not consistent with Cohen's notion of reality, which is in tension with the formal and the infinite. Indeed, realists accept the existence of wildly infinite sets, whose real ground (in Cohen's sense) they cannot explain (1971, p. 11). Cohen further finds disturbing the realist tenet "that if mathematics refers to a reality then human thought should resolve all mathematical questions" (2005, p. 2417). In particular, the realist hopes to achieve this resolution by means of axioms of infinity (1971, p. 12; 2005, p. 2418), but Cohen is pessimistic in regard to this project. This is due not only to Cohen's suspicion of large cardinal axioms, but also to his belief that large cardinal axioms cannot be of much use where some difficult number theoretic questions are concerned.

It is not surprising to find Cohen describing Gödel as a realist (2011, p. 438, 441), but it may raise some eyebrows to see Hilbert characterized by this term (2005, p. 2410, 2413). Cohen explains that Hilbert's naïve Realism motivated him to defend existing (supposedly real) mathematics by means of a formal proof of consistency—a sort of tactical detour against the finitist and constructivist critiques. Skolem is also a realist, in Cohen's interpretation, but one who, unlike Gödel and Hilbert, does not believe that axioms can found or adequately describe mathematics (2002, p. 1076; 2005, p. 2417).

Realist tendencies, according to Cohen, would be preferred by most mathematicians (1971, p. 11)—and yet "most of the famous mathematicians who have expressed themselves on the question have in one form or another rejected the Realist position"

(1971, p. 13). This attraction-rejection relationship with Realism is not a “bug” in Cohen’s account, but a “feature” that we need to unfold.

2.2 Formalism is no less problematic, as mathematics is informal in spirit and must be related to reality

The antipode of Realism is, for Cohen, *Formalism*. He understands Formalism as a reduction of mathematics to a computable game of signs that suspends their meanings. The game is controlled only by its consistency (1966, pp. 3–4, 12, 1971, p. 11, 2002, p. 1077, 2005, p. 2416). It is “the one totally precise (as distinct from correct) point of view” (1971, p. 11).

One glaring failure of Formalism is obviously the impossibility of a proof of consistency as shown by Gödel. But this is not the only way in which Formalism is deficient for Cohen. Indeed, “the greatest weakness of the Formalist position is to explain why the axioms of set theory, presumably reflecting no underlying reality, are able to prove arithmetical statements unprovable by more finitistic means” (1971, p. 11, see also 1971, p. 13). Indeed, formal set theoretic axioms “were never thought of as attempts to “explain” the rules of logic, but rather to write down those rules and axioms which appeared to correspond to what the contemporary mathematicians were using” (2002, p. 1074). But while Realism explains the success of axioms by means of their (unfounded, for Cohen) extrapolation from reality, Formalism offers no explanation at all. Furthermore, the existence, for formalists, of unresolvable number theoretic propositions makes their position even less enviable for Cohen than the position of the realist (1971, p. 12).

But the problems do not end there. There is a lot in mathematical practice that is not captured by formalization. While many objects, definitions, axioms, theories and arguments have formal and informal counterparts (e.g. 1966, pp. 8, 20, 26–28, 41–42, 54, 70, 90, 147), others do not. The full scope of induction in number theory applicable to *any* predicate (1966, pp. 21–25, 52), the categoricity of the integers (1966, p. 17), the truth of Gödel’s formally undecidable proposition (1966, p. 41), the very notion of truth (1966, p. 43), some meta-statements about formal systems (1966, p. 54), the existence of a model of ZF (1966, p. 79), and the very consistency of ZF (2002, p. 1088) have informal meanings or explanations, but not formal ones.

The notion of the informal is important to note, because, while Cohen articulates the real in contrast to the formal and the infinite, the formal itself is not articulated in contrast to the real, but in contrast to the informal, which supervenes and motivates mathematics, and therefore also formal mathematics. Indeed, a formal system can only be defined in informal (though precise) mathematical language (1966, p. 3), and the “spirit of tradition of mathematics” is informal (1966, pp. 11–12). This informal spirit explains why mathematicians “may feel that the “official” exposition of set theory, i.e., all of mathematics, using formal systems and particular axiom systems, has little relevance to their work as research mathematicians” (2002, p. 1072). Cohen even goes as far as to claim that “the famous antinomies in logic never played a role in mathematics simply because they were totally alien to the type of reasoning one normally uses” (1971, p. 10). It is therefore not surprising that “In order to think productively, one must use all the intuitive and informal methods of reasoning at one's disposal" (2002, p. 1078).

Cohen further associates Formalism with a proof theoretic approach, whereas "the mathematician would much rather speak about models of an axiom system than about the set of all formulas provable from those axioms" (1971, p. 13). This is, perhaps, what renders Gödel's formalist or proof theoretic approach to his proof of consistency of CH so alien to Cohen, making it "very technical, even partially philosophical" (2002, p. 1087). But the opposition between a formal and a model theoretic approach (the latter of which Cohen favors), does not mean that the latter cannot be formal*ized*. Indeed, Cohen's own model theoretic consistency argument can be revised to fit into the formal system ZF + Model Existence Axiom and rendered as a proof theoretical argument (1964, pp. 109–110). So, as in the case of the real, and as we observed earlier in this subsection, much of the informal has formal parallels.

The result is that the mathematician cannot be satisfied with either Realism or Formalism. As we stated above, Cohen believes that mathematicians would prefer Realism, but the problems of undecidability and large cardinals may cause them to "rush to the shelter of Formalism" (1971, p. 11). Still, "even if the formalist position is adopted, in actual thinking about mathematics one can have no intuition unless one assumes that models exist and that the structures are real." (2005, p. 2417). So the mathematician's "normal position will be somewhere between the two, trying to enjoy the best of two worlds" (1971, p. 11). Such a wavering position is attributed to Hilbert (2005,pp. 2413–2414), and balances Cohen's portrayal of Hilbert, noted above, as a realist.

In fact, Cohen characterizes himself as a "waverer" as well. While he states in no equivocal terms that he has "chosen the formalist position" (1971, p. 13), he does not reject all infinitist mathematics, because he feels that "we have an informal consistency proof for it" (1971, p. 14; see also 2002, p. 1088). This informal proof does not reach as far as large cardinal axioms, even though "experience has shown that they do not lead to contradictions and we have developed some kind of intuition that no such contradiction exists" (1971, p. 14). Moreover, Cohen cannot simply dismiss such axioms because of their relation to number theoretic problems.

His bottom line is that "there would be few operational distinctions between my view and the Realist position. Nevertheless, I feel impelled to resist the great aesthetic temptation to avoid all circumlocutions and to accept set theory as an existing reality" (1971, p. 15). This reluctant formalist position is reflected in his later writing as well: "Through the years I have sided more firmly with the formalist position. This view is tempered with a sense of reverence for all mathematics which has used set theory as a basis.... However, when axiom systems involving large cardinals or determinacy are used, I feel a loss of reality, even though the research is ingenious and coherent. (2005, 2416).

Let's recapitulate. The real is defined in opposition to the formal and to the infinite, or at least the large cardinals, as it is grounded in experience. Realism tries to think of mathematics as having an independent and knowable reality, but at the same time accepts large cardinals and tries to confront undecidability by looking for ever larger—that is ever unrealistic—cardinality axioms. This makes Realism not terribly realistic, in Cohen's terms.

On the other hand, we have Formalism, which views mathematics as a meaningless game of signs. But formalist approaches leave out too much useful and convincing informal mathematics, and leave too many questions open, including number theoretic questions that are real in Cohen's terms and that can sometimes be resolved by large cardinal axioms. But we must not confuse the informal with the real. There's a whole realm of mathematical thought which is informal as well as ungrounded in reality—the convincingly consistent part of infinitary mathematics, including a lot of model theory. This leaves Cohen and many other mathematicians (at least in his account) suspended between the two positions. So we need to figure out how to articulate a position suspended between Realism and Formalism, where Cohen and other mathematicians can hover.

3 Truth and Intuition

3.1 Truth is associated with the informal, but is not restricted to the real; nevertheless it is empirically human

Clearly, Cohen is uncomfortable with the historically received dichotomy between the real and the formal. In line with Lévi-Strauss's notion of the *bricoleur*, he does not toss them away so as to create a new conceptual system, but instead works with them by means of mediations and analogies to come up with richer conceptual systems. These systems will not resolve the inherent inconsistencies, but try to sort out some ad-hoc interim solutions that the *bricoleur* can use effectively.

In order to explore beyond the real and the formal, we need some mediating concepts. The first of these concepts in Cohen's work is *truth*. At the beginning of Cohen's book (1966, pp. 8–14), he makes a distinction (which is downplayed later on in the same book) between "valid" and "true"—the former referring to what follows some syntactic construction rules (formally provable), whereas the latter refers to what holds in all models of a certain system of axioms. In first order predicate calculus the two notions concur due to Gödel's completeness theorem (which is perhaps why Cohen sometimes replaces "valid" by "true" later on, as in the expression "all the true statements of set theory" in 1966, 19, where "set theory" refers here to the formal system), so formality and truth are not necessarily so far apart. Still, since models are contrasted with the formalist proof-theoretic approach in Cohen's view, this distinction tends to align truth with the informal.

This contrast is enhanced by statements that point to truths that are not formalizable. There are mentions of the "true power set" (1966, p. 94, 122) and "the true universe of all sets" (2002, p. 1076) as opposed to some models of the relevant formal systems. In fact, Cohen opens his book by stating that "the truth or falsity of the continuum hypothesis and other related conjectures cannot be determined by set theory as we know it today" (1966, p. 1), suggesting that he does not *foreclose* a notion of truth beyond what is valid in his contemporary set theory and what holds in all models recognized by his contemporary model theory (indeed, he writes that "a point of view which [he] feels may eventually come to be accepted is that CH is obviously false", even though it is true in some models, 1966, p. 151). On the other hand, the idea that large cardinal axioms *must* be true or false is associated with the idealist position (the one that he would later call realist) that sets "really "exist"" (1966, p. 80, scare quotes in the original).

Other truths that are not formal and do not hold in all models include Gödel's undecidable but informally provable statement (1966, p. 41), the axiom of the existence of a model of ZF (1966, p. 79, 2011, p. 436), and the negation of the continuum hypothesis (1966, 150–151—this is his informal argument against the continuum hypothesis). Moreover, Cohen states that such notions as consistency and ω-consistency are only approximations of truth (1966, pp. 43–44, 2002, p. 1099). Note that Cohen traces the position that mathematical truth cannot be captured by the axiomatic method already to Skolem (2005, p. 2411), which means that it does not depend on Gödel's incompleteness results.

So far, truth appears to exceed the formal, but also seems to advance beyond the limitations of the real, as it seems to endorse some sort of highly infinitary universe of sets, unattainable by axioms. In fact, truth seems to have some freedom to it, because Cohen considers his notion of forcing, which diverges from set theoretical truth, to be "a good notion of truth" (2002, p. 1094; also 2011, 437), suggesting that truth may come in varieties, as long as they preserve real truths or truths about reality (1965, pp. 43–44, Lemma 5; 2002, p. 1095). "In a somewhat exaggerated sense", claims Cohen, "it seemed that [in constructing the notion of forcing] I would have to examine the very meaning of truth and think about it in a new way" (2002, p. 1992). In fact, as we will see below, forcing is a notion of truth that tries to be as agnostic or uninformative as possible (1966, pp. 112–113). Moreover, the Boolean algebraic approach to forcing actually leaves many questions undecided, and yet is still "sufficiently similar to ordinary truth" (1966, 2002).

Therefore, Cohen's truth exceeds the formal and the real, and is at least partly dependent on human design. For example, forcing, conceived by Cohen as a form of truth, is based on a sequence of inductive decisions that are both constrained and allow for some constructed variety (more on that in the final section). This connects Cohen's notion of truth with the realization that "the only "true" science [i.e. mathematics] is itself of the same mortal, perhaps empirical, nature as all other human undertakings" (1971, p. 15). So truth may reach beyond the formal and beyond the real while, in practice, remaining mortally human and somewhat empirical.

3.2 Intuition is where the informal can be safely extrapolated away from the real

To explain the perplexing position of truth, we need to follow another term that fills the gap between the formal and the real in Cohen's work, namely *intuition,* the faculty that seems to host truth. Intuition is strongly related to the informal. Most often, it associates an informal idea to a formal object or argument. The interpretation of Gödel's undecidable statement as self-referential is intuitive (1966, p. 42), as are the interpretation of an ordinal as an equivalence class (1966, p. 56), the idea of the standard model of the integers (1966, p. 24), the view that ZF can express all normal definitions of sets (1966, p. 87), and the thinking behind the informal argument for set theoretic consistency (1971, pp. 14–15, 2011, p. 436). Of course, intuitive ideas are sometimes formalizable—even Cohen's formalized independence proof begins with an "intuitive" sketch (1963, p. 1143). He even characterizes his entire book as emphasizing "the intuitive motivations while at the same time giving as complete proofs as possible" (1966, preface).

Moreover, like the real, intuition tends to be associated with the finite and lower infinities. We clearly have an intuition of finite sets (2005, p. 2416), but we lack intuitive evidence for axioms of large cardinals or constructiblility (1971, pp. 11–12, 2005, p. 2416, 2418); at the very least, our intuition about them is not developed or communicable enough (1971, p. 15). The Gödel-Bernays system, with its proper classes, is also less intuitive, according to Cohen, than ZF (1966, p. 99). But this doesn't mean that we will never have an intuition of higher infinities—this is indeed what realists hope to develop (1971, p. 12), and we do have some intuition for some of the large cardinals in ZF (1971, p. 14). We even use such intuitions to prove some of the true but undecidable statements of ZF (1966, p. 45). Intuition is further guided by its distaste for impredicativity (1971, p. 14, 2002, p. 1081) and, more generally, properties that might lead to inconsistency (1966, p. 52).

The association between intuition and the informal is strengthened by the former's preference for models: "in actual thinking about mathematics one can have no intuition unless one assumes that models exist and that the structures are real" (2005, p. 2417) and "all our intuition comes from our belief in the natural, almost physical, model of the mathematical universe" (1966, p. 107). But intuition is not friendly to just any models: "one must work with standard models if one is to have any kind of reasonable intuitive understanding" (2002, p. 1081). The multiplicity of models is also a hurdle to intuition, but one that Cohen taught himself to surpass:

> one of the most difficult parts of proving independence results was to overcome the psychological fear of thinking about the existence of various models of set theory as being natural objects in mathematics about which one could use natural mathematical intuition (2002, 1072).

Like truth, intuition runs parallel to the formal and exceeds it well into the informal. It is associated with the finite and lower infinities, models and consistency, and in that sense tends to parallel the real. But we also saw above that intuition exceeds the real. Indeed, Cohen even acknowledges Gödel and Skolem as having deep intuitions (2011, pp. 437–438, 2005, p. 2411, 2413), even though, as Realists, they venture beyond what Cohen acknowledges as real. So both truth and intuition display tensions with Cohen's notion of reality. The problem is that if some sorts of intuition have "nothing to do with "real" sets", it becomes a challenge to explain "how a presumably incorrect intuition has led us to such a remarkable system" as set theory (1966, p. 150).

To resolve this challenge, Cohen invokes a "feel[ing] that our intuition about sets is inexhaustible" (1966, p. 150). More precisely, the decisive statement is this:

> If there is something infinite, perhaps it is the wonderful intuition we have which allows us to sense what axioms will lead to a consistent and beautiful system such as our contemporary set theory. … The ultimate response to CH must be looked at in human, almost sociological terms. We will debate, experiment, prove and

conjecture until some picture emerges that satisfies this wonderful taskmaster that is our intuition (2002, 1099).

This is how Cohen finally explains our informal ventures beyond the real. As we saw above, Cohen's notion of intuition is grounded in the real, but now we see that when it extrapolates away from the real, into set theory, it is guided by aesthetics and consistency. It is no longer real (except perhaps in scare quotes), but it is not restricted to the formal either, since formal set theory approximates some of these truths and intuitions but is incapable of expressing others. So while Cohen calls himself a formalist, I think it's better to call him an "intuitionist" in a specialized sense: not the foundationalist kind of intuitionist who restricts mathematics to the finite, countable or constructible, but one who trusts the inexhaustible aesthetic production of mathematically socialized humans, even when it ventures into unreal infinities—at least as long as it is guided by informal consistency.

Cohen finds it acceptable to draw conclusions from these intuitions for proving real, number theoretic propositions. Here, unlike Hilbertian formalists who would allow any extension of real mathematics as long as it is proven formally consistent (a project blocked in its original form by Gödel's incompleteness results), Cohen trusts only extensions of real mathematics where we have good intuitions of informal consistency, aesthetic coherence and a successful applications to the real. In this sense, Cohen is closer to "humanist" (e.g. Hersh, 1997) and "naturalistic" (in the sense of Maddy, 2007) philosophies of mathematics than to any foundationalist school.

However, Cohen was not optimistic about this kind of humanistic-naturalistic–"intuitionist" project. He explicitly believed that this project would resolve only a tiny fraction of mathematical problems. He believed that "the vast majority of statements about the integers are totally and permanently beyond proof in any reasonable system" (2005, p. 2418). According to him, our intuition would have nothing to hold onto with respect to some random patterns that happen to have some properties without any logical reason (he gives twin primes as a possible example). More generally, he may have been intuitively thinking about those non-generic statements that his forcing has no "strong reason" to mark as true (1966, p. 112), and at the same time might not be false (more on that in the final section). Or perhaps this pessimism is the result of his frustration with his attempts to prove Riemann's hypothesis. The latter is, indeed, the example that concludes his paper on Skolem (2005, p. 2418), where he suggests that we may have to settle for partial, finite, and empirical investigations of such questions. For Cohen, mathematical intuition was not likely to be enough to settle all real questions.

4 The one and the many

From the point of view of contemporary philosophers of mathematics, this attempt to handle the tension between Realism and Formalism is all too little too late. Cohen's position is lagging behind the professional discourse (e.g. some forms of structuralism, fictionalism, humanism, and later on some forms of naturalism and pluralism)—which is no surprise, given Cohen's indifference to this discourse. Indeed, suppose we accept a faculty of intuition guided by aesthetics and a sense of informal consistency, which extends beyond some sort of real mathematics (whose scope is not completely clear). Why would we allow such a faculty to impose its conclusions on real number theoretic questions? We cannot exclude the possibility that another, equally intuitive, aesthetic and consistent construction will prove opposite conclusions, and force us to retreat back to a more pluralistic formalism.[6] And if we do accept our current intuitions (whose "almost sociological" dimension Cohen seems to acknowledge) as authoritative, why reject outright the possibility that they reflect a set theoretic reality "swimming in an ethereal fluid beyond all direct human experience" (2002, p. 1099)?

But, as explained in the introduction, my purpose is not to extract from Cohen's work a new and exciting philosophy of mathematics. Rather, I show how the tension between two irreconcilable extremes—the real and the formal—is confronted in Cohen's texts by introducing the intermediary position of intuition and its truth. This intermediary position is not real in the strict sense, but extrapolates from the real something that, unlike the formal, is neither arbitrary nor an unduly limited simulacrum. Moreover, for Cohen, this intermediary position allows us to derive some real conclusions about numbers. This conceptual maneuver tells us something about how (at least some) mathematicians think. And since it is mathematicians who design mathematics, understanding how they understand mathematics (regardless of whether their understanding is philosophically refined) means understanding something about really existing mathematics. This, in turn, is relevant for the philosophy of mathematical practice.

Now, given that I take my inspiration here from Lévi-Strauss's characterization of "wild-" or "savage thought" (which is for him, recall, the structural ground of all human thought), it would be interesting to follow some of his other tools. For Lévi-Strauss, human thought does not only "resolve" oppositions by constructing intermediaries, it also handles them by creating analogies between "unresolved" oppositions and those that we are more accustomed to, and may consider "resolved".

Identifying analogies between oppositions is not easy methodologically, and sometimes Lévi-Strauss's work seems somewhat arbitrary, as it fails to clarify when we should consider a given pair of oppositions as analogous. Indeed, these pairs are usually only implicitly related by some sort of juxtaposition, but it is not clear (at least not to me) when one should consider two oppositions to be juxtaposed in a manner that establishes an analogy. Despite these difficulties, I will try to find in Cohen's work an analogy that might help us "resolve" the tension between the formal and the real. This will provide us with some further insight into mathematicians' indigenous conception of mathematics, which, as noted above, plays a role in designing really existing mathematics and is thus relevant for the philosophy of mathematical practice.

We have already noted that Cohen is wavering between the formalist and realist positions. As he put it: "I vacillated between two approaches: the model-theoretic, which I regarded as roughly more mathematical, and the syntactical-forcing, which I thought of as more philosophical" (2011, 437; recall that Cohen considers model

[6] However, as was noted by a reviewer, if one believes in the reality of the standard model of the integers, as Cohen seems to believe, Π_1 number theoretic statements, such as the Riemann hypothesis, would escape such a fate. This somewhat strengthens Cohen's position against this critique.

theory as informal, which places it somewhere between the intuitive and the real, while he associates syntactic proof theory with the formal).

Cohen's view of the tension between the formal and the real is the following: "one sees that the usual systems of mathematics such as the integers and real numbers, which presumably have only one model, cannot be described completely by any formal system of axioms" (1966, p. 17). In reality (or somewhere between reality and intuition), mathematical models are categorical and unique, but the plurality of formal systems does not quite grasp them. This problem extends to set theory itself: "the natural model for Z-F is the universe, which is not a set" (1965, p. 40)—once again, the intended model cannot be reduced to a single formally defined "real" or object-like set, only "approximated" by a chain of formally defined sets. In different ways, these are versions of a classical problem that appears often in Lévi-Strauss's work: the opposition between the one and the many.

The immediate context of this for Cohen is proofs of independence, and "the most natural way to give an independence proof is to exhibit a model with the required properties" (1966, 107). Since a model is nothing but "a set M together with an interpretation of some (possibly none) constant and relation symbols" (1966, p. 13), in order to "tolerate" the opposition of the real and the formal as a problem of the one and the many we might look for an analogous opposition in the context of models as sets, which we are more familiar with and may consider less bewildering.

Now, in a set theoretic context, "the existence of many possible models of mathematics is difficult to accept upon first encounter" (2002, p. 1072). But on subsequent encounters, the shift from the problem of the *formal : real*, via models, to sets is somewhat helpful. If we follow this shift, what we are facing is the existence of various different sets, which are all similar in some important respects to an intended set, but are nevertheless distinct. This version of the problem of the one and the many is much easier to digest—precisely because it suppresses the facts that one cannot quite identify *the* intended set which one is after and that the real thing one is after (if it can indeed be considered as real) might not "really" be a set. Nevertheless, portraying an analogy between an opposition that appears bewildering (*real : formal*) and an opposition that we accept much more casually (*intended model : other sets with similar properties which are not the intended model*) may help us view the former opposition as less onerous and unpalatable. The analogy shifts the problem, at least apparently, from that of a one (reality) that escapes the many (formalisms) which are all different in kind from that one, to the problem of finding one specific object among many objects of the same kind.

The same problem can be confronted by another analogy, the one between the opposition *formal : real* and the opposition *non-standard models : standard models*. We noted above that some aspects of the formal are less intuitive (and less trustworthy) than others, and are farther away from the real. The analogy between the last two oppositions carries us again into the safer realm of models or sets. Models that are constructed formally by stipulating constants (as in Löwenheim-Skolem constructions, 2005, p. 2411) or, more generally, non-standard models, bar "any kind of reasonable intuitive understanding" (2002, p. 1080) compared to standard models, which are simply sets with the "real" inclusion relation. In the framework of the analogy to the *formal : real*, non-standard models reflect the unrealistic pathologies of the formal, and the standard models reflect the intuitive truth of the real—again, without fully capturing the difficulties of the original opposition.

One final note. Cohen's entire understanding of the formal, proof theoretic approach appears to be mediated through models. Indeed, he presents his proof-theoretic consistency demonstrations as derivative translations of his model theoretic demonstrations (1964, pp. 109–110, 1965, pp. 53–54, 1966, pp. 147–148) and his understanding of Gödel's syntactic consistency of the axiom of choice and of the continuum hypothesis as dependent on a translation of Gödel's argument into model theoretic terms (2002, p. 1081). We have also quoted several of Cohen's statements that our intuition requires a model theoretic approach, and that we should "essentially forget that all proofs are eventually transcribed in this formal language" (2002, p. 1078). This makes it all the more curious that Cohen confesses that his original approach was proof theoretic, and that "models were later introduced as they appeared to simplify the exposition" (1966, pp. 147–148). A solution to the apparent contradiction may be that the emphasis on models helped Cohen resolve some conceptual difficulties, perhaps even the difficulties presented in this very section, and was not strictly necessary for him to come up with a proof of his independence theorems.

5 Cohen's conception in practice

To conclude this paper, I will try to show how Cohen's conceptions find an expression in his mathematical practice of proving consistency results. I will sketch the structure of his proofs, focusing mainly on the earlier presentations from 1963–1964 and 1966 (the so called "ramified forcing"), and analyze it by following some standard narratological axes. There is already some literature offering narratological analyses of mathematical proofs. They note analogies and disanalogies between narratives and proofs in order to shed light on philosophical (Thomas, 2007), educational (Dietiker, 2013; Solomon & O'Neill, 1998) and cognitive issues (Andersen, 2022; Carl et al., 2021). My purpose here, however, is not to make any general claim about the similarity of mathematics and narrative. I am using narratological analysis in order to get a more refined sense of specific proofs. In that sense, what I am doing here is more similar to the work of Netz (2009), the appendix in Wagner (2009a) and the suggestive account in Harris (2012).

This analysis will show how the tension between the formal and the real (mediated by intuition) is played out in Cohen's actual mathematical work. The "hero" of the analysis will be the set a (to be presented below), whose trajectory we will follow according to the axes of temporality, point of view and identity, which all reflect the wavering of the set a between the poles of the formal and the real. In a certain sense, we will consider a as a "character" in a narrative—a sort of *Bildungsroman*—which follows it as it grows beyond its formal and real underpinnings and reaches maturity as an object of intuition.

The task of Cohen's proof is to construct a model that satisfies some standard set theoretical axioms (specifically, ZF) as well as additional axioms whose consistency is to be proved (specifically, the negation of the continuum hypothesis and of the axiom of choice). Without getting into technicalities, we can summarize Cohen's strategy as follows (I am well aware that for someone who has never studied forcing this is probably non-digestible as such, so such readers are invited to read around what is too opaque). I emphasize that I am describing Cohen's original strategy for his independence proofs, not later reformulations.

- Take a countable, standard model (i.e. retaining the original belonging relation) M of ZF set theory, where all sets are formally constructible from the ZF axioms (so called "$V = L$").
- Add a set a of positive integers (or a bunch of sets a_δ) which is (or are) not in M to the model M, as well as all sets formally constructible from members of M and from a (or the sets a_δ) according to the ZF axioms. This resulting collection of old and new sets will be the set N. Since a is not a member of M, from the latter's point of view a is considered only as a formal symbol. Its actual constitution as a set (i.e. its members) is still to be decided at this point of the proof.
- For any finite set of decisions about the membership (or non-membership) of some integers in a (a so-called *forcing condition*), run a transfinite induction on all statements about N in the formal language of ZF (following the order of construction of such statements from each other) in order to assign some of these statements with truth-values:
  - o Those statements which have to be true by the properties of M, the previous truth-decisions in the induction, and the forcing conditions are considered true.
  - o Those which do not have to be true by the properties of M, the previous truth decisions, and any *finite extension* of the forcing conditions are considered false.
  - o Note that for any forcing condition, many statements about N will remain undecided.
- Now run another suitable induction on all statements about N (this time of order ω, not following the iterative construction of the statement from each other). For each statement, extend (if necessary) the forcing conditions such that the statement becomes either true or false according to the above truth-value assignment. In particular, for each integer, its membership in a will have a truth-value.
- The resulting N, built from M and a, will turn out to be a model of ZF. Restrictions on the sequence of forcing conditions that define the set a (or the sets a_δ) will guarantee the validity of the axioms whose consistency with ZF is to be proven.
- This will prove that starting from a standard model of ZF, we can construct another model with the desired properties. An amended translation of the argument into ZF will show the consistency of ZF and the desired property without assuming the existence of a standard model M.

5.1 The temporality axis

The most glaring concern with this strategy is that we assume that a is there even before we define which integers are members of a. In Cohen's words: "this requires giving names to the elements of N before we have actually chosen a and thus before we have N explicitly" (1966, 113). Note that this statement injects temporality ("before") into the proof. This temporality is explicit elsewhere as well, for example: "when N is finally constructed, the [formal] statements become actual statements about N when we replace [formal variables by the corresponding sets in N]" (1966, p. 115).

In other words, *first* we consider a as a formal variable inserted into formulas that will *eventually* express statements and define sets in the model N (recall that Cohen associates standard models with intuition and reality, as opposed to formal/proof theoretic approaches, so a gradually shifts from a formal mode of existence to a real one). These formulas, being strings of symbols rather than sets, are not elements in ZF, but are represented by elements of a "label space" S in ZF by some enumeration scheme (see 1966, p. 114). We are promised that "it *will eventually* turn out" (1966, p. 130) that the formulas will have the expected meanings. In fact, sometimes we are so certain about this eventuality, that we mix up the two kinds of notations (e.g. using the actual ordinals of the not-yet constructed model N instead of the formulas that *will eventually* express them in the not-yet-constructed N; 1963, p. 1145).

Since mathematics is often considered a-temporal, this temporal form of expression may seem out of place. Nevertheless, it is easy to make sense of such temporality. In literary theory, we are well aware that there are various temporalities involved. There's the temporality of the author, who wrote various parts of the story in a certain temporal order; the temporality of the narrator, who exposes events in a certain order; the temporalities of various characters, who experience events in a certain order; and the temporality of the reader, who may read and re-read the text in a certain order. If we consider a mathematical text to be, precisely, a text, we should not be surprised to find several temporalities as well. There's the temporality of the author's work and writing sequence; the temporality expressed by the order of statements in the narration of the text; the temporality expressed by the order of statement as they should appear in the logical progression of the proof; and, depending on one's approach, perhaps a temporality of mathematics itself, which may be an absolute simultaneity of all truths or a certain progression of construction events.

What we see here is that in Cohen's text (as in many other mathematical texts; see, for example, Tomalin, 2021), the temporality of the narration of the text and the temporality of an intended logical progression of the proof are not the same. This is the case not only in the sense that the narrator of the texts informs us about facts that are not yet proved, but in the sense that the meaning of terms, which is not yet available from the point of view of the logical progression of the proof, is used to guide the reader in advance. Moreover, the rhetoric of the text suppresses this mismatch in order to help us understand—bending logical precision in favor of intuitive understanding. It blends together the formal starting point and the "real" end point into a single intuitive picture in a manner that reflects Cohen's philosophical conceptions.

I do not claim that this is a conscious move on Cohen's part that is based on his philosophical views as reconstructed above. Indeed, this kind of maneuver is common among many mathematicians with various philosophical tendencies. But regardless of the purposes of this rhetorical move, the fact that in his proof "time is out of joint" fits Cohen's philosophical approach, wavering between the formal "past" and intuitive/real "future" of the formal symbol/set a. The informal intuition embraces that which is not yet real and that which cannot yet be expressed from a formalist point of view.

5.2 The point of view axis

Next, let's consider the points of view concerning a. First, as noted above, "a is not in M, so we cannot discuss [it] as a set" from the point of view of M (2002, p. 1995). We can, however, write formal expressions using a, and from the point of view of M, these are like statements involving a variable: "in analogy with field theory, we are actually dealing with the space of all (rational) functions of a, not actual sets" (2002, p. 1993; cf. 1966, p. 113).

However, the range of the "variable" a is problematic: the values it can take are subsets of the integers, but in which model? M or some undefined extension? (recall that the set a is not in M, but a is sometimes thought of as a variable set from the point of view of M). Moreover, the transition between viewing a as a variable set in M and as a formal variable is sometimes quite blurry (elements in the "label space" S of statements as sequences of symbols and corresponding sets in M are sometimes merged—see 1966, pp. 122–123, 1963, p. 1145; cf. Wagner, 2009b for similar ambiguities in a different context). So it turns out that the analogical view of a as a variable is not very stable. This is a fine expression of Lévi-Strauss's analogy principle: a problematic relation—a as a set with respect to M—is portrayed as analogous to a relation we are more familiar with—a as a variable added to M—even though the analogy is also problematic.

In principle, statements about the model N (which boil down to statements about a and M) may have a scope that exceeds what M can express. We can, however, "throw back questions about N to questions about forcing which can be formulated in M" (1963, p. 1147). Indeed, to ask if a statement about N is true is to ask about its relation to certain forcing conditions, which, given the inductive definitions, are equivalent to statements about a variable set a in M restricted by the forcing conditions (an important technical point is to make sure that whenever we discuss a set which is "too big" to be defined from the point of view of M, the information we need can be obtained from some reformulated statement about a set which is "small enough" to fit in M, e.g. 1966, pp. 122–123). So from the point of view of meaning too we are wavering in and out of what M can express.

None of these reflections should be taken for granted; indeed "one had to be sure that there was no contradiction in both working in and out of the model" (1966, p. 1093). And yet, all of this must be set aside: "in all honesty, I must say that one must essentially forget that all proofs are eventually transcribed in this formal language. In order to think productively, one must use all the intuitive and informal methods at one's disposal" (1966, p. 1078).

I should emphasize that I don't think there's anything mathematically wrong here. I'm not trying to nit-pick and accuse Cohen of imprecisions or errors. In fact, at least as far as I can see, he is a rather careful mathematical writer. My point is to highlight the rhetorical issues involved in mathematical practice and the conceptual tensions that they express and come to terms with. The set a is not in M (it is a formal symbol from the latter's point of view), but in Cohen's work, its real, finitary reflection in M (as a variable set in M subject to a finite forcing condition) plays a major role in the proof. We must be careful about these formal tensions, but Cohen warns us that we must also forget them to have an intuition of what is going on. Once again, an expression of Cohen's attempt to mediate the "real", finitary portions of a (expressed by his forcing conditions) and the information they provide with the infinite inductive formalism that constructs a as an infinite set.

5.3 The identity axis

The construction of a is curiously described by opposite terms: "forcing" and "choice" (e.g. 1963, p. 1147, Definition 8; 1966, p. 135). This apparent contradiction actually represents quite well the identity of a—it is at once something forced by a finitary reality and essentially undetermined from an infinitary point of view. Indeed, from the point of view of the constructed N or the end of the proof, a is completely determined (as explained in the proof-strategy summary above). But from the point of view of M, while every forcing condition in the forcing sequence is expressible and finite (and hence real in Cohen's terms), the entire sequence cannot be expressed. So from the latter point of view, "the set a will not be determined completely, but yet properties of a [i.e., statements about a in N] will be completely determined on the basis of very incomplete information about a" (2002, p. 1092).

In fact, the idea of forcing is to keep a generic, or, in a certain sense, determine as little as possible about it. In particular, if no *finite* forcing condition can force a statement about a to be true, then this statement will be false (e.g., a must be infinite, have infinitely many primes, etc.; see 1966, pp. 111–112; also 1965, p. 41). Moreover, every true statement about a as member of N corresponds to some forced statement about a as a variable set in M subject to a merely finitary condition.[7]

In technical terms, a is *constructed* to be *non-constructible* from M (1963, p. 1143, Theorem 1, 1965, p. 47), and in non-technical terms, it cannot be fully described by the vocabulary of M. But the genericity of a (in the above sense) means that such a full description is unnecessary—a is never really properly individuated (in fact, this lack of individuation allows Cohen to impose symmetries that prove the consistency of the negation of the axiom of choice; see 1965, pp. 50–51, 1966, pp. 136–142). This strange situation is required for Cohen's argument to work, because everything we can say about a should be determined from the point of view of M, whereas a itself is not determined from that point of view. Therefore, anything we can say about a, even after it is determined as a set, must leave anything but a finitary chunk of information about a underdetermined.

There are other ways to understand this underdeterminacy. For example, during the construction of a we go through all statements expressible in M. This means that we go over all mathematical open questions, and need to answer them all so as to complete the forcing process. Therefore, in a sense, the set a will be the "last" mathematical object to be determined, after we have solved all mathematical problems—at least relative to some model M.

[7] This may sound weird, because the entire list of members of a cannot be expressed in M or enforced by any finite condition, and yet it is true that a is equal to the collection of its members. But in Cohen's construction, this fact is expressed only by means of a tautology of the form $a = a$, and not by any informative expression about the composition of a as a collection of specifiable numbers.

From another technical point of view, that of Boolean algebra, a is not just undeterminable in some epistemological sense, but also ontologically undetermined. Indeed, "if one takes this approach, one need never actually choose a complete [forcing] sequence, but instead say that we have a Boolean valued model and the basic lemmas imply that this behaves sufficiently similar to ordinary truth" (2002, p. 1096).

Once we combine these points of view, I think it makes sense to think about a as simultaneously forced and chosen, determined by a construction and yet essentially undetermined. The set a has and lacks an identity, it has an intuitive "reality" and yet only a formal constitution based on finite (and hence real) subsets of its members. Once again, an example of Cohen's wavering philosophy in his hands-on practice.

The exercise that I performed here found its inspiration in literary theory. We considered a as a "character" in a sort of *Bildungsroman*, trying to come of age between its formative real and formal "experiences". But this would be a very peculiar—baroque, high-modernist or post-modernist—anti-linear *Bildungsroman*. In temporal terms, the only way to understand the character's past is by means of its future: its time is out of joint. In terms of point of view, it is always both in and out of sight: it is fully characterized by a point of view that cannot capture its identity. Finally, it is never quite its own individual self. It grows up to be a no-one or an every-one, a generic face in the crowd of sets, whose ambiguity (and the prevalence of this ambiguity in the world of sets) allows us to understand the "society" it inhabits.

Or at the very least, this is the literary analogy that my "savage thought" came up with in order to make sense of the oppositions expressed in Cohen's philosophy and mathematical practice.

6 Conclusion

In this paper I tried to analyze Cohen's philosophy of mathematics and how this philosophy is expressed in his mathematical practice. Due to Cohen's attitude to philosophy, I preferred not to attempt a consistent rationalization of Cohen's apparent incoherent views. Instead, I considered him as an indigenous "informant", applying a methodology from structural anthropology to understand his views.

I studied how the received tensions between formalist and realist philosophies leave Cohen in an uncomfortable position. Instead of replacing them by a new philosophical system, he follows the practice of a Lévi-Straussian *bricoleur*: he comes up with mediations and analogies to handle the insurmountable tension without thereby resolving this tension. He then applies these mediations and analogies in his work.

I believe that this is a good account of the relations between philosophy and mathematical practice. Following the structural-anthropological account, mathematicians are often more or less aware of philosophical tensions. But, not being philosophers (or at least not very often), they do not need to resolve these tensions—it's enough for them to manage them by mediations and analogies. This is expressed not only in their explicit philosophical reflections, but also in their mathematical practice. The status of the mathematical objects as reflected in their proofs is often just as shifty and full of tensions as their philosophical positions.

Now, these are highly presumptuous claims to make based on a single case study of a single mathematician, no matter how important or exciting his mathematical work may be. Indeed, if one believes Lévi-Strauss, there's really nothing to prove: mathematicians, like any other humans as conceived by Lévi-Strauss, must think and act this way, even when they think they are doing something else, or actually doing something else *on top of that way of thinking*. Cohen would then only illustrate what the Lévi-Straussian already knows. If, on the other hand, one doubts Lévi-Strauss's theories, then all we have is an anecdotal case study that can lead to no general conclusions.

Since even I find Lévi-Strauss's account to be too rigid and universalizing, I concede that all I brought is an anecdotal case study. Instead of proving a universal claim, it is meant to promote an ideological suggestion. I suggest that if we are indeed interested in the philosophy of mathematical practice, or in a naturalist second philosophy, or a humanist or pluralist philosophy of mathematics, then we should be open to reconstructing mathematics as a philosophically and practically inconsistent and unstable practice (cf. Mangraviti, 2023). We should be open to using its philosophical and practical incoherencies and ambiguities as a resource for explaining its successes (and failures), rather than try to resolve these incoherencies and ambiguities (cf., once again, Byers, 2007; Grosholz, 2007; Fisch 2017, chs. 5-7; this fits a possibly radicalized version of what Kant et al. 2021 call the "freeway explorer" approach to philosophy of mathematical practice). Instead of considering mathematics as the emblem of human rationality, we should be open to viewing it as one of many imperfectly rational human practices, held together by (rather than despite) their imperfect rationality.

And if this suggestion sounds too far-fetched, then I hope, at least, that the structural-analytic technique applied here might help make sense of other mathematical work.

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