Synthese (2023) 202:23
https://doi.org/10.1007/s11229-023-04240-0

ORIGINAL RESEARCH

Definitions in practice: An interview study

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Received: 19 August 2022 / Accepted: 22 June 2023 / Published online: 11 July 2023

Abstract
In the philosophy of mathematical practice, the aim is to understand the various aspects of this practice. Even though definitions are a central element of mathematical practice, the study of this aspect of mathematical practice is still in its infancy. In particular, there is little empirical evidence to substantiate claims about definitions in practice. In this article, we address this gap by reporting on an empirical investigation on how mathematicians create definitions and which roles and properties they attribute to them. On the basis of interviews with thirteen research mathematicians, we provide a broad range of relevant aspects of definitions. In particular, we address various roles of definitions and show that definitions are not just a product of mathematical factors, but also of social and contingent factors. Furthermore, we provide concrete examples of how mathematicians interact and think about definition. This broad empirical basis with a variety of examples provides an optimal starting point for future investigations into definitions in mathematical practice.

Keywords Concept · Definition · Mathematics · Mathematical practice · Interviews

1 Introduction

Definitions are a central element of mathematical practice. Almost every mathematical article contains a definition. From an inferential point of view, definitions are well understood. They are language extensions that are conservative and eliminable (Belnap, 1993; Gupta, 2019). In other words, definitions are language extensions such that the inferential power does not increase and such that every sentence from

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this extended language can be rewritten using the original language. As such, they are merely abbreviations.

In practice, definitions also have other functions and various qualities that are not captured by their ability to abbreviate. This is demonstrated by, for instance, the fact that for various concepts, there are equivalent characterizations. From an inferential point of view, the benefit of these characterizations is nihil. This suggests that, from a practice point of view, there is more to definitions than their logical, inferential side.

Although there is some literature addressing definitions in practice, they are not studied as systematically as a topic like proofs in practice[1]. The state of the art on definitions in practice has recently been described by Coumans (2021). It turns out that the literature on definitions in mathematical research practice is primarily analytical or case-study based. In particular, philosophers of mathematical practice often have to rely on their own intuitions regarding definitions in practice, without the availability of empirical evidence to gauge these claims. There is some literature describing experimental approaches to definitions in practice, but these focus mainly on educational settings.

In this article, we contribute to the investigation of definitions in practice by reporting on an interview study in which we investigated research mathematicians' perceptions of and interactions with definitions in practice[2]. In particular, we explore several themes that are discussed in the literature. First, we inquire into the role and functions of definitions in mathematical research practice. Second, we are interested in finding out what research mathematicians consider desirable properties of definitions. Third, we look into the processes by which research mathematicians create definitions.

This research constitutes one of the first empirical investigations into definitions in practice. In particular, this article provides empirical foundations for a broad range of topics. This enables philosophers of mathematical practice to gauge claims regarding definitions in practice. Furthermore, the presentation of the results contains various concrete examples of how mathematicians interact with and think about definitions. As such, this article serves as a window into mathematical practice.

The article is structured as follows. We describe the literature on definitions in practice in Sect. 2. Section 3 discusses methodological aspects of our interview study and the study's results are presented in Sect. 4. We conclude in Sect. 5 by discussing the results and presenting further lines of inquiry.

[1] For literature on proofs in practice, see (Andersen, 2020; Baldwin, 2018; Dawson, 2006; De Villiers, 1990; Detlefsen, 2008; Frans & Kosolosky, 2014; Geist et al., 2010; Hamami, 2022; Inglis & Aberdein, 2015; Kleiner, 1991; Larvor, 2012).

[2] In this interview study, we inquired into both concepts and definitions in mathematical practice. In this article, we report on our findings concerning definitions in practice. The results concerning concepts in practice are described in a parallel article.

2 State of the art

The literature on definitions in mathematical research practice has recently been described by Coumans (2021). As such, this section primarily constitutes a recap of Coumans' overview[3]. As mentioned in the introduction, definitions in practice is a multifaceted topic. In this section, we address several of these facets described in the literature. More concretely, we first look at literature on the various roles and functions of definitions. Then we look at (desirable) properties that are associated with definitions. After that, we discuss the process of creating definitions. Furthermore, as this research constitutes one of the first empirical studies on definitions in mathematical *research* practice, we review empirical work on definitions in practice. Most of this work has been done within the field of mathematics education. As such, the results from these educational studies are not directly applicable to investigating definitions in mathematical research practice. Nonetheless, these methods inspire our research design. Hence, we conclude this section by evaluating the suitability of various methods for investigating definitions in mathematical research practice.

2.1 The roles and functions of definitions

Definitions in practice were discussed by Cellucci (2018). He differentiated between two approaches to definitions, the *stipulative* and the *heuristic* conception, respectively. According to the stipulative conception, definitions are merely abbreviations that specify the meaning of a term. Cellucci (2018) described this approach as the conjunction of the following five properties[4]:

"(1) A definition merely stipulates the meaning of a term […] (2) A definition is an abbreviation […] (3) A definition is always correct […] (4) A definition can always be eliminated […] (5) A definition says nothing about the existence of the thing defined […]" (Cellucci, 2018, p. 608).

It can be argued that point (3) is ill-formulated. Cellucci explained that this item means that definitions are neither right nor wrong: they do not assert anything. In any regard, this stipulative approach conforms with the traditional, logical approach to definitions.

The alternative view, the heuristic conception, operates from the *heuristic conception of mathematics*. According to this conception, mathematical problem solving consists of providing plausible hypotheses that solve problems. These hypotheses are then problems themselves that are in need of plausible hypotheses that can explain them, and so on. The procedure of mathematical problem solving can continue indefinitely. In this view, definitions are also considered to be hypotheses that are presented in the course of problem solving. As such, the main objective for definitions is to solve problems.

[3] In addition, we also discuss some mathematics education research that is not discussed in (Coumans, 2021).

[4] Cellucci traces the origins of this approach back to Frege. In particular, he traces this to (Frege, 1967, 1979, 1980, 1984, 2013).

In addition to these two approaches by Cellucci, Ouvrier-Buffet (2013) introduced three conceptions that are based on the works of Aristotle, Popper and Lakatos. These conceptions describe the roles of definitions, possible actions and moves one can perform regarding definitions, and various associated principles. In contrast to prior usages of the term conception, Ouvrier-Buffet used it in a narrow, didactical interpretation, following the work of Balacheff (1995). The original description given by Ouvrier-Buffet, which is embedded in a highly educational framework, is beyond the scope of this text. Hence, we only give brief descriptions of these conceptions.

The Aristotelian conception considers definitions as tools for classification. One way to create such a definition is by considering examples and looking for differences. Two principles associated to this conception are that one should avoid (1) vicious circles and (2) metaphors. The Popperian conception is related to scientific theories and it considers definitions as tools for choosing between competing theories. Principles that steer this defining process include the reluctance to refute theories and various quality criteria for theories. Actions related to this conception include generating counterexamples and reducing the number of postulates for theories. In the Lakatosian conception, definitions are tools both for solving mathematical problems and for classifying. Many of the actions associated to this conception are lifted from *Proofs and Refutations* (Lakatos, 1976). For instance, one can use the methods of exception-barring and monster-barring to improve a definition.

Finally, several roles of definitions are mentioned in the mathematics education literature. An overview was given by Zaslavsky and Shir (2005). They mentioned that definitions can be used for "(1) introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties […] (2) constituting fundamental components for concept formation […] (3) establishing the foundation for proofs and problem solving […] and (4) creating uniformity in the meaning of concepts, which allows us to communicate mathematical ideas more easily" (Zaslavsky & Shir, 2005, p. 317)[5]. However, Zaslavsky and Shir based their overview on sources that did not investigate the roles of definitions for research mathematicians directly. They discussed how middle-school students or undergraduates interact with definitions and concepts (Borasi, 1992; Klausmeier & Feldman, 1975; Mariotti & Fischbein, 1997) or, for instance, how some proofs can justify definitions (Weber, 2002). In doing so, they sometimes simply assume a particular function of definitions or deduce it from their observations concerning students. As such, we only mention these roles, without discussing them further.

2.2 Properties of definitions

When it comes to the notion of 'definition', Zaslavsky and Shir (2005) differentiated between the *roles* and *features* of definitions. These features are then split into *imperative* and *optional* features. We discussed several roles in Sect. 2.1. In this section, we look at features of definitions. Zaslavsky and Shir (2005) presented several imperative features. For instance, they mentioned that the various clauses of a definition should not be contradictory and that definitions should be unambiguous. Further-

[5] For clarity, we have omitted the citations placed by Zaslavsky and Shir (2005) from this quote.

more, “a mathematical definition must be *invariant* under change of representation; and it should also be *hierarchical*, that is, it should be based on basic or previously defined concepts, in a *noncircular* manner” (Zaslavsky & Shir, 2005, p. 319). In addition, they mentioned minimality (i.e. the avoidance of redundancy) as a feature that some might find imperative whereas others might find it optional.

In the philosophy of mathematical practice, many features of definitions are discussed that may be considered optional. We will briefly discuss three of those. Sometimes definitions (or concepts) are praised for being natural. What naturalness entails precisely, is a topic of debate. Corfield (2003) presented three interpretations. One option is that naturalness refers to whether concepts can be encountered in “reasonably well frequented regions of mathematics” (Corfield, 2003, p. 224). Another interpretation links naturalness to inevitability: at some point, this particular concept would have been discovered anyway. The third refers to descriptions and definitions that do not depend on choices. Tappenden (2008) presented an alternative description in terms of “[carving] mathematical reality at the joints” (Tappenden, 2008, p. 264). He also argued that the Legendre symbol, although being seemingly artificial, is in fact a natural notation. To substantiate this claim, he looked at the Legendre symbol in more general contexts to demonstrate that it carves mathematical reality at the joints.

Another feature of definitions is the explanatory value of definitions. Lehet (2021a) proposed that explanatory definitions make the concept that is pinned down more accessible. As an example, she looked at CW-complexes that are complicated constructions that can be seen as a series of elementary constructions. In that sense, one can access the final product by looking at its constituting parts. Lehet also discussed various descriptions of the derivative and argued that some show the deeper meaning of the derivative by relating it to tangents and limits. Lehet (2021b) then further developed this notion of explanatory definitions by arguing that explanatory definitions are those that generate understanding.

Finally, fruitfulness of definitions and concepts was discussed by, e.g., Horty (2007), Shieh (2008), Tappenden (1995, 2012) and Yap (2011). When it comes to definitions, some are merely lists of properties that the definiendum has to satisfy. However, sometimes, definitions also present some internal structure that can lead to new knowledge, when analyzed. Horty (2007) tried to see how this can be compatible with the idea that definitions should not increase inferential power and Tappenden (2012) tried to argue that fruitfulness is related to mathematical beauty in the sense that definitions that expose some underlying structure are aesthetically pleasing. From a slightly different angle, Yap (2011) investigated whether the Legendre symbol or the modulus relation is more fruitful and came to the conclusion that in the context of proving the reciprocity law, the Legendre symbol is highly fruitful, but that the modulus relation is more generally applicable.

2.3 Process of defining

The most prominent description of how definitions are created is by Lakatos (1976). In *Proofs and Refutations*, he discussed various definitions of polyhedron in a hypothetical discussion between a teacher and his students. For the purpose of this article, we only highlight one important aspect of Lakatos' account, namely that defining is intricately intertwined with the process of proving. In particular, it is not clear that there is one final, perfect proof for a theorem. As such, one might encounter examples refuting the original statement or part of the suggested proof. In response to these refutations, the proof, the theorem and the definitions involved are updated. In that sense, definitions are sandwiched between prior intuitions about the concept to be defined and its role in the proof. For more details on the work of Lakatos, we refer to (Koetsier, 1991) and (Larvor, 1998).

We already saw that Lakatos' discussion was an inspiration for one of Ouvrier-Buffet's conceptions. However, she also described other aspects of the defining process. For instance, she described how the process of defining can be split up into four (perhaps overlapping) stages: in-action, zero, formalized and axiomatic. At the in-action stage, the mathematician is working with intuitions, ideas, examples and counterexamples in order to get an intuitive understanding of the concept of interest. In the "zero" stage, the mathematician has a first definition, a working definition that is the starting point of a series of improvements of the definition. At the formalized stage, there is a precise definition that can be used to communicate one's ideas and at the axiomatic stage, the definition has become part of a larger theory. Although these stages are inspired by interviews with mathematicians, Ouvrier-Buffet's results have a strong educational flavor and her theoretical framework originated from mathematics education. This makes these results less appropriate for the investigation of mathematical research practice. We discuss this in more detail in Sect. 2.4.

2.4 Use of empirical methods

Several empirical studies examined definitions in educational practice. The only empirical work that also addresses mathematical research practice, is in the work of Ouvrier-Buffet (2013, 2015). Her goal was to develop a framework for definitions, by which one can help students understand definitions and the process of defining. Her conceptions and the identification of different stages are a part of this framework. She conducted interviews with eight mathematicians "to enrich [her] model of the defining activity based upon three epistemological conceptions, and intrinsically to validate it" (Ouvrier-Buffet, 2015, p. 2216). For studying definitions in mathematical research practice, the reports on these interviews are to some extent lacking. First, the presentation of the interview results is interwoven with the validation of her educational framework. Therefore, it is not always clear which conclusions are derived from the educational framework and which are derived from the interviews. Second, when the results of the interviews are presented separately, this is done briefly without examples or descriptions. We mention several of her observations.

First, Ouvrier-Buffet identified various dimensions along which definitions can differ, namely:

- *the discrimination between the definitions one knows beforehand and the definitions one can deduce from other results;*
- *the distinction between the definitions which remain and will belong to the public domain and the local definitions which are used to shorten a talk.*

- *the working definitions: one starts from the intuition that one gets from objects and problems. With this kind of definitions, one can work with the mathematical objects, and the statement of these definitions can be put off.* (Ouvrier-Buffet, 2015, p. 2216)[6]

Second, she mentioned that mathematicians can have various goals associated with defining, namely: "to have a better understanding of a concept or a problem, to simplify, to generalize, to explore different linked frames or connected fields than the first one, [and] to communicate" (Ouvrier-Buffet, 2015, pp. 2216–2217). Third, she looked at the relation with proving and concluded that proofs dictate and that the definitions follow in the sense that definitions are adapted in order to make proofs work.

Before we go to the description of our methods, it is useful to look at experiments conducted in mathematics education with regard to defining and reflect on whether they can be of use for investigating mathematical research practice as well. A prime distinction in mathematics education is between the *concept image* and the *concept definition*. The concept image describes "the cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (Tall & Vinner, 1981, p. 152). This is contrasted with the concept definition which is "a form of words used to specify [a] concept" (Tall & Vinner, 1981, p. 152). Even though students are presented with a concept definition, the concept image need not concur with this definition. This is because the concept image is the result of a variety of factors not necessarily including the concept definition. As such, various experiments have been conducted on how to introduce mathematical definitions such that the concept image and concept definition are compatible. Often this is done by letting students "invent" the definition themselves (Koichu, 2012; Swinyard, 2011; De Villiers, 1998; Larsen & Zandieh, 2005).

There are two problematic factors that hamper using a similar approach to investigate how research mathematicians define. First, the definitions and concepts that are used in the aforementioned experiments, are all standard, elementary material. Research mathematicians, however, work on the frontier of mathematics. It is often not possible, for outsiders, to fully understand frontier-mathematics without going through extensive training. Second, in educational contexts, one can force the creation of a definition by giving hints and suggestions to the students. In that way, one can guarantee that after a session, the students have encountered the new definition. For mathematics research, it is difficult to schedule a moment where a mathematician invents or creates a definition.

In addition to experiments concerning the creation of definition, researchers have asked mathematics students or teachers what they expect from definitions and what features they value (Johnson et al., 2014; Zaslavsky & Shir, 2005). Furthermore, several examples of definitions are presented and the quality or acceptability of these definitions is discussed. In general, asking research mathematicians about their expectations of definitions is an insightful approach. Unfortunately, it is unfeasible

[6] Ouvrier-Buffet (2015) referred to these dimensions as different 'types' of definitions. Furthermore, she did not elaborate on these types of definitions.

for the interviewer to choose definitions to discuss for similar practical reasons as mentioned above concerning frontier-mathematics. As such, we conducted semi-structured interviews with research mathematicians to ask them how they define and how they think about definitions. It was expected that the respondents can then choose examples themselves and elaborate on these in a way that is meaningful and understandable for the interviewer[7].

3 Methods[8]

This article's results stem from an interview study into definitions and concepts in mathematical practice. Since the choice to address both notions in one interview study was motivated by the strong connection between the two notions, we discuss the relation between definitions and concepts in this section. Although the literature contains some dissensus as to the ontological status of definitions and concepts (see e.g. Cappelen and Plunkett, 2020; Gupta, 2019; Margolis and Laurence, 2019), the various accounts agree that concepts are often captured by definitions. Consider, for instance, Lakatos' (1976) *Proofs and Refutations*. As mentioned in Sect. 2, Lakatos described the process of proving through a rational construction in the form of a hypothetical dialogue between a teacher and his students. Over the course of the dialogue, mathematical definitions and concepts are discussed extensively. It is suggested that mathematicians often have an intended concept in mind and they attempt to capture this using a definition. However, if we zoom in, then there are actually two concepts at work. There is the concept as induced by the definition and the concept that is intended, i.e. that precedes the definition. When one has a definition, then one can look at the concept that it induces. When that concept coincides with the intended concept, then one has captured the intended concept. In that sense, one could argue that mathematicians try to create definitions whose induced concepts concur with the intended concept. In conclusion, one might summarize the relation between definitions and concepts as follows: *definitions are linguistic*[9]*entities that give rise to concepts, mental or otherwise and in doing so they can concur with or differ from pre-existing/intended concepts*[10]. Put more informally, concepts are the things themselves and definitions are ways to describe these things. The focus in this article is on definitions.

[7] The interviewer has a master's degree in mathematics.

[8] This methods section is largely identical to the methods section of the parallel article. Whereas this article focuses on definitions, the parallel article zooms in on mathematical concepts. To promote transparency, we have chosen to keep the methods section mostly identical and only change the parts that are different for these two articles. This is in line with the best practices concerning text-recycling as described in (Hall et al., 2021)

[9] The termed 'linguistic' is used in a tentative fashion. We mean to indicate that definitions are often written down or, more generally, communicated to *indicate* the meaning of a term, whereas concepts are the things to which these terms refer.

[10] This characterization is also used by Coumans (2021).

3.1 Data collection

We have interviewed[11] thirteen research mathematicians. Since the interviewer is Dutch, the preferred language of the interviews is Dutch. To this end, the research mathematicians were contacted via Dutch universities[12]. One of the participants was no longer employed at a Dutch university, but got the request via a newsletter. In the end, three of the interviews were conducted in English and the remaining in Dutch[13]. For the purposes of this study, we required that the respondents had already obtained their PhD and that they work as research mathematicians at a university. In the literature, there is no indication that mathematicians' areas of expertise or years of experience influence the way mathematicians think about or interact with definitions. Therefore, we did not select our sample on the basis of these factors. Doing so enabled us to observe possible influences of these factors on the participant's experience with definitions. To that end, we asked the participants what their area of expertise is and for how long they have been research mathematicians. Both these types of data have been displayed in more general categories to prevent identification of the individuals. More precisely, instead of the precise research topic, a more general term is given[14] and instead of the years of experience, a range is given. These data are presented in Table 1.

Table 1 The pseudonym of the participant, the area of expertise, and range indicating the years of experience

ID	Area of expertise	Years of experience (range)
RP01	Geometric analysis	[11,15]
RP02	Index theory	[11,15]
RP03	Algorithmic algebra & interdisciplinary research with lattices	[21,30]
RP04	Descriptive set theory	[31,40]
RP05	Calculus of variations	[5,10]
RP06	Mathematical logic	[21,30]
RP07	Tensor categories	[21,30]
RP08	Mathematical physics	[31,40]
RP09	Dynamical systems	[11,15]
RP10	Mathematical statistics	[41,50]
RP11	Mathematical physics and algebraic geometry	[5,10]
RP12	Non-commutative geometry	[16,20]
RP13	Number theory and geometry	[11,15]

[11] Also see (Brinkmann & Kvale, 2018).

[12] Representatives of four universities were contacted (e.g. secretaries or professors from mathematics institutes). These representatives were asked to forward the interviewer's request to their respective institutes.

[13] The analysis of the Dutch interviews was in Dutch. When we wanted to include a Dutch quote, we translated it into English.

[14] There exist various classifiers for mathematical disciplines. Some of them are very broad and others are very narrow. Hence, we chose to generate general terms to capture the participants' areas of expertise

The interviews were conducted via Zoom, and recorded, transcribed and pseudonomized per the informed consent protocol. The interviews lasted between 56 min and 1h53, where most lasted between 1h and 1h15. The interview guide can be found in the Appendix. There are some slight variations between the interviews with regard to the order and the precise formulation of interview questions as the interviews were semi-structured. The interviews were conducted and informally analyzed in sets of respectively four, four and five interviews. By doing the interviews in sets, we gained the flexibility to adjust the interview study if needed. By letting these groups consist of four or five interviews, we limited the influence of outliers.

Following the first four interviews, question 1 was added to the interview guide. The way the terms ‘definitions’ and ‘concepts’ were used in the first four interviews suggested that the usage of these terms need not always correspond to their theoretical meaning as described in the beginning of this section. To this end, the researchers decided to ask the participants to specify what they thought the relation between definitions and concepts was. This mismatch between the theoretical meaning of the terms and their usage in practice has implications for the analysis, as we will see in Sect. 3.2. Another noteworthy deviation from the interview guide is the way question 9 was presented. Whereas in its current formulation a concrete example is requested, the question was sometimes phrased more abstractly as: When you encounter a concept in the mathematical literature, what determines whether you spend time on this?

3.2 Analysis

The pseudonomized transcripts were analyzed using a thematic coding[15]. Coding was done in Atlas.ti 9 and transcription was performed with F4Transkript. Although both definitions and concepts were discussed during the interviews, the coding for these two notions was done separately. Here we describe the coding for the definition part.

After each cycle of respectively four, four and five interviews, the data were coded informally to inform the later interviews. Via this way, the lead researcher (VC) became acquainted with the data and the various themes involved. When all thirteen interviews were completed, the lead researcher (VC) started the final coding. The final coding process consisted of three steps. In the first step, the interviews were coded with eight general themes. After this initial, broad coding, the related quotes were coded in detail. These specific codes were then, in the third step, grouped and/or merged[16], based on commonalities (if relevant). This resulted in three-layered codes of the form ‘General theme – Group – Specific code’ or two-layered codes, in case no grouping was needed, like ‘General theme – Specific code’.

ourselves. By doing so, we could give a general indication of the area of expertise while protecting the anonymity of the interviewee.

[15] For more on coding, see (Boeije, 2010).

[16] This merging was based on discussions between the authors (VC and LC). In particular, LC also coded several representative interviews inductively and this coding was discussed as a form of researcher triangulation.

In the process, several of the general themes were also merged and we ended up with the following general themes[17]:

- Types – The identification of different types of or variations between definitions.
- Functions – The identification of different roles and functions of definitions.
- Quality – Remarks concerning what constitute good definitions.
- Process – Remarks concerning the process of defining.
- Naming – Remarks concerning the names of definitions and concepts.
- Occurrence – Remarks concerning how often mathematicians defined and the importance of definitions for mathematical practice.
- Creativity – Remarks on the role of creativity in defining.
- Interaction – Remarks on how mathematicians interact with definitions they encounter.

We conclude this methods section by discussing a result from the interviews that influenced the way we analyzed the interviews. As mentioned, the way the interviewees used the terms definition and concept did not always correspond to the aforementioned theoretical meanings. To this end, we asked RP05 through RP13 what they thought the relation between definitions and concepts is. In general, they discussed concepts as being more informal or intuitive than definitions. Definitions make concepts precise. Sometimes, definitions then refer to the precise formulation, but definitions can also refer to a technical interpretation of an intuitive concept, i.e. go beyond the wording. An example of the latter category is the physical concept of time. Although there is an intuitive and informal connotation, it has precise, technical operationalization in, for instance, general relativity theory and quantum mechanics. However, these operationalizations need not coincide and there can be debates as to which is the "right" operationalization.

When looking at how the research participants used these terms in the interviews, we see that they often concurred with the above specification. However, sometimes they used the terms 'definition' and 'concept' interchangeably. Consider, for instance, the following quote regarding the success of a particular notion.

RP06: Well, it turned out to be a more general concept that [...] provides structure in chaos. So that is a... A definition is somewhat of a pair of spectacles through which one can see. One can see the world from that perspective by using that concept, so to speak and using that, one can see different things [...].

While it is beyond the scope of this text to investigate whether mathematicians in practice use the terms 'definition' and 'concept' interchangeably, the quote shows that the meaning of these terms as described in the beginning of Sect. 3, is not as sharp in practice. This observation is relevant for this article. The way we see it, one concept can have multiple, equivalent definitions. In this article, we are interested in the particular definitions and not the underlying concept. The above observation shows that when an interviewee used the term 'definition' it can still refer to the concept it defines. To determine whether this is the case, the context in which these terms were used, was taken into account. For example, if a participant mentioned that the defini-

[17] Furthermore, there was a code for miscellaneous remarks.

tion of a group is a good definition because it is a structure that pops up in various situations, then this was interpreted as referring to groups themselves and not to the precise formulation.

4 Results

In this section, we present results[18] from our interviews concerning definitions and defining in mathematical practice. As mentioned in the introduction, definitions are multifaceted notions. Hence, this results section is structured by looking at various aspects of definitions in isolation. This entails that we first observe in Sect. 4.1 that interviewees identified variations between definitions. Hence, we describe various ways in which definitions can differ. Then we look at one particular dimension along which definitions can differ, namely their roles and functions. Although some of the roles we describe can have overlap, they embody different ways of thinking about definitions. Afterwards, we look at desirable features of definitions and the quality of definitions. Finally, we look at the process of creating definitions. In Sect. 5, the results concerning these aspects will be connected with both the literature and with each other.

4.1 Variations between definitions

Several of the interviewees commented that definitions can differ along various dimensions. We discuss four of these dimensions. First, definitions can differ with respect to the degree that it introduces a concept. Some definitions are primarily abbreviations in the sense that one can then use a single term or short expression to indicate a larger or more complex expression. Other definitions are more concerned with introducing a specific concept. For instance[19], RP01 remarked that "[s]ometimes it's just like a notational abbreviation if you like. A symbol that you would like to use instead of writing a long sentence. But sometimes it's more of like a concept, for example, for a space that has certain properties and that you need throughout your whole paper repeatedly that has some nice properties and for which you have like several examples and maybe that's the main object that you're studying. So then I make a definition for that space". RP05 gave examples of both types of definitions. An example of the abbreviation type is: "X is the space of all Lipschitz functions with Lipschitz constant less than 5", whereas the definition of a Lipschitz function is more concerned with introducing a concept.

Second, definitions can differ on the level of abstractness. Some mathematical concepts can be described in different ways, for instance, the notion of tensor product. One can define this concretely by specifying the set and the operations or one

[18] During the interviews, we investigated the lived experiences of several research mathematicians. Hence, the claims that we were able to collect were only of a personal nature. Hence, when a result describes how some mathematicians find property A important, this should not be interpreted as a claim about the majority of mathematicians.

[19] Throughout this results section, we provide various quotes. These quotes are not meant to convey some level of representativeness, but we included them only to illustrate certain perspectives.

can define this more abstractly via a categorical definition using universal properties. Similarly, one can define a groupoid as a small category in which all arrows are invertible or one can write down the specific axioms more concretely. These levels seem to suggest that definitions are either concrete or abstract. However, RP03 argued that it is a matter of degree.

Third, related to the abstractness of a definition is the level of detail. Definitions can be written down in various levels of detail. In a definition, one often uses previously introduced notions. The level of detail then refers to the ways these other notions are specified or worked out in that definition. The aforementioned example of groupoids defined as small categories in which all arrows are invertible, can be expanded by including what a small category is.

Fourth, Ouvrier-Buffet (2015) mentioned that mathematicians differentiate between definitions that are used in a local context and definitions that have become part of the public domain. In our interviews, we found evidence to support this claim. Some definitions are used in the context of one or a few articles; others become mainstream to a certain degree. For instance, RP09 remarked that she[20] often creates definitions, but that she "[does] not have the illusion that they are definitions that will be used by half the mathematical world in one hundred years. You see, it is really in the context of my research". Similarly, RP13 commented on one of the definitions she created that "this is a rather local definition in the sense that it is primarily relevant for our article and it is not something which we expect that it will be very useful like, for example, the definition of a vector space. In a way, that is a different kind of definition, that is really a foundation for a theory".

4.2 Roles of definitions

One way to differentiate between definitions consists of the dimensions described in the previous section. Another way to do this is by looking at the various roles and functions of definitions. Definitions turn out to have various roles and functions associated with them. However, this does not mean that all definitions fulfill these roles, but that some definitions do and others do not.

In Sect. 2, we already saw the role of definitions as expressed by the stipulative conception. According to this conception, definitions are abbreviations (Cellucci, 2018). This perspective on definitions was echoed in our interviews. One can use a definition to write a small expression to indicate a longer expression. For instance, in a calculation one might write a certain symbol abbreviating a longer equation. However, the length of an expression is not the only motivation for abbreviating. The prominence of that particular expression in an article is also important. As RP03 remarked, "when you notice that you're using a certain construction or tool very often, then at some point it is economical to say 'let's give it a name and be precise as to what it is' and then you can just use it".

In this subsection, we look at several others roles and functions that the interviewees attributed to definitions. Technically speaking, some of these roles and functions might exhibit some overlap. Our aim is not to give a classification of the different

[20] By default, we write 'she' when referring to a research participant, regardless of their gender.

roles and functions attributed to definitions, but to present various ways of thinking about definitions.

4.2.1 Capturing essences

One role of definitions is that they capture the essence of a concept. RP08 mentioned that "in defining a not yet rigorous concept, one has to try to capture the essence of something and to identify a mathematical structure that that concept already has inside itself". She mentioned that in mathematics this sometimes comes down to capturing the essence of a series of examples. As an example, she mentioned the definition of Hilbert spaces as capturing a particular set of spaces. When it comes to concepts in physics, one sometimes has a concept that is not yet mathematized, like force, and one has to mathematize it via a definition.

By creating a definition that captures a set of examples, one has two possible outcomes: either these examples are all possible examples of this definition or it turns out that there are other examples of the definition that were not yet recognized as such. For instance, RP08 referred to how there were various constructions of the real numbers. However, Hilbert's definition of the real numbers showed that all these constructions result in essentially the same structure. This is an example of the first case. As an example of the second case, she mentioned the classification of finite simple groups. Various finite simple groups were known already, but it turned out that there were previously undiscovered groups that were essentially different to those that were known[21].

Another example in which one tries to find the right concept through defining is given by RP09 who discussed trying to find a particular set of distributions that capture a particular natural phenomenon. In doing so, she combined various properties the distributions should have in order to represent the natural phenomenon. Once one has found an accurate representation of the natural phenomenon, one can predict the behavior of the system in question.

According to RP08, the process of capturing the essence is characterized by two notions: abstraction and precision. Through abstraction, one moves away from the set of examples and looks at them from a more general perspective. This essence should then be formalized in a mathematical language. Although nowadays this mathematical language is basically set theory, this is not essential to RP08. She remarked that there just has to be a decided-upon language in which one can formalize the essence.

4.2.2 Proving

Another role attributed to definitions is that they sometimes serve as foundations for proving. This is related to the function of capturing the essence of a concept. By capturing the essence of a concept, one is able to prove things regarding that concept. Nonetheless, there is a difference between thinking about definitions as foundations for proving and thinking about definitions as capturing the essence of a concept.

[21] This observation of definitions capturing a concept is about both concepts and definitions. As such, it is also discussed in the parallel article.

With respect to definitions as means for proving, RP03 mentioned that one of the most important things in mathematics is derivation using logical rules. However, to do so, she argued, one must have a certain foundation. Definitions provide that. They identify very clearly what is meant with a particular term, which allows mathematicians to make and derive precise statements concerning the concepts these definitions pin down.

RP07 related this to a notion of safety: “one wants to make sure that there are no contradictions”. She described how complex numbers were somewhat strange entities when they were first introduced without a proper definition. However, she claimed that when complex numbers were defined rigorously, one was able to see that there was nothing strange going on. She further described that working rigorously might not be that important for simple mathematics, however, “when mathematics becomes complicated, then you arrive in Hell’s kitchen”. RP07 observed that rigor is more important for definitions than for proofs, “for proofs one can still do a bit of hand waving, but if the definition is not right or is imprecise, then it becomes tricky”. She described being able to explain definitions to a proof checker as the ultimate test for the precision of a definition.

4.2.3 Communication and narratives

In addition to the previous roles, definitions have a strong communicatory value. In educational contexts, for instance, definitions can be used to convey ideas to students. In that sense, some definitions can be more suitable to generate intuition than others. More generally, when a researcher presents a new structure, she can use definitions to communicate to peers what she has in mind. Some of the participants described definitions as providing a solid basis for doing mathematics together.

The added benefit of using definitions becomes clear when looking at other disciplines where definitions are not used as much as in mathematics. RP03 described working in an interdisciplinary field and in that field, there is much confusion due to different meanings of terms. These differences originate from different backgrounds and cultural differences. For instance, sometimes, different terms are used to refer to the same notion and, sometimes, the same term has different meanings. RP03 described that for some terms, the meaning varies across countries. She further described how at a conference, she defined a notion in a precise way and that attendants of that conference did not use that definition, but relied on prior intuitions regarding the defined notion. This led to a misunderstanding regarding the core concept of her presentation. This suggests that different members of this interdisciplinary field, and hence, members of different disciplines, might have different conceptions of what a definition is and how one should work with definitions. RP03 continued to say that in mathematics there are fewer miscommunications due to the use of precise definitions.

A concrete example of how mathematicians use definitions to communicate came from RP09 who described discussing ‘solutions’ of a system of differential equations with a PhD student. At some point, RP09 intervened and asked the PhD student to elaborate on what the precise problem is: What are the precise conditions? She remarked that “if we are not clear about that, then we are not talking about the same thing and that just does not work”.

Another communicatory role of definitions concerns writing articles. When writing an article, one wants to convey a message. Definitions can help highlight which structures, objects or notions are important and which are not. It is argued that definitions are important for and interwoven with the narrative of an article. RP04 mentioned that “if you want to make an article, or tell something, […] then you make the definitions in such a fashion that it becomes a pretty story. Hence, choosing the definitions is a part of making the story”. To illustrate this, she referred to the first book of Euclid’s *Elements*. According to RP04, this is a story where the goal is to arrive at Pythogoras’ theorem. In comparison with literary stories, mathematical stories have more fixed criteria, but in the end one wants to make sure that there are no superfluous elements and that every remark has a function. In that sense, definitions are tools to help tell that story. Definitions, however, can also distort the story. For instance, RP09 mentioned that if one starts an article with a list of 28 definitions, then it is not clear what the main concepts of the article are. Hence, this particular usage of definitions is not beneficial to telling the story.

Another example of how narrative interacts with definitions was given by RP13. She described creating a definition on the basis of feedback from a referee. In her paper, she proved a lemma that introduced a particular number. Based on the remarks of referee, RP13 and her coauthors decided that it would be better to give that number a separate definition to write a clearer story and to, hopefully, give the number a more prominent place in the reader’s mental picture.

One might also look at narratives beyond the confounds of an article. Then definitions also have a spotlight function. This was described by RP04: “A definition can bring something to the foreground of which you think it is important. So, you encounter something and think: oh, this occurs more often, I will give this a name. And then [you] have pinned it down with that. You bring it under attention. That is an important thing”. This process of giving a definition to something which one thinks is important has a self-reinforcing effect. As RP04 remarked, by giving a name to a situation, one can try to raise awareness of a particular concept or phenomenon. However, after giving such a name to a concept, one can refer more easily to that concept, which makes it more accessible.

4.2.4 Understanding

In addition to helping other people understand what the definer is thinking of, definitions can also be used to promote a more technical interpretation of understanding[22]. The interviewees mentioned two important ways of how definitions promote understanding. The first is related to modularity[23]. Definitions allow mathematicians to bundle and group properties. By doing so, one can more easily oversee what is going on. RP06 formulated this by remarking that “one would like to make big problems,

[22] See (Inglis & Mejía-Ramos, 2021) for an interpretation of understanding related to mathematics.

[23] Modularity in mathematics, including modularity with respect to definitions, has been discussed by (Avigad, 2020).

big structures more manageable. So one wants to create order from chaos. Mathematics is [about] recognizing structures and the more you recognize, the more clearly you can see the connections and how to solve things. A definition helps you to group this or tackle them in a modular way". An explanation about why definitions can help one see more clearly was given by RP03. She argued that humans have difficulty stacking abstract concepts. She gave the example of recursion and on how many levels one can trace those, suggesting that deeper than three steps is difficult. She continued that "this means that if you think about abstract stuff where it becomes complicated and deep, then you have to create order in your mind by putting those things in a box so you don't have to think of and consider all the details if they are not needed for a certain line of thought".

The second way of providing understanding is that some definitions embody a certain perspective or way of thinking. The fruitfulness of these definitions then depends on the circumstances. RP11, for instance, described the notion of a scheme and that there are various ways to define those. However, "sometimes it is prettier to see it from a different perspective and then you can think about the functor of points. That is an equivalent definition […] the advantage is that one can generalize this better". Similarly, RP03 gave an example of describing permutation groups with bases and strong generating sets. She argued that this is a good definition and that it is not just a definition to describe something, but that it in fact "embodies an entire philosophy of how to look at permutation groups in an algorithmic fashion".

4.3 Quality and desirable properties

The third aspect of definitions we discuss concerns properties or features of definitions. In particular, we look at properties that make a definition "good". The interviewees mentioned several properties in this respect. For instance, they mentioned that definitions should be minimal, short, simple and powerful and that they should avoid redundancies. In addition to these properties, one might also interpret the remarks regarding the roles and functions of definitions as indicative of desirable properties of definitions. One can argue that good definitions are those that are successful in attaining their goals. As such, one might associate every identified role or function with a property. For instance, one might associate the communicatory role of definitions with the property of promoting communication.

In this subsection, however, we avoid the properties that are directly associated with roles and functions of definition. Instead, we want to highlight three alternative aspects related to the quality of definitions. First, based on the interviews, one might make the claim that good definitions are those that are suitable for a particular context. Second, there is a cluster of properties related to the accessibility of definitions. Definitions are better if they have features that contribute to its accessibility. Third, some of the interviewees used foundational factors to evaluate definitions.

4.3.1 Quality as alignment to the context

One might argue that the quality of definitions is highly related to its 'alignment to the context'. To illustrate what we mean by quality in terms of alignment to the context, recall the difference between abstract definitions and concrete ones. One might ask, given a concept with two equivalent definitions, one abstract and one concrete: Which one is preferable? The answer, it seems, is highly dependent on contextual factors. For instance, as RP02 described, categorical definitions in terms of universal properties "are probably good to prove general things, because it often yields short elegant proofs. […] So these abstract definitions are, I think, good for eventually giving short elegant proofs, but they are bad for generating intuition and a definition that is more concrete, is good for generating intuition". In that sense, which one is the better definition depends on what the mathematician in question wants to accomplish with the definition.

Similarly, RP03 described the influence of abstractness on the ability to calculate things. For instance, in the context of algorithms and representation theory, one is interested in calculating the matrices and, for instance, giving a basis for their tensor product. In that situation, the concrete definition is more suitable. However, when one is trying to make a more general claim about the representation of groups, then it is much more elegant to work with the universal properties. She further remarked that, "in principle, a definition is a way to talk about things, but one does not only want to talk about those things, but also address or even prove theorems. Then it depends on the context which aspect is more useful".

In addition to the context in terms of what goals one wants to attain, there is also the influence of the practitioners' background. In education, for instance, one aims to give a definition that is in line with the goals of a course. RP02 described, for instance, that she appreciated the definition of a topological space on the grounds of having significant experience with metric spaces and analysis. Furthermore, she gave the example of a colleague who introduced the notions of a set of n independent vectors and a spanning set of n vectors as injections and surjections, respectively, of $\mathbb{R}^n$ into that vector space. She commented that "that is indeed a funny definition, because it does not tell what's going on at all. It is a definition of the type 'probably useful for proving, but poor for generating intuition'".

The background of the practitioner also affects the appropriate level of detail for definitions. Some mathematicians preferred short definitions. However, the appropriate level of detail and the length of a definition are dependent on the mathematician's background. Consider, for instance, the notion of computational depth. RP06 mentioned that this concept was first introduced without a precise definition. However, over time, two equivalent definitions emerged. One was roughly half a page and the other a few lines. This makes the second definition preferable. However, the second one was defined in terms of computable functions and Kolmogorov complexity. If one is to spell these terms out, then that definition becomes lengthy as well. In that sense, the length of a definition depends on the notions a mathematician is already familiar with.

RP09 argued that in writing definitions, one takes the audience into account. In writing mathematical articles, "one is not addressing a random person, not even a random fellow mathematician, but you are often talking to a mathematician who already has a pretty good grasp what you are working on". As such, one does not have to spell out all the details. For introducing definitions to students, one might need to add more detail.

Hence, one might frame quality of definitions in terms of alignment with the context. If a definition supports the goals of a (group of) mathematician(s) and/or can be easily be expressed in the vocabulary of the mathematicians, then it is better than if it does not contribute to the goals and is difficult to interpret.

4.3.2 Quality of definitions and understanding

At the start of Sect. 4.3, we mentioned several properties of definitions like length and simplicity. Although these aspects can be interpreted as virtuous in and of themselves, one might also link these to understanding definitions. To be precise, definitions that make the underlying concept more accessible are preferred over definitions that do not (also see Lehet, 2021a).

In that sense, simple definitions are preferred over complicated ones, short definitions over lengthy ones, and intuitive definitions over opaque ones. Several of the respondents discussed the definition of a differential manifold in terms of atlases and transition maps. Although the concept of a manifold is a very valuable one, they expressed difficulty with getting familiar with the notion. RP02, for instance, mentioned that students have to spend several weeks to understand what it is about. Similarly, RP09 described that the definition of manifolds is filled with technical details, whereas the underlying idea is more easily understood.

Several of the participants voiced the importance of understanding. For instance, RP05 explained that she thinks in terms of images and she praised a definition for helping to generate an image in her mind. By contrast, she did not like an equivalent characterization because "the picture was way less clear". In a similar spirit, RP01 mentioned not liking a definition because it was difficult to understand what it means.

Hence, one might conclude that definitions that contribute to understanding or internalizing the underlying concept are preferable to those that do not.

4.3.3 Foundational preferences

In addition to the aforementioned properties associated with quality, several respondents evaluated definitions from a foundational perspective. RP04, for instance, preferred constructive definitions. For instance, she did not like the sequential definition of compactness for metric spaces[24] because, from a constructive point of view, the interval [0,1] is not compact according to that definition. Similarly, for the definition of prime number, there are (at least) two formulations. Prime numbers are numbers that have precisely two divisors, but one can also describe them by saying that prime numbers are those numbers that if they divide a product, they must divide one of its factors. She preferred the formulation in terms of having only two divisors because it is constructively acceptable. For a natural number, one can list all its divisors and then check whether it is prime of not. When described in terms of also dividing a factor when dividing a product, one quantifies over infinitely many products.

Another example of a foundational preference was given by RP07. As mentioned, one has the sequential definition for compactness on metric spaces, but one can also

[24] A metric space is compact if every sequence has a convergent subsequence.

use the open cover definition[25]. RP07 preferred the open cover definition because it is more intrinsic than the sequential one. The open cover definition does not refer to the natural numbers, whereas sequential compactness refers to sequences that are indexed by the natural numbers. In this sense, compactness in terms of nets, although perhaps less intuitive than sequences, is more intrinsic as it refers to all types of nets and not just the natural numbers. Similarly, RP07 expressed discontent with the tradition of introducing the sequence space of quadratically summable sequences instead of any set of functions from a set to the complex numbers with a finite quadratic norm. The second is a generalization of the first, but the first depends on the natural numbers, whereas the second is more intrinsic.

4.4 Creating definitions

The final aspects of definition we look at constitutes the process of creating definitions. To investigate this, the respondents were asked to give an example of a definition they created and elaborate on the process. Two respondents, RP08 and RP10, mentioned that they do not really create definitions themselves. RP10 explained that she was primarily involved in applying mathematics and when doing that, there is usually no need to introduce new notions. RP08 mentioned that although she had contributed to the development of an important definition, she is more engaged in trying to fully understand and appreciate existing definitions. The other respondents indicated that they create definitions themselves and that it is an important part of their work. However, RP03 is engaged in two fields. She mentioned that for one of these fields, she creates definitions in all of her articles. However, in the other field, new definitions are rare. In that field, one usually only works with established notions. This suggests that the importance of creating and presenting definitions differs across mathematical disciplines. In this section, we highlight several important aspects of creating definitions.

4.4.1 Stages

The respondents described several ways in which mathematicians create or improve definitions. One trajectory we observed contains the following stages. In the first stage, one is trying to understand the concept itself. For instance, when trying to prove a theorem of the form A implies B, one is trying to find the domain of validity. Stage two begins when one has obtained a reasonable understanding of what the concept should be. In stage two, the mathematician tries to describe the concept as precisely as possible. The result of this stage is a definition. Then, in stage three, the definition is adjusted to the audience. Figure 1 presents a schematic of these stages.

As an example of the refining stage, RP05 described that when she is writing an article, she is further refining the definitions "and maybe changing them in order to have everything clear and as readable as possible". Another example was given by RP03, who works in an interdisciplinary field. She remarked that when she has found a precise formulation, she thinks about how to reformulate the definition in

[25] A metric space or topological space is compact if every open cover has a finite subcover.

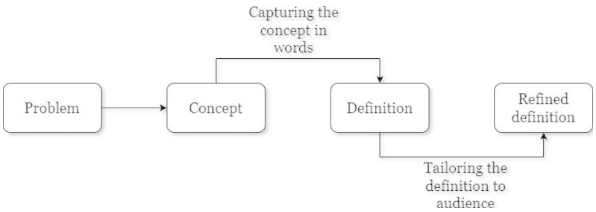

Fig. 1 One way of defining. For a particular problem context, a mathematician determines the needed concept. This concept is then captured in words, after which the formulation is tailored to the audience

such a way that scientists from other disciplines can also understand it. Therefore, she described two translation stages, one from the concept to the precise definition and one from the precise definition to a definition in terms that the intended audience understands. This situation is different in degree from the one described by RP09, who described that she takes the audience into account when deciding on the level of detail her definitions should have. In that situation, the intended audience consists of mathematicians who are familiar with the type of work that is described in her article. As such, for RP09, the second translation stage was less prominent.

This process of taking the audience into account seems to be mostly one-sided in the sense that the authors try to estimate what the definition should look like in order for it to be suitable for the audience. Alternatively, one might expect that the definition is formed in interaction with the intended audience. In our interviews, we observed little examples that this was the case. In Sect. 4.2.3, we described how RP03 experienced that one of her definitions was not fully understood by the audience at a presentation. This is one case of feedback regarding the definition. Furthermore, also mentioned in Sect. 4.2.3, RP13 described how she created a definition in response to feedback from a referee. This is an example of how social contacts influence the creation of definitions. In general, however, the interviewees did not mention these influences. This might suggest that definitions are usually not created through a dialogue with the intended audience, but it might also be the case that these interactions played an important role, but that the respondents for some reason did not mention this. One can imagine that interactions regarding definitions can take place on conferences, as is suggested by Ouvrier-Buffet (2015). She observed that according to the mathematicians she interviewed, “a definition will evolve when [mathematicians] will communicate their results”. This was not expressed directly by our respondents, although RP01 mentioned the interactions during conferences as tools for assessing the value of one’s work.

An alternative way of how definitions might be created, is by improving existing definitions. RP13 gave the example of generalizations of the notion of topological space. One existing generalization requires the use of so-called large cardinals. RP13 felt that these assumptions regarding large cardinals were too strong for the context. In response to this dissatisfaction, she started looking for alternative, more natural definitions.

The way a concept is created is sometimes also combined with the process of capturing it in words. In Sect. 4.2.1, we described how RP09, together with her collaborators, tried to find the right type of distribution for a natural phenomenon. This concept was specified by writing down particular properties this distribution should have. Then they assessed whether it corresponded to the natural phenomenon of interest. In that sense, finding the precise formulation and finding the right concept are sometimes intertwined activities.

4.4.2 Presentation

When it comes to the way mathematicians present definitions in articles, we make two observations. First, the place of the definition in the article depends on the story of the article. As mentioned in Sect. 4.2.3, RP09 referred to the situation where an article starts with 28 definitions. This is undesirable, as it does not provide a clear narrative structure. She further argued that if one can reverse the order of the sentences, one should have put more thought into the text's structure.

With respect to the placement of definitions in article, we observed two patterns. One structural pattern is that definitions sometimes precede theorems in which these definitions are used. For instance, RP01 mentioned introducing a property and then immediately proving a theorem concerning all objects that have that property. Similarly, as mentioned in Sect. 4.2.1, RP08 described how sometimes definitions capture a series of examples. Then there are two possibilities. "[T]he first possibility is that the theorem that follows the definition is that the examples are, in fact, all examples, up to isomorphism". The other possibility is that the definition captures examples that were not included in the original set of examples.

The other structural pattern is that presenting a definition is sometimes accompanied by discussing examples. For instance, RP02 mentioned that she prefers concrete definitions over abstract definitions. This motivated her to give examples and connections with other definitions to generate some intuition.

Second, mathematicians try to convince the reader of the value of their work. This can affect the way a definition is presented. RP02 gave an example that demonstrates this. She described solving a problem of the form $A=B$ by introducing a concept C and showing that $A=C$ and $B=C$. However, this concept C depended on several choices. RP02 showed that regardless of these choices, the equalities hold. However, she then found a formulation of C that did not depend on choices. As such, she decided to publish these results in two parts. Regarding this decision, she remarked that "that is how you try to sell it, because you also have to sell your work. First, our message was A is interesting and $A=B$ is interesting [...] but then we wrote down the prettier definition of C and then it turned out that [C itself] could be useful already. So that is how we explained it, like, this is the notion C. It is natural given the analogies with other definitions and it equals B. The conclusion that $A=C$ was provided as an illustration and [evidence that A] is also related to notion C". In conclusion, the presentation of definitions in articles is influenced by both the mathematical story one wants to tell as well as the desire to convince the reader of the value of this mathematical story.

4.4.3 Naming

Another important factor of creating definitions is coming up with a suitable name. From our interviews, we observed two patterns with regard to naming a concept. First, there is a tradition that names are used to give credit to the inventor of a definition. However, there are several examples where this is not the case. For instance, RP07 remarked that a concept is named after her. However, she did not come up with the concept, but her supervisor did. Although RP07 did make the concept precise, the origin for the concept lies with her supervisor. Another well-known example is given by RP08 who discussed Hilbert spaces. She remarked that it was, in fact, Von Neumann who created this notion and that it should actually be called Von Neumann spaces.

Second, names are used to give an indication of what the respective concepts entail. RP01 remarked that one does not always remember all the details of a given notion. That is why it is important to give a name that can “convey what is going on”. She concluded that names should describe the essence and should not be misleading.

RP06 stressed the importance of coming up with a good name as she reflected on how she named a concept she introduced. She mentioned that various suitable names turned out to be in use already. Hence, she spent an afternoon in the library, consulting various dictionaries to find a term that “captures the notion” and is pleasant to use. Over the following years, the introduced concept was picked up by the community. Regarding this, she mentioned that she thinks that “a part of the success of [RP06’s] definition of [name concept] is that it is a good word. It aligns with the intuition, it describes what it does and it has a ring to it. So that is important”. In contrast, she disapproves of the practice of using easy, general terms to create new notions. She referred to the practice of adding adjectives like good, strong, or weak, to an existing name. For instance, she mentioned a hypothetical case where one is looking at a certain class of groups and decides to call these ‘good groups’.

5 Conclusion and discussion

In this article, we presented results from an interview study on definitions in practice. In particular, we looked at differences between definitions, roles and functions of definitions, desirable properties of definitions, and the process of creating definitions. Our results suggest that there are various ways to differentiate between definitions. For instance, one can differentiate definitions that merely abbreviate from those that introduce a meaningful concept, abstract from concrete definitions, detailed definitions from less detailed definitions, and definitions for a local context from definitions for a broader context.

Furthermore, our results suggest that definitions can fulfill several roles. They can be used as means to abbreviate, means to capture the essence of a concept, means to promote rigor and proving, means to promote communication, and as means to promote understanding. From our interviews, we extracted several insights regarding ‘good’ definitions. Definitions that are aligned with the context are preferable to those that are not, definitions that promote accessibility are preferred to those that do not, and sometimes definitions are deemed better because of foundational features. Finally, when it comes to creating definitions, we observed a pattern consisting of three stages (finding the right concept, capturing the concept in words, adjusting the definition to the audience). Furthermore, we saw that the narrative of an article as well as promoting one's work affects the way definitions are presented and that naming is an important part of creating a definition.

In this section, we reflect on how our results relate to the literature and to each other, and on the methodological limitations of our interviews. We conclude by presenting lines for future investigation.

5.1 Relation with the literature and synthesis

In Sect. 4, we presented the results from the interview study. These results concern aspects of definitions that were also discussed in the literature. Furthermore, these aspects embody different perspectives by which one can assess the notion of definitions in practice. As such, we first evaluate how these results relate to the literature. Then we combine and compare these various perspectives to see what these results say about definitions in practice, in general.

5.1.1 Variety among definitions

Ouvrier-Buffet (2015) discussed several dimensions along which definitions can differ. For instance, she described a distinction between local and global definitions as "the distinction between the definitions which remain and will belong to the public domain and the local definitions which are used to shorten a talk" (Ouvrier-Buffet, 2015, p. 2216). We found evidence for the difference between local and global definitions, although we would split her distinction into two further dimensions. On the one hand, there are definitions that are merely abbreviations, whereas others are constitutive of more meaningful concepts. On the other, there are definitions that are used in the context of one or a few articles, whereas others are used more broadly. Ouvrier-Buffet combined these two dimensions and it may be argued that definitions in the context of one article, more often comprise abbreviations than the introduction of a meaningful concept, but this is not necessarily so.

Furthermore, we also observed two dimensions that were not mentioned by Ouvrier-Buffet, namely a definition's level of abstractness and level of detail.

5.1.2 Roles and functions of definitions

As to the roles and functions of definitions, in the literature we saw the functions of abbreviating, problem solving and the three conceptions by Ouvrier-Buffet (2013), which we might summarize as classifying, problem solving and selecting theories. Except for selecting theories, our interviewees mentioned all of these roles. Some respondents related classifying to definitions. For instance, RP08 described two situations of how a definition can capture all varieties of a particular notion. This might be interpreted as using definitions to classify. However, one might argue that in these cases, the function of definitions is not so much classifying itself, but capturing the essence of a particular concept.

For problem solving there seems to be a nuance as to the precise contribution of definitions. For instance, in Ouvrier-Buffet's Lakatosian conception, defining is approached as finding the domain of validity for a particular theorem. In that sense, the role of the definition is indicating the right concept for the theorem. However, from our interviews, we found that the precision associated with definitions allows mathematicians to write precise mathematical proofs. Furthermore, some definitions provide a fruitful way of thinking about a concept. This enables mathematicians to prove theorems as well. These are different interpretations of using definitions for problem solving.

In addition, our results corroborate Ouvrier-Buffet's conclusion that mathematicians define "to have a better understanding of a concept or a problem, to simplify, to generalize, to explore different linked frames or connected fields than the first one, [and] to communicate" (Ouvrier-Buffet, 2015, pp. 2216–2217). Finally, we quoted a list of functions definitions can have, as described in the mathematics education literature. All of these items were mentioned in the interviews. This suggests that there might be some similarity between the way research mathematicians and mathematics students interact with or think about definitions.

5.1.3 Features of definitions

Regarding the features of definitions, several of the imperative features described by Zaslavsky and Shir (2005) were not mentioned by research participants. For instance, it was not mentioned that definitions should be hierarchical or that they should be invariant under change of representation. However, the respondents did mention that definitions should be unambiguous and minimal. A possible explanation for why some of these imperative features were not mentioned is that they are interwoven with the notion of definition. As an example, mathematicians often associate proofs with arguments that demonstrate the validity of the associated theorem. If an argument does not justify the theorem, then we would not call it a proof. In that sense, if one is asked about what properties a good proof should have, one might forget to mention the features that are inherent to proofs. Similarly, it might have been the case that the respondents forgot to mention some of the imperative features of definitions.

Some respondents also described optional features as discussed in Sect. 2.2. RP02 gave the example of a definition that was preferred over another because it was natural in the sense that it did not depend on particular choices. Furthermore, RP07 described preferring particular formulations because they were intrinsic, which according to her is what naturalness entails. Whether a particular definition is useful or fruitful was also mentioned as a sign of quality. Usefulness then also refers to providing useful perspectives. As discussed in Sect. 4.2.3, definitions can embody a particular way of thinking and sometimes these ways of thinking can be fruitful. This relates to the third optional feature we discussed, explanatory value. Some of the respondents preferred definitions because these gave intuitions as to what the underlying concept was. This might promote understanding.

5.1.4 Creating definitions

With regard to the process of defining, we find similarities between the stages described in Sect. 4.3 and the phases presented by Ouvrier-Buffet (2015). The "in-action" phase loosely corresponds to the first stage described in Sect. 4.3, where the mathematician tries to determine the right concept. The remaining three phases mentioned by Ouvrier-Buffet loosely correspond to stages two and three. These are the stages where the concept is translated into a definition (stage two) and the definition is tailored to the audience (stage three). Furthermore, our results suggest that finding a good name is an important part of defining. This is not mentioned by Ouvrier-Buffet (2015).

If we compare our findings concerning the process of defining with Lakatos' *Proofs and Refutations* (1976), we find that whereas Lakatos emphasized the back-and-forth between proofs and counterexamples, and in effect, the back-and-forth between definitions/concepts and counterexamples, our respondents did not mention this prominently. Some described it in a way that is reminiscent of Lakatos' work. RP04 for instance described the method of finding the right concept as trial-and-error. She described that "one just tries it and then you prove things about it and then at some point you think: actually, this is not really elegant, I can better change the definition a little and then it will go much easier". However, several mathematicians described the process of defining as a linear process, starting with the problem, finding a concept, formulating a definition and then refining the definition for communication. This is not to say that the process actually is linear, only that some mathematicians perceived it to be linear. In fact, these types of conclusions are limitations of the method, as we will discuss in Sect. 5.2.

5.1.5 Synthesis

Whereas the previous results all zoomed in on particular aspects of definitions, like the creation of definitions or the role of a definition, this interview study has also contributed to the notion of definitions in practice in general. In this respect, we make the following two observations.

First, our results show that the set of definitions in practice is a heterogeneous one. We saw that definitions can have various roles, various dimensions on which they can differ, and various ways in which they can be created. This suggests that the term 'definition in practice' may be construed too broadly. Our findings suggest that the study of definitions in practice might benefit from a typology of different types of definitions and the corresponding expectations of those types. This heterogeneity is amplified by the fact that, as the results from this study suggest, definitions seem to be context-dependent. Not only the type of definition determines its qualities, but also the context in which a definition is given. For instance, some definitions might be useful from the point of view of algebraic topology, but less so from the perspective of combinatorics. This suggests that characterizing 'definitions in practice' is a nuanced endeavor.

The results of our interview study give rise to a preliminary typology on the basis of a definition's role. For instance, one might talk of abbreviatory definitions, explanatory definitions, essence capturing definitions, etc. All these types of definitions come with certain expectations. Hence, this article would be a good starting point for developing a typology of definitions.

Second, what this interview study also showed, is that definitions are not solely the result of mathematical factors. In the process of creating definitions, social factors and contingent factors also contribute to the development of definitions. Partly, this is explained by the fact that definitions can have a strong communicatory role as well. As such, the precise formulation of a definition is the result of various considerations, both strictly mathematical and audience-focused in nature. The relevance of audiences in mathematical practice is also discussed by Ashton (2021). She argues that mathematicians write proofs with their ‘universal’ audience in mind. Similar consideration might also hold for how mathematicians define. However, our interviews suggest that the audience they have in mind is more concrete than a universal audience.

Another example that demonstrates that definitions are not just the result of mathematical factors concerns the name of definitions. Our interviews suggest that, in addition to finding the right definition, some mathematicians also try to find the right name for the concept. In the search for the name considerations like ‘does the name capture what we intend with the concept’ and ‘does the name sound good’ also influence the process.

5.2 Methodological reflections

In the previous section, we discussed the diversity of definitions. However, we are not able to make quantitative statements regarding the occurrence of various types of definitions. Interview studies are typical examples of qualitative research. The aim of this type of research is not making generalizable, statistically relevant claims. This research is primarily aimed at theory building. Nonetheless, one might argue that there is still some epistemic weight to these interviews in light of methodological triangulation (cf. Löwe and Van Kerkhove, 2019).

As the use of interviews in the philosophy of mathematical practice is not standard practice yet, our study is relevant for other philosophers of mathematical practice who intend to use interviews. In general, we think of interviews as an effective method to inquire into definitions in practice through the experience of the practitioners. However, there are some limitations. Most importantly, interviews are not suited to discuss mathematical content in depth. It takes time to become acquainted with the concepts and definitions of particular mathematical research topics. As such, it is unlikely that an interviewer will understand the details of the respondents’ research over the course of an interview. This entails that when discussing a particular example, the interviewee has to translate her work in a way such that the interviewer can get an idea of what is going on. In doing so, the interviewee implicitly makes a choice of which aspects of her research are important. This is not problematic for interviews per se, but it limits the interviewer’s options to take a critical stance to the perspective of the interviewee.

A further complication arises when assessing the process of defining. When the interviewee explains which factors affect the way she defines, this is a rationalization of the process. If it were possible to conduct an observation study, one might obtain a different picture. This suggests that the information the researcher obtains is secondary in a sense: it has been interpreted by the respondent. As such, what we are measuring is not how mathematicians actually define, but how they experience their defining, which is important in its own right and at least indicative of the defining process itself.

5.3 Impact and future work

The main goal of this study is to provide empirical results with respect to definitions in practice. In particular, we aim to enable philosophers of mathematical practice to gauge their intuitions regarding definitions in practice. As such, the results on the various aspects of definitions in practice constitute a sort of window into mathematical practice. Hence, the outcomes can contribute to various discussions in the philosophy of mathematical practice and give rise to various lines for further inquiry.

First, a prominent topic in the philosophy of mathematical practice is explanation and understanding (D'Alessandro, 2019; Inglis & Mejía-Ramos, 2021). The scope of this debate has for a long time focused on mathematical proof. However, recently it has been suggested that other artefacts can also be explanatory (D'Alessandro, 2017). Lehet (2021a, b) discusses the notion of explanatory definitions and argues that explanatory definitions are definitions that generate understanding. The results in this article suggest various phenomena that are related to understanding in mathematics, like definitions as vehicles of modularity and definitions as embodiments of useful perspectives. This inspired the work of Coumans et al. (2022) who developed an account of what it means for definitions to be explanatory using these aforementioned useful perspectives.

Second, in this article we observed several structural patterns regarding the way definitions are presented. Andersen et al. (2021) has conducted an investigation of how mathematicians write research papers. Identifying these structural patterns contributes to this investigation. Furthermore, one can investigate these patterns by looking at mathematical publications using corpus analysis. In this article, it was mentioned that some mathematicians present examples to illuminate definitions. Investigating a variety of examples following definitions in mathematical articles, might shed light on what the role of these examples is and more generally, how definitions are embedded in mathematical publications. This helps us better understand the practices of writing mathematical articles.

Third, an interesting topic in the philosophy of mathematical practice is the influence of mathematical cultures (Larvor, 2016). RP08 and RP10 mentioned that they do not create definitions themselves whereas others mentioned that they define in all of their articles. In contrast to the other respondents, RP10 mentioned that she was mostly concerned with applying mathematics. This suggests that there are differences in the way mathematicians from different disciplines deal with definitions. Studying how a mathematician's area of expertise influences their perception of and interaction with definitions, might contribute to understanding the influence of mathematical cultures on mathematical practices.

In conclusion, definitions in mathematical research practice constitutes an interesting research topic. Via this article we not only hope to have contributed to understanding definitions in practice, but also to have provided an opportunity for philosophers of mathematical practice to gauge their intuitions regarding definitions in practice and a useful starting point for further research into definitions in practice.

Appendix – interview guide

Introduction of the interview study.

Asking what the participants' area of expertise and years of experience are.

1. What would you say, is the relation between 'definitions' and 'concepts'? To what extent is providing/creating definitions (either for new concepts or already existing concepts) a part of your activities as a mathematician?
2. Think of the last definition that you made. Why did you make that definition?
3. Please describe the process of providing/creating that definition. What factors steered this process?
4. What would you say is an example of a good definition? Why do you think this is a good definition?
5. What would you say is an example of a bad definition? Why do you think this is a bad definition?
6. What, according to you, is the purpose or function of definitions?
7. When there are multiple equivalent ways of defining a concept, do you sometimes prefer one of these definitions to the others? If so, please give an example and explain your preference. If not, why not?
8. What is a concept that you investigate? How did you come to study that concept or why do you study that concept?
9. What is the most recent concept that you encountered in your work? What determined whether you would spend time to investigate that concept?
10. Would you say that some concepts are more valuable than others?
11. Can you give an example of a concept that you think is valuable? What makes this valuable?
12. Can you give an example of a concept that you think is not valuable? What makes this concept not valuable? The interview concludes with a presentation of Corfield's Snook example[26] and by asking for the participant's thoughts on that example.

Acknowledgements We are grateful to Serge Horbach for helpful comments on earlier drafts of this article.

Authors' contribution Vincent Coumans developed the interview guide, conducted the interviews, coded and analyzed the interviews, wrote the first version of this article and contributed to writing the final version.

Luca Consoli coded three interviews for researcher triangulation, contributed to writing the final version of this article and provided critical feedback on products by Vincent Coumans.

[26] Corfield describes a hypothetical case of the introduction of a mathematical concept named Snook and its possible acceptance by the mathematical community. He writes: "If I define a snook to be a set with three binary, one tertiary and a couple of quaternary operations, satisfying this, that and the other equation, I may be able to demonstrate with unobjectionable logic that all finite snooks possess a certain property, and then proceed to develop snook theory right up to noetherian centralizing snook extensions. But, unless I am extraordinarily fortunate and find powerful links to other areas of mathematics, mathematicians will not think my work worth a jot. By contrast, my articles may well be in demand if I contribute to the understanding of Hopf algebras, perhaps via noetherian centralizing Hopf algebra extensions" (Corfield, 2003, p. 11).

Declarations

The authors declare that they have no conflict of interest.

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